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Chapter 1

Preliminaries

The chapter explores the graph theoretic terminology and notations that will appear in the subsequent chapters. We adopt the basic definitions and notations as in Graph Theory [20], written by J.A. Bondy and U.S.R. Murty. This chapter includes three sections. The first section deals with basic definitions and notations that may appear in the forthcoming chapters. In the second section various graph theoretic operations are discussed. Third section incorporates some basic results and theorems which are used in the forthcoming chapter to study the roots of polynomials.

1.1 Basic terminology

A **graph** G is an ordered pair (V, E) consisting of the disjoint sets V of vertices and E of edges, together with an incidence function $\psi : E \to V \times V$ which associates each edge of G with an unordered pair of vertices of G. A graph having finite number of vertices and edges is called a **finite graph**. The number of vertices and number of edges of a finite graph G are called the **order** and **size** of G respectively. Two or more edges having same end vertices are called **multiple edges** and an edge with identical end vertices is called a **loop**. A graph is **simple** if it has no multiple edges or loops.

The end vertices of an edge are said to the **incident** with the edge. Two **vertices are adjacent** if they are incident with a common edge and two **edges are adjacent** if they are incident to a common end vertex. Two adjacent vertices are said to be neighbors of each other. The set of all neighbors of a vertex $v \in V$ is called the **neighbor set of** v and is denoted by $N(v)$. The number of vertices in $N(v)$ is called the **degree of** v. Vertices of degree 1 are called **pendent vertices**. A graph having all the vertices with same degree is called a **regular graph**. A subset S of the set of vertices of a graph G in which any two distinct vertices are adjacent is called a **clique** in G.

Let G be a graph of order n. Then the **adjacency matrix** of G is a $n \times n$ matrix in which the ij^{th} entry becomes 1 or 0 according as the pair of vertices v_i and v_j are adjacent or not in G.

A **complete graph** is a simple graph in which all the pairs of vertices are adjacent. A graph is **bipartite** if its vertex set can be partitioned into two subsets X and Y so that any edge of G has one end vertex in X and the other in Y. If each vertex of X is joined to every vertex of Y in a bipartite graph, it is called a **complete bipartite graph**.

A **complete m-partite graph** $K_{n_1,n_2,\ldots,n_m}$ is a graph whose vertex set can be partitioned into m non empty sets $V_i, i = 1, 2, \ldots, m$ such that every vertex in V_i is adjacent to every vertex in V_j for every $i \neq j$ and $i, j \in \{1, 2, \ldots, m\}$.

A **walk** is an alternating sequence $v_0e_1v_1e_2 \ldots v_{i-1}e_iv_i \ldots v_n$ of vertices and edges in which the vertices v_{i-1} and v_i are the end points of the edge e_i. The length of a walk is the number of edges in the walk. A **path** is a walk having all the vertices distinct. A path on n vertices is denoted by P_n. A **trail** is a walk where all the edges are distinct. A closed trail in which all the vertices are distinct is called a **cycle**. A cycle of length n is denoted by C_n. A graph G is **connected** if for each pair of vertices u and v in $V(G)$, there is a u-v path in G. A **disconnected graph** is a graph which is not connected. A graph is **acyclic** if it contains no cycles. A connected acyclic graph is called a **tree**.

The **distance** between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest u-v path in G. The maximum distance between any pair of vertices of G is called the diameter of G. The **Hosoya polynomial**[26] of G is defined as $H(G, x) = \sum_{j=1}^{l} d(G, j)x^j$ where $d(G, j)$ denote the number of pairs of vertices in G having distance j apart and l denote the diameter of the graph.

A **Wheel graph** $W_n, n > 3$ is obtained by taking the join of the cycle C_{n-1} and K_1. A **helm,** $H_n, n > 3$ is obtained from a wheel graph W_n by adding pendent edges to every vertices on the wheel rim. A **web graph** $WB_n, n > 3$ is obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. WB_n has $3n - 2$ vertices and $3(n - 1)$ edges. A **shell graph** S_n where $n \geq 3$ is obtained from the cycle graph C_n by adding the edges corresponding to the $(n - 3)$ concurrent chords of the cycle. The vertex at which all the chords are concurrent is called the apex of the shell. A **bow graph** is a double shell with same apex in which each shell has any order.

A **butterfly graph** is a bow graph along with exactly two pendent edges at the apex. A **friendship graph** F_n is the one point union of n copies of the cycle C_3. A **Tadpole** $T_{(n,l)}$ is a graph obtained by attaching a path P_l to one of the vertices of the cycle C_n by a bridge. The n**- barbell graph** $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge. The **Lollipop graph** $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path P_n with a bridge.

A **bistar graph** $B_{m,n}$ is obtained by connecting the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by a bridge. The **bipartite Cocktail party graph** B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$. The **Windmill graph** $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common. An **armed crown** $C_n \bigodot P_m$ is a graph obtained by attaching a path P_m to every vertex of the cycle C_n.

A simple k-regular graph G on n vertices is said to be **strongly regular of type** (n, k, λ, l) if there exists integers λ, l such that any adjacent pair of vertices of G have exactly λ common neighbors and any non-adjacent pair of vertices of G have exactly l common neighbors.

A **rooted tree**[13] is a tree in which one of the vertices is distinguished as the root. According to the distance of other vertices from the root vertex, there is a hierarchy on the vertices of a rooted tree. The distance of a vertex v from the root is called the depth or level of the vertex. The height of a rooted tree is the greatest depth of a vertex of the tree. Considering a path from the root to a vertex w, if a vertex v immediately precedes w , then v is called the parent of w and w is called the child of v. Vertices having same parent are called siblings.

An **m-ary tree** ($m \geq 2$) is a rooted tree in which every vertex has m or fewer number of children. A **complete m-ary tree** is an m-ary tree in which every internal vertices has exactly m children and all leaves are of same distance from the root.

The **derivative of a graph** G is a graph obtained from G by deleting all the pendent vertices of G. A **caterpillar**[39] is a tree graph whose derivative is a path graph . Consequently, a caterpillar $P_n(m_1, m_2, \ldots, m_n)$ is obtained by attaching m_i pendent edges to the vertex v_i of a path P_n where $i \in \{1, 2, \ldots, n\}$. A **star like tree graph** $S(n_1, n_2, \ldots, n_k)$[24] is a graph having only one vertex w of degree greater than 2 such that deletion of w results in a disjoint union of the path graphs $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$. The star like tree graphs are used to represent proteins which will have generally 20 branches where each branch indicates the presence of one of the 20 natural amino acids.

Let G and H be two graphs with incidence functions ψ_G and ψ_H respectively. Then G and H are **isomorphic**[33] if there exists bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$ where $u, v \in V(G)$ and $e \in E(G)$.

1.2 Graph operations

The **splitting graph** [12] $S(G)$ of a graph G is obtained by adding new vertices v' to G corresponding to each vertex v of G and then joining the vertex v' to all vertices of G adjacent to v in G. The **shadow graph** $Sh(G)$ of a graph G is obtained by taking two copies of G, say G_1 and G_2 and joining each vertex of

G_1 to the neighbors of the corresponding vertex of G_2. The **Mycielski graph,** $\mu(G)$ [22] of a graph G contains G itself as an isomorphic subgraph together with $n+1$ additional vertices; a vertex v_i corresponding to each vertex u_i of G and another vertex w. Each v_i is connected by an edge to w and for each edge u_iu_j of G, $\mu(G)$ includes two additional edges u_iv_j and v_iu_j.

Consider the graph $G(V, E)$ and let $w \notin V$. Then the graph $G' = G + w$ is a graph obtained from G by including the vertex w in G and joining it to all other vertices of G. If H and K are two graphs, then the **join,** $H \vee K$ is the graph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K) \cup \{uv : u \in V(H), v \in V(K)\}$.

The **corona of two graphs**[13] K and H is formed from one copy of K and $|V(K)|$ copies of H where the i^{th} vertex of K is adjacent to every vertex in the i^{th} copy of H[35]. It is denoted by $K \circ H$. The **Cartesian product**[13] of two graphs G and H is the graph $G \Box H$ with vertex set $V(G) \times V(H)$ and the vertices (u, v) and (x, y) are adjacent if and only if $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

A **rooted graph** is a graph in which one vertex is distinguished as a root. The **rooted product**[3] of a graph G and a rooted graph H is obtained as follows: Take $|V(G)|$ copies of H and for each vertex v_i of G, identify v_i with the root vertex of the i^{th} copy of H. The **tensor product**[13] of two graphs K and H is the graph $K \times H$ with vertex set $V(K) \times V(H)$ and the vertices (u, v) and (x, y) are adjacent if and only if $ux \in E(K)$ and $vy \in E(H)$.

1.3 Polynomials

The following theorems can be used to study the number of real roots of polynomials.

Theorem 1.3.1 (de Gua's Theorem [42]). *If the polynomial $f(x)$ lacks $2m$ consecutive terms then it has no less than $2m$ imaginary roots. If $2m+1$ consecutive terms are missing then, if they are between terms of different signs, the polynomial has no less than $2m$ imaginary roots, whereas, if the missing terms are between terms of same sign, the polynomial has no less than $2m+2$ imaginary roots.*

Theorem 1.3.2 (S. Kakeya [40]). *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with real coefficients satisfying $a_0 < a_1 \leq a_2 \leq \ldots \leq a_n$, then all the zeros of $p(z)$ lie in $|z| \leq 1$.*

Theorem 1.3.3. *[37] Consider the cubic equation $ax^3 + bx^2 + cx + d = 0$. Then the discriminant of the cubic equation is given by $\Delta = b^2c^2 - 4ac^3 - 4b^d + 18abcd - 27a^2d^2$. If $\Delta > 0$, the equation has three real distinct roots; if $\Delta = 0$, the equation has three real roots in which one of them is a multiple root; if $\Delta < 0$, the equation has one real root and two imaginary roots.*

A polynomial $f(x_1, \ldots, x_n)$ is said to be **stable** [28] with respect to a region $\Omega \in \mathbb{C}^n$ if no root of f lies in Ω. Polynomials which are stable with respect to the closed right half plane and with respect to the open unit disk are called Hurwitz polynomial and Schur polynomial respectively. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems[15].

Let $\mathcal{G}$ be the set of finite graphs on n vertices and $R[x]$ the polynomial ring over the real numbers. Then a graph polynomial is a function $P : \mathcal{G} \rightarrow R[x]$ such that for any two graphs $G, H \in \mathcal{G}$, if G is isomorphic to H, then $P(G) = P(H)$. A graph polynomial encodes information about the graph and enables algebraic methods for extracting this information.

With the introduction of edge difference polynomial[21] in 1878, J.J. Sylvester initiated the study of graph polynomials which was further studied by J. Petersen in 1891. Since then many graph polynomials were introduced among which matching polynomial[9], chromatic polynomial[16], Hosoya polynomial[26] and domination polynomial[36] are well popularized.