D 91508 (Pages : 3) Name..................................

Reg. No..............................

# THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

M.Sc. Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions from this part.*

*Each question carries* 2 *marks.*

1. Prove or disprove: A continuous bijection from a compact space onto a Haus-dorff space is a Homeomorphism.
2. Prove that the unit circle $S^1$ is compact.
3. Give an example of a space which is connected but not locally connected .
4. True or false : Local connectedness is productive. Justify your answer.
5. Prove that in a Hausdorff space limits of all nets are unique.
6. Let S be a family of subsets of a set X with the finite intersection property. Show that there exists a filter on X having S as a sub-base.
7. Prove that every totally bounded metric space is bounded.
8. Prove that every complete metric space with no isolated points is uncountable.

$(8 \times 2 = 16)$

## Part B

*Answer any* **four** *questions.*

*Each question carries* 4 *marks.*

9. If for every closed subset A of a topological space X has the property that every continuous real valued function on A has a continuous extension to X, then show that X is normal.
10. Prove that every quotient space of a locally connected space is locally connected.

Turn over

11. Let A be a subset of a space X and let $x \in X$. Show that $x \in \bar{A}$ if and only if there exists a net in A which converges to $x$ in X.

12. Let $\mathcal{B}$ be a family of non-empty subsets of a set X. Prove that there exists a filter on X having $\mathcal{B}$ as a base if and only if $\mathcal{B}$ has the property that for any $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$.

13. Let S be any set and let Y be the set of all bounded functions from S into X. For $f, g \in Y$, define $e(f, g) = sup \{d(f(s), g(s)) : s \in S\}$. Prove that the metric $d$ is complete if the metric $e$ is complete.

14. Prove that an open subspace of a metrically topologically complete space is metrically topologically complete.

(4 × 4 = 16)

## Part C

*Answer* **either** A **or** B *of each question.*
*Each question carries* 12 *marks.*

### Unit I

15. (A) (a) Prove that every regular, Lindeloff space is normal.

(b) State and prove Wallace's theorem.

(B) State and prove Tietze extension theorem.

### Unit II

16. (A) (a) Let Y be a separable space and let I = [0, 1). Show that the product space $Y^I$ is separable.

(b) Show that a product space is locally connected if and only if each co-ordinate space is locally connected and all except finitely many of them are connected.

(B) (a) Prove that a product of spaces is connected if and only if each co-ordinate space is connected.

(b) Prove that a subset C is a path component of a space X if and only if C is a maximal subset of X w.r.t. the property of being path connected.

## Unit III

17. (A) (a) Let $S : D \to X$ be a net in a topological space and let $x \in X$. Show that $x$ is a cluster point of S if and only if there exists a subnet of S which converges to $x$ in X.

(b) Let X be the topological product of an indexed family of spaces $\{X_i : i \in I\}$. Let $\mathscr{F}$ be a filter on X and $x \in X$. If for each $i \in I$, the filter $\pi_i(\mathscr{F})$ converges to $\pi_i(x)$ in $X_i$, then show that the filter $\mathscr{F}$ converges to $x$ in X.

(B) (a) Prove that a topological space is Hausdorff if and only if no filter can converge to more than one point in it.

(b) Let X, Y be topological spaces, $x \in X$ and $f : X \to Y$ a function. Show that $f$ is continuous at $x$ if and only if whenever a net S converges to $x$ in X, the net $f \circ S$ converges to $f(x)$ in Y.

## Unit IV

18. (A) (a) Prove that every contraction of a complete metric space into itself has a unique fixed point.

(b) Prove that every compact metric space is complete. Is the converse true ? Justify.

(B) (a) Prove that in a complete metric space, a subset of first category cannot have any interior points.

(b) Let A be a subset of a metric space $(X, d)$. Show that if A is totally bounded w.r.t d, then for every $\varepsilon > 0$, A can be covered by finitely many open balls with centres in A and of radii less than $\varepsilon$ each.

$(4 \times 12 = 48)$

**D 91506** **(Pages : 3)** **Name.......................................**

**Reg. No...................................**

# THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

M.Sc. Mathematics

MAT 3C 12—FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries 2 marks.*

1. Define a complete metric space. Show that $\mathbb{R}$ with usual metric is complete.

2. For $1 \le p < r < \infty$, show that $\| x \|_r \le \| x \|_p$ and $l^p \subset l^r$.

3. Let E be a non-empty convex subset of a normed space X over $\mathbb{K}$. If $E^0 \neq \theta$, and $b$ belongs to the boundary of E in X, then prove that there is a non-zero $f \in X'$ such that $\text{Re}\, f(x) \le \text{Re}\, f(b)$ for all $x \in \overline{E}$.

4. If X is a finite dimensional normed space, prove that any linear map from X to a normed space Y is continuous.

5. Prove that in a Banach space X, every absolutely summable series is summable.

6. Let X be a normed space over $\mathbb{K}$. If $0 \neq a \in X$, show that there is some $f \in X'$ such that $f(a) = \| a \|$ and $\| f \| = 1$.

7. State Bounded Inverse Theorem. Show by an example that completeness of the spaces cannot be omitted.

8. Define a projection map on a normed space X. Give an example.

(8 × 2 = 16 marks)

## Part B

*Answer any* **four** *questions.*
*Each question carries 4 marks.*

9. Show that $l^{\infty}$ with the supremum metric is complete.

10. Let X be a normed space, Y be a closed subspace of X and $Y \neq X$. Let $r$ be a real number such that $0 < r < 1$. Prove that there is some $x_r \in X$ with $\| x_r \| = 1$ and $r \leq \text{dist}(x_r, Y) \leq 1$.

11. Define operator norm in $BL(X, Y)$ and prove that it is a norm.

12. Show by an example that, a linear map on a linear space X may be continuous with respect to some norm on X but discontinuous with respect to another norm on X.

13. Consider $Z = \{ (x(1), x(2)) \in X : x(1) = x(2) \}$, and define $g \in Z'$ by $g(x(1), x(2)) = x(1)$. Find any *two* Hahn Banach extensions of $g$.

14. Let $X = Y = C_{\infty}$ with the norm $\| \|_{\infty}$. For $x = (x(1), x(2), \ldots\ldots) \in X$, define $F(x)(j) = jx(j), j = 1, 2, 3\ldots\ldots$ Prove that F is a closed map but it is not continuous.

(4 × 4 = 16 marks)

## Part C

*Answer* **either** *A* **or** *B of each question.*
*Each question carries 12 marks.*

15. A (a) State and prove Baire's theorem for metric spaces.

(b) For $1 \leq p < \infty$, show that the metric space $l^p$ is complete.

B (a) Let E be a measurable subset of $\mathbb{R}$. For $1 \leq p < \infty$, show that the set of all simple measurable functions on E which are zero outside subsets of finite measure is dense in $L^p(E)$.

(b) Let Y be a finite dimensional subspace of a normed space X. Prove that Y is complete and hence closed in X.

16. A (a) Show that every bijective linear map from a finite dimensional normed space X to a normed space Y is a homeomorphism. Also prove that all norms on X are equivalent and X is complete in each.

(b) There exists a discontinuous linear map on an infinite dimensional normed linear space X. Prove or disprove.

B (a) Let X be a normed space over $\mathbb{K}$, E be a non-empty open convex subset of X and Y be a subspace of X such that $E \cap Y = \theta$. Prove that there exists an $f \in X'$ such that $f(x) = 0$ for every $x \in Y$ but $\operatorname{Re} f(x) \neq 0$ for every $x \in E$.

(b) Let $M = (K_{ij})$ be an infinite matrix with scalar entries such that $\sup\left\{\sum_{j=1}^{\infty} |K_{ij}| : i = 1, 2, 3 \ldots\ldots\right\} < \infty$. For $x \in l^{\infty}$, let $M(x) \in l^{\infty}$ be defined by $(Mx)(i) = \sum_{j=1}^{\infty} K_{ij}\, x(j)$. Show that M is a continuous linear map.

17. A (a) Let X and Y be normed spaces and $X \neq \{0\}$. Prove that $BL(X, Y)$ is a Banach space in the operator norm if and only if Y is a Banach space.

(b) State and prove Hahn Banach extension theorem.

B (a) Let X be normed space and Y be a closed subspace of X. Prove that X is a Banach space if and only if Y and X/Y are Banach spaces in the induced norm and quotient norm, respectively.

(b) If Y is a proper dense subspace of a Banach space X, then prove that Y is not a Banach space in the induced norm.

18. A State and prove, closed graph theorem.

B (a) Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Prove that F is continuous and open.

(b) Give an example to show that open mapping theorem may not hold if the normed spaces X and/or Y are not complete.

(4 × 12 = 48 marks)

**D 91505** (Pages : 3) Name..........................................

Reg. No..................................

# THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

M.Sc. Mathematics

MAT 3C 11—COMPLEX ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries 2 marks.*

1. Find the image of $\left\{z : |\operatorname{Im} z| < \pi/2\right\}$ under the exponential function.

2. Define Möbius transformation. Discuss the fixed points of a Möbius transformation.

3. Evaluate $\int_\gamma \frac{e^{iz}}{z^2}\, dz, \gamma(t) = e^{it}, 0 \le t \le 2\pi.$

4. Find the power series expansion of $\sqrt{z}$ about $z = 1$.

5. Find $\int_\gamma \left(\frac{z}{z-1}\right)^n dz$ for all positive integers $n$, where $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$.

6. Evaluate $\int_{|z|=2} \frac{z}{z^2+1}\, dz.$

7. Prove that $\frac{\sin z}{z}$ has a removable singularity.

8. Find the poles of $f(z) = \frac{z^2+1}{z(z-1)}$ and find the residue at a pole.

(8 × 2 = 16 marks)

## Part B

*Answer any* **four** *questions.*
*Each question carries 4 marks.*

9. Explain Stereographic projection.

Turn over

10. If $f$ is a bounded entire function, then prove that it is a constant.

11. State and prove the Fundamental theorem of Algebra.

12. Prove that if $\gamma$ is a closed rectifiable curve in G such that $\gamma \sim 0$ then $n(\gamma; w) = 0$. For all $w$ in $C - G$.

13. Prove that an entire function has a pole at infinity of order $m$ if and only if it is a polynomial of degree $m$.

14. Let $D = \{z : |z| < 1\}$ and suppose that $f$ is analytic on D with $|f(z)| \leq 1$ for $z$ in D, and $f(0) = 0$. Then prove that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z$ in the disk D.

(4 × 4 = 16 marks)

## Part C

*Answer* **either** *(A)* **or** *(B) of each of the following questions.*
*Each question carries 12 marks.*

### Unit-I

15. A) (i) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Then prove that :

a) For each $k \geq 1$, the series $\sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)a_n (z-a)^{n-k}$ has radius of convergence R.

b) $f$ is infinitely differentiable on B ($a$ ; R).

c) For $n \geq 0, a_n = \frac{1}{n!} f^{(n)}(a)$.

(ii) If G is open connected and if $f : G \longrightarrow C$ is differentiable with $f'(z) = 0$ for all $z$ in G, then prove that $f$ is constant.

B) (i) Define cross ratio. If $z_2, z_3, z_4$ are distinct points and T is any Möbius transformation, then prove that

$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any point $z_1$.

(ii) If $z_2, z_3, z_4$ are distinct points in $C_\infty$ and $\omega_2, \omega_3, \omega_4$ are also distinct points in $C_\infty$, then prove that there is one and only one Möbius transformation such that $Sz_2 = \omega_2, Sz_3 = \omega_3, Sz_4 = \omega_4$.

(iii) State and prove Symmetry Principle.

UNIT-II

16. (A) (i) Let $\gamma:[a,b]\to C$ be a rectifiable path and $\varphi:[c,d]\to[a,b]$ be a continuous non-decreasing function with $\varphi(c)=a, \varphi(d)=b;$ then prove that for any function continuous on $\{\gamma\}, \int_\gamma f = \int_{\gamma o \varphi} f.$

(ii) Let G be open in C and let $\gamma$ be a rectifiable path in G with initial and end points $\alpha$ and $\beta$ respectively. If $f$ : G —> C is a continuous function with a primitive F : G —> C, then prove that $\int_\gamma f = F(b) - F(a).$

(B) (i) Let $f$ and $g$ be analytic on a region G. Then prove that $f \equiv g$ if and only if $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

(ii) Let $f$ be analytic on an open connected set G and $f$ be not identically zero. Then prove that each zero of $f$ has finite multiplicity.

UNIT-III

17. (A) (i) Let G be an open set and let $f$ : G —> C be a differentiable function. Then prove that $f$ is analytic on G.

(ii) State and prove open mapping theorem.

(B) (i) Let G be an open subset of the plane and let $f$ : G —> C an analytic function. If $\gamma$ is a closed rectifiable curve in G such that $n(\gamma;w)=0$ for all $w$ in C – G. Then prove that for $a \in G - \{\gamma\}, n(\gamma;a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)}dz.$

(ii) State and prove Morera's theorem.

UNIT-IV

18. (A) (i) State and prove Residue theorem.

(ii) Show that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$

(B) (i) Let $f(z) = \frac{1}{z(z-1)(z-2)}$ ; give the Laurent expansion of $f(z)$ in each of the annuli : a) $ann(0 ; 0, 1)$ b) $ann(0 ; 1, 2)$ c) $ann(0 ; 2, \infty)$

(ii) State and prove the Argument principle.

(4 × 12 = 48 marks)

**D 91502** (Pages : 3) Name.......................................

Reg. No...................................

# THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

M.Sc. Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *the questions.*
*Each question carries 2 marks.*

1. Write an example for a Hausdorff topological space.
2. State the Urysohn's lemma.
3. If a space X is locally connected, then show that components of open subsets of X are open in X.
4. Show that first countability is preserved under continuous open functions.
5. If limits of all nets in a space are unique, then prove that the space is Hausdorff.
6. Define filter on a set X. Prove that a filter has the finite intersection property.
7. Define a Cauchy sequence in a metric space. Using example show that the same sequence can be a Cauchy sequence with respect to one metric and not a Cauchy sequence with respect to another metric on the same set.
8. Define equivalence of Cauchy sequences in metric space. Prove that this equivalence is an equivalence relation.

($8 \times 2 = 16$ marks)

## Part B

*Answer any* **four** *questions.*
*Each question carries 4 marks.*

9. Prove that in a Hausdorff space X, a singleton subset $\{x\}$ and a finite subset F of X can be separated by two disjoint open sets in X.
10. Prove that every compact Hausdorff space is a $T_3$ space.

**Turn over**

11. Prove that a product of topological spaces is path-connected if and only if each co-ordinate space is path connected.

12. Define eventual and cofinal subsets of a directed set. Prove that in general a cofinal subset is not an eventual subset.

13. Let $S$ be a family of subsets of a set X. Prove that there exists a filter on X having $S$ as a sub-base if and only if $S$ has the finite intersection property.

14. Prove that every totally bounded metric space is bounded.

($4 \times 4 = 16$ marks)

## Part C

*Answer* **either** *(A)* **or** *(B) part of the following questions.*
*Each question carries* 12 *marks.*

15. (A) (a) Let X be a completely regular space. Suppose F is a compact subset of X, C is a closed subset of X and $F \cap C = \phi$. Then prove that there exists a continuous function from X into the unit interval [0,1] which takes the value 0 at all points of F and the value 1 at all points of C.

(b) Suppose $\mathcal{D}$ is a decomposition of a space X each of whose members is compact and suppose the projection $p : X \longrightarrow \mathcal{D}$ is closed. Then prove that the quotient space $\mathcal{D}$ is Hausdorff or regular according as X is Hausdorff or regular.

(B) (a) Prove that $T_4$ spaces are completely regular.

(b) Suppose a topological space X has the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Then prove that X is normal.

16. (A) Prove that for a topological space X, the following statements are equivalent :

(i) X is locally connected.

(ii) Components of open subsets of X are open in X.

(iii) X has a base consisting of connected subsets.

(iv) For any $x \in X$ and every neighbourhood N of $x$, there exists a connected open neighbourhood M of $x$ such that $M \subset N$.

(B) Prove that metrisability is a countably productive property.

17. (A) (a) Let $S : D \longrightarrow X$ be a net in a topological space and let $x \in X$. Prove that $x$ is a cluster point of S if and only if there exists a subnet of S which converges to $x$ in X.

(b) Prove that a subset A of a space X is closed if and only if limits of nets in A are in A.

(B) (a) Prove that any family which does not contain the empty set and which is closed under finite intersections is a base for a unique filter.

(b) For a topological space X, prove that the following statements are equivalent :

(i) X is compact.

(ii) Every filter on X has a cluster point in X.

(iii) Every filter on X has a convergent subfilter.

18. (A) (a) Prove that every compact metric space is complete.

(b) Write an example to show that a bounded metric need not be totally bounded.

(B) (a) Prove that in a complete metric space the intersection of countably many open dense sets is dense.

(b) Prove that every contraction of a complete metric space into itself has a unique fixed point.

($4 \times 12 = 48$ marks)

**D 91500-A** (Pages : 4) Name..................................

Reg. No...............................

# THIRD SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2020

(CCSS)

M.Sc. Mathematics

MAT 3C 12—FUNCTIONAL ANALYSIS

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries 2 marks.*

1. Define $n^{\text{th}}$ Dirichlet Kernel $D_n(t)$ and show that $\int_{-\pi}^{\pi} D_n(t)\,dt = 2\pi$.

2. Show that the closure of a convex subset of a normed space is convex.

3. Show that the linear functional $f$ on $c$ defined by $f(x) = \lim_{s \to \infty} x(s), x \in c$ is continuous and $\|f\| = 1$, where $c$ is the normed space of all convergent sequences in $k$ with the norm $\| \|_\infty$.

4. Show that every non-zero linear functional on a normed space X maps open subsets of X onto open subsets of K.

5. Let C (T) be the Banach space of all continuous functions on a metric space T with sup norm and let $C_0(T)$ be the subspace $C_0(T) = \{x \in C(T) :$ for every $\in > 0,$ there is a compact set E C T such that $|x(t)| < E$ for all $t \notin E\}$.

   Show that $C_0(T)$ in a Banach Space.

6. State Hahn-Banach separation theorem.

7. Show that the inverse of a bijective linear closed map on a normed space is closed.

8. Let X be a normed space and $f : X \to K$ be linear. Show that $f$ is closed iff $f$ is continuous.

(8 × 2 = 16 marks)

## Part B

*Answer any* **four** *questions.*
*Each question carries 4 marks.*

9. Show that the metric space $L^{\infty}([a, b])$ is not separable.

10. Show that the intersection of a countable number of dense open subsets of a complete metric space X is dense in X.

11. Let X and Y be normed spaces and Z be a closed subspace of X. Show that if $\tilde{F} \in BL(X/Z, Y)$ and if we let $F(x) = \tilde{F}(x + Z)$ for $x \in X$, then $F \in BL(X, Y)$ and $\|F\| = \|\tilde{F}\|$.

12. Let X be a normed space. Show that if every absolutely summable series of elements in X is summable in X, then X is a Banach space.

13. Let X and Y be normed spaces. Show that if Z is a closed sub space of X, then the quotient map $Q : X \to X/Z$ is continuous and open.

14. Show that two comparable complete norms on a linear space are equivalent.

(4 × 4 = 16 marks)

## Part C

*Answer* **either** (A) **or** (B) *of the following questions.*
*Each question carries 12 marks.*

15. (A) Let E be a measurable subset of R. Show that, for $1 \leq p \leq \infty$, the metric space $L^p(E)$ is complete.

(B) (a) Show that every finite dimensional subspace of a normed space is complete.

(b) Show that an infinite dimensional subspace of a normed space X may not be closed in X.

16. (A) (a) Let X and Y be normed space for $F \in BL(X, Y)$, let $\|F\| = \sup\{\|F(x)\| : x \in X, \|x\| \le 1\}$.

(i) Show that $\| \ \|$ is a norm on BL (X, Y).

(ii) If $X \ne \{0\}$,

show that $\|F\| = \sup\{\|F(x)\| : x \in X, \|x\| = 1\} = \sup\{\|F(x)\| : x \in X, \|x\| < 1\}$.

(b) Let X be a subspace of B (T) with the sup norm, $1 \in X$ and *f* be a linear functional on X. Show that if $\text{Re}\, x \in X$ whenever $x \in X$ and if *f* is positive, then show that *f* is continuous and $\|f\| = f(1)$.

(B) (a) Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range of F is finite dimensional. Show that F is continuous iff the zero space Z (F) of F is closed in X.

(b) Let Y be a subspace of a normed space X which is not a hyperspace in X. Show that the complement $Y^c$ is connected.

17. (A) (a) Let Y be a subspace of a normed space X and $g \in Y'$. Show that there is some $f \in X'$ such that $f/_{Y} = g$ and $\|f\| = \|g\|$.

(b) Let $X = K^2$ with the norm $\| \ \|_\infty$. Let $Y = \{(x(1), x(2)) \in X : x(1) = x(2)\}$, and define $g \in Y'$ by $g(x(1), x(2)) = x(1)$. Determine the Hahn-Banach extensions of *g* to X.

(B) (a) Let Y be a closed subspace of a Banach Space X. Show that $X/_{Y}$ is a Banach space in the quotient norm.

(b) State and prove Uniform boundedness principle.

18. (A) (a) Let X and Y be Banach spaces and $F : X \to Y$ be a closed linear map. Show that F is continuous.

(b) Show that the result in Part (a) may not hold if both X and Y are not Banach spaces.

(B) (a) Let X and Y be normed space and $F : X \to Y$ be linear. Show that F is an open map iff there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.

(b) Let X be a normed space and E be a subset of X. Show that E is bounded in X iff $f(E)$ is bounded in K for every $f \in X'$.

(4 × 12 = 48 marks)

**D 91045** **(Pages : 3)** **Name....................................**

**Reg. No.................................**

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3E 04—PROBABILITY THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A

*Answer* **all** *questions.*
*Each question has weightage* 1.

1. A continuous random variable X has pdf $f(x) = kx, 0 < x \leq 5$. Find $P(2 \leq X \leq 4)$.

2. What do you meant by probability distribution of a random variable ?

3. Define quantile of a random variable.

4. Let (X, Y) have pdf $f(x,y) = \frac{1}{2}xy, 0 < y < x, 0 < x \leq 2$. Check whether X and Y are independent.

5. If $X_1$ and $X_2$ are iid random variables having pdf $f(x)$ and distribution function $F(x)$, obtain the conditional distribution of $X_{1:2}$ given $X_{2:2}$.

6. Define almost sure convergence of a sequence of random variables.

7. State Borel-Cantelli Lemma.

8. State Lindberg-Feller central limit theorem.

(8 × 1 = 8 weightage)

**Turn over**

## Part B

*Answer any* **six** *questions.*
*Each question has weightage 2.*

9. Let X be a random variable having pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x2}{2}}, -\infty < x < \infty$. Find the pdf of $Y = X^2$.

10. Find the expected number of throws of a fair die until a 6 is obtained.

11. For the random variable with pdf $f(x) = \frac{e^{-x} x^{\lambda}}{\lambda!}, x > 0$, where $\lambda \geq 0$ is an integer, show that $P[0 < X < 2(\lambda + 1)] > \frac{\lambda}{\lambda + 1}$.

12. (X,Y) have joint pdf $f(x,y) = 2, 0 < x < y < 1$. Find $E(X|Y)$ and $P\left(X < \frac{1}{2} \middle| Y = \frac{3}{4}\right)$.

13. Define distribution function of a bivariate random variable. What are its properties ? Check whether the function given below is a bivariate distribution function.

$$F(x,y) = \begin{cases} 0, & x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1, & \text{otherwise} \end{cases}$$

14. Let $X_1, X_2, \ldots, X_n$ be iid random variables with common pdf $f(x) = 1, 0 < x < 1$. Obtain the pdf of range.

15. Let $\{X_n\}$ be a sequence of independent random variables each having pdf $f(x) = \frac{1}{\theta}, 0 < x < \theta$. Show that $X_{n:n} \xrightarrow{L} X$, where X is a degenerate random variable.

16. Let $\{X_n\}$ be a sequence of random variables with $P[X_n = e^n] = \frac{1}{n^2}, P[X_n = 0] = 1 - \frac{1}{n^2}$. Check the convergence in probability and convergence in $r^{th}$ mean of $\{X_n\}$.

17. Let $\{X_n\}$ be a sequence of iid random variables with pdf $f(x) = \frac{1}{\pi} \frac{1}{(1 + x^2)}, -\infty < x < \infty$. find the limiting distribution of $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

(6 × 2 = 12 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question has weightage 5.*

18. (a) If moments of order $t$ exists for a random variable X, then show that moments of order $0 < s < t$ exist.

    (b) State and prove Lyapunov inequality.

19. (a) Let $X_1, X_2, X_3$ be three iid random variables each with distribution function $F(x) = 1 - e^{-x}, x > 0$.

    Show that $X_1 + X_2 + X_3, \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \frac{X_1}{X_1 + X_2}$ are independent.

    (b) Let (X, Y, Z) have joint pdf $f(x,y,z) = 6(1 + x + y + z)^{-4}, x, y, z > 0$. Find the pdf of X + Y + Z.

20. Let (X, Y) have joint pdf $f(x,y) = x + y, 0 < x, y < 1$. Find the correlation co-efficient between X and Y.

21. State and prove Lindeberg-Levy central limit theorem.

(2 × 5 = 10 weightage)

**D 91044** (Pages : 3) Name.....................................

Reg. No.....................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION, NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3E 03—MEASURE AND INTEGRATION

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A (Short Answer Questions)

*Answer* **all** *questions.*
*Each question has weightage* 1.

1. Define measurable function. Show that the characteristic function $\chi_E$ is measurable iff E is measurable.
2. State Fatou's Lemma. Give an example for strict inequality in this Lemma.
3. If $f \in L^1(\mu)$, then show that $\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.
4. State Lusin's theorem.
5. If $\mu$ is a positive $\sigma$-finite measure on a $\sigma$-algebra M in a set X. Then prove that there exist a function $w \in L^1(\mu)$ then show that $0 < w(x) < 1 \; \forall \; x \in X$.
6. Define Jordan decomposition of a measure $\mu$ and if $\mu = \lambda_1 - \lambda_2$ where $\lambda_1, \lambda_2$ are positive measures, then prove $\lambda_1 \geq \mu^+$ and $\lambda_1 \geq \mu^-$.

**Turn over**

7. Define the $x$-section and $y$-section of a function $f(x, y)$ and show that they are measurable if $f(x, y)$ measurable.

8. Define product measure. And find the measure of a rectangle $A \times B$.

(8 × 1 = 8 weightage)

## Part B (Short Essays)

*Answer any* **six** *questions.*
*Each question has weightage* 2.

9. State and prove Lebesgue's Monotone convergence Theorem.

10. Let $f_n : X \to [0, \infty]$ be measurable, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then show that $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$.

11. State and prove the bilinearity property of Lebesgue integrable functions.

12. Let X be a locally compact Hausdorff space in which every open set is σ-compact and λ a positive Borel measure on X such that $\lambda(K) < \infty \; \forall$ compact set K. Then prove that λ is regular.

13. If μ is a complex measure on X, then prove that $|\mu|(X) < \alpha$.

14. State and prove Hahn Decomposition Theorem.

15. Define the $x$-section and $y$-section of a set E and show that they are measurable if E is measurable.

16. Let $m_k$ denote Lebesgue measure on $R^k$. If $k = r + s, r \geq 1, s \geq 1$ then $m_k$ is the completion of the product measure $m_k \times m_s$. Prove.

17. Give an example to show that the condition $f \in L^1(\mu \times \nu)$ cannot be dropped from Fubini's theorem to obtain the conclusion of the theorem.

(6 × 2 = 12 weightage)

## Part C (Essays)

*Answer any* **two** *questions.*
*Each question has weightage 5.*

18. State and prove Riesz Representation Theorem.

19. State and prove the Lebesgue-Radon-Nikodym theorem.

20. State and prove Fubini's theorem.

21. Let $[X, \zeta]$ and $[Y, \Gamma]$ be two measurable spaces. Show that $\zeta \times \Gamma$ is the smallest monotone class containing all elementary sets.

(2 × 5 = 10 weightage)

**D 91043** (Pages : 2) Name.....................................

Reg. No.................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3E 02—CRYPTOGRAPHY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage



## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A (Short Answer Questions)

*Answer* **all** *questions.*
*Each question has weightage* 1.

1. Define the terms Cryptosystem, monoalphabetic cryptosystem, polyalphabetic cryptosystem with example.
2. Compare between Substitution Cipher and Permutation Cipher.
3. Prove that $a \bmod m = b \bmod m$ if and only if $a \equiv b \pmod m$.
4. State Jensen's inequality.
5. Prove that $H(X/Y) \leq H(X)$, with equality if and only if X and Y are independent.
6. Explain the terms round key mixing and whitening in SPN.
7. Define hash family.
8. Write short note on differential cryptanalysis.

(8 × 1 = 8 weightage)

## Part B (Short Essay)

*Answer any* **two** *questions.*
*Each question has weightage* 2.

9. Use Hill Cipher with key $K = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$ to encrypt the message GIVE THEM TIME.
10. Define involutory key. Find the number of involutory keys in the Permutation Cipher for $m = 3, 4$.
11. Explain Cryptanalysis of Affine Cipher.

**Turn over**

12. Prove that if a cryptosystem has perfect secrecy and $|K| = |C| = |P|$, then every cipher text is equally probable.

13. Define unicity distance of a cryptosystem. Calculate it for Substitution Cipher.

14. Prove that $H(X,Y) \leq H(X) + H(Y$, with equality if and only if X and Y are independent random variables.

15. State and prove the Piling-up-lemma.

16. Explain Random Oracle Model Hash functions.

17. Describe Message Authentication code(MAC) and explain HMAC.

(6 × 2 = 12 weightage)

## Part C (Essay)

*Answer any* **two** *questions.*
*Each question has weightage* 5.

18. (a) Define different types of Stream Ciphers and explain the methods of keystream generation.

    (b) How Cryptanalysis done in Stream Cipher.

19. (a) Prove that the Affine Cipher achieves perfect secrecy if every key is used with equal probability 1/312.

    (b) Compute H(K/C) and H(K/P,C) for the Affine Cipher assuming that keys are used equiprobably and plaintexts are equiprobable.

20. Describe and analyse Data Encryption Standard(DES) and Advanced Encryption Standard(AES)

21. Define iterated hash functions and explain the generic construction and the Merkle-Damgard Construction methods of the same.

(2 × 5 = 10 weightage)

**D 91042** (Pages : 3) Name..................................

Reg. No..............................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3E 01—CODING THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A

*Answer* **all** *questions.*
*Each question carries weightage* 1.

1. Define Complete Maximum Likelihood Decoding.

2. Show that $C = \{0000, 1100, 0011, 1111\}$ is a linear code and its distance is $d = 2$.

3. Let C be a code with parity check matrix $H = \begin{bmatrix} 11 \\ 11 \\ 01 \\ 10 \\ 01 \end{bmatrix}$. Find a generator matrix of $C^{\perp}$.

4. State Hamming bound for a code C of length $n$.

5. Does there exist a linear code of length $n = 9$, dimension $k = 2$ and distance $d = 5$.

6. Prove that Golay code $n = 23$, $k = 12$ and $d = 7$ is perfect.

7. Find the sum and product of the polynomials $f(x) = x^5 + x^6 + x^7$, $g(x) = 1 + x^2 + x^3 + x^4$.

8. Find the generator polynomial of the dual cyclic code C of length $n = 6$ and having generator polynomial $g(x) = 1 + x + x^2$.

(8 × 1 = 8 weightage)

## Part B

*Answer* **six** *questions.*
*Each question carries weightage* 2.

9. Find a generator matrix for the code $C = \{0000, 1110, 0111, 1001\}$ in the standard form.

10. Find the distance of the linear code C with parity check matrix $H = \begin{bmatrix} 0111 \\ 1110 \\ 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}$.

11. Construct an SDA assuming CMLD for the code $C = \{0000, 1001, 0101, 1100\}$.

12. Give a generator and parity check matrix of Hamming code of $n = 7$.

13. Show that an extended Hamming code of length $n = 8$ is a self dual code or $C = C^{\perp}$.

14. Give a generator matrix of Extended Golay code $C_{24}$.

15. Prove that every cyclic code contains a unique idempotent polynomial which generates the code.

16. Find a basis and a generator matrix of the cyclic code C of length $n = 7$ with generator polynomial $g(x) = 1 + x + x^3$. Also encode the message $a(x) = 1 + x^2$.

17. State and prove the properties of minimal polynomials in a finite field.

(6 × 2 = 12 weightage)

## Part C

*Answer* **two** *questions.*
*Each question has weightage* 5.

18. (a) Prove that a code of distance $d$ will correct all error patterns of weight $\leq \left[\frac{(d-1)}{2}\right]$. More over there is at least one error pattern of weight $1+\left[\frac{(d-1)}{2}\right]$ which C will not correct.

(b) List the distinct cosets of C = {000,111}.

19. Decode the word $w$ = 001001001101, 101000101000 in $C_{24}$.

20. Let G (1, 3) be the generator matrix of RM (1, 3). Decode the received vector $w$ = 10101011.

21. Let $w$ be a received word with syndromes $w(\beta) = s_1$ = 0111 and $s_3$ = 1010 = $w(\beta^3)$ in two error correcting BCH code $C_{15}$. Find the error pattern.

(2 × 5 = 10 weightage)

**D 91041** **(Pages : 4)** Name.......................................

Reg. No...................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION, NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3C 14—PDE AND INTEGRAL EQUATIONS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A

*Answer* **all** *questions.*
*Each question has weightage* 1.

1. Describe the transversity condition and its importance in existence and uniqueness theorem.

2. Prove that $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ is parabolic and find its canonical form.

3. Derive the canonical form of one dimensional wave equation.

4. Show that negative and zero eigen values does not exist for the eigen value problem

$$\frac{d^2X}{dx^2} + \lambda X = 0, 0 < x < L, X(0) = X(L) = 0.$$

5. State the Dirichlet's problem .What do you mean by a problem of third kind in partial differential equation.

6. State and prove a necessary condition for the existence of a solution to the Neumann problem.

**Turn over**

7. Explain Volterra equation of second kind with an example.

8. If $I_n(x) = \int_a^x (x-\xi)^{n-1} f(\xi)\, d\xi$ then show that $\frac{d^n}{dx^n} I_n = (n-1)!\, f(x)$.

(8 × 1 = 8 weightage)

## Part B

*Answer any* **six** *questions.*
*Each question has weightage* 2.

9. Solve the equation $u_x + u_y + u = 1$, subject to the initial condition $u = \sin x$ on $y = x + x^2, x > 0$.

10. Find a co-ordinate system $s = s(x, y), t = t(x, y)$ that transforms the equation $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$ into its canonical form. Also find the general solution.

11. Consider the Cauchy Problem $u_{tt} - u_{xx} = 0, -\infty < x < \infty, t > 0$.

$$u(x, 0) = f(x) = \begin{cases} 0, -\infty < x < -1, \\ x + 1, -1 \le x \le 0, \\ 1 - x, 0 \le x \le 1, \\ 0, 1 < x < \infty. \end{cases}$$

$$u_t(x, 0) = g(x) = \begin{cases} 0, -\infty < x < -1 \\ 1 - 1 \le x \le 1, \\ 0, 1 < x < \infty. \end{cases}$$

(a) Evaluate $u$ at the point $\left(1, \frac{1}{2}\right)$.

(b) Discuss the smoothness of the solution $u(x, y)$.

12. Solve the initial boundary value problem $u_t - u_{xx} = 0, 0 < x < \pi, t > 0\ u(0,t) = u(\pi, t) = 0, t \geq 0$

$$u(x,0) = f(x) = \begin{cases} x, 0 \leq x \leq \dfrac{\pi}{2} \\ \pi - x, \dfrac{\pi}{2}, \leq x \leq \pi \end{cases}.$$

13. State and prove the mean value principle.

14. Let $u \in C_H$ be a solution of heat equation $u_t = k\Delta u, t > 0$ in $Q_T$. Prove that $u$ achieves its maximum on $\partial_P Q_T$.

15. Prove that the characteristic numbers of a Fredholm equation with a real symmetric kernel are all real.

16. Show that $y'' + A(x)y' + B(x)y = f(x)$ with initial conditions $y(a) = y_0, y'(a) = y_0'$ can be transformed to a Volterra's equation of second kind.

17. Transform the problem $y'' + y = x, y(0) = 1$ and $y'(1) = 0$ to a Fredholm integral equation.

(6 × 2 = 12 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question has weightage* 5.

18. a) Find a function $u(x, y)$ that solves the Cauchy problem $x^2 u_x + y^2 u_y = u^2, u(x, 2x) = x^2, x \in R$.

b) Check whether the transversality condition holds.

c) Is the solution defined for all $x$ and $y$ ?

19. Solve the problem $u_{tt} - u_{xx} = t^7, -\infty < x < \infty, t > 0$ subject to the conditions

$$u(x,0) = 2x + \sin x, -\infty < x < \infty$$
$$u_t(x,0) = 0, -\infty < x < \infty.$$

20. Solve the following wave equation using the method of separation of variables :

$$u_{tt} - 4u_{xx} = 0, 0 < x < 1, t > 0$$
$$u_x(0, t) = u_x(1, t) = 0, t \geq 0$$
$$u(x, 0) = f(x) = \cos^2 \pi x, 0 \leq x \leq 1$$
$$u_t(x, 0) = g(x) = \sin^2 \pi x \cos \pi x, 0 \leq x \leq 1.$$

21. Solve the Fredholm equation $y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi)\, y(\xi) d\xi$ using the iterative method.

(2 × 5 = 10 weightage)

**D 91040** **(Pages : 3)** **Name..................................**

**Reg. No...............................**

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3C 13—FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A

*Answer* **all** *questions.*
*Each question carries weightage* 1.

1. Show that two cosets $[x]$ and $[y]$ of a linear space E either coincide or they are disjoint sets.
2. Define a normed space with an example.
3. Show that if $z \in I[x,y]$ then $\|x-y\| = \|x-z\| + \|z-y\|$, $x, y, z$ all belonging to a linear space E.
4. Define inner product space. State and prove Pythagorean theorem.
5. Prove that for every $x, y$ in a linear space H with inner product $\langle .,. \rangle$ $|\langle x\, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} . \langle y, y \rangle^{1/2}$.
6. Prove that for any normed space X , the dual space X* is a Banach space.
7. State the Hahn-Banach Theorem.
8. Show that L (X, Y) is a normed space.

(8 × 1 = 8 weightage)

Turn over

## Part B

*Answer* **six** *questions.*
*Each question has weightage 2.*

9. State and prove Minkowski's inequality.

10. Prove that C $[a, b]$ is a Banach space.

11. Show that if O is an open set then $F = O^C$ is closed and vice-versa.

12. State and prove Bessel's inequality. Also prove that for any Hilbert space H with $x \in H$ and any orthonormal system $\{e_i\}_1^{\infty}$ there exists a $y \in H$ such that $y = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$.

13. State and prove Parseval's identity.

14. Let H be a Hilbert space and M be a convex closed set in H. Show that there exists a unique $y \in M$ such that $\rho(x, M) = \|x - y\|$.

15. Let L be a closed subspace of a normed space X. Show that for the subspaces $L^{\perp}$ and $\left(L^{\perp}\right)_{\perp}$ of X* $\left(L^{\perp}\right)_{\perp} = L$.

16. Show that $Af = \int_0^1 K(t, \tau)\, d\tau$, where K is a constant function of two variables, is a linear operator on C [0, 1]. Also, show that $\|A\| = \max_t \int_0^1 |K(t, \tau)|\, d\tau$.

17. Let $K(t, \tau)$ be a continuous function of two variables on $[0,1] \times [0,1]$. Prove that the operator $Kx = \int_0^1 K(t, \tau)\, x(\tau)\, d\tau$ is a compact operator on C [0, 1].

(6 × 2 = 12 weightage)

## Part C

*Answer* **two** *questions.*
*Each question carries weightage* 5.

18. State and prove Holder's inequality.

19. (a) State and prove Pythagorean theorem.

 (b) State and prove Riesz representation theorem.

20. (a) Let $L \hookrightarrow X$ be a subspace of a normed space X and let $x \in X$ such that $dist\,(x, L) = d > 0$.

 Show that there exists $f \in X^*$ such that $\|f\| = 1, f(L) = 0$ and $f(x) = d$.

 (b) The $l_p$ spaces are reflexive. Justify the statement.

21. Show that L (X, Y) is a Banach space.

(2 × 5 = 10 weightage)

**D 91039** (Pages : 3) Name.....................................

Reg. No.....................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION, NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3C 12—COMPLEX ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A

*Answer* **all** *questions.*
*Each question has weightage* 1.

1. Explain the concept of radius of convergence of a power series ? Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a^n z^n$.

2. Prove that $\sum a_n$ an converges if $\sum a_n$ converges absolutely.

3. Let $\gamma : [0, 2\pi] \to C$ and define $f(z) = \frac{1}{z}$; for $z \neq 0$ find $\int_{\gamma} f(z)\, dz$.

4. Prove that a Mobius transformation is the composition of translation, dilation and the inversion.

5. Identify the type of singularity of the function $\frac{\sin z}{z}$ at $z = 0$.

**Turn over**

6. If $\sum a_n$ and $\sum b_n$ converges absolutely then prove that $\sum c_n$, where $c_n = \sum_{n=0}^{n} a_k b_{n-k}$ converges absolutely.

7. Show that $f(z) = \tan z$ is analytic in C except for simple poles at $z = \pi 2 + n\pi$, for each integer $n$. Determine the singular part of $f$ at each of these poles.

8. Define the residue of the function $f(z)$ at the singularity $a$.

(8 × 1 = 8 weightage)

## Part B

*Answer any* **six** *questions.*
*Each question has weightage* 2.

9. If G is open and connected and $f : G \to C$ is differentiable with $f'(z) = 0$ for all $z \in G$, then prove that $f$ is constant.

10. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though Cauchy-Riemann equations are satisfied.

11. Find the image of $\{z : \text{Re}(z) = 0\}$ and $\{z : \text{Im}\, z = \pi/2\}$ under the exponential function.

12. State and prove Liouville's theorem.

13. Give the power series expansion of $\log z$ about $z = i$ and find its radius of convergence.

14. Let $\gamma$ be the closed polygon $[1-i, 1+i, -1+i, -1-i, 1-i]$. Find $\int_{\gamma} \frac{1}{z} dz$.

15. If $z = a$ is an isolated singularity of $f$ and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ be the Laurent Expansion in $ann(a; 0, R)$ then prove that $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \le -1$.

16. Evaluate $\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$.

17. Find the Laurent series expansion of $e^{\frac{1}{z}}$.

(6 × 2 = 12 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question has weightage* 5.

18. (a) Let $u$ and $v$ be real-valued functions defined on a region G and suppose that $u$ and $v$ have continuous partial then prove that $f : G \to C$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if it satisfies the C-R equations.

(b) Show that the real part of the function $\frac{1}{z^2}$ is always positive.

19. (i) Prove that a Mobius transformation carries circles into circles.

(ii) If $z_1, z_2, z_3, z_4$ are four distinct points in $C_\infty$, then prove that their cross ratio is real if and only if all four points lie on a circle.

20. (a) State and prove general form of Cauchy's theorem.

(b) Let $\gamma$ and $\sigma$ be the two polygons $[1, i]$ and $[1, 1+i, i]$. Express $\gamma$ and $\sigma$ as paths and calculate $\int_\gamma f$ and $\int_\sigma f$ where $f(z) = |z|^2$.

21. State and prove the argument principle.

(2 × 5 = 10 weightage)

**D 91038** (Pages : 2) Name..........................................

Reg. No......................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3C 11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

## General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section / Part.*
2. *The minimum number of questions to be attended from the Section / Part shall remain same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to maximum weightage of the Section / Part.*

## Part A

*Answer* **all** *questions.*
*Each question has weightage* 1.

1. Suppose X is a vector space, and dim X = $n$. Show that the set E of $n$ vectors in X spans X if and only if E is independent.

2. Show that if I is the identity operator on $\mathbb{R}^n$, then $\det[I] = \det(e_1, e_2, \ldots, e_n) = 1$.

3. Define a tangent vector. Calculate the tangent vector of the curve $\gamma(t) = (\cos^2 t, \sin^2 t)$.

4. Show that, if $f(x, y)$ is a smooth function, its graph $\{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y)\}$ is a smooth surface with atlas consisting of the single regular surface patch $\sigma(u,v) = (u, v, f(u,v))$.

5. Show that, if $\sigma(u,v)$ is a surface patch, the set of linear combinations of $\sigma_u$ and $\sigma_v$ is unchanged when $\sigma$ is reparametrized.

6. Calculate the first fundamental forms of the surface $\sigma(u,v) = (u - v, u + v, u^2 + v^2)$.

7. Show that $\|\sigma_u \times \sigma_v\| = \left(\|\sigma_u\|\|\sigma_v\| - (\sigma_u \cdot \sigma_v)^2\right)^{1/2}$.

8. Show that the second fundamental form of a plane is zero.

(8 × 1 = 8 weightage)

**Turn over**

## Part B

*Answer any* **six** *questions.*
*Each question has weightage* 2.

9. Show that a linear operator on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

10. Show that if $A,B \in L(\mathbb{R}^n,\mathbb{R}^m)$ and $c$ is a scalar, then $\|A+B\| \le \|A\|+\|B\|, \|cA\| \le |c|\|A\|$ Also show that $L(\mathbb{R}^n,\mathbb{R}^m)$ is a metric space with the distance between A and B defined as $\|A-B\|$.

11. Prove that det([B][A]) = det[B]det[A], for nxn matrices [A] and [B].

12. Compute the curvature of the circular helix $\gamma(\theta)=(a\cos\theta, a\sin\theta, b\theta);\theta \in \mathbb{R}$.

13. Define a surface. Show that any open disc in the $xy$-plane is a surface.

14. Define orientable surface. Show that Mobius band is not orientable.

15. Show that $\sigma(u,v)=(\operatorname{sech}u\cos v, \operatorname{sech}u\sin v, \tanh u)$ is a regular surface patch for $S^2$.

16. Show that Gaussian curvature of a ruled surface is negative or zero.

17. Calculate the principal curvatures of the torus

$$\sigma(\theta,\varphi)=((a+b\cos\theta)\cos\varphi,(a+b\cos\theta)\sin\varphi, b\sin\theta).$$

(6 × 2 = 12 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question has weightage 5.*

18. State and prove the Inverse function theorem.

19. Show that $f$ is a local diffeomorphism if and only if, for all $p \in S$, the linear map $D_p f : T_p S \to T_{f(p)}\tilde{S}$ is invertible, where S and $\tilde{S}$ are surfaces and $f : S \to \tilde{S}$ a smooth map.

20. Let $\gamma:(\alpha,\beta)\to\mathbb{R}^2$ be a unit-speed curve, let $S_0 \in (\alpha,\beta)$ and let $\varphi_0$ be such that $\dot{\gamma}(s_0)=(\cos\varphi_0,\sin\varphi_0)$. Then there is a unique smooth function $\varphi:(\alpha,\beta)\to\mathbb{R}$ such that $\varphi(s_0)=\varphi_0$ and $\dot{\gamma}(s)=(\cos\varphi(s),\sin\varphi(s))$ holds for all $s \in (\alpha,\beta)$.

21. Prove that a smooth map $f : S_1 \to S_2$ is a local isometry if and only if the symmetric bilinear forms $\langle,\rangle_p$ and $f^*\langle,\rangle_p$ on $T_p S_1$ are equal for all $p \in S_1$.

(2 × 5 = 10 weightage)

**D 90738** (Pages : 3) Name..................................

Reg. No..............................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (SUPPLEMENTARY) EXAMINATION, NOVEMBER 2020

(CUCSS)

Mathematics

MT 3C 15—PDE AND INTEGRAL EQUATIONS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

## Part A

*Answer* **all** *questions.*
*Each question carries 1 weightage.*

1. Find the partial differential equation of all spheres of radius $r$, having center in the $xy$ plane.

2. Show that $(x-z)(y-z)=1$ is a singular integral of $z = px + qy - 2\sqrt{pq}$.

3. Verify that the Paffian differential equation $yzdx + 2xzdy - 3xydz = 0$ is integrable and find the corresponding integrals.

4. Find the complete integral of $4(1+z^3) = 9z^4pq$.

5. Determine the characteristic curve for solving the equation $z_x - zz_y + z = 0$ for every $y$ and $x > 0$ with the initial conditions $x_0 = 0, y_0 = s, z_0 = -2s, -\infty < s < \infty$.

6. What is the range of influence of the point ?

7. Describe the second and third boundary value problem of the Laplace equation.

8. Show that the solution of the Neumann problem is unique up to the addition of constant.

9. Classify the partial differential equation $xy\dfrac{\partial^2 z}{\partial x^2} - \left(x^2 - y^2\right)\dfrac{\partial^2 z}{\partial x \partial y} - xy\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x}y - \dfrac{\partial z}{\partial y}x = x^2 - y^2$.

10. Show that $y(x) = \cos(2x)$ is the solution of the integral equation $y(x) = \cos(x) + 3\int_0^{\pi} K(x, \xi)\, y(\xi)\, d\xi$

where $K(x, \xi) = \begin{cases} \sin(x)\cos(\xi), & 0 \le x \le \xi \\ \cos(x)\sin(\xi), & \xi \le x \le \pi \end{cases}$.

11. Define different types of kernel of an integral equation with an example.

12. Find the eigenvalue and the eigen functions of the homogeneous integral equation $y(x) = \lambda \int_0^1 \sin(\pi x)\cos(\pi\xi)\, d\xi$.

13. Find the solution of the integral equation $g(x) = x + \int_0^1 x\xi^2 g(\xi)\, d\xi$.

14. Determine the resolvent kernel associated with $k(x, \xi) = e^{x+\xi}$ in (0, 1), in the form of a power series in $\lambda$.

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each question carries 2 weightage.*

15. Find the general integral of the equation $(y + z)\, p + (z + x)\, q = x + y$.

16. Using Charpit's method find the complete integral of $(p^2 + q^2) = z^2 (x + y)$.

17. Solve the equation $p^2 x + q^2 y = z$ by Jacobi's method.

18. Reduce the equation $y^2 \dfrac{\partial^2 z}{\partial x^2} + x^2 \dfrac{\partial^2 z}{\partial y^2} = 0$ into canonical form.

19. Show that the surfaces $x^2 + y^2 + z^2 = c^{\frac{2}{3}}$ can form a family of equipotential surfaces, and find the general form of the corresponding potential function.

20. State the one dimensional vibration of an infinite string and derive D'Alambert's solution.

21. State the Heat conduction problem for finite rod of length $l$ with the initial temperature $f(x)$ with heat source F $(x, t)$. Show that if the solution exists then it is unique.

22. Transform the boundary value problem $\frac{d^2y}{dx} + xy = 1, y(0) = y(1) = 0$ into an integral equation.

23. Show that the characteristic functions of the symmetric kernel corresponding to distinct characteristic numbers are orthogonal.

24. Solve $y(x) = \sin(x) + 2\int_0^x e^{x-\xi}\, y(\xi)\, d\xi$ by iterative method.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question carries 4 weightage.*

25. Define compatibility of system of first order PDE and establish the necessary and sufficient condition to obtain the one parameter family of common solutions.

26. Determine the characteristic of the equation $pq = z$ which passes through the parabola $x = 0$, $y^2 = z$.

27. Using Riemann method find the solution of the non-homogeneous wave equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} + f(x, t) = 0$$

with the initial condition $z(x, 0) = f(x), z_t(x, 0) = g(x)$.

28. Show that the integral equation $y(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+\xi)\, y(\xi)\, d\xi$ possesses no solution for $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$.

(2 × 4 = 8 weightage)

**D 90737** (Pages : 3) Name.......................................

Reg. No...................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (SUPPLEMENTARY) EXAMINATION, NOVEMBER 2020

(CUCSS)

Mathematics

MT 3C 14—FUNCTIONAL ANALYSIS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

## Part A

*Answer* **all** *the questions.*

*Each question carries weightage 1.*

1. Let $V = C[-1,1]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$ for $f, g \in V$. Let $f(x) = x$, $g(x) = \frac{3x^2 - 1}{2}$. Check whether $f$ and $g$ are orthogonal ?

2. Show that the metric space $l^\infty$ is not separable.

3. State the finite intersection property.

4. Show that for $p \geq 1$, the sequence space $l^p \subset l^\infty$.

5. Define the quotient norm on the quotient space $X/Y$, where Y is a closed subspace of a normed space X.

6. Show by an example that a linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X.

7. Is norm a bounded linear functional on a normed space X ? Justify your answer.

8. State the Hahn-Banach extension theorem.

Turn over

9. State the Taylor-Foguel Theorem.

10. Is $C_{00}$, the space of all scalar sequences having only finitely many non-zero terms as a subspace of $l^{\infty}$, a closed set ? Justify your answer.

11. State the Bounded inverse theorem.

12. Is the parallelogram equality satisfied in $l^1$ ? Justify your answer.

13. Give an example of a discontinuous linear map.

14. State the Parseval formula in a Hilbert space.

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each question carries weightage* 2.

15. Give an example to show that the open mapping theorem may not hold true if the normed spaces X and Y are not Banach spaces.

16. Show that the set of all polynomials in one variable is dense in $C[a, b]$ with sup metric.

17. Define a strictly convex normed space. Give an example of a space which is not strictly convex.

18. Show that a linear map F from a normed space X to a normed space Y is a homeomorphism if there are $\alpha, \beta > 0$ such that

$$\beta \| x \| \leq \| F(x) \| \leq \alpha \| x \|$$

for all $x \in X$.

19. State and prove the Bessel's inequality.

20. Show that a Banach space cannot have a denumerable Hamel basis.

21. Let X and Y be normed spaces. If Z is a closed subspace of X, then show that the quotient map Q from X to $X/Z$ is continuous and open.

22. State and prove the Riesz-Fischer theorem.

23. If all functionals vanish on a given vector, then show that the vector must be zero.

24. Let X be a Banach space, Y a normed linear space and $T_n \in BL(X, Y)$ such that the sequence $(T_n(x))$ is Cauchy in Y for every $x \in X$. Show that the sequence $(\|T_n\|)$ is bounded.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question carries weightage 4.*

25. Consider the norms $\|x\|_1 = |x_1| + |x_2| + .... + |x_n|$ and $|x|_\infty = \max\{|x_1|, |x_2|, ....., |x_n|\}$ for for $x = (x_1, x_2, ...., x_n) \in R^n$.

   a) Prove that $\|x\| = \frac{1}{3}\|x\|_1 + \frac{2}{3}\|x\|_\infty$ defines a norm on $R^n$.

   b) Sketch the open unit ball in $R^2$ with respect to the norm in part (a).

   c) In $C[0,1]$ with supremum norm, compute $d(f, g)$ for $f(x) = 1$ and $g(x) = x$.

26. State and prove the closed graph theorem.

27. Let H be a non-zero Hilbert space over K. Then show that the following conditions are equivalent.

   a) H has a countable orthonormal basis.

   b) H is linearly isometric to $K^n$ for some $n$ or to $l^2$.

   c) H is separable.

28. Let X and Y are normed spaces and $X \neq 0$. Then show that :

   (a) $BL(X, Y)$, the set of all bounded linear maps from X to Y is a Banach space in the operator norm if and only if Y is a Banach space.

   (b) The dual $X'$ of every normed space X is a Banach space.

(2 × 4 = 8 weightage)

**D 90736** (Pages : 3) Name.......................................

Reg. No..................................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (SUPPLEMENTARY) EXAMINATION, NOVEMBER 2020

(CUCSS)

Mathematics

MT 3C 13—COMPLEX ANALYSIS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

## Part A

*Answer* **all** *questions.*

*Each question carries weightage* 1.

1. Show that real and imaginary parts of an analytic function satisfy Laplace Equation.

2. If $f(z) = u + iv$ is an analytic function, then calculate the value of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2$.

3. Find the linear fractional transform that maps 1, 2 and 3 to 0, $i$ and $-i$, respectively.

4. If $f(z) = z^2$ and $\gamma$ is the parabola given by $2t + i(t^2 + 1), t \in \mathbb{R}$, then find $\int_\sigma f(z)\,dz$, where $\sigma$ is the part of $\gamma$ from $i$ to $2 + i2$.

5. If $f(z)$ and $g(z)$ have algebraic orders $h$ and $k$ at $z = a$, show that $fg$ has the order $h + k$.

6. Determine the nature of the singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z = 0$ and $z = \infty$.

7. State Argument principle.

8. State Schwarz Lemma.

Turn over

9. Find the residue of $\frac{e^z}{(z+3)(z-2)^2}$ at $z = 2$.

10. If $P(z)$ is a non-constant polynomial in C, then prove that $P(z)$ has a root in C.

11. Every bilinear transformation which has only one fixed point $\alpha$ can be put in the form $\frac{1}{w-z} = \frac{1}{z-\alpha} + \lambda$, where $\lambda$ is constant.

12. For $f(z) = \frac{z+3}{z(z^2 - z - 2)}$, write down the Laurent series expansion of $f$ in the annulus region $1 < |z| < 2$.

13. Show that if $f$ is a non-constant analytic function on a region $\Omega$, then $|f(z)|$ has no maximum in $\Omega$.

14. Show that if $f$ is an elliptic function, then $f'$ is also an elliptic function.

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each question carries weightage* 2.

15. Every Mobius transformation maps circles or straight lines into circles or straight lines.

16. State and prove the symmetry principle.

17. Find an analytic function $f(z) = u + iv$ whose real part $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

18. Prove *or* disprove : A function which has no singularity in the finite part of the plane or at infinity is constant .

19. Define winding number. Prove that $n(\gamma, a) = n(\gamma, b)$, where $a, b$ belongs to same region determined by the closed curve $\gamma$ in $\mathbb{C}$.

20. State and prove Morera's theorem.

21. Every entire elliptic function is a constant function.

22. By the method of residues, evaluate $\int_0^\infty \frac{\cos ax}{\left(x^2+b^2\right)^2}\,dx$, where $a>0, b>0$.

23. Prove that all the rots of $z^7-5z^3+12=0$ lies between the circles $|z|=1$ and $|z|=2$.

24. Let $f$ be analytic in $B(a;R)$. Show that $f$ has a power series expansion in $B(a;R)$.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question carries weightage 4.*

25. State and prove Laurent's series expansion for an analytic function $f(z)$ in an annulus $R_1<|z-a|<R_2$.

26. State and prove Cauchy's Theorem for a Rectangle.

27. Prove that any *two* bases of the same module are connected by a unimodular transformation.

28. Prove that $\begin{vmatrix} \wp'(u) & \wp(u) & 1 \\ \wp'(z) & \wp(z) & 1 \\ -\wp'(u+z) & \wp(u+z) & 1 \end{vmatrix}=0.$

(2 × 4 = 8 weightage)

**D 90735** (Pages : 3) Name..................................

Reg. No...............................

# THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (SUPPLEMENTARY) EXAMINATION, NOVEMBER 2020

(CUCSS)

Mathematics

MT3C12—MULTIVARIABLE CALCULUS AND GEOMETRY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage



## Part A

*Answer* **all** *the questions.*
*Each question carries weightage 1.*

1. If $A \in L(\mathbb{R}^n, R^m)$, then prove that $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $R^m$.

2. Suppose $f$ maps a convex open set $E \subset \mathbb{R}^n$ into $R^m$, $f$ is differentiable in E, and there is a real number M such that $\|f'(x)\| \leq M$ for every $x \in E$. Then prove that $|f(b) - f(a)| \leq M|b - a|$ for all $a, b \in E$.

3. State the implicit function theorem.

4. Determine a parametric representation for the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ that does not involve radicals.

5. Find the representation of the helix helix $\alpha(t) = (a \cos t) e_1 + (a \sin t) e_2 + bte_3$, where $a, b$ are constants and $t \in \mathbb{R}$, in terms of arc length.

6. Show that the curve given by the equation $r = 2 \cos \theta - 1$, $0 \leq \theta \leq 2\pi$, in polar co-ordinates is regular.

7. Is it possible to define Gauss map for any orientable surface ? If so, why ?

8. What are quadrics ? Give an example of a quadric which is not a surface.

Turn over

9. Show that any re-parametrization of a regular curve is regular.

10. Calculate the arc-length of the space curve given by $\gamma(t) = (t, t^2, t^3)$, where $-\infty < t < \infty$, starting at $\gamma(0) = 0$.

11. Compute the torsion of the circular helix $\gamma(\theta) = (a \cos\theta, a \sin\theta, b\theta)$, where $-\infty < \theta < \infty$ and $a, b$ are constants.

12. Show that any geodesic has constant speed.

13. Name all the possible geodesics on the circular cylinder $x^2 + y^2 = 1$.

14. Differentiate between first fundamental form and second fundamental form.

(14 × 1 = 14 weightage)

## Part B

*Answer any* **seven** *questions.*
*Each question carries weightage* 2.

15. Suppose X is a vector space of dimension $n$. Prove that a set E of $n$ vectors in X spans X if and only if E is independent.

16. Prove that a linear operator A on a finite dimensional vector space X is one-to-one if and only if the range of A is all of X.

17. Let $\Omega$ be the set of all invertible linear operators on $R^n$. If $A \in \Omega$, $B \in L(R^n)$ and $\|B - A\| \|A^{-1}\| < 1$ then prove that $B \in \Omega$.

18. If X is a complete metric space, and if $f$ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $f(x) = x$.

19. Show that a parametrized curve has a unit speed re-parametrization if and only if it is regular.

20. Let $\gamma$ be a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then show that $\gamma$ is part of a circle.

21. Let $f: \mathbf{S}_1 \to \mathbf{S}_2$ be a diffeomorphism. If $\sigma_1$ is an allowable surface patch on $\mathbf{S}_1$, then prove that $f \circ \sigma_1$ is an allowable surface patch on $\mathbf{S}_2$.

22. Show that Mobius band is not an orientable surface.

23. Prove that any normal section of a surface is a geodesic.

24. Prove that an isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

(7 × 2 = 14 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question carries weightage* 4.

25. State and prove the inverse function theorem.

26. Let $\gamma$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature so that the torsion $\tau$ of $\gamma$ is defined. Then, prove that the image of $\gamma$ is contained in a plane if and only if $\tau$ is zero at every point of the curve.

27. Determine the geodesics on the unit sphere $S^2$ by solving the geodesic equations.

28. Let P be a point of a flat surface S and assume that P is not an umbilic. Then prove that there is a patch of S containing P that is a ruled surface.

(2 × 4 = 8 weightage)

**C 63436** **(Pages : 3)** Name.................................

Reg. No...............................

# THIRD SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper XV—LINEAR PROGRAMMING AND ITS APPLICATIONS

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries* 4 *marks.*

I. (a) Define closed sets in $E_n$, the $n$-dimensional Euclidean space. Prove that the intersection of two closed sets is a closed set.

(b) State the implicit function theorem.

(c) Prove that the transportation problem has a triangular basis.

(d) Let $f(X, Y)$ be such that both $\max_X \min_Y f(X, Y)$ and $\min_Y \max_X f(X,Y)$ exist. Then prove that

$\max_X \min_Y f(X, Y) \leq \min_Y \max_X f(X,Y).$

(4 × 4 = 16 marks)

## Part B

*Answer any* **four** *questions without omitting any unit.*
*Each question carries* 16 *marks.*

Unit I

II. (a) Define convex hull of a set. Prove that the convex hull of the set S is the set of all convex linear combinations of points in S.

(b) Fine the convex hull of the set of points {(1, 0, 0), (0, 1, 0), (0, 0, 1)} in $E_3$.

III. (a) Let $f(X)$ be defined in a convex domain $K \subseteq E_n$ and be differentiable. Then prove that $f(X)$ is a convex function if and only if $f(X_2) - f(X_1) \geq (X_2 - X_1)' \nabla f(X_1)$ for all $X_1, X_2 \in K$, where $\nabla f(X_1)$ denotes the Hessian of $f(X_1)$.

(b) Prove that $f(x) = x^2, x \in \mathbb{R}$ is a convex function.

IV. (a) Prove that the set $S_F$ of feasible solutions, if not empty, is a closed set bounded from below and so has atleast one vertex.

Turn over

(b) Solve graphically the linear programming problem :

Minimize $z = x_1 + 3x_2$ subject to $x_1 + x_2 \geq 3, -x_1 + x_2 \leq 2, x_1 - 2x_2 \leq 2, x_1 \geq 0, x_2 \geq 0.$

Unit II

V. (a) Define canonical form of equations. Write the advantages of putting an equation in the canonical form.

(b) Use Simplex method to solve the problem :

Maximize $f(X) = 5x_1 + 3x_2 + x_3$ subject to the constraints $2x_1 + x_2 + x_3 = 3, -x_1 + 2x_3 = 4,$ $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

VI. (a) Define dual of a linear programming problem. Prove that the dual of the dual of a linear programming problem is the original problem.

(b) Write the dual of the linear programming problem :

Maximise $x_1 + 6x_2 + 4x_3 + 6x_4$ subject to $2x_1 + 3x_2 + 17x_3 + 80x_4 \leq 48,$ $8x_1 + 4x_2 + 4x_3 + 4x_4 = 21, x_2, x_3 \geq 0, x_1, x_4$ unrestricted in sign.

VII. (a) Define transportation matrix. Write an example of a transportation matrix.

(b) Solve the transportation problem for minimum cost with the cost coefficients, demands and supplies as given in the following table. Obtain three optimal solutions :

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
|---|---|---|---|---|---|
| $O_1$ | 1 | 2 | – 2 | 3 | 70 |
| $O_2$ | 2 | 4 | 0 | 1 | 38 |
| $O_3$ | 1 | 2 | – 2 | 5 | 32 |
| | 40 | 28 | 30 | 42 | |

Unit III

VIII. (a) Describe the Caterer problem in general terms.

(b) A batch of four jobs can be assigned to five different machines. The set up time for each job on each machine is given in the following table. Find an optimal assignment of jobs to machines which will minimise the total set-up time :

Machines

| | | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| | 1 | 10 | 11 | 4 | 2 | 8 |
| | 2 | 7 | 11 | 10 | 14 | 12 |
| Jobs | 3 | 5 | 6 | 9 | 12 | 14 |
| | 4 | 13 | 15 | 11 | 10 | 7 |

IX. (a) Write the summary of the transportation algorithm.

(b) Solve the following integer programming problem by cutting plane method :

Minimise $4x_1 + 5x_2$ ; subject to $3x_1 + x_2 \geq 2$, $x_1 + 4x_2 \geq 5$, $3x_1 + 2x_2 \geq 7$, $x_1, x_2$ non-negative integers.

X. (a) Describe the different characteristics by which games can be classified.

(b) Solve graphically the game whose pay-off matrix is :

$$\begin{pmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{pmatrix}.$$

(4 × 16 = 64 marks)

**C 63435** (Pages : 2) Name..................................

Reg. No..................................

## THIRD SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper XIV—DIFFERENTIAL GEOMETRY

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

### Part A

*Answer* **all** *questions.*
*Each question carries* 4 *marks.*

1. (a) Define dot product of two vectors in $\mathbb{R}_p^{n+1}$ and find a vector $(p,v) \in \mathbb{R}_p^2$ so that $\|(p,v)\| = 2$ where $p = (1, 1)$.

   (b) For $f(x_1, x_2) = x_1^2 + x_2^2 - 1$ describe $\nabla f(p)$ at $p \in f^{-1}(0)$.

   (c) Find the acceleration and speed of the parametrized curve $\alpha(t) = (\cos t, \sin t)$.

   (d) Find the normal curvature of the hyperboloid $-x_1^2 + x_2^2 + x_3^2 = 1$ at $v = (p,(1,0,0))$ with $p = (0, 0, 1)$.

(4 × 4 = 16 marks)

### Part B

*Answer any* **four** *questions without omitting any unit.*
*Each question carries* 16 *marks.*

Unit I

2. (a) Define gradient of a smooth function $f : \mathrm{U} \to \mathbb{R}$ where $\mathrm{U} \subseteq \mathbb{R}^{n+1}$.

   (b) Show that the gradient of $f$ at $p \in f^{-1}(c)$ is tangent to all vectors tangent to $f^{-1}(c)$ at $p$ where $c \in \mathbb{R}$.

3. (a) Define Lagrange multiplier.

   (b) State and prove the theorem on existence of Lagrange multipliers.

4. (a) Define oriented $n$-surface.

   (b) Prove that each oriented $n$-surface has exactly two orientations.

   (c) Give an example of an orientation on a connected 2-surface.

Unit II

5. (a) Define geodesic in an $n$-surface.

   (b) State and theorem on existence of maximal geodesics on an $n$-surface and sketch the proof.

6. (a) Define parallel vector field along a parametrized curve $\alpha$.

   (b) Show that for any parametrized curve $\alpha$ in an $n$-surface S there exists a parallel vector field along $\alpha$.

7. (a) Define Weingarten map $L_p$.

   (b) Prove that $L_p(v).w = v.L_p(w)$ for all $v, w \in S_p$.

Unit III

8. (a) Define second fundamental form of an $n$-surface S at $p \in S$.

   (b) Let S be a compact oriented $n$-surface. Show that there is a point $p \in S$ such that the second fundamental form at $p$ is definite.

9. (a) Define the differential $d\phi$ of a smooth map $\phi : U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^n$.

   (b) Show that the matrix of $d\phi_p : \mathbb{R}^n_p \to \mathbb{R}^m_{\phi(p)}$ relative to the standard basis of $\mathbb{R}^n_p$ and $\mathbb{R}^m_{\phi(p)}$ is the Jacobian matrix of $\phi$ at $p$.

   (c) Show that if $\psi : U \to U$ is a smooth map then $d(\phi \circ \psi) = d\phi \circ d\psi$.

10. (a) Define local parametrization of an $n$-surface.

    (b) Show that for any $n$-surface S in $\mathbb{R}^{n+1}$ there exists a local parametrization of S.

(4 × 16 = 64 marks)

**C 63434** (Pages : 2) Name..............................

Reg. No..............................

# THIRD SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper XIII—TOPOLOGY—II

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries* 4 *marks.*

I. (a) A topological space X has the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Then show that X is a normal space.

(b) If the evaluation function of a family of functions is one-to-one then prove that the family distinguishes points.

(c) Define directed set. Write an example for a directed set.

(d) Define ultrafilter. Prove that an ultrafilter converges to a point if and only if that point is a cluster point of it.

($4 \times 4 = 16$ marks)

## Part B

*Answer any* **four** *questions without omitting any unit.*
*Each question carries* 16 *marks.*

Unit I

II. (a) Prove that a subset of a topological space is a box if and only if it is the intersection of family of walls.

(b) Let $C_i$ be a closed subset of a space $X_i$ for $i \in I$. Then prove that $\prod_{i \in I} C_i$ is a closed subset of $\prod_{i \in I} X_i$ with respect to the product topology.

III. (a) If the product is non-empty, then prove that each co-ordinate space is embeddable in it.

(b) Prove that a topological product is path-connected if and only if each co-ordinate space is path-connected.

IV. (a) Prove that the projection functions are open.

(b) Let (X, $d$) be a metric space and let $\lambda$ be any positive real number. Prove that there exists a metric $e$ on X such that $e(x,y) \leq \lambda$ for all $x, y \in X$ and $e$ induces the same topology on X as $d$ induces.

**Turn over**

## Unit II

V. (a) Prove that a topological space is completely regular if and only if the family of all continuous real-valued functions on it distinguishes points from closed sets.

(b) Prove that a second countable space is metrisable if and only if it is $T_3$.

VI. (a) Suppose $S : D \to X$ is a net in a topological space and let $x \in X$. Then prove that $x$ is a cluster point of S if and only if there exists a subnet of S which converges to $x$ in X.

(b) Prove that a subset of a topological space X is closed if and only if limits of nets in A are in A.

VII. (a) Prove that every filter is contained in an ultrafilter.

(b) Prove that a topological space is compact if and only if every ultrafilter in it is convergent.

## Unit III

VIII. (a) Prove that a continuous image of a countably compact space is countably compact.

(b) Prove that sequential compactness is a countably productive property.

IX. (a) Prove that an open subspace of a locally compact regular space is locally compact.

(b) Prove that among all Hausdorff compactifications of a Tychnoff space, the Stone-Cech compactification is the largest compactification upto a topological equivalence.

X. (a) Prove that every compact metric space is complete.

(b) Prove that a non-empty complete metric space is of second category.

($4 \times 16 = 64$ marks)

C 63433 (Pages : 2) Name..................................

Reg. No..................................

# THIRD SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper XII—FUNCTIONAL ANALYSIS

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries* 4 *marks.*

1. (a) Define Schauder basis. Give example.

(b) Let $E_1$ and $E_2$ be subsets of a normed linear space X with $E_1$ open. Show that $E_1 + E_2$ is open.

(c) Let X be an inner product space and $\{x_1, \ldots x_n\}$ be an orthogonal set in X. Then show that

$$\|x_1 + \ldots + x_n\|^2 = \|x_1\|^2 + \ldots + \|x_n\|^2 .$$

(d) Let $X = R^2$, $x_1 = (1, 1)$, $x_2 = (1, 2)$. Show that $x_1$ and $x_2$ are linearly independent. Use Gram Schmidt process to find an orthonormal set $\{u_1, u_2\}$ such that span $\{x_1, x_2\} = \text{span}\{u_1, u_2\}$.

($4 \times 4 = 16$ marks)

## Part B

*Answer any* **four** *questions without omitting any unit.*
*Each question carries* 16 *marks.*

Unit I

2. (a) Let X be a normed linear space. If Z is an hyperspace in X, then show that there is a linear functional $f$ on X such that $Z = Z(f)$.

(b) Let X be a normed space. Show that X is finite dimensional if and only if every closed and bounded subset of X is compact.

3. (a) Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on a linear space X, show that $\|\cdot\|$ is stronger than $\|\cdot\|'$ if and only if there is some $\alpha > 0$ such that $\|\cdot\|' \leq \alpha \|\cdot\| \forall x \in X$.

(b) Show that for $1 \leq p \leq \infty$, the metric space $L^p(E)$ is complete for any measurable subset E of R.

4. (a) State and prove Riesz lemma.

(b) Let $X = K^3$. For $x = (x(1), x(2), x(3)) \in X$. Show that $\|x\| = \left[\left(|x(1)|^2 + |x(2)|^2\right)^{\frac{3}{2}} + |x(3)|^3\right]^{\frac{1}{3}}$ is a norm on $K^3$.

Turn over

Unit II

5. (a) Show that every linear map on a finite dimensional normed space is continuous.

(b) State and prove Hahn-Banach separation theorem.

6. (a) Show that a linear functional $f$ on a normed space X is continuous if and only if the zero space of $f$ is closed in X.

(b) State and prove Polarization identity.

7. (a) Let $H = L^2([0, 1])$. Find an orthonormal basis for H.

(b) State and prove Schwarz inequality.

Unit III

8. (a) Show that a Banach space cannot have a denumerable (Hamel) basis.

(b) State and prove uniform boundedness principle.

9. (a) Show that every normed space can be embedded as a dense subspace of a Banach space.

(b) State and prove open mapping theorem.

10. (a) Show that the closed graph theorem may not hold for normed spaces.

(b) Let X and Y be Banach spaces. Let $F \in BL(X, Y)$ be bijective, then show that $F^{-1} \in BL(Y, X)$.

($4 \times 16 = 64$ marks)

C 66482 (Pages : 3) Name.................................

Reg. No.................................

# THIRD SEMESTER M.Sc. (SDE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper XI—COMPLEX ANALYSIS

(2008 Admissions)

Time : Three Hours Maximum : 80 Marks

## Part A

*Answer* **all** *questions.*
*Each question carries 4 marks.*

1. (a) Find the linear transformation which carries $0, i, -i$ into $1, -1, 0$.

   (b) If the piecewise differentiable closed curve $\gamma$ does not pass through the point $a$, then prove that the value of the integral $\int_{\gamma} \frac{dz}{z-a}$ is a multiple of $2\pi i$.

   (c) Find the poles and residues of the function $\frac{1}{\left(z^2-1\right)^2}$.

   (d) Give the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ in $0 < |z| < 1$.

(4 × 4 = 16 marks)

## Part B

*Answer any* **four** *questions without omitting any unit.*
*Each question carries 16 marks.*

Unit I

2. a) Prove that an analytic function in a region $\Omega$ whose modulus is constant must reduce to a constant in $\Omega$.

   b) Prove that the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle or on a straight line.

3. a) Find the linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point $-2$ into the origin, and the origin into $i$.

   b) Describe the Riemann surface associated with the function $w = \cos z$.

Turn over

4. (a) Prove that the line integral $\int_{\gamma} pdx + qdy$ defined in a region $\Omega$ depends only on the end points of arc $\gamma$ if and only if there exists a function $U(x, y)$ in $\Omega$ with the partial derivatives $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.

(b) Let $f(z)$ be analytic on the set $R^1$ obtained from a rectangle R by omitting a finite number of inferior points $\mathcal{G}j$. If $\lim_{z \to G_j} (z - G_j) f(z) = 0$ for all $j$, then prove that $\int_{\partial R} f(z)\, dz = 0$ where $\partial R$ is the boundary of R.

Unit II

5. (a) Suppose that $\phi(\mathcal{G})$ is continuous on the arc $\gamma$. Prove that the function $F_n(z) = \int_{\gamma} \frac{\phi(G)}{(G - z)^n} dG$ is analytic in each of the regions determined by $\gamma$. Also prove that $F_n'(z) = nF_{n+1}(z)$.

(b) If $p(z)$ is a non-constant polynomial, then prove that there exists a complex number $a$ with $p(a) = 0$.

6. (a) Let $z_j$ be the zeros of a function $f(z)$ which is analytic in a disk $\Delta$ and does not vanish identically, each zero being counted as many times as its order indicates. For every closed curve $\gamma$ in $\Delta$ which does not pass through a zero, prove that

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

where the sum has only a finite number of non-zero terms.

(b) Prove that an analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity.

7. (a) State and prove Rouche's theorem.

(b) Evaluate the integral $\int_0^{\infty} \frac{\cos x}{x^2 + a^2} dx$, where $a$ is real.

Unit III

8. (a) Prove that the arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of $\log r$.

(b) Suppose that $u(z)$ is harmonic for $|z| < R$, and continuous for $|z| \le R$. Prove that

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta$$

for all $|a| < R$.

9. (a) If the functions $f_n(z)$ are analytic and non-zero in a region $\Omega$, and if $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of $\Omega$, then prove that $f(z)$ is either identically zero or never equal to zero in $\Omega$.

(b) Develop $\frac{1}{1+z^2}$ in powers of $z+1$.

10. (a) Prove that an elliptic function without poles is a constant.

(b) Briefly describe about the Weierstrass $\wp$-function $\wp(z)$. Also prove that

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z).$$

(4 × 16 = 64 marks)