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NON-VANISHING OF MODULAR L-FUNCTIONS INSIDE THE CRITICAL STRIP

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CERTIFICATE

I hereby certify that the thesis entitled "**Non-vanishing of modular L-functions inside the critical strip**" is a bonafide work carried out by **Mr. Sandeep E. M.**, under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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NOTATIONS

Symbol	Description
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of rational integers
$\mathcal{P}$	The set of rational prime numbers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{H}$	The complex upper half plane
$\mathbb{D}$	The open unit disc
$GL_2^+(\mathbb{R})$	The group of all 2×2 real matrices with positive determinant
$SL_2(\mathbb{Z})$	The group of all 2×2 integer matrices with determinant 1
$\mathcal{D}$	The fundamental domain for the usual $SL_2(\mathbb{Z})$ action on $\mathbb{H}$
$a\|b$	a divides b
(a, b)	Greatest common divisor of the integers a and b
$\bar{z}$	Complex conjugate of a complex number z
$\Re(z)$	Real part of z
$\Im(z)$	Imaginary part of z
$\|z\|$	Absolute value of z
Γ	The Euler-gamma function
$\zeta(s)$	The Riemann zeta function

$\lfloor x \rfloor$	The greatest integer $\leq x$
$\dim V$	Dimension of the vector space V
$K \subseteq X$	K is a subset of X
$K^{\mathsf{c}} := X \setminus K$	The complement of K in X.

We mention a few asymptotic notations next.

1) $f(s) = \mathcal{O}(g(s)),\ s \in S$ or equivalently, $f(s) \ll g(s),\ s \in S$ means there exists a constant c such that $|f(s)| \leq c|g(s)|$ for all $s \in S$.

2) $f(s) = o(g(s)),\ s \to s_0$ means $\lim_{s \to s_0} \frac{f(s)}{g(s)} = 0$.

3) $f(s) \asymp g(s)$ means $f(s) \ll g(s)$ & $g(s) \ll f(s)$, $s \in S$.

4) $f(s) \sim g(s),\ s \to s_0$ means $\lim\limits_{s \to s_0} \frac{f(s)}{g(s)} = 1$.

Few Remarks The letters m, n, M, N would usually denote positive integers. The letters a, b, c, d would usually denote integers unless specified otherwise. The symbol τ would be reserved for the Ramanujan tau-function. Complex numbers would be denoted using $z = x + iy$ unless in the context of L-functions, where we would stick to the conventional notation $s = \sigma + it$. However, in Chapter 4, we would use $s = \sigma + i\beta$ (with or without sub-scripts). The symbols ϵ and δ (with or without subscripts, including other variants like δ' etc) may be used to denote arbitrary positive real numbers, usually small though. However, in Chapters 3 & 4, we assign a meaning to ϵ (with or without subscripts), where it could take negative values too. For a complex $s = \sigma + it$ and real numbers a and b, sometimes, we use the notation $\{a < \sigma < b\}$ to denote $\{s \in \mathbb{C} \mid a < \Re(s) < b\}$.

Contents

Chapter 1

Introduction

1.1 Review of the work of Euler and Riemann

L functions are one of the classical objects of interest for number theorists and were studied at least from the time of Euler. Euler initiated the study of the famous *Riemann-Zeta function*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

for real $s > 1$ and realised that this function is intimately connected to the prime numbers ($\mathcal{P}$) in the following way:-

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} \qquad (s > 1). \tag{1.1}$$

This identity is an analytic equivalent for the unique prime factorisation of any natural number. As $s \to 1$ from the right, it follows that $\zeta(s) \to \infty$ since the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ diverges. The right hand side of the identity (1.1) led Euler to a proof of the fact that $\sum_{p\in\mathcal{P}} \frac{1}{p}$ diverges from which the infinitude of primes also follows.

Inspired by Euler, in his epoch-making memoir of 1860 [Rie59], Riemann showed that the further study of the distribution of primes lies in the study of $\zeta(s)$ for a complex variable s, in particular, in the study of the zeros of the meromorphic continuation of $\zeta(s)$. The proof of the *prime number theorem*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$$

by Hadamard and de la Vallée Poussin has at its heart the non-vanishing of ζ function on the line $\Re(s) = 1$, thus using Riemann's findings.

In addition, Riemann also showed that this meromorphic continuation, which we shall again call ζ, satisfies a certain functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$

which could be thought of as a consequence of a certain transformation law satisfied by the Jacobi θ function

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}, \quad z \in \mathbb{H}.$$

The functional equation and Euler product of $\zeta(s)$ along with its non-vanishing nature on $\Re(s) = 1$ shows that all the (non-trivial) zeros of $\zeta(s)$ has to lie in the region $\{0 < \Re(s) < 1\}$. In [Rie59], Riemann also made an ingenious conjecture about these zeros.

Riemann Hypothesis All the non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.

Inspired by Riemann's zeta function, many similar L-functions were introduced into number theory like Dedekind zeta function, L-function of an elliptic curve, Hecke L-function, Artin L-function etc. In 1921, Hamburger showed that any Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

satisfying such a functional equation is essentially a zeta function under some regularity conditions. These ideas were greatly generalised by Hecke in 1936 to the context of automorphic forms. Essentially, he showed that if a Dirichlet series satisfies a certain functional equation, then they are of the form $L(f, s)$ for some modular form f with respect to $SL_2(\mathbb{Z})$ and of integral weight k under similar analytic conditions on the Dirichlet series.

Appearance of modular forms can be traced back to 19$^{\text{th}}$ century in the works of Jacobi, Gauss, Kronecker, Klein, Poincaré etc. They appeared naturally in the theory of elliptic functions and binary quadratic forms. In the beginning of 20$^{\text{th}}$ century, the contributions from Ramanujan, Mordell, Hecke, Petersson etc resulted in the systematic development of its theory.

1.2 Introduction

Let $k \geq 12$ be an even positive integer such that $k \neq 14$. Let S_k denote the space of holomorphic cusp forms of weight k with respect to the full modular group $SL_2(\mathbb{Z})$. Let

$$f(z) \ = \ \sum_{n \geq 1} a_f(n) e^{2\pi i n z},$$

be an arithmetically normalised Hecke eigenform[1] in S_k. Its associated L-series

$$L(f,s) := \sum_{n \geq 1} \frac{a_f(n)}{n^s},$$

defines a holomorphic function in the region $\Re(s) > \frac{k+1}{2}$ and has an Euler product expansion here. Further, it can be analytically continued (uniquely) to $\mathbb{C}$ as an entire function, which we denote again by $L(f,s)$. This analytic continuation also satisfies the following functional equation

$$(2\pi)^{-s}\Gamma(s)L(f,s) = (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)L(f,k-s).$$

By virtue of the Euler product, functional equation and the knowledge of non-vanishing on $\{\Re(s) = \frac{k+1}{2}\}$, it is known to have all its non-trivial zeros lying inside the critical

[1] Hereafter, in this Chapter, by a Hecke eigenform, we mean an arithmetically normalised Hecke eigenform

strip $\{\frac{k-1}{2} < \Re(s) < \frac{k+1}{2}\}$.

This thesis studies the non-vanishing aspects of L-functions of Hecke eigenforms inside the critical strip. By the functional equation, it suffices to study the region $\frac{k}{2} \leq \Re(s) < \frac{k+1}{2}$. Let $\mathcal{B}_k$ denote the orthogonal basis (with respect to the standard Petersson inner product) of Hecke eigenforms in S_k. Grand Riemann Hypothesis, in this context, predicts that for any $f \in \mathcal{B}_k$, $L(f,s) \neq 0$ for all s satisfying $\frac{k}{2} < \Re(s) < \frac{k+1}{2}$. However, the existence of even one such Hecke eigenform is not currently known.

1.3 Survey of Literature

Let us first consider the question of non-vanishing of $L(f,s)$ at real points inside the critical strip. From the functional equation, it follows that

$$L(f, k/2) = 0 \quad \text{if } k = 2 \bmod 4.$$

However, if you consider the interval $(\frac{k}{2}, \frac{k+1}{2})$, it is not known yet whether an f exists whose L-value is non-zero here. Although, in cases where $\dim S_k = 1$, i.e., when $12 \leq k \leq 26$ and $k \notin \{14, 24\}$, it is known due to Murty ([RM83], Theorem 6) that $L(f,s)$ is monotonically increasing on $(\frac{k}{2}, \frac{k+1}{2})$. Also, in the two dimensional space S_{24}, he shows the existence of an f satisfying $L(f,s) \neq 0$ on $s \in (\frac{k}{2}, \frac{k+1}{2})$.

In a different direction, one may start with an arbitrary choice of s in the critical strip (not lying in the critical line) and ask if one can find a Hecke eigenform f which satisfies $L(f,s) \neq 0$. Kohnen, in [Koh97], answered this by showing the non-vanishing of a certain sum of L functions evaluated at a given point s, for all weights k sufficiently large, depending on $\Im(s)$.

Theorem 1.3.1. *[Koh97] Let $t_0 \in \mathbb{R}$ and $\delta > 0$. Then, there exists a constant $C = C(t_0, \delta) > 0$ such that for $k \geq C(t_0, \delta)$, the sum*

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)}{\langle f, f \rangle} \neq 0 \tag{1.2}$$

for $s = \sigma + it_0$, where $\frac{k-1}{2} < \sigma < \frac{k}{2} - \delta$ or $\frac{k}{2} + \delta < \sigma < \frac{k+1}{2}$.

Here, $L^*(f,s)$ denotes the completed L-function (entire) associated to f. As a corollary, it follows that given any point s on the horizontal line segments

$$\left\{\Im(s) = t_0, \frac{k-1}{2} < \Re(s) < \frac{k}{2} - \delta\right\} \cup \left\{\Im(s) = t_0, \frac{k}{2} + \delta < \Re(s) < \frac{k+1}{2}\right\},$$

there is at least one form[2] f in B_k such that $L(f,s) \neq 0$ for $k \gg_{t_0,\delta} 1$. In particular, given a real $\sigma \in (\frac{k}{2} + \delta, \frac{k+1}{2})$, one can find an $f \in \mathcal{B}_k$ such that $L(f, \sigma) \neq 0$ as long as k is large enough ($k \gg_\delta 1$). Later, in [CK18], Kohnen, along with Choie, improve upon this and indicate a proof of the non-vanishing for **all** k (multiples of 4).

Theorem 1.3.2. *[CK18] Let $k \geq 12$ be an integer such that $4|k$. Then, for a given real $\sigma \in [\frac{k}{2}, \frac{k+1}{2}]$, one can find an $f \in \mathcal{B}_k$ such that $L(f,\sigma) \neq 0$.*

Slightly digressing to the case of half integral weight cusp forms, in [CK18], Choie

[2]Choice of f may depend on the value of σ and t_0 considered

and Kohnen also show that for a given $\sigma \in \mathbb{R}$, there exists at least one Hecke eigenform in the eigen subspace $S^{(+,2)}_{k_1+\frac{1}{2}}(4)$ with non-vanishing L value at σ, for an integer $k_1 \geq 4$, where $S^{(+,2)}_{k_1+\frac{1}{2}}(4)$ is the 1-eigen subspace of $S_{k_1+\frac{1}{2}}(4)$ under the Fricke Involution W_4. They also provide a similar result inside $S^{(-,2)}_{k_1+\frac{1}{2}}(4)$.

1.4 Motivation for the thesis problem

1.4.1 Kohnen's cusp form

In order to prove (1.2), Kohnen uses a certain holomorphic form in S_k which is dual (with respect to the Petersson inner product) to the (completed) L function, which is sometimes called the *kernel function* (kernel to the linear functional L^*). This follows from Riesz representation theorem by viewing $L(f,s)$ (or $L^*(f,s)$) as a complex linear functional on the Hilbert space S_k. Note that the dual $f_{k,\bar{s}}$ satisfies

$$\langle g, f_{k,\bar{s}} \rangle = c_k L^*(g,s), \quad \text{for all } g \in S_k \tag{1.3}$$

for some constant c_k.

Similar approaches involving the respective kernel functions in the appropriate setting have been widely used in proving non-vanishing results of L-functions on an average by various authors: for higher levels and primitive character modulo level in [Rag05], for half integral weights of level $4N$ with (even) character modulo $4N$ in [RS14]. A kernel to the product of L-values of integral weight Hecke cusp forms at two complex points has also been studied in [CKZ20] which we shall return to, later on.

In all these scenarios, the approach has been to prove that the first Fourier coefficient of the kernel is non-zero for sufficiently large weights. Note that by the definition of kernel, we have

$$f_{k,s}(z) = C_{k,s} \sum_{f \in \mathcal{B}_k} L^*(f,s) \frac{f(z)}{\langle f, f \rangle}, \quad z \in \mathbb{H} \tag{1.4}$$

which is valid for any complex s satisfying $1 < \Re(s) < k-1$ (although we are primarily interested only in the situation $\Re(s) \in [\frac{k}{2}, \frac{k+1}{2}]$), where $C_{k,s}$ is a complex constant depending on k and s. From (1.4), it follows that the expression in (1.2) is (upto a constant) the first Fourier coefficient of this kernel.

1.4.2 Non-vanishing of L-values on an average

In connection with the Theorem (1.3.1) of Kohnen, it would be nice if one could remove the dependency of k on $\Im(s)$ and prove the non-vanishing of the sum in (1.2) at s for all k (here s lies inside the critical strip), and thus prove GRH at least for a weighted sum of Hecke eigenforms, if not individually. However, this thesis doesn't attempt to prove this.

Our first work (in Chapter 3) begins with the observation that in order to obtain such non-vanishing results on an average, one needn't rely on the Fourier expansion of the dual. In fact, it suffices to prove the non-vanishing of $f_{k,s}(z)$ at any point $z = z_0 \in \mathbb{H}$.

For example, when we evaluate $f_{k,s}(z)$ at $z = i$, for $\Re(s) := \sigma$ in $[\frac{k-1}{2}, \frac{k+1}{2}]$, we are, in fact, obtaining a different weighted sum of the form

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)f(i)}{\langle f, f \rangle}.$$

In Chapter 3, we prove the non-vanishing of the sums of the above form as a real valued function of s on the interval $[\frac{k-1}{2}, \frac{k+1}{2}]$ for all $k \geq 12$ such that $4|k$ and comment on certain consequences.

1.4.3 Counting Hecke eigenforms with non-vanishing L-value

In the context of the corollary to Theorem (1.3.1), it might be interesting to ask the following question.

Problem 1. *Given an integer k and a complex point s inside the critical strip, how many Hecke eigenforms in S_k have non-vanishing L values at s?*

For this purpose, let us define

$$N_k(s) := \#\{f \in \mathcal{B}_k \mid L(f,s) \neq 0\}. \tag{1.5}$$

We now mention a few asymptotic results known in the literature in this direction. When $4|k$, Luo ([Luo15], (4)) showed that

$$N_k(k/2) \gg k, \quad (k \to \infty).$$

That is, a positive proportion of Hecke eigenforms exist in S_k whose L-values are non-vanishing at the central critical point $s = \frac{k}{2}$ as $k \to \infty$ through multiples of 4. Prior to that, in [Sen00], the author proved[3] the lower bound

$$N_k(k/2) \gg_\delta k^{1-\delta}$$

when $4|k$, assuming the **Lindelöf hypothesis** in the k-aspect for $L(f, s)$, the method of which also involves Kohnen's kernel function. Note that GRH predicts that for an arbitrary k,

$$N_k(s) = \dim S_k = \frac{k}{12} + \mathcal{O}(1) \tag{1.6}$$

for all s satisfying $\frac{k}{2} < \Re(s) < \frac{k+1}{2}$.

Recently, in [CKZ20], the authors extended Theorem (1.3.1) to the simultaneous non-vanishing of L-values (on an average) *at two points* inside the critical strip. More precisely, given positive real numbers T and δ, they proved the non-vanishing of the sum

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f, s_1) L^*(f, s_2)}{\langle f, f \rangle}$$

for large enough $k \gg_{T,\delta} 1$ when $(s_1, s_2) \in R'_{T,\delta}$, where[4],

$$R'_{T,\delta} \ := \ \Big\{ (s_1 = \frac{k}{2} + \epsilon_1 + i\beta_1, s_2 = \frac{k}{2} + \epsilon_2 + i\beta_2) \in \mathbb{C}^2 \ |$$

[3]conditional

[4]Our notation $R'_{T,\delta}$ is the same as $R_{T,\delta}$ used in [CKZ20]

$$-T \leq \beta_1, \beta_2 \leq T,\ \delta \leq |\epsilon_1|, |\epsilon_2| < \frac{1}{2}\Big\}.$$

Here, they utilise a cusp form which is dual to the double product of L^* values of an arbitrarily given Hecke eigenform f with respect to Petersson Inner product (compare this with (1.3))

$$\langle E^*_{s_1,k-s_1}(z, s_2), f\rangle = L^*(f, s_1)L^*(f, s_2), \tag{1.7}$$

and compute the q-series expansion of $E^*_{s_1,k-s_1}(z, s_2)$, where $q := e^{2\pi iz}$, $(s_1, s_2) \in R'_{T,\delta}$, and extract the conditions when its first Fourier coefficient is non-vanishing. Again, as a consequence, they observe that for a given $(s_1, s_2) \in R'_{T,\delta}$, there exists a Hecke eigenform f in S_k such that $L(f, s_1)L(f, s_2) \neq 0$, when k is sufficiently large. In this context, we may also ask the following related question.

Problem 2. *Given a weight k and complex points s_1, s_2 such that $\frac{k-1}{2} \leq \Re(s_1), \Re(s_2) \leq \frac{k+1}{2}$, is it possible to quantify the following numbers*

$$N_k(s_1, s_2) := \#\{f \in \mathcal{B}_k \ \mid L(f, s_1) \cdot L(f, s_2) \neq 0\}, \tag{1.8}$$

in terms of k?

We had already motivated a need for evaluating the kernel function at a point $z = i$ on the imaginary axis so as to obtain a Riemann-Hypothesis kind of result on an (weighted) average for all weights k. In Chapter 4, we will show more applications that come out of evaluating the kernel function at any arbitrary point $z = it$ on the imaginary axis. We provide one such application in determining the lower bounds for L values in Chapter 3. In Chapter 4, we show nicer applications of this idea in addressing Problem (1) and Problem (2). Thus, we are in fact, exploring for results on non-vanishing of L-functions by studying within the modular forms and without resorting to usual analytic techniques like sieve methods, mollifiers etc.

1.5 Organisation of the thesis

This dissertation comprises of 5 chapters. In the introductory chapter, i.e., **Chapter 1**, we provide the motivation behind our work and provide a survey of various known works in regard to the non-vanishing aspects of L-functions of modular forms inside the critical strip. In the **Chapter 2**, we quickly gather the preliminaries in the theory of modular forms and L-functions. We also list few key lemmas and results from analytic number theory which we shall often need for computation/estimation purposes.

The study of the cusp form in (1.4) is evidently very much crucial to this thesis and is described in the first section of **Chapter 3**. Here, we gather few of its known properties in order to make the thesis as self-contained as possible. In the Section (3.2), we evaluate $f_{k,s}(z)$ at $z = i$ when s is a real parameter, i.e., $s = \sigma$, inside the critical strip and derive an asymptotic formula for $f_{k,\sigma}(i)$, a weighted sum of L-values of Hecke eigenforms at σ, in terms of k, as $k \to \infty$ through multiples of 4. In order to separate the main term and error term, we adapt a technique of Rankin and Swinnerton-Dyer [RSD70].

Theorem 1.5.1. *Let $4|k$. The cusp form $f_{k,\sigma}(z)$ in S_k satisfies the asymptotic relation*

$$f_{k,\sigma}(i) = 4 + \mathcal{O}(2^{-\frac{k}{4}})$$

at $z = i$ for all values of $\sigma \in [\frac{k-1}{2}, \frac{k+1}{2})$.

Further, by computation, we also provide an explicit lower bound for $f_{k,\sigma}(i)$ for all $k \geq 12$, $4|k$ in Section (3.3).

Theorem 1.5.2. *Let k be an integer as above and divisible by 4. Then, $f_{k,\sigma}(i)$ is real valued and satisfies*

$$f_{k,\sigma}(i) \ \geq 2.745.$$

One can also deduce Theorem (1.3.2) (due to Choie and Kohnen), from our Theorem (1.5.2). We discuss this in brief in Section (3.4). As an application of Theorem (1.5.2), we derive the following lower bound (Corollary (1.5.3)) for the maximum value of $|L(f,\sigma)|$ as f varies over $\mathcal{B}_k$. This is shown in Section (3.5).

Corollary 1.5.3. *Given $\sigma = \frac{k}{2} + \epsilon \ \in [\frac{k-1}{2}, \frac{k+1}{2}]$ and an arbitrarily small $\delta > 0$, for sufficiently large $k \geq K_\delta$, where $4|k$, we have*

$$\max_{f \in \mathcal{B}_k} |L(f,\sigma)| \gg_\delta \ k^{-(\sigma - \frac{k}{2}) - 1 - \delta}.$$

It should be noted that better lower bounds could be available for $\max_{f\in\mathcal{B}_k} |L(f,\sigma)|$ (cf. [Sou08], Theorem 3). In **Chapter 4**, we generalise our approach to other points on the imaginary axis and provide certain partial answers to Problem (1) and Problem (2) in Corollaries (1.5.7) and (1.5.8).

For this, first we obtain an asymptotic expression for $f_{k,s}(it)$ which is valid for any $t \geq 1$ and for any s on the critical strip with an apriori fixed imaginary part. As earlier, we adapt the method of Rankin and Swinnerton-Dyer [RSD70].

Theorem 1.5.4. *Let β be an arbitrary but fixed real number and let s be a complex number such that $\frac{k-1}{2} \leq \Re(s) \leq \frac{k+1}{2}$ and $\Im(s) = \beta$. Then, for all $t \geq 1$, the cusp form $f_{k,s}(it)$ satisfies the asymptotic relation $(k \to \infty)$*

$$f_{k,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e^{-2\pi nt} + (-1)^{k/2} 2\frac{(2\pi)^{k-s}}{\Gamma(k-s)} \sum_{n\geq 1} n^{k-s-1} e^{-2\pi nt} + \mathcal{O}_\beta\left(\frac{1}{t^{k-2}}\right).$$

Remark 1.5.5. *Note that the Corollary* (1.5.3) *gives a lower bound for* $\max_{f\in\mathcal{B}_k} |L(f,\sigma)|$ *for σ real. Using the asymptotic expression in Theorem (1.5.4), we prove that in fact,* $\max_{f\in\mathcal{B}_k} L(f,\sigma)$ *is a positive quantity and thus, may drop the modulus in the Corollary* (1.5.3).

For certain complex points s_1 and s_2 lying inside the critical strip with $|\Im(s_j)|$ atmost T, we study the Mellin Transform of f_{k,s_1} with respect to s_2 and prove the following:-

Corollary 1.5.6. *Let T be an arbitrary but fixed positive real number and let $0 < \delta, \delta' \leq 1/2$ be arbitrary but fixed positive reals. Let $R_{T,\delta,\delta'}$ denote the region of points*

$(s_1, s_2) \in \mathbb{C}^2$, $s_1 = \frac{k}{2} + \epsilon_1 + i\beta_1, s_2 = \frac{k}{2} + \epsilon_2 + i\beta_2$ *satisfying*

$$\begin{aligned} &\bullet\ -T \leq \beta_j \leq T \text{ for } j = 1, 2, \\ &\bullet\ 0 < |\epsilon_1| < |\epsilon_1| + \delta' \leq |\epsilon_2| \leq \frac{1}{2}, \\ &\bullet\ |\epsilon_1| + |\epsilon_2| \geq \frac{1}{2} + \delta. \end{aligned} \tag{1.9}$$

Then, there exists a constant $C = C(T, \delta, \delta') > 0$ *depending only on* T, δ, δ' *such that for* $k \geq C(T, \delta, \delta')$, *we have*

$$L^*(f_{k,s_1}, s_2) \gg_{T,\delta'} k^{|\epsilon_1|+|\epsilon_2|},$$

for any pair $(s_1, s_2) \in R_{T,\delta,\delta'}$.

We mention here that an identity for $L^*(f_{k,s_1}, s_2)$ in terms of well-known functions like ratios of Gamma functions, ζ functions, hypergeometric functions etc., is known ([KKR19], Theorem (1)) when $s_1 + s_2 \in 2\mathbb{Z} + 1 \cap (1, k-1)$, $1 < \Re(s_j) < k - 1$ and $\Re(s_1) > \Re(s_2) + 1$. However, the objective of the authors in [KKR19] has been to generalise a similar identity obtained for the periods of $f_{k,n}$ when n is an integer (see [KZ84], Theorem 1). Their approach, similar to that in [KZ84], was to write the Mellin transform of f_{k,s_1} with respect to s_2 as sum of certain term-wise integrals, obtained by splitting the series f_{k,s_1} in a suitable way. We too split the series f_{k,s_1} in order to estimate $L^*(f_{k,s_1}, s_2)$, although we consider points s_1, s_2 within the critical strip and our focus is to address the questions posed regarding Problem (1) and Problem (2).

Corollary 1.5.7. *Let T be an arbitrary but fixed positive real number and let $0 < \delta, \delta' \leq 1/2$ and δ'' be arbitrary but fixed positive reals. Let $(s_1, s_2) \in R_{T,\delta,\delta'}$. Then, for $k \geq C(T, \delta, \delta')$, we have*

$$N_k(s_1, s_2) \gg_{T,\delta',\delta''} k^{|\epsilon_1|+|\epsilon_2|-\delta''}.$$

This in fact improves the result of Choie, Kohnen and Zhang (Corollary 3.2, [CKZ20]) although we impose further restrictions on the choice of (s_1, s_2). As a corollary to Corollary (1.5.7), we obtain an asymptotic lower bound for $N_k(s_1)$ in terms of k when s_1 is δ-bounded away from the critical line.

Corollary 1.5.8. *Let T be an arbitrary but fixed positive real number and let $0 < \delta, \delta' \leq 1/2$ and δ'' be arbitrary small but fixed positive reals. Let $s_1 = \frac{k}{2} + \epsilon_1 + i\beta_1$ satisfy $|\beta_1| \leq T$ and $\delta \leq |\epsilon_1| \leq \frac{1}{2} - \delta'$. Then, for $k \geq C(T, \delta, \delta')$, we have*

$$N_k(s_1) \gg_{T,\delta',\delta''} k^{\frac{1}{2}+|\epsilon_1|-\delta''}.$$

In **Chapter 5**, we discuss some problems for further research. The problems are described briefly.

Chapter 2

Preliminaries

In this chapter, we provide the essential background to the theory of holomorphic modular forms (of integral weight k with respect to $SL_2(\mathbb{Z})$) and their associated L-functions.

For $z \in \mathbb{C}$, $z \neq 0$ we choose $\arg z \in (-\pi, \pi]$ and denote

$$\log z = \log|z| + i \arg z$$

as the principal branch of the logarithm which is real for positive z, where $i := \sqrt{-1}$. Further, for any s in $\mathbb{C}$, we define $z^s := \exp(s \log z)$.

Note that $GL_2^+(\mathbb{R})$ acts on $\mathbb{H} \cup \{i\infty\}$ under the fractional linear transformation given by

$$\gamma \circ z := \frac{az+b}{cz+d}, \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R}).$$

Here,

$$\gamma \circ i\infty := \begin{cases} \dfrac{a}{c} & \text{if } c \neq 0, \\ i\infty & \text{if } c = 0. \end{cases}$$

For brevity, we omit the $\circ$ and simply denote $\gamma \circ z$ as γz.

2.1 Modular forms of integral weight

Let k be an integer. Consider the following weight k action (right) of $GL_2^+(\mathbb{R})$ on the $\mathbb{C}$-vector space $\{f : \mathbb{H} \to \mathbb{C} \mid f \text{ is holomorphic on } \mathbb{H}\}$.

$$(f|_k\gamma)(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

where $\gamma z = \gamma \circ z$ as defined earlier.

Definition 2.1.1. *(Modular form of weight k with respect to $SL_2(\mathbb{Z})$) A (holomorphic) modular form of weight k with respect to $SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that*

(Modularity)

$$(f|_k\gamma)(z) = f(z) \quad \forall\, \gamma \in SL_2(\mathbb{Z}) \text{ and } \forall\, z \in \mathbb{H}. \tag{2.1}$$

(Holomorphic at** $i\infty$**) *f has a Fourier expansion (around $i\infty$) of the form*

$$f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z}.$$

Let

$$\Gamma_\infty := \left\{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \mid n \in \mathbb{Z} \right\}.$$

One can see that $\Gamma_\infty \subseteq SL_2(\mathbb{Z})$. Thus, f is 1-periodic and defines a holomorphic function on $\Gamma_\infty \backslash \mathbb{H}$ which is bi-holomorphic to the punctured unit disc $\mathbb{D} \backslash \{0\}$ under $z \mapsto q(z) := e^{2\pi i z}$. Thus, f has a Laurent series expansion $\sum_{n \in \mathbb{Z}} a_f(n) q^n$ around $q = 0$. The condition (ii) stipulates that $a_f(n) = 0$ if $n < 0$ in this Laurent expansion, thus implying that the singularity at $i\infty$ is neither an essential singularity nor a pole and that f could be extended holomorphically to the cusp $i\infty$. This Fourier expansion of f is sometimes also called its q-expansion. Further, such a modular form f is said to be a *cusp form* with respect to $SL_2(\mathbb{Z})$ if $a_f(0) = 0$.

The collection of modular forms of weight k with respect to $SL_2(\mathbb{Z})$ form a $\mathbb{C}$-vector space, denoted by $M_k(SL_2(\mathbb{Z}))$ (in future, simply by M_k) and the collection of cusp forms form its vector subspace and we denote it by $S_k(SL_2(Z))$ (or simply by S_k). M_k is finite dimensional. In fact, $M_k = \{0\}$ when k is odd or $k < 0$ or $k = 2$. When $k = 0$, $M_k = \mathbb{C}$. The first non-trivial example of a modular form is the Eisenstein Series defined by

$$E_k(z) := \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (cz + d)^{-k},$$

valid for even $k \geq 4$. Note that E_k has a Fourier expansion given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k denotes the k^{th} *Bernoulli number* defined as coefficients of the series

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \quad \text{and,}$$

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

We list below the q-expansions of E_4 and E_6:-

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n\geq 1} \sigma_3(n) e^{2\pi i n z} \\ E_6(z) &= 1 - 504 \sum_{n\geq 1} \sigma_5(n) e^{2\pi i n z}. \end{aligned} \tag{2.2}$$

Similarly, we have $S_k = \{0\}$ if $k < 12$ or $k = 14$. The first example of a cusp form is the Ramanujan-Delta function, a weight 12 cusp form given by

$$\begin{aligned} \Delta(z) &:= \frac{E_4^3(z) - E_6^2(z)}{1728}, \\ &= \sum_{n\geq 1} \tau(n) q^n, \end{aligned}$$

where $\tau(n)$ is the famous Ramanujan τ-function. This cusp form also has the following product representation

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{2.3}$$

Its Fourier coefficients were first studied in 1916 by Ramanujan, who made pioneering conjectures on $\tau(n)$ regarding its multiplicative nature and on its upper bound.

The following explicit dimension formula for modular forms is known:-

Theorem 2.1.2. *(a) For even $k \geq 4$,*

$$\dim\ M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \bmod 12 \\ \\ \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \bmod 12 \end{cases}$$

$$dim\, S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1 & k \equiv 2 \bmod 12 \\ \\ \lfloor \frac{k}{12} \rfloor & k \not\equiv 2 \bmod 12 \end{cases}$$

Given any fundamental domain $\mathcal{D}'$ under the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$, one can define an inner product (due to H. Petersson) on S_k:-

$$\langle f, g \rangle := \int_{\mathcal{D}'} f(z)\overline{g(z)}\Im(z)^k d\mu(z),$$

where $d\mu(z) := \frac{dxdy}{y^2}$ denotes the $SL_2(\mathbb{R})$- invariant hyperbolic measure on $\mathbb{H}$. Note that the integrand $f(z)\overline{g(z)}\Im(z)^k$ is $SL_2(\mathbb{Z})$ invariant due to the modularity of f and

g and due to the relation

$$\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2},$$

for any $\gamma \in SL_2(\mathbb{Z})$. Under this inner product, S_k becomes a Hilbert space (since it is finite dimensional). Now, let $\mathcal{D}$ denote *the standard (open) fundamental domain of* $\mathbb{H}$ *under the action of* $SL_2(\mathbb{Z})$

$$\mathcal{D} := \left\{ z \in \mathbb{H} \mid |z| > 1 \;\&\; |\Re(z)| < \frac{1}{2} \right\}.$$

Due to the exponential decay of f and g as $y \to \infty$, over $\mathcal{D}$, this integral is easily seen to be absolutely convergent. One can see that this inner product is well-defined on the space of modular forms M_k as well, as long as at least one of them, without loss of generality, say f is a cusp form. The fact that $\lim_{z \to i\infty} g(z) = a_g(0)$ is finite allows for this.

2.1.1 Valence Formula

Let $N_f(z)$ denote the order of zero of f at the point $z \in \mathbb{H}$. Also, let

$$N_f(i\infty) := \min \{ n \in \mathbb{Z} \mid a_f(n) \neq 0 \}$$

denote the order of the zero at $q = 0$ in the Fourier expansion $\sum_{n \geq 0} a_f(n) q^n$.

One notices by the *modularity condition* (2.1) that f is uniquely determined if one knows the value of f in $\mathcal{D}$ and on its boundary $\partial\mathcal{D}$ in $\mathbb{H}$. The Valence formula counts the zeros of f in the closure of the fundamental domain and expresses it in terms of its weight.

Theorem 2.1.3. *(Valence Formula) Let f be a non-zero modular form of weight k for the full modular group $SL_2(\mathbb{Z})$. Then,*

$$N_f(i\infty) + \frac{1}{2}N_f(i) + \frac{1}{3}N_f(\rho^2) + \sum_{\substack{z\neq i,\rho^2 \\ z\in\mathcal{D}\cup\partial\mathcal{D}}}^{*} N_f(z) = \frac{k}{12}. \tag{2.4}$$

where $*$ means that the summation counts each orbit representative only once and $\rho := e^{i\pi/3}$.

2.1.2 Hecke Operators

We define the n^{th} Hecke Operator T_n on M_k, $n \geq 1$ as

$$\begin{aligned} T_n f &:= n^{\frac{k}{2}-1} \sum_{\substack{ad=n \\ a>0}} \sum_{0\leq b<d} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \\ &= \frac{1}{n} \sum_{\substack{ad=n \\ a>0}} a^k \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right). \end{aligned}$$

If $f(z) = \sum_{m=0}^{\infty} a_f(m)q^m$, then,

$$T_n(f) = \sum_{m=0}^{\infty} a_{T_n f}(m)q^m,$$

where, $a_{T_nf}(m)$, the m^{th} Fourier coefficient of $T_n(f)$ satisfies, for all $m, n \geq 1$,

$$a_{T_nf}(m) = \sum_{d|(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right).$$

Thus, it clearly follows that Hecke operators maps S_k to S_k. Further, one can verify that they are self adjoint on S_k, i.e.,

$$\langle T_n f, g\rangle = \langle f, T_n g\rangle$$

for all $f, g \in S_k$. Also for $m, n \geq 1$,

$$T_m T_n = \sum_{d|(m,n)} d^{k-1} T_{\frac{mn}{d^2}} = T_n T_m.$$

This commuting family of self-adjoint operators on the finite dimensional Hilbert space S_k admits an orthogonal basis (with respect to Petersson inner product) consisting precisely, the simultaneous eigenfunctions for the whole family $\{T_n\}_{n\geq 1}$. It can be seen easily that the eigenvalues $\{\lambda_f(n)\}_{n\geq 1}$ of such a simultaneous eigenfunction f satisfy the following relation with the Fourier coefficients of f

$$a_f(m)\lambda_f(n) = \sum_{d|(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right) \tag{2.5}$$

for all $m, n \in \mathbb{N}$ and thus, by putting $m = 1$ above, they are proportional to their Fourier coefficients with $a_f(1)$ being the proportionality constant, which forces $a_f(1)$ to be non-zero for such f (for non-trivial S_k). Hence, we divide by $a_f(1)$ to *normalise* these forms and the collection of normalised simultaneous eigenfunctions with respect to the class $\{T_n\}_{n\geq 1}$ are called *Hecke eigenforms* and they form a (unique) orthogonal basis for S_k, known as the Hecke basis. We denote it by $\mathcal{B}_k$. Note that for such an f, the proportionality constant $a_f(1) = 1$. Hence, the eigenvalues of Hecke eigenforms (with respect to the class $\{T_n\}$) are exactly, their Fourier coefficients and hence, we get

$$a_f(m)a_f(n) = \sum_{d|(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right) \tag{2.6}$$

Moreover, these are the only forms in S_k whose Fourier coefficients satisfy the above multiplicative relation (2.6).

Remark 2.1.4. *One could indeed talk about above notions in the whole space M_k. For example, the arithmetically normalised Eisenstein series $\tilde{E}_k := -\frac{B_k}{2k}E_k$ satisfies*

$$T_n\tilde{E}_k = \sigma_{k-1}(n)\tilde{E}_k$$

for all $n \geq 1$ and hence, is an Hecke eigenform in M_k that is not a cusp form. However, in this thesis, we only work in the subspace S_k.

Example 2.1.5. *The Discriminant function Δ is an example of a Hecke eigenform in S_{12}. Note that $T_n\Delta = \tau(n)\Delta$ for all $n \geq 1$.*

We also remark here that (2.6) is equivalent to the following two conditions put together.

$$
\begin{aligned}
&(i)\ a_f(m)a_f(n) = a_f(mn) \text{ if } \gcd(m,n)=1,\\
&(ii)\ a_f(p^{\nu+1}) = a_f(p)a_f(p^{\nu}) - p^{k-1}a_f(p^{\nu-1})\ \forall\, p \in \mathcal{P}\ \&\ \nu \geq 1.
\end{aligned} \tag{2.6$'$}
$$

2.1.3 Poincaré Series

For $m \geq 0$, we define the m$^{\text{th}}$ Poincaré series of weight k as

$$
P_{k,m}(z) := \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} (cz+d)^{-k} e^{2\pi i n \frac{a_0 z + b_0}{cz+d}}. \tag{2.7}
$$

Here, for a given co-prime pair (c,d) in the summation, we fix any pair (a_0, b_0) in $\mathbb{Z}^2$ satisfying $a_0 d - b_0 c = 1$. We list here few well-known fundamental facts about them without providing any proof. We refer to [IK04] for further details.

Proposition 2.1.6. *[IK04]*

1. *If* $m = 0$, *then* $P_{k,0} = E_k$.

2. *If* $m \geq 1$, *then* $P_{k,m}$ *is a cusp form in* S_k.

Lemma 2.1.7. *Let* $f = \sum_{n\geq 0} a_f(n) q^n \in M_k$, *then*

$$
\langle f, P_{k,m} \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m). \tag{2.8}
$$

Corollary 2.1.8. *The set* $\{P_{k,m}\}_{m\geq 1}$ *spans* S_k.

2.1.4 L-functions

For a modular form f of weight k, it is known that $a_f(n) = \mathcal{O}(n^{k-1})$. In particular, if f is a cusp form, then $a_f(n) = \mathcal{O}\left(n^{k/2}\right)$. We provide a quick sketch of its proof. First, we note the fact that $\Im(z)^{k/2}|f(z)|$ is $SL_2(\mathbb{Z})$-invariant for $f \in M_k$. Thus, one may work on the standard fundamental domain. Since f is a cusp form, we have $f(z) = \mathcal{O}(e^{-2\pi y})$. Hence, $y^{k/2}|f(z)| \to 0$ as $y \to \infty$. Thus, we see that $y^{k/2}|f(z)|$ is bounded on $\mathcal{D} \cup \partial\mathcal{D}$. Now, from Cauchy's integral formula (for derivatives), we have

$$a_f(n) = e^{2\pi ny} \int_0^1 f(x+iy)e^{-2\pi inx}dx$$

valid for all $y > 0$. In particular, by substituting $y = 1/n$ and collecting the above facts, one can arrive at the bound

$$a_f(n) = \mathcal{O}(n^{k/2}). \tag{2.9}$$

Associated to a cusp form f, one can define the following Dirichlet series

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \tag{2.10}$$

which is valid on the right half plane $\Re(s) > \frac{k}{2} + 1$ from the estimate (2.9). Now, consider the Mellin transform of f, which we denote as

$$L^*(f,s) := \int_0^{\infty} f(it)t^{s-1}dt. \tag{2.11}$$

Since f has exponential decay $f(x+iy) \ll e^{-2\pi y}$ as $y \to \infty$, it follows that the integrand in (2.11) is $\mathcal{O}\left(t^{\sigma-1}e^{-2\pi t}\right)$ and hence, the integral

$$I_1 := \int_1^\infty f(it)t^{s-1}dt$$

is absolutely convergent for any $s \in \mathbb{C}$. One can differentiate (with respect to s) under the integral sign and deduce that this integral is entire (to justify, one can apply dominated convergence theorem). Now, note that by modularity of f,

$$f(it) = (it)^{-k} f\left(\frac{i}{t}\right).$$

Thus,

$$I_2 := \int_0^1 f(it)t^{s-1}dt = (i)^{-k}\int_1^\infty f(it)t^{k-s-1}dt$$

which is easily seen to be absolutely convergent and entire by earlier arguments. Now, it follows that the integral

$$\int_0^\infty f(it)t^{s-1}dt = I_1 + I_2$$

is also an entire function.

We use this entire function to analytically extend the Dirichlet series given in (2.10). By noting that

$$L^*(f,s) = \int_0^\infty \sum_{n\geq 1} a_f(n)e^{-2\pi n t}t^{s-1}dt$$

$$= \sum_{n\geq 1} a_f(n) \int_0^\infty e^{-2\pi n t} t^{s-1} dt$$

$$= (2\pi)^{-s}\Gamma(s) L(f,s), \tag{2.12}$$

on $\Re(s) > \frac{k}{2} + 1$, we have a candidate for analytically extending $L(f,s)$ to the whole complex plane, namely $\dfrac{(2\pi)^s}{\Gamma(s)} L^*(f,s)$. By Proposition (2.2.5), this expression is entire. Now, since the Dirichlet series $L(f,s)$ and $\frac{(2\pi)^s}{\Gamma(s)} L^*(f,s)$ agree on the open set $\left\{\Re(s) > \frac{k}{2} + 1\right\}$, the above analytic continuation for $L(f,s)$ is unique too. Hence, without ambiguity, we retain the same notation for the analytic continuation too. Thus, for all $s \in \mathbb{C}$, we define

$$L(f,s) := \frac{(2\pi)^s}{\Gamma(s)} L^*(f,s). \tag{2.13}$$

It follows from Proposition (2.2.5) that $L(f,s)$ has simple zeros at $s = 0, -1, -2, \ldots$. These are called the *trivial zeros* of $L(f,s)$.

A function $f : \mathbb{N} \mapsto \mathbb{C}$ (not identically zero) is called

- *multiplicative*: if $f(mn) = f(m)f(n) \ \forall \ \gcd(m,n) = 1$.
- *completely multiplicative*: if $f(mn) = f(m)f(n) \ \forall \ m, n \in \mathbb{N}$.

Let us now recall a theorem in analytic number theory.

Theorem 2.1.9. *Let f be a multiplicative arithmetical function such that the series $\sum f(n)$ is absolutely convergent. Then, the sum of the series can be expressed as an* *absolutely convergent infinite product*

$$\sum_{n\geq 1} f(n) = \prod_{p\in\mathcal{P}} \{1 + f(p) + f(p^2) + \dots\}$$

where $\mathcal{P}$ denotes the set of primes. If f is completely multiplicative, we have

$$\sum_{n\geq 1} f(n) = \prod_{p\in\mathcal{P}} \frac{1}{1 - f(p)}.$$

This theorem essentially says that for a cusp form f, the Dirichlet series $\sum_{n\geq 1} \frac{a_f(n)}{n^s}$ has an *Euler product* expansion given by

$$\prod_{p\in\mathcal{P}} \left(1 + \frac{a_f(p)}{p^s} + \frac{a_f(p^2)}{p^{2s}} + \dots\right)$$

in its region of absolute convergence $\left\{\Re(s) > \frac{k}{2} + 1\right\}$ if $\frac{a_f(n)}{n^s}$ is multiplicative. If $a_f(n)$ satisfy (ii) in (2.6$'$), it follows that

$$\sum_{\nu=0}^{\infty} \frac{a_f(p^\nu)}{p^{\nu s}} = \left(1 - \frac{a_f(p)}{p^s} + \frac{p^{k-1}}{p^{2s}}\right)^{-1} \quad \forall\, p \in \mathcal{P}.$$

Hence for cusp forms f whose $a_f(n)$ satisfy (2.6$'$), we have

$$\sum_{n\geq 1} \frac{a_f(n)}{n^s} = \prod_{p\in\mathcal{P}} \left(1 - \frac{a_f(p)}{p^s} + \frac{p^{k-1}}{p^{2s}}\right)^{-1}. \tag{2.14}$$

on $\Re(s) > \frac{k}{2} + 1$. Thus, L-series of a Hecke eigenform has an Euler product expansion in its region of absolute convergence[1] (and is consequently, non-zero there). Denote by α_p and β_p, the roots of the polynomial $X^2 - a_f(p)X + p^{k-1}$. Then,

$$L(f,s) = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}. \tag{2.15}$$

One can recover $a_f(n)$ from (2.15) by noting that for each prime p and for all $m \geq 0$ integer,

$$a_f(p^m) = \frac{\beta_p^{m+1} - \alpha_p^{m+1}}{\beta_p - \alpha_p}.$$

Recall that

$$L^*(f,s) = \int_1^\infty f(it)\left(t^s + (-1)^{k/2} t^{k-s}\right)\frac{dt}{t}.$$

From this, by replacing s by $k-s$, one can easily see that $L(f,s)$ satisfies a *functional equation* given by

$$(2\pi)^{-(k-s)}\Gamma(k-s)L(f,k-s) = (-1)^{\frac{k}{2}}(2\pi)^{-s}\Gamma(s)L(f,s), \tag{2.16}$$

which is valid on $s \in \mathbb{C}$.

Ramanujan-Petersson Conjecture in the context of cuspidal Hecke eigenforms says that for any $n \geq 1$,

$$a_f(n) \ll n^{\frac{k-1}{2}+\epsilon} \text{ for any } \epsilon > 0. \tag{2.17}$$

[1] It will be revealed after (2.19) that the region of absolute convergence is $\Re(s) > \frac{k+1}{2}$

As a consequence of his proof [Del72] of the Weil conjectures in 1972, P. Deligne proved that

$$a_f(n) \leq \sigma_0(n) n^{\frac{k-1}{2}} \tag{2.18}$$

thus proving the above Ramanujan-Petersson Conjecture, since $\sigma_0(n) \ll_\epsilon n^\epsilon$. In addition, (2.18) also shows that the implied constant in (2.17) is independent of the choice of f. This shows that $L(f,s)$ of any Hecke eigenform f in S_k is absolutely convergent on the right half plane $\Re(s) > \frac{k+1}{2}$. If f is just a cusp form, by writing it as linear combination of Hecke eigenforms, we still have

$$a_f(n) \ll_{f,\epsilon} n^{\frac{k-1}{2}+\epsilon}, \tag{2.19}$$

although, in this case, the implicit constant depends on f too. To summarise, we note that L-series of Hecke cusp eigenforms $\sum_{n\geq 1} \frac{a_f(n)}{n^s}$ satisfy the following:

1. has an Euler product expansion in its region of absolute convergence,
2. admits an analytic continuation to the whole $\mathbb{C}$,
3. has a functional equation, and
4. Each $a_f(n)$ satisfies the Ramanujan-Petersson Conjecture.

These L-series are known to lie in a much bigger class of meromorphic functions known as the *Selberg Class* $\mathcal{S}$ [KP12] (Upon re-writing $L(f,s)$ in terms of $\tilde{L}(f,s') := \sum_{n\geq 1} \frac{\lambda_f(n)}{n^{s'}}$, where $\lambda_f(n) := a_f(n) n^{-(\frac{k-1}{2})}$ and $s' := s - \frac{k-1}{2}$, one can see that $\tilde{L}(f,s')$

is absolutely convergent on the right half plane $\Re(s') > 1$. However, we revert back to the previously used notation $L(f,s)$). Few examples of other elements in $\mathcal{S}$ are

1. Riemann Zeta function $\zeta(s)$,
2. Dirichlet L-function $L(s,\chi)$, where χ is a primitive Dirichlet character,
3. Dedekind zeta-function $\zeta_K(s)$ for a number field K over $\mathbb{Q}$.

Note that the Euler product and functional equation forces that $L(f,s) \neq 0$ on $\Re(s) > \frac{k+1}{2}$ and $\Re(s) < \frac{k-1}{2}$ except for the trivial (simple) zeros $s = 0, -1, -2, \ldots$. The fact that there are no zeros for $L(f,s)$ on the vertical line $\Re(s) = \frac{k+1}{2}$ is also known and may be found, for example, in Chapter 10, [Gold06] or [Ran39] (the latter is an adaptation of Merten's proof of non-vanishing of $\zeta(s)$ on the vertical line $\Re(s) = 1$). The remaining region $\left\{\frac{k-1}{2} < \Re(s) < \frac{k+1}{2}\right\}$ is known as the *critical strip* of the L-function $L(f,s)$. Grand Riemann Hypothesis (GRH) in this context predicts that if $L(f,s) = 0$ for some s in the critical strip, then $\Re(s) = \frac{k}{2}$. The line $\Re(s) = \frac{k}{2}$ is known as the *critical line*. In fact, the following non-vanishing region for $L(f,s)$ is already known. We assume from now onwards that k is an even integer greater than or equal to 12.

Theorem 2.1.10. *[IK04] Let f be a Hecke eigenform in S_k. There exists an absolute constant $c > 0$ such that $L(f,s)$ has no zeros in the region*

$$\left\{\Re(s) \geq \frac{k+1}{2} - \frac{c}{\log(|t| + k + 3)}\right\}$$

except possibly a simple real zero $\sigma_0 < \frac{k+1}{2}$.

2.1.5 Symmteric square L-function

For f a normalised Hecke eigenform in S_k, consider the series

$$A_f(s) := \sum_{n\geq 1} \frac{a_f(n)^2}{n^s}.$$

By (2.17), the above series is absolutely convergent for $\Re(s) > k$, and we define the *symmetric square L-function* of f as

$$L(\mathrm{Sym}^2(f), s) := \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} A_f(s). \tag{2.20}$$

It is known that $L(\mathrm{Sym}^2(f), s)$ has an Euler product expansion namely

$$L(\mathrm{Sym}^2(f), s) = \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s}))(1-\beta_p^2 p^{-s})},$$

where α_p and β_p are as in (2.15). Further, after multiplying by an appropriate completion factor, let

$$L^*(\mathrm{Sym}^2(f), s) := \pi^{-3s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-k}{2}+1\right) L(\mathrm{Sym}^2(f), s).$$

$L(\text{Sym}^2(f), s)$ can be extended to a holomorphic function [Shi75] on the whole complex plane which also satisfies the following functional equation

$$L^*(\text{Sym}^2(f), 2k-1-s) = L^*(\text{Sym}^2(f), s).$$

Critical strip of the symmetric square L function is known to be $\{k-1 < \sigma < k\}$ and the critical line is $\left\{\sigma = k - \frac{1}{2}\right\}$. The next theorem relates $L(\text{Sym}^2(f), k)$ to the Petersson norm of f.

Theorem 2.1.11. *[CS17] Let f be a Hecke eigenform in S_k. Then,*

$$\langle f, f \rangle = \frac{2\Gamma(k)}{\pi(4\pi)^k} L(\mathit{Sym}^2(f), k). \tag{2.21}$$

2.2 Analytic Tools

2.2.1 The zeta function

For $\Re(s) > 1$, the **Riemann Zeta function** is defined as the infinite series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.22}$$

It is absolutely and uniformly convergent on every half plane of the form $\Re(s) > 1+\delta$ and therefore, defines a holomorphic function in the half plane $\Re(s) > 1$. As described in Chapter 1, its has the following product form (Euler product):-

$$\sum_{n\geq 1}\frac{1}{n^s}=\prod_{p\in\mathcal{P}}\frac{1}{1-p^{-s}} \qquad (\Re(s)>1). \tag{2.23}$$

In 1859, in his only paper in number theory [Rie59], Riemann proved the following.

Theorem 2.2.1. *(*B. Riemann*) (i) The function $\zeta(s)$ has a meromorphic continuation into the whole complex plane, whose only singularity is a simple pole at $s=1$.*

(ii) The meromorphically extended ζ function satisfies the following functional equation for all $s\in C$.

$$\pi^{-s/2}\Gamma(s/2)\zeta(s)=\pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

We mention an estimate for ζ which is valid on $\Re(s)>1$.

Lemma 2.2.2. *For $s=\sigma+it$, where $\sigma>1$, we have*

$$|\zeta(s)|\geq\left|\frac{\zeta(2\sigma)}{\zeta(\sigma)}\right|.$$

Proof. From the Euler product (2.23), and the fact that $|p^s-1|\leq p^\sigma+1$, we get

$$\left|\frac{\zeta(2\sigma)}{\zeta(\sigma)\zeta(s)}\right|=\left|\prod_{p\in\mathcal{P}}\frac{p^{2\sigma}(p^\sigma-1)(p^s-1)}{(p^{2\sigma}-1)p^\sigma p^s}\right|\leq 1.$$

□

2.2.2 Gamma Function

Definition 2.2.3. *For $\Re(s) > 0$, we define the* ***Euler***[2] ***Gamma function*** *as*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt.$$

This integral converges near $t = 0$ since t^{s-1} is integrable and for large t, the exponential decay guarantees the convergence. It is holomorphic on $\Re(s) > 0$ and can be meromorphically continued to the whole $\mathbb{C}$ as detailed below:-

Proposition 2.2.4. *The function $\Gamma(s)$ can be meromorphically continued to the whole complex plane into a function whose poles (all are simple) are exactly the non-positive integers and it satisfies the functional equation*

$$\Gamma(s+1) = s\Gamma(s). \tag{2.24}$$

Proposition 2.2.5. *The function $\dfrac{1}{\Gamma(s)}$ is entire with simple zeros at $s = 0, -1, -2, \ldots$ and it vanishes nowhere else.*

Remark 2.2.6. *By Proposition* (2.2.5)*, it follows that $\Gamma(s)$ is never zero.*

The following lemma is due to [Gau59].

[2]in the honour of L. Euler who initiated the study of this function.

Lemma 2.2.7. *(Gautschi's inequality) For $x > 0$ and $s \in (0, 1)$,*

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}. \tag{2.25}$$

Lemma 2.2.8. *a. For $a, b \in \mathbb{C}$,*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \tag{2.26}$$

as $z \to \infty$, along any curve joining 0 and ∞, provided $z \neq -a, -a-1, \ldots; z \neq -b, -b-1, \ldots$.

b. (Real Stirling's Formula) As $x \to \infty$, we have

$$\Gamma(x) \sim \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}. \tag{2.27}$$

c. [OLBC10] The following inequality is true:-

$$|\Gamma(x+iy)| \geq (\cosh \pi y)^{\frac{-1}{2}} \Gamma(x),\ x \geq \frac{1}{2}. \tag{2.28}$$

Proof. Part (a) follows from the asymptotic expansion

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1+\frac{(a-b)(a+b-1)}{2}\frac{1}{z}+\frac{1}{12}\binom{a-b}{2}\left(3(a+b-1)^2-a+b-1\right)\frac{1}{z^2}+\ldots$$

(see 6.1.47 in page 257 of [AS65]) valid for z as above. Part (b) is fairly standard. □

Definition 2.2.9. *For $z_1, z_2 \in \mathbb{C}$ with $\Re(z_j) > 0$ for $j = 1, 2$, we define the **Beta function** as*

$$B(z_1, z_2) := \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt$$

We list few other representations of $B(z_1, z_2)$ which would be used in the thesis.

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = 2\int_0^{\pi/2} (\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}dt. \tag{2.29}$$

2.2.3 Few preliminary lemmas

If ω_1 and ω_2 are two non-zero complex numbers whose ratio is not real, we denote

$$\Omega(\omega_1, \omega_2) := \{m\omega_1 + n\omega_2 \mid m, n \in Z\}.$$

Lemma 2.2.10. *Let $\alpha \in \mathbb{R}$. Then, the infinite series*

$$\sum_{\substack{\omega\in\Omega(\omega_1,\omega_2)\\ \omega\neq 0}} \frac{1}{\omega^\alpha}$$

converges absolutely if and only if $\alpha > 2$.

Proposition 2.2.11. *(Lipschitz's Summation Formula) For $\Re(s) > 1$ and $z \in \mathbb{H}$, we* *have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^s} = e^{-\pi i s/2}\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1} n^{s-1}e^{2\pi i n z}. \tag{2.30}$$

Lemma 2.2.12. *Let $x > 1$ and $r \geq 1$,*

$$\sum_{\substack{a\in\mathbb{Z}\\ a\geq r}} \frac{1}{a^x} \;\leq\; \zeta(x)/r^{x-1}.$$

Proof. We have

$$\sum_{a\geq r}\frac{1}{a^x} \;\leq\; \sum_{m\geq 1}\sum_{n=0}^{r-1}\frac{1}{(mr+n)^x} \;\leq \sum_{m\geq 1}\frac{r}{(mr)^x} \;=\; \frac{\zeta(x)}{r^{x-1}}.$$

□

Lemma 2.2.13. *(i) For $z_1, z_2, s \in \mathbb{C}$,*

$$(z_1z_2)^s = \begin{cases} z_1^s z_2^s e^{-2\pi i s} & \textit{if} \quad \arg z_1 + \arg z_2 > \pi, \\ z_1^s z_2^s & \textit{if} \quad -\pi < \arg z_1 + \arg z_2 \leq \pi, \\ z_1^s z_2^s e^{2\pi i s} & \textit{if} \quad \arg z_1 + \arg z_2 \leq -\pi. \end{cases} \tag{2.31}$$

(ii) If $z, s = \sigma + it \in \mathbb{C}$,

$$|z^s| = |z|^\sigma e^{-t\arg z}, \tag{2.32}$$

$$\overline{z^s} = \bar{z}^{\bar{s}}. \tag{2.33}$$

Proof. The first easily follows from the fact that

$$\arg z_1 z_2 = \begin{cases} \arg z_1 + \arg z_2 - 2\pi & \text{if} \quad \arg z_1 + \arg z_2 > \pi, \\ \arg z_1 + \arg z_2 & \text{if} \quad -\pi < \arg z_1 + \arg z_2 \leq \pi, \\ \arg z_1 + \arg z_2 + 2\pi & \text{if} \quad \arg z_1 + \arg z_2 \leq -\pi. \end{cases}$$

The second part follows from the observation $|z^s| = \left|e^{(\sigma+it)(\log|z|+i\arg z)}\right| = e^{\sigma\log|z|-t\arg z}$. The third follows from direct computation by noting $\arg\bar{z} = -\arg z$. □

Lemma 2.2.14. *For $\sigma > 1$ real and $z = x + iy \in \mathbb{H}$, there exists a constant $C(\sigma) > 0$ such that*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} = C(\sigma) y^{-\sigma} \max(1, y). \tag{2.34}$$

Proof. We refer to the proof of Lemma (3.5.9) in [CS17]. (Note that since $|z+n| = |\bar{z}+n|$, one may as well extend this result to $y < 0$ by replacing y in the above result with $|y|$). □

The implicit constant $C(\sigma)$ is described in Lemma (3.5.9) of [CS17] in terms of an integral and we further wish to estimate it in order to obtain an explicit bound in the case $y \geq 1$ and $\sigma \geq 3$.

Remark 2.2.15. *For $\sigma \geq 3$ and $\Im(z) = y \geq 1$,*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} < \frac{7}{y^{\sigma-1}}. \tag{2.35}$$

Proof. From Lemma (3.5.9) in [CS17], one explicitly obtains that

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} \leq \frac{1}{y^\sigma} + \frac{4}{y^{\sigma-1}} \int_0^\infty \frac{1}{(u^2+1)^{\sigma/2}} du.$$

We estimate this explicitly when $y \geq 1$ and $\sigma \geq 3$, using (2.29) and Lemma (2.2.7).

$$\begin{aligned}\int_0^\infty \frac{1}{(u^2+1)^{\sigma/2}} du = \int_0^{\pi/2} (\cos\theta)^{\sigma-2} d\theta = \frac{\Gamma(\frac{1}{2})}{2} \frac{\Gamma((\sigma-1)/2)}{\Gamma(\sigma/2)} \\ \leq \sqrt{\frac{\pi}{2(\sigma-2)}}.\end{aligned}$$

Note here that $\Gamma(1/2) = \sqrt{\pi}$. Hence, we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} \leq \frac{1}{y^\sigma} + \frac{4}{y^{\sigma-1}} \sqrt{\frac{\pi}{2(\sigma-2)}} < \frac{7}{y^{\sigma-1}}. \tag{2.36}$$

□

Lemma 2.2.16. *(i) For any fixed $a > 0$,*

$$\sum_{m\geq 1} m^a e^{-mz} = \mathcal{O}_a \left(\frac{1}{z}\right)^{a+1} \textit{ for each } z > 0.$$

(ii)

$$\max_{z>0} z^{a+1} \sum_{m\geq 1} m^a e^{-mz} \ll \left(\frac{a+1}{e}\right)^{a+1} \qquad (a\to\infty).$$

Proof. For fixed $z > 0$, consider the function $f_z(x) = x^a e^{-xz}$ on the domain $\{x \in \mathbb{R} \mid x \geq 0\}$. This function is increasing on the interval $[0, a/z]$ and decreasing on $[a/z, \infty)$. Let $N := \left\lfloor \frac{a}{z} \right\rfloor$. Hence, the lower Riemann sum upto $\frac{a}{z}$ satisfies

$$\sum_{n=0}^{N-1} f_z(n) + f_z(N)(a/z - N) < \int_0^{a/z} f_z(x)dx. \text{ Similarly,}$$
$$\sum_{n=N+2}^{\infty} f_z(n) + f_z(N+1)(N+1-a/z) < \int_{a/z}^{\infty} f_z(x)dx.$$

From the above two inequalities, it follows that

$$\sum_{m\geq 0} f_z(m) < \int_0^{\infty} x^a e^{-xz}dx + f_z(N)(N+1-a/z) + f_z(N+1)(a/z-N) \leq \frac{\Gamma(a+1)}{z^{a+1}} + f_z(a/z).$$

In other words,

$$z^{a+1} \sum_{m\geq 1} m^a e^{-mz} < \Gamma(a+1) + z\left(\frac{a}{e}\right)^a. \tag{2.37}$$

Similarly, by considering the upper sums, one can find that

$$\Gamma(a+1) - z\left(\frac{a}{e}\right)^a < z^{a+1} \sum_{m\geq 1} m^a e^{-mz}.$$

Thus, it follows that

$$\lim_{z\to 0} z\sum_{m\geq 1}(mz)^a e^{-mz} = \Gamma(a+1).$$

On the other hand, clearly, $\lim_{z\to\infty} z\sum_{m\geq 1}(mz)^a e^{-mz} = 0$. And, on any compact interval $[\delta, T]$ with $\delta > 0$, due to Weierstrass M-test, the partial sums $-\sum_{1\leq m\leq n} m^{a+1}e^{-mz}$ converge uniformly and absolutely and hence, derivative of $\sum_{m\geq 1} m^a e^{-mz}$ is the sum of its termwise derivatives.

Now, applying the product rule on $g_a(z) := z^{a+1}\sum_{m\geq 1} m^a e^{-mz}$, we see that

$$\frac{d}{dz}g_a(z) = z^a\left((a+1)\sum_{m\geq 1} m^a e^{-mz} - z\sum_{m\geq 1} m^{a+1}e^{-mz}\right) < 0,$$

if $z \geq a+1$. Hence, we can see that the maximum of $g_a(z)$ in $[\delta, T]$ occurs at some point $z_0 < a+1$ (we choose $T \gg a+1$ sufficiently large and $\delta > 0$ sufficiently small). Thus, by (2.37),

$$\max_{\delta\leq z\leq T} z^{a+1}\sum_{m\geq 1} m^a e^{-mz} < \Gamma(a+1) + (a+1)\left(\frac{a}{e}\right)^a \ll \left(\frac{a+1}{e}\right)^{a+1}.$$

We remark that $\Gamma(a+1) = o\left((\frac{a+1}{e})^{a+1}\right)$. This concludes the proof. $\square$

Chapter 3

A weighted sum of L-functions of Hecke eigen forms

In this chapter, we study the dual cusp form that was discussed in the Chapter 1, Introduction. Further, we present an asymptotic relation ($k \to \infty$) for this cusp form when evaluated at the point i in terms of the weight k. Note that this function is defined for a fixed complex parameter, say s, as it is the dual to the L value (at s). Further, we also obtain GRH on the real interval $[\frac{k-1}{2}, \frac{k+1}{2}]$ for a weighted sum of Hecke eigenforms (instead of a single Hecke eigenform) for all $k \geq 12, 4|k$. As a corollary, we deduce a lower bound for the maximum value of $|L(f, \sigma)|$, where f runs over the Hecke basis $\mathcal{B}_k$.

3.1 Kernel function and its properties

In [Koh97], W. Kohnen showed that for a given point s inside the critical strip (outside the critical line), non-vanishing of L values of Hecke eigenforms holds on an average for large enough weights $k \gg_{\Im(s)} 1$.

Theorem 3.1.1. *[Koh97] Let $t_0 \in \mathbb{R}$ and $\delta > 0$. Then, there exists a constant $C = C(t_0, \delta) > 0$ such that for $k \geq C(t_0, \delta)$, the sum*

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f, s)}{\langle f, f \rangle} \neq 0$$

for $s = \sigma + it_0$, where $\frac{k-1}{2} < \sigma < \frac{k}{2} - \delta$ or $\frac{k}{2} - \delta < \sigma < \frac{k+1}{2}$.

As a consequence, Kohnen showed the existence of an $f \in \mathcal{B}_k$ such that $L^*(f, s) \neq 0$ for $k \gg_{\Im(s)} 1$. For the proof, Kohnen used a cusp form which we will study in detail in this Section. It is also essential for the future results proved in this thesis. For a fixed $s \in \mathbb{C}$, the function

$$\begin{aligned} L^*(\ , s) : S_k &\to \mathbb{C} \\ f &\mapsto \int_0^\infty f(it) t^{s-1} dt \end{aligned}$$

is a linear functional on the Hilbert space S_k and hence, by the Riesz representation theorem, there is a unique cusp form ϕ_s in S_k, called *the kernel function to $L^*(\ , s)$*, satisfying

$$\langle f, \phi_s \rangle = L^*(f, s) \quad \forall\, f \in S_k. \tag{3.1}$$

We now provide a candidate for ϕ_s. Let $s \in \mathbb{C}$ such that $1 < \sigma := \Re(s) < k-1$. Then, for $z \in \mathbb{H}$, we define

$$R_{k,s}(z) := \gamma_k(s) \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s}, \tag{3.2}$$

where $\gamma_k(s) := \frac{1}{2} e^{\frac{\pi i s}{2}} \Gamma(s)\Gamma(k-s)$.

Lemma 3.1.2. *[Koh97] For $s \in \mathbb{C}$ such that $1 < \sigma < k-1$, the series*

$$\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} \tag{3.3}$$

defines a holomorphic function of $z \in \mathbb{H}$. Moreover, it is a cusp form of weight k.

Proof. Although the idea of the proof is briefly sketched in [Koh97], it is worthwhile to present it here as it provides an opportunity to describe the properties of this cusp form. To prove the holomorphicity, it suffices to show the following:-

The above series is absolutely and uniformly convergent on every strip[1] of the form

$$S_\epsilon = \{z = x + iy \mid |x| \leq \tfrac{1}{\epsilon},\ y \geq \epsilon\}$$

[1]Note that any compact subset $K \subset \mathbb{H}$ is contained in one of the S_ϵ.

for a fixed s in $\{1 < \Re(s) < k-1\}$, where ϵ could be any arbitrary positive real.

We first prove so when s satisfies $2 < \sigma < k-2$.

$$\begin{aligned}
&\Big| \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} \Big| \\
\leq & \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} |cz+d|^{-k} \left|\frac{az+b}{cz+d}\right|^{-\sigma} e^{t \arg\left(\frac{az+b}{cz+d}\right)} \\
\leq & \, e^{\pi|t|} \left(\sum_{\substack{(a,b)\in Z^2 \\ (a,b)\neq(0,0)}} |az+b|^{-\sigma} \right) \times \left(\sum_{\substack{(c,d)\in Z^2 \\ (c,d)\neq(0,0)}} |cz+d|^{-(k-\sigma)} \right).
\end{aligned} \tag{3.4}$$

To arrive at the second inequality above, we have used (2.32) and for the later step, we note that $\arg\left(\frac{az+b}{cz+d}\right) < \pi$ since $\mathbb{H}$ is invariant under the action of $SL_2(\mathbb{Z})$. By standard techniques, (for example, see [Apo76] Theorem 1.15), one can find a constant $M = M(\epsilon) > 0$ independent of the choice of $z \in S_\epsilon$ such that

$$|cz+d|^2 > M(c^2+d^2). \tag{3.5}$$

From Lemma (2.2.10), one sees that the series $\displaystyle\sum_{\substack{(a,b)\in Z^2 \\ (a,b)\neq(0,0)}} \frac{1}{(az+b)^s}$ and $\displaystyle\sum_{\substack{(c,d)\in Z^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^{k-s}}$ converge absolutely and uniformly on S_ϵ since $\sigma > 2$ and $k-\sigma > 2$. Thus, the series in (3.3) defines a holomorphic function on $\mathbb{H}$ for a fixed s in $\{2 < \Re(s) < k-2\}$.

Next, we observe below that in (3.3), if we replace s by $k-s$, the series remains invariant upto a scalar.

$$
\begin{aligned}
&\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} \\
&= \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-k} \left(\frac{az+b}{cz+d}\right)^{k-s} \\
&= e^{\pi i(k-s)} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (az+b)^{-k} \left(\frac{-cz-d}{az+b}\right)^{-(k-s)} \\
&= e^{\pi i(k-s)} \sum_{\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \in SL_2(\mathbb{Z})} (az+b)^{-k} \left(\frac{-cz-d}{az+b}\right)^{-(k-s)} \\
&= e^{\pi i(k-s)} \sum_{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})} (c'z+d')^{-k} \left(\frac{a'z+b'}{c'z+d'}\right)^{-(k-s)}.
\end{aligned} \tag{3.6}
$$

Here, we have used Lemma (2.2.13) in the second equality and the fact that

$$
\begin{aligned}
\phi_0 : SL_2(\mathbb{Z}) &\to SL_2(\mathbb{Z}) \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \qquad \text{is a bijective map.}
\end{aligned}
$$

Then, it suffices to prove the claim in the left half of the line of symmetry $\sigma = \frac{k}{2}$. In particular, it remains to prove absolute and uniform convergence on S_ϵ for a fixed s satisfying $1 < \sigma < 2 + \delta$. However, we remark that the following argument works for any fixed s with $1 < \sigma < \frac{k}{2} - 1$.

For a given co-prime integer pair $(c, d) = 1$, let (a_0, b_0) be a fixed but arbitrary pair such that $a_0 d - b_0 c = 1$. For $z \in \mathbb{H}$ and $\Re(s) > 1$, we have by (2.30),

$$\sum_{n \in \mathbb{Z}} \left(\frac{a_0 z + b_0}{cz + d} + n \right)^{-s} = e^{-i\frac{\pi s}{2}} \frac{(2\pi)^s}{\Gamma(s)} \sum_{n \geq 1} n^{s-1} e^{2\pi i n \frac{a_0 z + b_0}{cz + d}}. \tag{3.7}$$

Now, we let (c, d) run over all co-prime integer pairs and define

$$\tilde{R}_{k,s}(z) := \gamma_k(s) \sum_{\substack{(c,d) \in Z^2 \\ (c,d)=1}} (cz + d)^{-k} \left\{ e^{-i\frac{\pi s}{2}} \frac{(2\pi)^s}{\Gamma(s)} \sum_{n \geq 1} n^{s-1} e^{2\pi i n \frac{a_0 z + b_0}{cz + d}} \right\}. \tag{3.8}$$

Then,

$$\begin{aligned} |\tilde{R}_{k,s}(z)| &\leq |\gamma_k(s)| \sum_{\substack{(c,d) \in Z^2 \\ (c,d)=1}} |cz + d|^{-k} \left\{ e^{\frac{\pi t}{2}} \frac{(2\pi)^\sigma}{|\Gamma(s)|} \sum_{n \geq 1} n^{\sigma-1} e^{-2\pi n \frac{y}{|cz+d|^2}} \right\} \\ &\leq |\gamma_k(s)| e^{\frac{\pi t}{2}} \frac{(2\pi)^\sigma}{|\Gamma(s)|} \alpha_\sigma \frac{1}{y^\sigma} \sum_{\substack{(c,d) \in Z^2 \\ (c,d)=1}} |cz + d|^{-k+2\sigma}. \end{aligned} \tag{3.9}$$

using Lemma (2.2.16). Here, α_σ is the implied constant in Lemma (2.2.16). Using (3.5) and Lemma (2.2.10), when $\frac{k}{2} - \sigma > 1$, one sees that the series in (3.9) is bounded above uniformly for z in S_ϵ. Hence, we may combine all the constants in (3.9) into one and rewrite as

$$|\tilde{R}_{k,s}(z)| \leq C(k, s) \frac{1}{y^\sigma} \ \leq \ C(k, s) \left(\frac{1}{\epsilon^\sigma} \right), \tag{3.10}$$

where $C(k,s)$ is a positive real valued constant depending only on k and s. $\tilde{R}_{k,s}(z)$ is now easily seen to be uniformly convergent for all $z \in S_\epsilon$ and for all s with $1 < \sigma < \frac{k}{2} - 1$. Thus, $\tilde{R}_{k,s}(z)$ is holomorphic as a function of z on $\mathbb{H}$ for any fixed s such that $\{1 < \sigma < \frac{k}{2} - 1\}$.

On $\{1 < \sigma < \frac{k}{2} - 1\}$, $\tilde{R}_{k,s}$ can be shown to be equal to $R_{k,s}$ by rearranging using the Lipschitz Summation Formula (2.30):-

$$\begin{aligned}
\tilde{R}_{k,s}(z) &= \gamma_k(s) \sum_{\substack{(c,d)\in Z^2\\(c,d)=1}} (cz+d)^{-k} \sum_{n\in\mathbb{Z}} \left(\frac{a_0 z + b_0}{cz+d} + n\right)^{-s} \\
&= \gamma_k(s) \sum_{\substack{(c,d)\in Z^2\\(c,d)=1}} \sum_{n\in\mathbb{Z}} (cz+d)^{-k} \left(\frac{(a_0+nc)z + (b_0+nd)}{cz+d}\right)^{-s} \\
&= \gamma_k(s) \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} \\
&= R_{k,s}(z). \qquad (3.11)
\end{aligned}$$

Consider the sums in the second and third equalities here. Every matrix of the form $\begin{pmatrix} a_0+nc & b_0+nd \\ c & d \end{pmatrix}$ lies in $SL_2(\mathbb{Z})$. Hence, all terms in the former sum are present in the latter sum. Viceversa, for a given co-prime pair $(c,d) = 1$, if you consider any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $ad - bc = 1$, then clearly, a and b satisfy the relations $ad \equiv a_0 d \bmod c$ and $bc \equiv b_0 c \bmod d$. (Recall that we had fixed a_0 and b_0 for a given co-prime pair (c,d)). This forces that $a = a_0 + nc$ for some $n \in \mathbb{Z}$ and the determinant condition $ad - bc = 1$ tells that b also has to satisfy $b = b_0 + nd$ for the same n. Thus,

$R_{k,s}(z)$ (and $\tilde{R}_{k,s}$) defines a holomorphic function of $z \in \mathbb{H}$ for a fixed s satisfying $1 < \sigma < k-1$.

As for modularity, since $SL_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, it suffices to check the following:-

- $R_{k,s}(z+1) = R_{k,s}(z)$,
- $R_{k,s}(\frac{-1}{z}) = z^k R_{k,s}(z) \; \forall \; z \in \mathbb{H}$.

These follow easily, since

$$\begin{aligned} R_{k,s}\left(\frac{-1}{z}\right) &= \gamma_k(s) \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} \left(c\frac{-1}{z} + d\right)^{-k} \left(\frac{a\frac{-1}{z} + b}{c\frac{-1}{z} + d}\right)^{-s} \\ &= \gamma_k(s) z^k \sum_{\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \in SL_2(\mathbb{Z})} (dz - c)^{-k} \left(\frac{bz - a}{dz - c}\right)^{-s} \\ &= z^k R_{k,s}(z). \end{aligned}$$

and the maps

$$\begin{aligned} \phi_1 : SL_2(\mathbb{Z}) &\to SL_2(\mathbb{Z}) & \qquad \phi_2 : SL_2(\mathbb{Z}) &\to SL_2(\mathbb{Z}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \end{aligned}$$

are bijections and the above rearrangements are valid for $1 < \sigma < k-1$. Also, from (3.6), it follows that $\dfrac{R_{k,s}(z)}{\gamma_k(s)} = e^{\pi i(k-s)}\dfrac{R_{k,k-s}(z)}{\gamma_k(k-s)}$ which implies the following *functional equation* for $R_{k,s}$ when $s \in \mathbb{C}$ such that $1 < \sigma < k-1$:-

$$R_{k,s}(z) = (-1)^{k/2} R_{k,k-s}(z). \tag{3.12}$$

It remains to prove that $R_{k,s}$ is a cusp form. Without loss of generality, let $z \in \mathcal{D} \cup \partial\mathcal{D}$. From the first inequality in (3.10), it follows immediately that $\lim_{y\to\infty} R_{k,s}(z) = 0$ when $1 < \sigma < \frac{k}{2} - 1$. For the case $\frac{k}{2} - 1 \leq \sigma \leq \frac{k}{2}$, we split the series $R_{k,s}(z)/\gamma_k(s)$ as

$$\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-s} = \left(\sum_{ac=0} + \sum_{ac\neq 0}\right)(cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-s}$$

$$= \sum_{d\in\mathbb{Z}} (z+d)^{-k}\left(\frac{-1}{z+d}\right)^{-s} + \sum_{b\in\mathbb{Z}} (z+b)^{-s} + \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \\ ac\neq 0}} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-s}.$$

The first two terms above are the contributions of $a = 0$ and $c = 0$ respectively. $\Bigg($Note that $a = 0$ forces the matrix to be $\left(\begin{smallmatrix} 0 & -1 \\ 1 & d \end{smallmatrix}\right)\Bigg)$. The last sum ($ac \neq 0$) can be bounded above by

$$\ll e^{\pi|t|} \sum_{a\geq 1}\sum_{b\in\mathbb{Z}} \frac{1}{|az+b|^{\sigma}} \sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{1}{|cz+d|^{k-\sigma}},$$
$$\ll_{\sigma,|t|} \left(\frac{1}{y^{\sigma-1}}\right)\left(\frac{1}{y^{k-\sigma-1}}\right),$$

$$= \ll_{\sigma,|t|} \left(\frac{1}{y^{k-2}}\right),$$

which follows immediately by applying Lemma (2.2.14). The contributions of sum with $ac = 0$ may also be similarly estimated to finally get

$$R_{k,s}(z)/\gamma_k(s) \ll_{\sigma,|t|} \frac{1}{y^{k-\sigma-1}}.$$

which lets us conclude that $\lim_{y\to\infty} R_{k,s}(z) = 0$ and that it is a cusp form. $\square$

The next Proposition shows that this cusp form is indeed (upto a constant) the cusp form ϕ_s that we were seeking in (3.1).

Proposition 3.1.3. *[Koh97] Let $f \in S_k$ and $s \in \mathbb{C}$ with $1 < \sigma < k-1$. Then,*

$$\langle f, R_{k,\bar{s}}\rangle = c_k L^*(f, s), \tag{3.13}$$

where $c_k = \frac{(-1)^{k/2}\pi(k-2)!}{2^{k-2}}$.

The proof involves the usage of Lipschitz's Summation Formula and the inner product relation (2.8) of the Poincare series with any cusp form. We refer to Lemma 1 in [Koh97] for details. We next mention an explicit formula for the n^{th} Fourier coefficient of $R_{k,s}$ without proof.

Lemma 3.1.4. *[Koh97]* $R_{k,s}(z) = \sum_{n\geq 1} r_{k,s}(n)q^n,$ *where*

$$r_{k,s}(n) = (2\pi)^s\Gamma(k-s)n^{s-1} + (-1)^{k/2}(2\pi)^{k-s}\Gamma(s)n^{k-s-1}$$
$$+\frac{1}{2}(-1)^{k/2}(2\pi n)^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\lambda_{k,s}(n).$$

Here,

$$\lambda_{k,s}(n) := \sum_{\substack{(a,c)\in Z^2, ac>0\\ (a,c)=1}} c^{-k}\left(\frac{c}{a}\right)^s \Big\{ e^{2\pi ina'/c}e^{\pi is/2}{}_1F_1\left(s,k;-2\pi in/ac\right)$$
$$+ e^{-2\pi ina'/c}e^{-\pi is/2}{}_1F_1\left(s,k;2\pi in/ac\right)\Big\},$$

where $a' \in \mathbb{Z}$ *is an inverse of a modulo c, and* ${}_1F_1(\alpha,\beta;z)$ *is Kummer's degenerate hypergeometric function.*

Now, we have set up the sufficient background for the theorems to be proved in this chapter and the next. Firstly, we give the idea of the proof of Theorem (3.1.1). One can see from (3.13) that[2]

$$\begin{aligned} R_{k,s}(z) &= \sum_{f\in\mathcal{B}_k}\frac{\langle R_{k,s}, f\rangle}{\langle f,f\rangle}f(z) \\ &= c_k\sum_{f\in\mathcal{B}_k}\frac{L^*(f,s)}{\langle f,f\rangle}f(z). \end{aligned} \tag{3.14}$$

[2]The equation (3.14) fixes a small typo (s instead of $\bar{s}$) in the left hand side of the first displayed equation in pg 188 in [Koh97]

By comparing the first Fourier coefficients on either sides, one can see that

$$\begin{aligned} r_{k,s}(1) &= (2\pi)^s\Gamma(k-s) + (-1)^{k/2}(2\pi)^{k-s}\Gamma(s) \\ &+ \frac{1}{2}(-1)^{k/2}(2\pi)^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\lambda_{k,s}(1) \\ &= c_k \sum_{f\in\mathcal{B}_k} \frac{L^*(f,s)}{\langle f,f\rangle}. \end{aligned}$$

Kohnen proves that for large enough $k \gg_{t_0,\delta} 1$, the middle term cannot vanish and thus, proves the non-vanishing of the sum $\sum_{f\in\mathcal{B}_k} \frac{L^*(f,s)}{\langle f,f\rangle}$.

In Section (3.3), when restricted to the specific case $t_0 = 0$, we obtain the non-vanishing of a different[3] weighted sum on the interval $[\frac{k-1}{2}, \frac{k+1}{2}]$ for all weights $k \geq 12, 4|k$. We show an asymptotic relation first.

3.2 A certain asymptotic relation

For $1 < \sigma < k-1$, we define *the kernel function* as

$$\begin{aligned} f_{k,s}(z) &:= 2\frac{R_{k,s}(z)}{\Gamma(s)\Gamma(k-s)} \\ &= e^{i\frac{\pi}{2}s} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z})} z^{-s}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{3.15}$$

[3] different from that considered by Kohnen

Note that the functional equation of $L^*(\ , s)$ carries over to $f_{k,s}$, i.e.,

$$f_{k,s}(z) = (-1)^{k/2} f_{k,k-s}(z). \tag{3.16}$$

We also remark here that from eqns (3.14) and (3.15), it follows that

$$\begin{aligned} f_{k,s} &= \sum_{f \in \mathcal{B}} \langle f_{k,s}, f \rangle \frac{f}{\langle f, f \rangle} \\ &= \frac{(-1)^{k/2} \pi \Gamma(k-1)}{2^{k-3} \Gamma(s) \Gamma(k-s)} \sum_{f \in \mathcal{B}} L^*(f, s) \frac{f}{\langle f, f \rangle}. \end{aligned} \tag{3.17}$$

We assume $\frac{k-1}{2} \leq \sigma \leq \frac{k+1}{2}$ from now on. We restrict our attention to kernel functions with *real parameter* σ and study its asymptotic behaviour at $z = i$ when $4|k$. Note that $f_{k,\sigma}(i)$ is a weighted sum of L values of Hecke eigenforms evaluated at σ, a real point inside the critical strip. Our first result in this thesis is the following:-

Theorem 3.2.1. *Let* $4|k$. *The cusp form* $f_{k,\sigma}(z)$ *in* S_k *satisfies the asymptotic relation*

$$f_{k,\sigma}(i) = 4 + \mathcal{O}(2^{-\frac{k}{4}}) \qquad (k \to \infty)$$

at $z = i$ *for all values of* $\sigma \in [\frac{k-1}{2}, \frac{k+1}{2}]$.

Proof. For given $1 < \sigma < k-1$, due to the absolute convergence of $f_{k,\sigma}$ on $\mathbb{H}$, one may rearrange the series as follows:-

$$\begin{aligned} f_{k,\sigma}(z) &= e^{i\frac{\pi}{2}\sigma} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-\sigma} \\ &= e^{i\frac{\pi}{2}\sigma} \sum_{\substack{M\geq 1 \\ N \geq 1}} \sum_{a^2+b^2=\boldsymbol{M}} \sum_{\substack{c^2+d^2=\boldsymbol{N} \\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-\sigma}. \end{aligned}$$

where we run both the pairs (a, b) and (c, d) over those integral pair of points ($\subseteq \mathbb{Z}\times\mathbb{Z}$) which lie on all the concentric circles of the form $x^2 + y^2 = r^2$, where r^2 runs over all natural numbers subject to the condition $ad - bc = 1$. (There may be empty terms too, for example, when M is of the form $3 \bmod 4$). Here, we adapt the method used by F.K.C. Rankin and Swinnerton-Dyer in [RSD70].

For $M_0, N_0 \in \mathbb{N}$, we define

$$T_{\boldsymbol{M_0},\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{a^2+b^2=\boldsymbol{M_0}} \sum_{\substack{c^2+d^2=\boldsymbol{N_0} \\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

$$T_{\boldsymbol{M_0},\geq\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{a^2+b^2=\boldsymbol{M_0}} \sum_{\substack{N\in\mathbb{N} \\ N\geq\boldsymbol{N_0}}} \sum_{\substack{c^2+d^2=N \\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

$$T_{\geq\boldsymbol{M_0},\boldsymbol{N_0},\sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{\substack{M\in\mathbb{N} \\ M\geq\boldsymbol{M_0}}} \sum_{a^2+b^2=M} \sum_{\substack{c^2+d^2=\boldsymbol{N_0} \\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

$$T_{\geq \boldsymbol{M_0}, \geq \boldsymbol{N_0}, \sigma}(z) := \quad e^{i\frac{\pi}{2}\sigma} \sum_{\substack{M \in \mathbb{N} \\ M \geq \boldsymbol{M_0}}} \sum_{a^2+b^2=M} \sum_{\substack{N \in \mathbb{N} \\ N \geq \boldsymbol{N_0}}} \sum_{\substack{c^2+d^2=N \\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-\sigma},$$

As given in the theorem, we assume $4|k$. We now proceed to split the series $f_{k,\sigma}(i)$ into a main term and an error term. We first evaluate $T_{1,1,\sigma}(i)$.

$T_{1,1}(z)$ is formed by summing $z^{-\sigma}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over the collection $\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$. Thus, we get

$$T_{1,1,\sigma}(i) = \; 2e^{i\frac{\pi}{2}\sigma}\left\{(i)^{-\sigma} + (-i)^{-k}\left(\tfrac{1}{-i}\right)^{-\sigma}\right\} = \; 4.$$

Next, we note that

$$|T_{\geq 2, \geq 1, \sigma}(i)| \leq \sum_{\substack{M \in \mathbb{N} \\ M \geq \boldsymbol{2}}} \sum_{a^2+b^2=M} \sum_{\substack{N \in \mathbb{N} \\ N \geq \boldsymbol{1}}} \sum_{\substack{c^2+d^2=N \\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} |ci+d|^{-k+\sigma}|ai+b|^{-\sigma}. \tag{3.18}$$

For a fixed $M \geq 2$, the number of integral pairs (a, b) satisfying $a^2+b^2 = M$ is atmost $2(2\sqrt{M}+1)$, which is $\leq 5\sqrt{M}$. Now, given a pair (a, b) as above and given $N \geq 1$, the number of pairs (c, d) satisfying both $c^2 + d^2 = N$ and $ad - bc = 1$ is atmost 2 (since c now satisfies a quadratic equation and d gets automatically fixed once a, b and

c are fixed). Hence,

$$|T_{\geq 2,\geq 1,\sigma}(i)| \leq \sum_{\substack{M\in\mathbb{N}\\ M\geq \mathbf{2}}} \frac{5}{M^{(\sigma-1)/2}} \sum_{N\geq 1} \frac{2}{N^{(k-\sigma)/2}},$$
$$\ll \frac{\zeta((\sigma-1)/2)\zeta((k-\sigma)/2)}{2^{\sigma/2}} \ll 2^{-k/4}.$$

This follows from Lemma 2.2.12. Note that

$$f_{k,\sigma}(i) = T_{1,1,\sigma}(i) + T_{\geq 2,\geq 1,\sigma}(i) + T_{1,\geq 2,\sigma}(i).$$

In the remaining term, for any pair (a,b) satisfying $a^2+b^2 = 1$, it can be seen that $|ai+b|^\sigma = 1$. Hence,

$$|T_{1,\geq 2,\sigma}(i)| \leq \sum_{a^2+b^2=1} \sum_{\substack{N\in\mathbb{N}\\ N\geq \mathbf{2}}} \sum_{\substack{c^2+d^2=N\\ ad-bc=1}} |ci+d|^{-k+\sigma},$$
$$\ll \sum_{N\geq 2} \frac{1}{N^{(k-\sigma)/2}} \ll \zeta\left(\frac{k-\sigma}{2}\right)/2^{(k-\sigma)/2} \ll 2^{-k/4}.$$

□

3.3 An explicit estimate for kernel evaluated at i

In addition to the above asymptotic relation, one also has an explicit inequality valid for all $4|k$, $k\geq 12$ as shown below:-

Theorem 3.3.1. *Let $k \geq 12$ be an integer divisible by 4. Then, $f_{k,\sigma}(i)$ is real valued and satisfies*

$$f_{k,\sigma}(i) \ \geq 2.745.$$

Proof. From eqns (3.17), it follows that

$$f_{k,\sigma}(i) = \frac{(-1)^{k/2}\pi\Gamma(k-1)}{2^{k-3}\Gamma(\sigma)\Gamma(k-\sigma)} \sum_{f\in\mathcal{B}} \frac{L^*(f,\sigma)}{\langle f, f\rangle} f(i) \tag{3.19}$$

Since f is a Hecke eigenform, its Fourier coefficients are real valued and hence, from the Fourier expansion, it follows that $f(i)$ (in fact, even $f(it)$) is real valued, whence, same follows for $L^*(f,\sigma)$. Thus, it easily follows that $f_{k,\sigma}(i)$ ($f_{k,\sigma}(it)$) is real valued. By (3.16), it suffices to prove the theorem for $\sigma \geq k/2$.

We define

$$\begin{aligned} T_{main,\sigma}(z) &:= T_{1,1,\sigma}(z) + T_{1,2,\sigma}(z) + T_{2,1,\sigma}(z), \\ T_{error,\sigma}(z) &:= f_{k,\sigma}(z) - T_{main,\sigma}(z). \end{aligned} \tag{3.20}$$

For the rest of the proof, we drop the parameter σ as it is clear from the context.

3.3.1 The main term $T_{main}(i)$

The matrices which form the term $T_{1,2}(i)$ are $\left\{\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right\}$.

$$T_{1,2}(i) = 2e^{i\frac{\pi}{2}\sigma}\Bigg\{ (i+1)^{-k}\left(\frac{i}{i+1}\right)^{-\sigma} + (i+1)^{-k}\left(\frac{-1}{1+i}\right)^{-\sigma} + \\ (-i+1)^{-k}\left(\frac{i}{-i+1}\right)^{-\sigma} + (-i+1)^{-k}\left(\frac{1}{-i+1}\right)^{-\sigma}\Bigg\}.$$

Now, consider $\left(\frac{i}{1+i}\right)^{-\sigma}$. Since $\arg\frac{1}{1+i} < 0$, we see that

$$\left(\frac{i}{1+i}\right)^{-\sigma} = i^{-\sigma}\left(\frac{1}{1+i}\right)^{-\sigma} = i^{-\sigma}\left(1+i\right)^{\sigma},$$

by applying Lemma (2.2.13) (i). Similarly, it follows that

(i) $\left(\frac{-1}{1+i}\right)^{-\sigma} = (-1)^{-\sigma}(1+i)^{\sigma}$,

(ii) $\left(\frac{i}{1-i}\right)^{-\sigma} = (i)^{-\sigma}(1-i)^{\sigma}$ &

(iii) $\left(\frac{1}{1-i}\right)^{-\sigma} = (1-i)^{\sigma}$.

Hence, we get

$$T_{1,2}(i) = 2\left\{(i+1)^{-k+\sigma} + (i+1)^{-k+\sigma}\,e^{-i\frac{\pi}{2}\sigma} + (-i+1)^{-k+\sigma} + (-i+1)^{-k+\sigma}\,e^{i\frac{\pi}{2}\sigma}\right\}.$$

Clearly, the first and third terms are conjugates of each other and so are, the second and fourth terms. Thus, $T_{1,2}(i)$ (also real valued) simplifies as

$$\begin{aligned} T_{1,2}(i) &= 2\left\{\frac{e^{i\frac{\pi}{4}(k-\sigma)}+e^{-i\frac{\pi}{4}(k-\sigma)}}{2^{(k-\sigma)/2}}+\frac{e^{i\frac{\pi}{4}(k+\sigma)}+e^{-i\frac{\pi}{4}(k+\sigma)}}{2^{(k-\sigma)/2}}\right\}, \\ &= \frac{4}{2^{(k-\sigma)/2}}\left\{\cos\left(\frac{\pi}{4}(k-\sigma)\right)+\cos\left(\frac{\pi}{4}(k+\sigma)\right)\right\}, \\ &= \frac{8}{2^{(k-\sigma)/2}}(-1)^{k/4}\left(\cos\frac{\pi k}{8}\cos\frac{\pi\epsilon}{4}-\sin\frac{\pi k}{8}\sin\frac{\pi\epsilon}{4}\right), \end{aligned}$$

where we have substituted $\sigma=\frac{k}{2}+\epsilon$, $0\le\epsilon\le\frac{1}{2}$. After few simplifications, $T_{1,2}(i)$ is obtained as

$$T_{1,2}(i)=\begin{cases} \frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\cos\frac{\pi\epsilon}{4} & k\equiv 0 \bmod 16, \\ \frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\sin\frac{\pi\epsilon}{4} & k\equiv 4 \bmod 16, \\ -\frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\cos\frac{\pi\epsilon}{4} & k\equiv 8 \bmod 16, \\ -\frac{2^{\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}}\sin\frac{\pi\epsilon}{4} & k\equiv 12 \bmod 16. \end{cases}$$

Similarly, the term $T_{2,1}(i)$ is formed by the matrices in $\left\{\pm\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}, \pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right\}$. So,

$$T_{2,1}(i)=2e^{i\frac{\pi}{2}\sigma}\left\{(i)^{-k}\left(\frac{i-1}{i}\right)^{-\sigma}+(-i)^{-k}\left(\frac{i+1}{-i}\right)^{-\sigma}+(i+1)^{-\sigma}+(i-1)^{-\sigma}\right\}.$$

Proceeding as earlier, one obtains

$$T_{2,1}(i) = \begin{cases} \frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 0 \bmod 16, \\ -\frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 4 \bmod 16, \\ -\frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 8 \bmod 16, \\ \frac{2^{-\frac{\epsilon}{2}}}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 12 \bmod 16. \end{cases}$$

Adding $T_{1,1}, T_{1,2}$ and $T_{2,1}$ at $z = i$, we get $T_{main}(i)$ as

$$T_{main}(i) = \begin{cases} 4 + \frac{\left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 0 \bmod 16, \\ 4 + \frac{\left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 4 \bmod 16, \\ 4 - \frac{\left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \cos \frac{\pi\epsilon}{4} & k \equiv 8 \bmod 16, \\ 4 - \frac{\left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right)}{2^{\frac{k}{4}-3}} \sin \frac{\pi\epsilon}{4} & k \equiv 12 \bmod 16. \end{cases}$$

We next find a lower bound for $T_{main}(i)$ which is valid for all $k \geq 12, 4|k$. Clearly, $T_{main}(i) \geq 4$ in the first two cases as $\epsilon \in [0, \frac{1}{2}]$. Further, the function $f_1(\epsilon) := \left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right) \cos \frac{\pi\epsilon}{4}$ is decreasing on $[0, 1/2]$. By virtue of this, we see that in the third case, for $k \geq 24$,

$$\left| \frac{1}{2^{\frac{k}{4}-3}} \left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right) \cos \frac{\pi\epsilon}{4} \right| \leq 0.25$$

and thus, $T_{main}(i) \geq 3.75$. Similarly, the function $f_2(\epsilon) := \left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right) \sin \frac{\pi\epsilon}{4}$ is increasing on $[0, 1/2]$. To see this, note that the derivative

$$f_2'(\epsilon) = \left(2^{\frac{\epsilon}{2}} + 2^{\frac{-\epsilon}{2}}\right) \tfrac{\log 2}{2} \sin \tfrac{\pi\epsilon}{4} + \left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right) \tfrac{\pi}{4} \cos \tfrac{\pi\epsilon}{4} > 0 \text{ on } (0, \tfrac{1}{2}].$$

since each term individually is positive. Hence, in the fourth case where $k \equiv 12 \bmod 16$, we have

$$\left| \frac{1}{2^{\frac{k}{4}-3}} \left(2^{\frac{\epsilon}{2}} - 2^{\frac{-\epsilon}{2}}\right) \sin \frac{\pi\epsilon}{4} \right| \leq 0.134$$

and hence, $T_{main}(i) \geq 3.866$. To summarize,

$$T_{main}(i) \geq \begin{cases} 4 & k \equiv 0 \bmod 16, \\ 4 & k \equiv 4 \bmod 16, \\ 3.75 & k \equiv 8 \bmod 16, \\ 3.866 & k \equiv 12 \bmod 16. \end{cases} \tag{3.21}$$

3.3.2 Estimating the error term

Recall from (3.20) that

$$T_{error}(i) = T_{1,\geq 5}(i) + T_{2,\geq 5}(i) + T_{\geq 5,1}(i) + T_{\geq 5,2}(i) + T_{2,2}(i) + T_{\geq 5,\geq 5}(i).$$

For fixed $N \in \mathbb{N}$, $T_{N,1}(i)$ is formed precisely from the matrices $\pm \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where a runs over all $\mathbb{Z}$ such that $a^2 + 1 = N$. Thus, we get

$$\begin{aligned} T_{\geq 5,1}(i) &= \sum_{N \geq 5} T_{N,1}(i) \\ &= 2e^{i\frac{\pi}{2}\sigma} \sum_{|a| \geq 2} \left\{ i^{-k} \left(\frac{ai-1}{i} \right)^{-\sigma} + (i+a)^{-\sigma} \right\} \\ &= 4e^{i\frac{\pi}{2}\sigma} \sum_{|a| \geq 2} (a+i)^{-\sigma}. \end{aligned}$$

Thus,

$$|T_{\geq 5,1}(i)| \leq 4 \sum_{|a| \geq 2} \left(\frac{1}{1+a^2} \right)^{\frac{k}{4}+\frac{\epsilon}{2}} \leq 8 \sum_{a=2}^{\infty} \frac{1}{a^{\frac{k}{2}}} \leq \frac{\zeta(\frac{k}{2})}{2^{\frac{k}{2}-4}}.$$

where we have used the Lemma (2.2.12) in the last inequality.

For fixed $M \in \mathbb{N}$, $T_{1,M}(\theta)$ is formed precisely by the matrices in

$$\pm \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & c \end{pmatrix} | c \in \mathbb{Z}, c^2 + 1 = M \right\}.$$

Using this,

$$\begin{aligned} T_{1,\geq 5}(i) &= \sum_{M \geq 5} T_{1,M}(i) \\ &= 2e^{i\frac{\pi}{2}\sigma} \sum_{|c| \geq 2} \left\{ (ci+1)^{-k} \left(\frac{i}{ci+1} \right)^{-\sigma} + (i+c)^{-k} \left(\frac{-1}{i+c} \right)^{-\sigma} \right\}. \end{aligned}$$

Thus,

$$\left|\sum_{M\geq 5} T_{1,M}(i)\right| \leq 4\sum_{|c|\geq 2}\left(\frac{1}{1+c^2}\right)^{\frac{k}{4}-\frac{\epsilon}{2}} \leq 8\sum_{c=2}^{\infty}\frac{1}{c^{\frac{k}{2}-\epsilon}} \leq \frac{\zeta\left(\frac{k-1}{2}\right)}{2^{\frac{k-9}{2}}},$$

using Lemma (2.2.12). Next, we consider terms of the form $T_{N,2}$ where $N \geq 5$. Note that these are formed by matrices in $SL_2(\mathbb{Z})$ which are one among

$$\pm\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} a & b \\ 1 & -1 \end{pmatrix}, \text{ where } |a| \geq 1, |b| \geq 1 \text{ and } a^2+b^2 = N. \tag{3.22}$$

We claim that terms of the form $T_{N,2}$ where N is NOT a sum of squares of consecutive integers do not survive; (for example, $T_{2,2}, T_{10,2}, T_{17,2}, T_{26,2}$). Indeed, for any $\gamma = \begin{pmatrix} a & b \\ * & * \end{pmatrix}$ from the above collection, by the determinant condition, it follows that a and b must satisfy either

$$|a+b| = 1 \quad \text{OR} \quad |a-b| = 1. \tag{3.23}$$

Clearly then, such a pair a and b can be neither simultaneously odd nor simultaneously even. Clearly, $a \neq b$ follows, which further implies $||a|-|b|| \geq 1$. Thus,

$$1 \leq ||a|-|b|| \leq \min\{|a+b|, |a-b|\} = 1.$$

Thus, in both the cases in (3.23), their absolute values differ by 1 exactly. Hence, if a

and b satisfy the conditions in (3.22), then, $b \in \{a-1, a+1, -a-1, -a+1\}$. Now, further respecting the determinant conditions therein, we conclude that $T_{N,2}$ is formed only out of the following set of matrices:

$$\left\{ \pm \begin{pmatrix} a & a-1 \\ 1 & 1 \end{pmatrix} \mid a \in \mathbb{Z}, a^2+(a-1)^2 = N \right\} \cup \left\{ \pm \begin{pmatrix} a & -(a+1) \\ 1 & -1 \end{pmatrix} \mid a \in \mathbb{Z}, a^2+(a+1)^2 = N \right\} \tag{3.24}$$

Thus, we see that

$$\begin{aligned} T_{\geq 5,2}(i) = 2e^{i\frac{\pi}{2}\sigma}\Big\{ & \sum_{\substack{a\in\mathbb{Z} \\ a^2+(a-1)^2\geq 5}} (i+1)^{-k}\left(\frac{ai+a-1}{i+1}\right)^{-\sigma} \\ & + \sum_{\substack{a\in\mathbb{Z} \\ a^2+(a+1)^2\geq 5}} (i-1)^{-k}\left(\frac{ai-(a+1)}{i-1}\right)^{-\sigma}\Big\}. \end{aligned}$$

So,

$$\begin{aligned} |T_{\geq 5,2}(i)| &\leq 2\left\{ \sum_{|a|\geq 2} \frac{1}{|i+1|^{k-\sigma}}\frac{1}{|ai+a-1|^{\sigma}} + \sum_{|a|\geq 1} \frac{1}{|i-1|^{k-\sigma}}\frac{1}{|ai-(a+1)|^{\sigma}} \right\}, \\ &= \frac{8}{2^{(k-\sigma)/2}} \sum_{n\geq 1}\left(\frac{1}{n^2+(n+1)^2}\right)^{\sigma/2} \leq \frac{8}{2^{k/2}}\zeta(\sigma) \leq \frac{8}{2^{k/2}}\zeta(k/2). \end{aligned}$$

Our next term in $T_{error}(i)$ is of the form $\sum_{M\geq 5} T_{2,M}(i)$. The terms of the form $T_{2,M}(i)$ are formed by matrices which are one among

$$\left\{ \pm \begin{pmatrix} 1 & 1 \\ c & d \end{pmatrix} \mid c,d \in \mathbb{Z}, c^2+d^2 = M \right\} \cup \left\{ \pm \begin{pmatrix} 1 & -1 \\ c & d \end{pmatrix} \mid c,d \in \mathbb{Z}, c^2+d^2 = M \right\}$$

As above, we see that $||d|-|c||=1$ and hence, given c as above, the choices of d lie in $\{c+1, c-1, -c+1, -c-1\}$ and further, in view of the determinant condition, can be narrowed down further. Hence, we see that

$$\left\{\pm\begin{pmatrix} 1 & 1 \\ c & c+1 \end{pmatrix},\ \pm\begin{pmatrix} 1 & -1 \\ c+1 & -c \end{pmatrix} \mid c\in\mathbb{Z}, c^2+(c+1)^2=M\right\}.$$

Thus, we get

$$T_{2,\geq 5}(i) = 2e^{i\frac{\pi}{2}\sigma}\sum_{\substack{c\in\mathbb{Z}\\ c^2+(c+1)^2\geq 5}}\left\{(ci+c+1)^{-k}\left(\frac{i+1}{ci+c+1}\right)^{-\sigma} + ((c+1)i-c)^{-k}\left(\frac{i-1}{(c+1)i-c}\right)^{-\sigma}\right\}.$$

Thus, similar to previous computation, we get

$$\begin{aligned}|T_{\geq 5,2}(i)| &\leq 4\left\{\sum_{|c|\geq 1}\frac{1}{|i+1|^{\sigma}}\frac{1}{(c^2+(c+1)^2)^{(k-\sigma)/2}}\right\},\\ &\leq \frac{8}{2^{k/2}}\zeta((k-1)/2).\end{aligned}$$

The only remaining term to be estimated in the error term is $T_{\geq 5,\geq 5}(i)$. We have

$$\begin{aligned}|T_{\geq 5,\geq 5}(i)| \leq& \sum_{M\geq 5}\sum_{a^2+b^2=M}\sum_{N\geq 5}\sum_{\substack{c^2+d^2=N\\ det\begin{pmatrix} a & b\\ c & d\end{pmatrix}=1}} |ci+d|^{-(k-\sigma)}\,|ai+b|^{-\sigma}\\ &\leq \left(\sum_{N\geq 5}\sum_{c^2+d^2=N}|ci+d|^{-(k-\sigma)}\right)\left(\sum_{M\geq 5}\sum_{a^2+b^2=M}|ai+b|^{-\sigma}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}|T_{\geq 5,\geq 5}(i)| &\leq \left(\sum_{N\geq 5}\frac{4\sqrt{N}}{N^{\frac{k}{4}-\frac{\epsilon}{2}}}\right)\left(\sum_{M\geq 5}\frac{4\sqrt{M}}{M^{\frac{k}{4}+\frac{\epsilon}{2}}}\right)\\ &\leq \frac{16}{5^{\frac{k}{2}-3}}\zeta\left(\frac{k-2}{4}\right)\zeta\left(\frac{k-3}{4}\right),\end{aligned}$$

where, we have again used Lemma 2.2.12. Thus, we have the following numerical estimates:

$$|T_{error}(i)| \leq \begin{cases} 0.22148 & k \equiv 0 \bmod 16, \\ 0.05402 & k \equiv 4 \bmod 16, \\ 0.01346 & k \equiv 8 \bmod 16, \\ 1.12052 & k \equiv 12 \bmod 16. \end{cases} \tag{3.25}$$

From (3.21) and (3.25), when $4|k$, clearly, $f_{k,\sigma}(i) > 0$ and the lower bound mentioned in the statement of the theorem is obtained as

$$f_{k,\sigma}(i) \geq T_{main}(i) - |T_{error}(i)| > 3.866 \; - \; 1.12052 \; = \; 2.74548.$$

This completes the proof. $\square$

3.4 Non-vanishing at real points

We have now established GRH on the interval $[\frac{k-1}{2}, \frac{k+1}{2}]$ for a weighted sum of Hecke eigenforms for all $k \geq 12, 4|k$. An attempt to establish the same for complex points is currently not being done here. Next, we discuss in brief an application which immediately follows from the previous proof.

Remark 3.4.1. *Recall (3.19). For all $k \geq 12$ such that $4|k$,*

$$2.745 \leq f_{k,\sigma}(i) = \frac{\pi\Gamma(k-1)}{2^{k-3}\Gamma(\sigma)\Gamma(k-\sigma)} \sum_{f \in \mathcal{B}} \frac{L^*(f,\sigma)}{\langle f, f\rangle} f(i).$$

From this, we also observe that for given $\sigma \in [\frac{k-1}{2}, \frac{k+1}{2}]$, there exists at least one Hecke eigenform $f \in S_k$ (dependent on σ) whose L-value is real and non-zero. However, as mentioned in the Chapter 1 (Introduction), there is an easier proof hinted in [CK18] by Choie and Kohnen. We briefly discuss it here.

Theorem 3.4.2. *[CK18] Let $k \geq 12, 4|k$. For $\sigma \in [\frac{k-1}{2}, \frac{k+1}{2}]$, there exists at least one Hecke eigenform $f \in S_k$ whose L-value is real and non-zero.*

Proof. Note that since $4|k$, we may write $k = 12q + 4r$ where $q \in \mathbb{N}$ and $r \in \{0, 1, 2\}$. Since $\Delta^q E_4^r$ is a modular form of weight k, we may write

$$L^*(\Delta^q E_4^r, \sigma) = \int_0^\infty \Delta^q(it) E_4^r(it) t^{\sigma-1} dt.$$

It follows from (2.2) that $E_4(it) > 0$ for all $t > 0$ since all Fourier coefficients are positive. From the product expansion (2.3) of Δ, we also know that $\Delta(it) > 0$ (non-vanishing even follows from Valence Formula (2.4)). Hence, we get

$$0 < L^*(\Delta^q E_4^r, \sigma) = \sum_{f \in \mathcal{B}_k} c_f L^*(f, \sigma).$$

where $\Delta^q E_4^r(z) = \sum_{f \in \mathcal{B}_k} c_f \ f$. From this, the theorem clearly follows. $\square$

3.5 Applications to L-functions

Theorem (3.3.1) has also the following consequence on lower bounds for the values of L-function in the critical strip.

Corollary 3.5.1. *Given $\sigma = \frac{k}{2} + \epsilon \ \in [\frac{k-1}{2}, \frac{k+1}{2}]$ and an arbitrarily small $\delta > 0$, for sufficiently large k, where $4|k$, we have*

$$\max_{f \in \mathcal{B}_k} |L(f, \sigma)| \gg_\delta k^{-\epsilon - 1 - \delta}.$$

Proof. It is well known that for any normalised Hecke eigenform $f \in S_k$,

$$k^{-\delta_1} \ll_{\delta_1} L(\text{Sym}^2(f), k) \ll_{\delta_1} k^{\delta_1}$$

holds for arbitrarily small $\delta_1 > 0$ ([Luo17], p.4). Recalling Theorem (2.1.11), we see that

$$\langle f, f\rangle \gg_{\delta_1} \frac{\Gamma(k)}{(4\pi)^k k^{\delta_1}}. \tag{3.26}$$

From Theorem (3.3.1), it follows that when $4|k$,

$$\begin{aligned}
2.745 < f_{k,\sigma}(i) &= \frac{\pi\Gamma(k-1)}{2^{k-3}\Gamma(\sigma)\Gamma(k-\sigma)} \sum_{f\in\mathcal{B}_k} \frac{L^*(f,\sigma)}{\langle f, f\rangle} f(i), \\
&\ll_{\delta_1} \frac{1}{k}\frac{(2\pi)^{k-\sigma}}{\Gamma(k-\sigma)} k^{\delta_1} \sum_{f\in\mathcal{B}_k} L(f,\sigma) f(i), \\
&\ll_{\delta_1} \frac{(2\pi)^{k-\sigma}}{\Gamma(k-\sigma)} k^{\delta_1} \max_{f\in\mathcal{B}_k} \left| L(f,\sigma) \sum_{n\geq 1} a_f(n) e^{-2\pi n} \right|.
\end{aligned}$$

Using the Ramanujan-Petersson estimate (2.17), for arbitrarily small $\delta_2 > 0$, we have

$$\sum_{n\geq 1} a_f(n) e^{-2\pi n} \ll_{\delta_2} \sum_{n\geq 1} n^{\frac{k-1}{2}+\delta_2} e^{-2\pi n}. \tag{3.27}$$

Using Lemma (2.2.16), the right hand side of (3.27) is estimated to be

$$\ll_{\delta_2} \left(\frac{\frac{k+1}{2}+\delta_2}{2\pi e}\right)^{\frac{k+1}{2}+\delta_2}.$$

By Stirling's estimate (2.27), we have $\Gamma(k-\sigma) \gg (\frac{k-\sigma}{e})^{k-\sigma}\sqrt{\frac{1}{k-\sigma}}$. Hence,

$$1 \ll_{\delta_1,\delta_2} \left(\frac{2\pi e}{k-\sigma}\right)^{k-\sigma} (k-\sigma)^{1/2} k^{\delta_1} \max_{f\in\mathcal{B}} |L(f,\sigma)| \left(\frac{\frac{k+1}{2}+\delta_2}{2\pi e}\right)^{\frac{k+1}{2}+\delta_2}$$

for large enough $k \gg 1$. Writing $\sigma = \frac{k}{2} + \epsilon$, this simplifies to

$$1 \ll_{\delta_1,\delta_2} k^{1+\delta_1+\delta_2+\epsilon} \max_{f\in\mathcal{B}} |L(f,\sigma)|.$$

Thus, for $k \gg 1$ and $\sigma \geq \frac{k-1}{2}$, we have

$$\max_{f\in\mathcal{B}_k} |L(f,\sigma)| \gg_\delta k^{-(\sigma-\frac{k}{2})-1-\delta},$$

where, we take $\delta_1 = \delta_2$ and define $\delta := 2\delta_2$.

We remark here that better lower bounds could be available for $\max_{f\in\mathcal{B}_k} \ |L(f,\sigma)|$ than what has been presented in the above corollary. For example, when $4|k$ and $\sigma = \frac{k}{2}$, we indeed have,

Theorem 3.5.2. *[Sou08] For large $k \equiv 0 \bmod 4$, there exists an $f \in \mathcal{B}_k$ with*

$$L(f,\frac{k}{2}) \geq \exp\left((1+o(1))\,\frac{\sqrt{2\log k}}{\log\log k}\right).$$

However, our purpose was to show applications of Theorem (3.3.1) to the L values of Hecke eigenforms (in regards to non-vanishing and obtaining lower bounds). In the next chapter, we provide more interesting applications in this regard. $\square$

Chapter 4

Counting Hecke eigenforms with non-vanishing L-value

4.1 Introduction

We begin with a notation. Let

$$N_k(s) := \{f \in \mathcal{B}_k \mid L(f,s) \neq 0\}. \tag{4.1}$$

Recall that, as a consequence of Theorem (3.1.1), Kohnen showed that $N_k(s) \geq 1$, for complex points s on the line segments inside the critical strip $\{\frac{k-1}{2} < \Re(s) < \frac{k}{2} - \delta, \Im(s) = t_0\}$ and $\{\frac{k}{2} + \delta < \Re(s) < \frac{k+1}{2}, \Im(s) = t_0\}$ for all weights $k \gg_{t_0,\delta} 1$. However, one could ask if there are more than one Hecke eigenform in S_k whose L

value at s is non-zero. At the central critical point $s = \frac{k}{2}$, W. Luo ([Luo15], (4)) showed that

$$N_k(k/2) \gg k,$$

as $k \to \infty$, $4|k$. Prior to that, in [Sen00], the author proved the lower bound

$$N_k(k/2) \gg_{\delta''} k^{1-\delta''}$$

when $4|k$, assuming the Lindelöf hypothesis in the k-aspect for $L(f,s)$, using Kohnen's kernel function. We note here that if $k \equiv 2 \bmod 4$, then the L-function vanishes at $s = k/2$ since the functional equation has root number -1.

Equally interesting is the question of estimating the number of Hecke eigenforms in S_k whose L-value is simultaneously non-vanishing at two given points. For a given weight k and points s_1 & $s_2 \in \mathbb{C}$, let us define

$$N_k(s_1, s_2) := \#\{f \in \mathcal{B}_k \mid L(f, s_1) \cdot L(f, s_2) \neq 0\}. \tag{4.2}$$

We mention here a recent result in this direction due to Y. Choie, W. Kohnen and Y. Zhang:-

Theorem 4.1.1. *[CKZ20] For any fixed positive real numbers T & δ, let the region $\mathcal{R}'_{T,\delta}$ of points $(s_1, s_2) \in \mathbb{C}^2$ be such that*

- $\Im(s_j) \in [-T, T]$,

- $\frac{k-1}{2} < \Re(s_j) < \frac{k+1}{2}$,
- $\left|\Re(s_j) - \frac{k}{2}\right| \geq \delta$,

holds for both $j = 1, 2$. *Then, there exists a constant* $C(T, \delta) > 0$ *depending only on* T & δ *such that for all weights* $k > C(T, \delta)$, *the following function*

$$\sum_{f \in \mathcal{B}_k} \frac{L^*(f, s_1) L^*(f, s_2)}{\langle f, f \rangle}$$

does not vanish at any pair $(s_1, s_2) \in \mathcal{R}'_{T,\delta}$.

Analogous to the kernel function considered by Kohnen, here they consider the dual to a product of L^* values as shown below:-

$$\langle E^*_{s_1, k-s_1}(z, s_2), f \rangle = L^*(f, s_1) L^*(f, s_2),$$

and find an expression for the first Fourier coefficient of $E^*_{s_1,k-s_1}(z, s_2)$ (as a function of z), valid for $(s_1, s_2) \in \mathcal{R}'_{T,\delta}$ ([CKZ20], Lemma (4.6)) and extract the conditions when it is non-vanishing. Here, f is any Hecke eigenform in S_k and $E^*_{s_1,k-s_1}(z, s_2)$ is the completed double Eisenstein series (for definition, see [CKZ20], Theorem (4.1)) which is an element of S_k as a function of z, valid for any fixed pair $(s_1, s_2) \in \mathbb{C}^2$.

As a corollary to Theorem (4.1.1), one can immediately see the following: For $k > C(T, \delta)$ and for any pair $(s_1, s_2) \in \mathcal{R}'_{T,\delta}$,

$$N_k(s_1, s_2) \geq 1.$$

In this chapter, we provide certain partial answers to both these questions, i.e., estimating $N_k(s_1, s_2)$ and $N_k(s)$. For this purpose, we first evaluate $f_{k,s}(it)$ on the imaginary axis when $t \geq 1$ (in Theorem (4.1.2)). For this, as earlier, we take cue from the idea of Rankin and Swinnerton-Dyer in [RSD70].

Theorem 4.1.2. *Let $s = \sigma + i\beta$ be a complex number such that $\frac{k-1}{2} \leq \sigma \leq \frac{k+1}{2}$. Then, for all even $k \geq 12$ and $t \geq 1$, the cusp form $f_{k,s}(it)$ satisfies the following explicit relation:-*

$$f_{k,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1} n^{s-1}e^{-2\pi nt} + (-1)^{k/2}2\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1} n^{k-s-1}e^{-2\pi nt} + C_\beta\left(\frac{1}{t^{k-2}}\right),$$

where, $|C_\beta| \leq 300e^{\frac{\pi}{2}|\beta|}$.

Proof. The computations performed in the proof are similar to those followed in the proof of Theorem (3.2.1). Firstly, we re-arrange $f_{k,s}(it)$ as below:-

$$f_{k,s}(it) = T_{1,1,s}(it) + T_{\geq 2,1,s}(it) + T_{1,\geq 2,s}(it) + T_{\geq 2,\geq 2,s}(it). \tag{4.3}$$

Now, $T_{1,1,s}(it)$ is formed by the matrices in $\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\}$. Hence, we get

$$\begin{aligned} T_{1,1,s}(it) &= 2e^{i\frac{\pi}{2}s}\left\{(it)^{-s} + (it)^{-k}\left(\frac{-1}{it}\right)^{-s}\right\} \\ &= \frac{2}{t^s} + \frac{2(-1)^{k/2}}{t^{k-s}}. \end{aligned}$$

The last equality is true since $(it)^{-s} = e^{-\frac{i\pi}{2}s}t^{-s}$ and $(\frac{-1}{it})^{-s} = e^{-i\frac{\pi}{2}s}t^{s}$ using Lemma (2.2.13).

The term $T_{1,\geq 2,s}(it)$ is formed by the matrices of the form

$$\left\{\pm\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c\in\mathbb{Z}, c^2+1\geq 2\right\}\bigcup\left\{\pm\begin{pmatrix} 0 & -1 \\ 1 & c \end{pmatrix} \mid c\in\mathbb{Z}, c^2+1\geq 2\right\}.$$

It shows that

$$T_{1,\geq 2,s}(it) = 2e^{i\frac{\pi}{2}s}\sum_{|c|\geq 1}\left\{(c+it)^{-k}\left(\frac{-1}{c+it}\right)^{-s} + (cit+1)^{-k}\left(\frac{it}{cit+1}\right)^{-s}\right\}.$$

Let

$$\begin{aligned} T^{main}_{1,\geq 2,s}(it) &:= 2e^{i\frac{\pi}{2}s}\sum_{|c|\geq 1}(c+it)^{-k}\left(\frac{-1}{c+it}\right)^{-s}, \\ T^{error}_{1,\geq 2,s}(it) &:= 2e^{i\frac{\pi}{2}s}\sum_{|c|\geq 1}(cit+1)^{-k}\left(\frac{it}{cit+1}\right)^{-s}. \end{aligned}$$

The first term above may be simplified as

$$\begin{aligned} T^{main}_{1,\geq 2,s}(it) &= 2e^{-i\frac{\pi}{2}s}\sum_{|c|\geq 1}(c+it)^{-(k-s)} \\ &= 2(-1)^{k/2}\left\{e^{i\frac{\pi}{2}(k-s)}\sum_{c\in\mathbb{Z}}(c+it)^{-(k-s)} - e^{i\frac{\pi}{2}(k-s)}(it)^{-(k-s)}\right\} \\ &= 2(-1)^{k/2}\left\{\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1}n^{k-s-1}e^{-2\pi nt} - \frac{1}{t^{k-s}}\right\}. \end{aligned}$$

The last equality follows from the Lipschitz Summation Formula (2.30). Similarly, we simplify $T^{error}_{1,\geq 2,s}(it)$ as below:-

$$\begin{aligned} T^{error}_{1,\geq 2,s}(it) &= \frac{2}{(it)^k} e^{i\frac{\pi}{2}(s)} \sum_{|c|\geq 1} (c - \frac{i}{t})^{-(k-s)} \\ &= \frac{2}{t^k} e^{i\frac{\pi}{2}(k-s)} \sum_{|c|\geq 1} (c + \frac{i}{t})^{-(k-s)} \\ &= \frac{2}{t^k} \frac{(2\pi)^{k-s}}{\Gamma(k-s)} \sum_{n\geq 1} n^{k-s-1} e^{-2\pi n/t} - \frac{2}{t^s}. \end{aligned} \tag{4.4}$$

The term $T^{error}_{1,\geq 2,s}(it)$ is easily seen to be bounded above by $\mathcal{O}_\beta(t^{-k})$. One can see this from below observations:

$$\left| \left(\frac{it}{cit+1} \right)^{-s} \right| = \left| \left(c + \frac{1}{it} \right)^{s} \right| = \left(c^2 + \frac{1}{t^2} \right)^{\sigma/2} e^{-\beta \arg(c + \frac{1}{it})}.$$

Applying modulus on the definition of $T^{error}_{1,\geq 2,s}(it)$, we get

$$\begin{aligned} \left| T^{error}_{1,\geq 2,s}(it) \right| &\leq \frac{2}{t^k} \sum_{|c|\geq 1} \left(c^2 + \frac{1}{t^2} \right)^{-(k-\sigma)/2} e^{-\beta\left(\frac{\pi}{2} + \arg(c + \frac{1}{it})\right)}, \\ &\leq e^{\frac{\pi}{2}|\beta|} \frac{4}{t^k} \zeta(k-\sigma). \end{aligned} \tag{4.5}$$

The last inequality follows by noting that $-\frac{\pi}{2} < \frac{\pi}{2} + \arg(c + \frac{1}{it}) < \frac{\pi}{2}$, as $t > 0$.

For fixed $N \in \mathbb{N}$, $T_{N,1}(z)$ is formed precisely from the matrices in

$$\left\{ \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \mid a \in \mathbb{Z}, a^2 + 1 = N \right\}.$$

Performing similar computations as above, one can see that

$$T_{\geq 2,1,s}(it) = \sum_{N \geq 2} T_{N,1}(it) = 2e^{i\frac{\pi}{2}s} \sum_{|a| \geq 1} \left\{ (it)^{-k} \left(a + \frac{i}{t} \right)^{-s} + (a + it)^{-s} \right\}.$$

Let $T^{error}_{\geq 2,1,s}(it)$ denote the first sum $2e^{i\frac{\pi}{2}s} \sum_{|a|\geq 1} (it)^{-k} \left(a + \frac{i}{t}\right)^{-s}$. From its definition, we obtain the following equality and bound:-

$$\begin{aligned} T^{error}_{\geq 2,1,s}(it) &= \frac{2}{(it)^k} \left\{ \frac{(2\pi)^s}{\Gamma(s)} \sum_{n \geq 1} n^{s-1} e^{-2\pi n/t} - t^s \right\}. \\ |T^{error}_{\geq 2,1,s}(it)| &\leq \frac{2}{t^k} \sum_{|a| \geq 1} \frac{1}{(a^2 + \frac{1}{t^2})^{\sigma/2}} e^{-\beta\left(\arg(a + \frac{i}{t}) - \frac{\pi}{2}\right)} \\ &\leq 4e^{\frac{\pi}{2}|\beta|} \frac{\zeta(\sigma)}{t^k}, \end{aligned} \tag{4.6}$$

where, we note that $-\frac{\pi}{2} < \arg(a + \frac{i}{t}) - \frac{\pi}{2} < \frac{\pi}{2}$. The second sum, denoted by $T^{main}_{\geq 2,1,s}(it)$ may also be simplified using the Lipschitz Summation Formula as earlier to obtain

$$T^{main}_{\geq 2,1,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)} \sum_{n \geq 1} n^{s-1} e^{-2\pi n t} - \frac{2}{t^s}.$$

We denote the main term (of $f_{k,s}(it)$) as

$$T_{main,s}(it) := T_{1,1,s}(it) + T^{main}_{\geq 2,1,s}(it) + T^{main}_{1,\geq 2,s}(it).$$

Remaining terms in the summation (4.3) are brought into the error term ((4.8)). Thus,

$$T_{main,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1} n^{s-1}e^{-2\pi nt} + 2(-1)^{k/2}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1} n^{k-s-1}e^{-2\pi nt}. \quad (4.7)$$

We remark here that the term on the RHS is the contribution of the terms with $ac = 0$ in the equation

$$f_{k,s}(z) = e^{i\frac{\pi}{2}s}\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z})} (cz+d)^{-k}\left(\frac{az+b}{cz+d}\right)^{-s}.$$

which was previously considered by Kohnen (pp 186, [Koh97]). Note that the term above vanishes if $s = \frac{k}{2}$ and $k \equiv 2 \bmod 4$.

Next we bound the error term. Note that

$$T_{error,s}(it) := T^{error}_{\geq 2,1,s}(it) + T^{error}_{1,\geq 2,s}(it) + T_{\geq 2,\geq 2,s}(it). \quad (4.8)$$

The final term in (4.3) is bounded above by

$$
\begin{aligned}
|T_{\geq 2,\geq 2,s}(it)| &\leq\ e^{-\frac{\pi}{2}\beta}\sum_{M\geq 2}\sum_{a^2+b^2=M}\sum_{N\geq 2}\sum_{\substack{c^2+d^2=N\\ det\begin{pmatrix} a & b\\ c & d\end{pmatrix}=1}} |cit+d|^{-k}\left|\left(\frac{ait+b}{cit+d}\right)^{-s}\right| \\
&\leq \sum_{M\geq 2}\sum_{a^2+b^2=M}\sum_{N\geq 2}\sum_{\substack{c^2+d^2=N\\ ad-bc=1}} |cit+d|^{-k+\sigma}\,|ait+b|^{-\sigma}\, e^{\beta\left(\arg\left(\frac{ait+b}{cit+d}\right)-\frac{\pi}{2}\right)} \\
&\leq e^{\frac{\pi}{2}|\beta|}\sum_{\substack{a^2+b^2\geq 2\\ (a,b)=1}}\frac{1}{|ait+b|^{\sigma}}\sum_{\substack{c^2+d^2\geq 2\\ (c,d)=1}}\frac{1}{|cit+d|^{k-\sigma}}. \qquad (4.9)
\end{aligned}
$$

For any $a\in\mathbb{N}$ and $b\in\mathbb{Z}$, $\quad |ait+b|^{\sigma}=|-ait+b|^{\sigma}$. Hence,

$$
\sum_{\substack{a^2+b^2\geq 2\\ (a,b)=1}}\frac{1}{|ait+b|^{\sigma}} = 2\sum_{a=1}^{\infty}\frac{1}{a^{\sigma}}\sum_{\substack{b\in\mathbb{Z}\setminus\{0\}\\ (a,b)=1}}\frac{1}{|it+\frac{b}{a}|^{\sigma}} \ = 2\sum_{a=1}^{\infty}\frac{1}{a^{\sigma}}\sum_{\substack{0<r<a\\ (r,a)=1}}\sum_{q\in\mathbb{Z}}\frac{1}{|it+q+\frac{r}{a}|^{\sigma}}.
$$

By (2.35), the inner sum in the right hand side is $\mathcal{O}\left(\frac{1}{t^{\sigma-1}}\right)$ if $t\geq 1$ (up until this, we only used $t>0$). This follows since Euler-phi function $\phi(a)<a$ and $\sum_{a=1}^{\infty}\frac{1}{a^{\sigma-1}}$ converges. More precisely,

$$
\sum_{\substack{a^2+b^2\geq 2\\ (a,b)=1}}\frac{1}{|ait+b|^{\sigma}} < 14\frac{\zeta(\sigma-1)}{t^{\sigma-1}},\ \text{and,}
$$

$$
\sum_{\substack{c^2+d^2\geq 2\\ (c,d)=1}}\frac{1}{|cit+d|^{k-\sigma}} < 14\frac{\zeta(k-\sigma-1)}{t^{k-\sigma-1}}.\ \text{Hence, by (4.9), we get}
$$

$$
T_{\geq 2,\geq 2,s}(it) < e^{\frac{\pi}{2}|\beta|}\frac{14^2\times\zeta^2(4.5)}{t^{k-2}} < e^{\frac{\pi}{2}|\beta|}\frac{250}{t^{k-2}}. \qquad (4.10)
$$

(Note that $k \geq 12 \implies \sigma, k-\sigma \geq 5.5$). Equations (4.5) and (4.6), along with the above fact implies that as long as $k \geq 12$ and $t \geq 1$, the term

$$|T_{error,s}(it)| \leq e^{\frac{\pi}{2}|\beta|}\frac{300}{t^{k-2}}.$$

The theorem follows immediately. In addition, $\forall\ t \geq 1,\ f_{k,s}(it)$ satisfies the following EXPLICIT ESTIMATE for all even $k \geq 12$:-

$$f_{k,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1} n^{s-1}e^{-2\pi nt} + 2(-1)^{k/2}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1} n^{k-s-1}e^{-2\pi nt} + C_\beta\left(\frac{1}{t^{k-2}}\right), \tag{4.11}$$

where, $|C_\beta| \leq 300e^{\frac{\pi|\beta|}{2}}$. □

4.2 A lower bound for Mellin transform of kernel function

We focus our attention to further consequences of the Theorem (4.1.2). Let $s_1,\ s_2$ be two complex points inside the critical strip. In this section, we aim to obtain a lower bound in terms of k for the Mellin transform of the kernel function f_{k,s_1} evaluated at s_2, i.e., $L^*(f_{k,s_1}, s_2)$. We will be imposing further conditions on s_j as we go along.

Recall (from (3.16)) that f_{k,s_1} satisfies the functional equation

$$f_{k,s_1} = (-1)^{k/2} f_{k,k-s_1}.$$

Along with the functional equation of L-function i.e., (2.16), this says

$$\begin{aligned} L^*(f_{k,s_1}, s_2) &= (-1)^{k/2} L^*(f_{k,k-s_1}, s_2) \\ &= L^*(f_{k,k-s_1}, k-s_2) \\ &= (-1)^{k/2} L^*(f_{k,s_1}, k-s_2). \end{aligned} \tag{4.12}$$

Hence, without loss of generality, it is sufficient to study s_1 and s_2 on the right half of the critical strip. We assume so from here onwards. We now state our assumptions on s_1 and s_2 precisely. Fix an arbitrary positive real $T > 0$ (as the height) and fix δ and δ' arbitarily to be small positive reals. Let s_1 and s_2 satisfy

- $-T \leq \Im(s_j) \leq T$ for $j = 1, 2$,
- $\Re(s_1) + \Re(s_2) - k \geq \frac{1}{2} + \delta,$ (4.13)
- $\frac{k}{2} < \Re(s_1) < \Re(s_1) + \delta' \leq \Re(s_2) \leq \frac{k+1}{2}.$

From eqns (4.3), (4.7) and (4.8), we see that

$$L^*(f_{k,s_1}, s_2)$$
$$= \int_0^\infty f_{k,s_1}(it) t^{s_2-1} dt$$

$$
\begin{aligned}
&= \int_0^\infty \left(2(-1)^{k/2} \frac{(2\pi)^{k-s_1}}{\Gamma(k-s_1)} \sum_{n\geq 1} n^{k-s_1-1} e^{-2\pi nt} + 2\frac{(2\pi)^{s_1}}{\Gamma(s_1)} \sum_{n\geq 1} n^{s_1-1} e^{-2\pi nt} + T_{error,s_1}(it) \right) t^{s_2-1} dt \\
&= 2(-1)^{k/2} \frac{(2\pi)^{k-s_1}}{\Gamma(k-s_1)} \sum_{n\geq 1} n^{k-s_1-1} \int_0^\infty e^{-2\pi nt} t^{s_2-1} dt \\
&\qquad + 2\frac{(2\pi)^{s_1}}{\Gamma(s_1)} \sum_{n\geq 1} n^{s_1-1} \int_0^\infty e^{-2\pi nt} t^{s_2-1} dt + L^*(T_{error,s_1}, s_2) \\
&= 2\frac{(2\pi)^{s_1-s_2}}{\Gamma(s_1)} \Gamma(s_2)\zeta(s_2 - s_1 + 1) \\
&\qquad + 2(-1)^{k/2} \frac{(2\pi)^{k-s_1-s_2}}{\Gamma(k-s_1)} \Gamma(s_2)\zeta(s_2 - (k-s_1) + 1) + L^*(T_{error,s_1}, s_2).
\end{aligned}
\tag{4.14}
$$

Interchanging the summation and integration in the second last step is justified by applying Fubini-Tonelli theorem by noting that $\sum_{n\geq 1} \int_0^\infty |n^{s_1-1} e^{-2\pi nt} t^{s_2-1}| \, dt < \infty$ since $\Re(s_2) > \Re(s_1)$ and $\sum_{n\geq 1} \int_0^\infty \left|n^{k-s_1-1} e^{-2\pi nt} t^{s_2-1}\right| dt < \infty$ since $\Re(s_2) > \Re(k-s_1)$.

(The notation $L^*(T_{error,s_1}, s_2)$ makes sense as the Mellin Transform of T_{error,s_1}. It is well-defined as all the other integrals in the equality are finite). Similarly, we define,

$$
L^*(T_{main,s_1}, s_2) := 2\frac{(2\pi)^{s_1-s_2}}{\Gamma(s_1)} \Gamma(s_2)\zeta(s_2-s_1+1) + 2(-1)^{k/2} \frac{(2\pi)^{k-s_1-s_2}}{\Gamma(k-s_1)} \Gamma(s_2)\zeta(s_2-(k-s_1)+1).
$$

Let $s_j = \frac{k}{2} + \epsilon_j + i\beta_j$ for $j = 1, 2$, where we have, $0 < \epsilon_1 < \epsilon_1 + \delta' \leq \epsilon_2 \leq \frac{1}{2}$ and $\sum_{j=1}^2 \epsilon_j \geq \frac{1}{2} + \delta$. (Note that this forces $\epsilon_1 \geq \delta$). From (2.28) and Lemma (2.2.7), we see that

$$
\left| \frac{\Gamma(s_2)}{\Gamma(k-s_1)} \right| \geq \frac{1}{\sqrt{\cosh \pi\beta_2}} \frac{\Gamma(\frac{k}{2} + \epsilon_2)}{\Gamma(\frac{k}{2} - \epsilon_1)},
$$

$$\begin{aligned}
&\geq e^{-\frac{\pi|\beta_2|}{2}} \left(\frac{k}{2} + \epsilon_2 - 1\right)^{\epsilon_1+\epsilon_2}, \\
&\geq \frac{k^{\epsilon_1+\epsilon_2}}{3} e^{-\frac{\pi T}{2}}. \qquad (4.15)
\end{aligned}$$

Similarly, one can see that

$$\frac{\Gamma(s_2)}{\Gamma(s_1)} \leq e^{\frac{\pi T}{2}} k^{\epsilon_2-\epsilon_1}.$$

Next, we find a lower bound for $\zeta(s_1 + s_2 - k + 1)$. By Lemma (2.2.2), we get

$$|\zeta(s_1 + s_2 - k + 1)| \geq \frac{\zeta(2(1 + \epsilon_2 + \epsilon_1))}{\zeta(1 + \epsilon_2 + \epsilon_1)} > \frac{\zeta(3)}{\zeta(1.5)} = c_1.$$

Now, the function $(x - 1)\zeta(x)$ is bounded in the interval $1 \leq x \leq 2$. So, let $c_2 := \max_{1\leq x\leq 2}(x - 1)\zeta(x)$. Then,

$$|\zeta(s_2 - s_1 + 1)| \leq \zeta(\epsilon_2 - \epsilon_1 + 1) \leq \zeta(\delta' + 1) \leq \frac{c_2}{\delta'},$$

since, $\epsilon_2 - \epsilon_1 \geq \delta'$. Thus, one sees that

$$\begin{aligned}
|L^*(T_{main,s_1}, s_2)| \geq 2\Bigg\{&(2\pi)^{-(\epsilon_1+\epsilon_2)} \left|\frac{\Gamma(s_2)}{\Gamma(k - s_1)}\zeta(s_1 + s_2 - k + 1)\right| \\
&- (2\pi)^{\epsilon_1-\epsilon_2} \left|\frac{\Gamma(s_2)}{\Gamma(s_1)}\zeta(\epsilon_2 - \epsilon_1 + 1)\right|\Bigg\}, \\
\geq &\frac{c_1}{3\pi} k^{\epsilon_1+\epsilon_2} e^{-\frac{\pi T}{2}} - \frac{2c_2}{\delta'} e^{\frac{\pi T}{2}} k^{\epsilon_2-\epsilon_1}. \qquad (4.16)
\end{aligned}$$

Next, we consider the term $T_{error,s_1}(it)$. We claim that its component $T_{\geq 2,\geq 2,s_1}(it)$

satisfies the modularity under the action of $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, i.e., for all $t > 0$,

$$T_{\geq 2,\geq 2,s_1}(it) = (it)^{-k} T_{\geq 2,\geq 2,s_1}(i/t).$$

It follows from the below observations:-

$$\begin{aligned}
T_{\geq 2,\geq 2,s_1}(it) &= e^{i\frac{\pi}{2}s_1} \sum_{a^2+b^2\geq 2} \sum_{\substack{c^2+d^2\geq 2 \\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (cit+d)^{-k} \left(\frac{ait+b}{cit+d}\right)^{-s_1} \\
&= e^{i\frac{\pi}{2}s_1} \sum_{a^2+b^2\geq 2} \sum_{\substack{c^2+d^2\geq 2 \\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=1}} (it)^{-k}(-d(i/t)+c)^{-k} \left(\frac{-b(i/t)+a}{-d(i/t)+c}\right)^{-s} \\
&= (it)^{-k} T_{\geq 2,\geq 2,s_1}(i/t).
\end{aligned}$$

Thus,

$$\begin{aligned}
L^*(T_{\geq 2,\geq 2,s_1}, s_2) &= \int_1^\infty T_{\geq 2,\geq 2,s_1}(it) \left(t^{s_2} + (-1)^{k/2} t^{k-s_2}\right) \frac{dt}{t} \\
L^*(T_{\geq 2,\geq 2,s_1}, s_2) &\leq 250 e^{\frac{\pi T}{2}} \left(\int_1^\infty \frac{1}{t^{k-2}} \left(t^{\sigma_2} + t^{k-\sigma_2}\right) \frac{dt}{t}\right) \\
&\leq 250 e^{\frac{\pi T}{2}} \left(\frac{1}{k-\sigma_2-2} + \frac{1}{\sigma_2-2}\right) \\
&\leq 2000 e^{\frac{\pi T}{2}} \left(\frac{1}{k}\right). \qquad (4.17)
\end{aligned}$$

for all $k \geq 12$. However, other terms in $T_{error,s}(it)$, i.e., $T^{error}_{\geq 2,1,s}(it)$ and $T^{error}_{1,\geq 2,s}(it)$ are not invariant under S and hence, we estimate individually. Note that

$$L^*(T_{error,s_1}, s_2) = L^*(T_{\geq 2,\geq 2,s_1}, s_2) + \left(\int_0^1 + \int_1^\infty\right)\left(T^{error}_{\geq 2,1,s_1}(it) + T^{error}_{1,\geq 2,s_1}(it)\right) t^{s_2-1}dt.$$

Here, using the bounds in (4.5) and (4.6), we get

$$\begin{aligned}\left|\int_1^\infty \left(T^{error}_{\geq 2,1,s_1}(it) + T^{error}_{1,\geq 2,s_1}(it)\right) t^{s_2-1}dt\right| &\leq 10e^{\frac{\pi T}{2}}\left(\int_1^\infty t^{\sigma_2-1-k}dt\right), \\ &\leq \frac{40e^{\frac{\pi T}{2}}}{k}, \end{aligned} \tag{4.18}$$

again, for all $k \geq 12$. These estimates, however are not much useful when $0 < t < 1$. However, we do the following. From eqns (2.28) and (4.4), we see that

$$\left|T^{error}_{1,\geq 2,s_1}(it)\right| \leq \frac{2}{t^k}(\cosh \pi\beta_1)^{\frac{1}{2}}\frac{(2\pi)^{k-\sigma_1}}{\Gamma(k-\sigma_1)}\sum_{n\geq 1} n^{k-\sigma_1-1}e^{\frac{-2\pi n}{t}} + \frac{2}{t^{\sigma_1}}.$$

From Lemma (2.2.16), it follows that there exists absolute constants M_0 and K_0 such that

$$\sum_{n\geq 1} n^{k-\sigma_1-1}e^{-2\pi nu} \leq M_0\left(\frac{k-\sigma_1}{e}\right)^{k-\sigma_1}\frac{1}{(2\pi u)^{k-\sigma_1}},$$

for $k \geq K_0$ uniformly for $u \geq 1$. (Here we have replaced $\frac{1}{t}$ by u). Thus,

$$\begin{aligned}&\left|\int_0^1 T^{error}_{1,\geq 2,s_1}(it)t^{s_2-1}dt\right| \\ &\leq 2(\cosh \pi\beta_1)^{\frac{1}{2}}\frac{(2\pi)^{k-\sigma_1}}{\Gamma(k-\sigma_1)}\int_1^\infty \sum_{n\geq 1} n^{k-\sigma_1-1}e^{-2\pi nu}u^{k-\sigma_2}\frac{du}{u} + \frac{2}{\sigma_2-\sigma_1},\end{aligned}$$

$$\ll e^{\frac{\pi T}{2}} \frac{1}{\Gamma(k-\sigma_1)} \left(\frac{k-\sigma_1}{e}\right)^{k-\sigma_1} \frac{1}{\sigma_2-\sigma_1} + \frac{2}{\sigma_2-\sigma_1}.$$
$$\ll e^{\frac{\pi T}{2}} \sqrt{(k-\sigma_1)}\ \frac{1}{\delta'} \leq \frac{e^{\frac{\pi T}{2}}}{\delta'}\sqrt{k}.$$

where we have used (4.13) and the fact

$$\frac{1}{\Gamma(k-\sigma_1)} \left(\frac{k-\sigma_1}{e}\right)^{k-\sigma_1} \ll \sqrt{k-\sigma_1},$$

which follows from the Stirling's estimates in Lemma(2.2.8) b. Thus,

$$\int_0^1 T^{error}_{1,\geq 2,s_1}(it)t^{s_2-1}dt = \mathcal{O}_{T,\delta'}(k^{1/2}). \tag{4.19}$$

Similarly, it follows that

$$\int_0^1 T^{error}_{\geq 2,1,s_1}(it)t^{s_2-1}dt = \mathcal{O}_{T,\delta'}(k^{1/2}). \tag{4.20}$$

From eqns (4.8), (4.17), (4.18), (4.19) and (4.20), it follows that

$$L^*(T_{error,s_1}, s_2) = \mathcal{O}_{T,\delta'}(\sqrt{k}).$$

Along with the above and the eqns (4.14) and (4.16), we see

$$|L^*(f_{k,s_1}, s_2)| \geq \frac{c_1}{3\pi} e^{-\frac{\pi T}{2}} k^{\epsilon_1+\epsilon_2} - \frac{2c_2}{\delta'} e^{\frac{\pi T}{2}} k^{\epsilon_2-\epsilon_1} - C(T,\delta')k^{1/2}.$$

Thus, it follows that

$$L^*(f_{k,s_1}, s_2) \gg_{T,\delta'} k^{\epsilon_2+\epsilon_1}. \tag{4.21}$$

for sufficiently large $k \gg_{T,\delta,\delta'} 1$ since $\epsilon_2 + \epsilon_1 \geq \frac{1}{2} + \delta$. Given an arbitrary pair of points s_1, s_2 satisfying $\epsilon_1 + \epsilon_2 \geq \frac{1}{2} + \delta$ on the strict right side of the critical line, i.e., $\left\{\frac{k}{2} < \Re(s_j) \leq \frac{k+1}{2}\right\}$, one can always choose the point with smaller real part as the kernel parameter and the other one to be the Mellin transform parameter (unless both points lie on the same vertical line, in which case, clearly, our result doesn't hold). And, even in situations where either of (or both) $s_1,\ s_2$ assume values strictly to the left of the critical line, by virtue of (4.12), one can perform the appropriate reflection $s_j \mapsto k - s_j$, thus effectively reducing it to a question about a pair of points on the right half of the critical strip. We now precisely state the final inference as a corollary to Theorem (4.1.2).

Corollary 4.2.1. *Let T be an arbitrary but fixed positive real number and let δ, δ' be arbitrary small but fixed positive reals. Let $R_{T,\delta,\delta'}$ denote the region of points $s_1 = \frac{k}{2} + \epsilon_1 + i\beta_1, s_2 = \frac{k}{2} + \epsilon_2 + i\beta_2 \in \mathbb{C}^2$ satisfying*

- $-T \leq \beta_j \leq T$ *for* $j = 1, 2$,
- $0 < |\epsilon_1| < |\epsilon_1| + \delta' \leq |\epsilon_2| \leq \dfrac{1}{2}$,
- $|\epsilon_1| + |\epsilon_2| \geq \dfrac{1}{2} + \delta.$

Then, there exists a constant $C = C(T, \delta, \delta') > 0$ depending only on T, δ, δ' such that *for $k \geq C(T, \delta, \delta')$, we have*

$$L^*(f_{k,s_1}, s_2) \gg_{T,\delta'} k^{|\epsilon_1|+|\epsilon_2|}.$$

Recall from the (3.17),

$$\begin{aligned} L^*(f_{k,s_1}, s_2) &= \int_1^\infty f_{k,s_1}(it)\left(t^{s_2} + (-1)^{k/2} t^{k-s_2}\right)\frac{dt}{t}. \\ &= \frac{(-1)^{\frac{k}{2}}\pi\Gamma(k-1)}{2^{k-3}\Gamma(s_1)\Gamma(k-s_1)} \sum_{f\in\mathcal{B}} \frac{L^*(f, s_1)}{\langle f, f\rangle} L^*(f, s_2). \end{aligned} \tag{4.22}$$

From eqns (4.21) and (4.22), we immediately observe that given a pair of points s_1, s_2 in the critical strip such that ϵ_j satisfies the conditions in Corollary (4.2.1), there is at least one eigenform $f \in \mathcal{B}_k$ whose L-function is simultaneously non-vanishing at both s_1 & s_2 for $k \gg_{T,\delta,\delta'} 1$. Nevertheless, Theorem (4.1.1) provides the same result for a much larger region than this inside the critical strip.

4.3 Simultaneously non-vanishing L-values of Hecke eigenforms

However, in this context, keeping in mind the Riemann Hypothesis, it seems interesting to ask whether one could quantify the number $N_k(s_1, s_2)$ in terms of k. We provide the following theorem in this regard.

Corollary 4.3.1. *Let T be an arbitrary but fixed positive real number and let δ, δ' be arbitrary small but fixed positive reals. Let $R_{T,\delta,\delta'}$ denote the region of points $\left(s_1 = \frac{k}{2} + \epsilon_1 + i\beta_1, s_2 = \frac{k}{2} + \epsilon_2 + i\beta_2\right) \in \mathbb{C}^2$ satisfying*

$$\begin{aligned} &\bullet\ -T \le \beta_j \le T \text{ for } j = 1,2, \\ &\bullet\ 0 < |\epsilon_1| < |\epsilon_1| + \delta' \le |\epsilon_2| \le \frac{1}{2}, \\ &\bullet\ |\epsilon_1| + |\epsilon_2| \ge \frac{1}{2} + \delta. \end{aligned} \tag{4.23}$$

Then, there exists a constant $C = C(T, \delta, \delta') > 0$ depending only on T, δ, δ' such that for $k \ge C(T, \delta, \delta')$, we have

$$N_k(s_1, s_2) \gg_{T,\delta',\delta''} k^{|\epsilon_1|+|\epsilon_2|-\delta''},$$

where, $\delta'' > 0$ is an arbitrarily small positive real number.

We note that the restriction $\Re(s_1) < \Re(s_2)$ is not really a restriction since $N_k(s_1, s_2) = N_k(s_2, s_1)$ by its definition. As an immediate corollary, we obtain an asymptotic lower bound for $N_k(s_1)$ in terms of k. Points on the (right) edge of the critical strip lie inside the following known non-vanishing region of L functions of Hecke eigenforms (2.1.10) ([IK04], Theorem (5.39)):-

$$\left\{ \Re(s) \ge \frac{k+1}{2} - \frac{c}{\log(k + |t| + 3)} \right\}, \tag{4.24}$$

where $c > 0$ is an absolute constant. By substituting $s_2 = \frac{k+1}{2}$ in Corollary (4.3.1), we get the following corollary:-

Corollary 4.3.2. *Let T be an arbitrary but fixed positive real number and let δ, δ' be arbitrary small but fixed positive reals. Let $s_1 = \frac{k}{2} + \epsilon_1 + i\beta_1$ satisfy*

- $-T \leq \beta_1 \leq T,$
- $\delta \leq |\epsilon_1| \leq \dfrac{1}{2} - \delta'.$

Then, there exists a constant $C = C(T, \delta, \delta') > 0$ depending only on T, δ, δ' such that for $k \geq C(T, \delta, \delta')$, we have

$$N_k(s_1) \gg_{T,\delta',\delta''} k^{\frac{1}{2}+|\epsilon_1|-\delta''},$$

where, $\delta'' > 0$ is an arbitrarily small positive real number.

Finally, we provide a proof for the Corollary (4.3.1) in the next section.

4.3.1 Proof of Corollary 4.3.1

Without loss of generality, as earlier, we assume s_1 and s_2 on the right half of the critical strip. From (4.22), we get

$$k^{\epsilon_2+\epsilon_1} \ll_{T,\delta'} \frac{1}{k2^{k-3}} \frac{\Gamma(k)}{|\Gamma(s_1)\Gamma(k-s_1)|} \sum_{f\in\mathcal{B}} \left| \frac{L^*(f,s_1)L^*(f,s_2)}{\langle f,f\rangle} \right|, \tag{4.25}$$

for large enough k (as mentioned in (4.21)). By the Phragmén-Lindelöf theorem [Rad59], we have

$$L(f, s_1) \ll_{\delta''} k^{1/2-\epsilon_1+\delta''} \quad \text{for an arbitrarily small } \delta'' > 0. \tag{4.26}$$

From (3.26), we get

$$\sum_{f \in \mathcal{B}} \left| \frac{L^*(f, s_1)L^*(f, s_2)}{\langle f, f \rangle} \right| \ll_{\delta'''} N_k(s_1, s_2)(2\pi)^{-(\sigma_1+\sigma_2)} \left|\Gamma(s_1)\Gamma(s_2)\right| k^{1-(\epsilon_2+\epsilon_1)+2\epsilon} \frac{(4\pi)^k k^{\delta'''}}{\Gamma(k)}.$$

By (2.28), we estimate $\dfrac{\Gamma(s_2)}{\Gamma(k - s_1)}$ to re-write (4.25) as

$$k^{\epsilon_2+\epsilon_1} \ll_{T,\delta',\delta'',\delta'''} N_k(s_1, s_2)k^{-(\epsilon_2+\epsilon_1)+2\delta''+\delta'''} \left| \frac{\Gamma(s_2)}{\Gamma(k - s_1)} \right| \ll_T k^{2\delta''+\delta'''} N_k(s_1, s_2).$$

Thus,

$$N_k(s_1, s_2) \gg k^{\epsilon_2+\epsilon_1-\delta''},$$

(by first putting $\delta''' = \delta''$ and then, replacing δ'' by $\delta''/3$) where the implied constant is independent of k (depends on T, δ', δ''). Thus, there is a constant $C(T, \delta, \delta') > 0$ such that for $k \geq C(T, \delta, \delta')$, the number of Hecke eigenforms in S_k whose L^*-value is simultaneously non-vanishing at any two points s_1, s_2 satisfying the conditions in (4.13) is at least $k^{\epsilon_2+\epsilon_1-\delta''}$.

Again, by virtue of (4.12), one observes that

$$N_k(s_1, s_2) = N_k(s_1, k - s_2) = N_k(k - s_1, k - s_2) = N_k(k - s_1, s_2).$$

This allows us to conclude that the lower bound for $N_k(s_1, s_2)$ could be extended for $(s_1, s_2) \in R_{T,\delta,\delta'}$ too.

Remark 4.3.3. *It can be observed from the calculations above that any improvement in the convexity bounds would naturally lead to an equal improvement in the Corollaries* (4.3.1) *and* (4.3.2). *For example, under the assumptions of Lindelöf Hypothesis (which asserts that,*

$$L(f, s) \ll_{\delta''} k^{\delta''} \qquad \text{for any } \delta'' > 0$$

holds for any s with $\Re(s) \geq \frac{k}{2}$), we get that

$$N_k(s_1, s_2) \gg_{T,\delta',\delta''} k^{1-\delta''}.$$

Chapter 5

Conclusion

In this thesis, we established the non-vanishing of a weighted sum of L-functions of Hecke eigenforms evaluated at real points inside the critical strip for all $k \geq 12, 4|k$ and as a consequence, derived a lower bound for $\max_{f \in \mathcal{B}_k} L(f, \sigma)$ for large weights.

More importantly, we also provided estimates for the number of Hecke eigenforms $N_k(s_1, s_2)$ whose L-values are non-vanishing simultaneously at two given points s_1 and s_2 inside the critical strip satisfying some conditions and as a consequence, provided an estimate for the number of Hecke eigenforms $N_k(s_1)$ whose L-values are non-vanishing at a given point s_1 inside the critical strip outside the critical line.

For both these, we used the dual cusp form studied by W. Kohnen in [Koh97] and arrived at both the above inferences by evaluating this dual at points on the imaginary axis. We provided asymptotic expansion for the dual at any point $z = it$ on the imaginary axis and used its Mellin transform to estimate lower bounds for the above quantities. For these, we used the convexity bounds for L functions in the weight aspect and also familiar bounds for Petersson norm. In order to arrive at the asymptotic expansion, we adapted a method used by Rankin and Swinnerton-Dyer in [RSD70] to split the series into a main term and an error term.

Future Work Here we will mention some related problems for further research. Recall that by showing $f_{k,\sigma}(i)$ is non-zero, we had shown the non-vanishing of the below weighted sum of L-values of Hecke eigenforms

$$\sum_{f\in\mathcal{B}_k}\frac{L^*(f,\sigma)}{\langle f,f\rangle}f(i).$$

for all $k\geq 12,\ 4|k$ for any real σ inside the critical strip. It is a worthwhile problem to see upto what extent one could generalize similar results from real σ to the complex points s inside the critical strip for all $k\geq 12$, independent of the choice of $\Im(s)$.

In relation with the Corollaries (4.3.1) and (4.3.2), it could be interesting to improve the count $N_k(s_1)$ using other analytic techniques like mollified averages that was used by W. Luo. There are various problems one can ponder in this regard.

In relation with the Theorem (3.1.1) of Kohnen, it is also interesting to study the height of the rectangular region of non-vanishing of the sum

$$\sum_{f\in\mathcal{B}_k}\frac{L^*(f,s)}{\langle f,f\rangle}$$

in terms of k. For example, one can study the growth of the constant $C(t_0,\delta)$ in Theorem (3.1.1) in terms of $|t_0|$, or in turn, study $|t_0|$ as a function of k and infer that the non-vanishing result of Kohnen holds for all points $s = \sigma + it$ in the rectangle defined by

$$\left\{ \frac{1}{2} + \delta < \sigma < 1,\ |t| \leq t_0(k) \right\}$$

for sufficiently large k. It could be a nice problem to further improve the height of this zero-free region in terms of the weight k.

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1. M. Manickam, V. K. Murty and E. M. Sandeep, *A weighted average of L-functions of modular forms*, C. R. Math. Rep. Acad. Sci. Canada, **43** (2021), 63–77.

2. M. Manickam, V. K. Murty and E. M. Sandeep, *Counting Hecke eigenforms with non-vanishing L-value*, Bull. Aust. Math. Soc. (2022), 1-20, doi:10.1017/S0004972721000927.