ON THE STUDY OF SOME PROBLEMS IN SPECTRAL SETS, DUGGAL TRANSFORMATIONS, AND ALUTHGE TRANSFORMATIONS

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SAJI MATHEW

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALICUT
KERALA
INDIA

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALICUT

Dr. M. S. Balasubramani Calicut University
Professor 28^{th} March 2008

CERTIFICATE

Certified that the work presented in this thesis is a bonafide work done by Mr. Saji Mathew under my guidance in the Department of Mathematics, University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree.

Dr. M. S. Balasubramani

Supervising Guide

DECLARATION

I declare that the work presented in this thesis is based on the original work done by me under the guidance of Dr. M. S. Balasubramani, Professor, Department of Mathematics, University of Calicut and has not been included in any other thesis submitted previously for the award of any degree either to this university or to any other university/institution.

University of Calicut

28 March 2008 Saji Mathew

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Contents

Chapter 0

Introduction

As the title suggests, this thesis deals with some problems in spectral sets, Duggal transformations, and Aluthge transformations of bounded linear operators on Hilbert spaces.

The notion of spectral sets was introduced by J. von Neumann in 1951. If T is a bounded linear operator on a Hilbert space, a closed proper subset X of the complex plane $\mathbb{C}$ is called a spectral set for T if X contains the spectrum of T and $\| f(T) \| \leq \| f \|_\infty$, for all rational functions f having poles off $\check{X}$, where $\check{X}$ denotes the closure of X in the Reimann sphere $\mathfrak{S}$, and $\| f \|_\infty$ the norm of f in the C^*-algebra $C(\partial \check{X})$.

If $T = U\,|T|$ is the polar decomposition of a bounded linear operator T on a Hilbert space, where U is a partial isometry such that T and U have the same kernel, then the operator $\widetilde{T} = |T|^{1/2}\,U\,|T|^{1/2}$ is known as the Aluthge transformation of T. Aluthge transformation was first studied by A. Aluthge in 1990 in the paper [1] in relation with the $p-$hyponormal and $\log-$hyponormal operators. Aluthge transformation has received considerable attention in recent years. One reason is the connection of Aluthge transformation with the invariant subspace problem. Il Bong Jung, Eungil Ko, and Carl Pearcy proved in [22] that

T has a nontrivial invariant subspace if and only if $\widetilde{T}$ does. Another reason is related with the iterated Aluthge transformation. In [22], Jung, Ko, and Pearcy defined the n^{th} Aluthge transform $\widetilde{T}^{(n)}$ for each non-negative integer n, as $\widetilde{T}^{(n)} = (\widetilde{\widetilde{T}^{(n-1)}})$ and $\widetilde{T}^{(0)} = T$. They conjectured that for every bounded linear operator on a Hilbert space, the Aluthge sequence $\{\widetilde{T}^{(n)}\}_{n=0}^{\infty}$ is norm convergent to a quasinormal operator. M. Chō, I. B. Jung, and W. Y. Lee in [10], showed that the conjecture is not true in the case of infinite dimensional Hilbert spaces. The finite dimensional case is under study and yet to be resolved completely. Aluthge transformation is very useful in the study of non-normal operators.

For a bounded linear operator T on a Hilbert space with polar decomposition $T = U\,|T|$, the operator $\widehat{T} = |T|\,U$ is the Duggal transformation of T. For each non-negative integer n, the n^{th} Duggal transformation $\widehat{T}^{(n)}$ can be defined as $\widehat{T}^{(n)} = (\widehat{\widehat{T}^{(n-1)}})$ and $\widehat{T}^{(0)} = T$. In 2003, Ciprian Foias, Il Bong Jung, Eungil Ko, and Carl Pearcy initiated the study of Duggal transformations in [16], and proved several analogous results for Aluthge transformations and Duggal transformations. They named the transformation after B. P. Duggal who inspired them to study this transformation. The volume of work done on Duggal transformations, is considerably less, compared to that on Aluthge transformations.

Though the scope of this study embraces Duggal transformations and Aluthge transformations, it devotes more attention to Duggal transformations. Spectral sets of operators on Hilbert spaces are studied, and some relation between spectral sets and the above mentioned transformations are established. The main results of this work are on the convergence of the norms of the Duggal iterates, the convergence of the Duggal iterates and spectral sets, contractivity and positivity of the maps between the Riesz Dunford algebras determined by an operator and its Aluthge transformation and Duggal transformation, the polar decomposition of Duggal transformations and Aluthge transformations, the minimal spectral sets, and $n-$level spectral sets.

Apart from the introduction, the thesis contains four chapters. Chapter 1 is devoted to the basic definitions and results that are necessary for the study. This chapter covers some topics on bounded linear operators on Hilbert spaces, rational and holomorphic functional calculi, C^*-algebras, and spectral sets.

Aluthge and Duggal transformations form the object of study in chapter 2. T. Yamazaki in [35] proved that for every bounded linear operator T on a Hilbert space, the sequence of the norms of the Aluthge iterates of T converges to the spectral radius $r(T)$. Derming Wang in [33] gave another proof of this result. In an attempt to prove, the analogue of the result of Yamazaki, that for every bounded linear operator T on a Hilbert space, the sequence of the norms of the Duggal iterates of T converges to the spectral radius $r(T)$, it is proved that for certain classes of operators, the sequence of the norms of the Duggal iterates converges to the spectral radius. This result is given as theorem 2.2.2. Further investigation leads to an example of a finite dimensional operator showing that there exist bounded linear operators on Hilbert spaces such that the sequence of the norms of the Duggal iterates does not converge to the spectral radius, and this is a striking difference of Duggal transformation compared to Aluthge transformation. This example is exhibited in section 2.2.3.

After Yamazaki's result in 2002 on the convergence of the sequence of the norms of the Aluthge iterates, in 2003, T. Ando and T. Yamazaki in [3], proved that in the case of a 2×2 matrix, the sequence of the iterated Aluthge transformations itself converges. In 2006, Jorge Antezana, Enrique R. Pujals and Demetrio Stojanoff [4], proved the convergence of iterated Aluthge transformation sequence for diagonalizable matrices. In section 2.2.3, an example is constructed showing that the analogous results for these three results fail to hold in the case of Duggal transformations. In 2004, T. Ando proved in [2] that if A is an $n \times n$ matrix over $\mathbb{C}$, then the convex hull of the spectrum $\sigma(A)$ equals the numerical range $W(A)$ if and only if A and $\widetilde{A}$ have the same numerical range . In section 2.3.1, the study argues that this result does not hold if the Aluthge transformation is replaced by the Duggal transformation.

If T is a bounded linear operator on a Hilbert space, there is an algebra of bounded linear operators determined by the Riesz Dunford functional calculus associated with the operator T, known as the Riesz Dunford algebra determined by T. In section 2.3.2, the thesis analyzes the Aluthge and the Duggal transformations of the operator T when the partial isometry U in the polar decomposition of T happens to be a coisometry. The homomorphisms between the Riesz Dunford algebras determined by T, $\widetilde{T}$, and $\widehat{T}$ are studied. The notion of $n-$level spectral set is introduced between spectral sets and complete spectral sets, for every positive integer n. The thesis shows that if $T = U|T|$ is the polar decomposition of T, and U a coisometry, then T and $\widehat{T}$ have the same collection of spectral sets, have the same collection of complete spectral sets, and have the same collection of $n-$level spectral sets. Further the thesis proves that if T is an invertible operator, and if for some n, the n^{th} Duggal iterate $\widehat{T}^{(n)}$ is normal, then T is normaloid; in fact, in such cases, $f(T)$ is normaloid for every rational function f having poles off $\sigma(T)$.

In [14], Ken Dykema and Hanne Schultz proved that if $\mathcal{H}$ is any Hilbert space, then the Aluthge transformation map $T \to \widetilde{T}$ is continuous on the space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. In section 2.3.3, the thesis examines the continuity of the Duggal transformation map $T \to \widehat{T}$. Using the continuity of the Duggal transformation map $T \to \widehat{T}$ on the set of invertible operators, results regarding the relation between the spectral sets of an operator and the spectral sets of the limit of the sequence of the Duggal iterates, are obtained. The following theorem is one of these results. If T is an invertible operator on a Hilbert space, then the sequence of Duggal iterates $\{\widehat{T}^{(n)}\}_{n=0}^{\infty}$ can converge to an invertible operator if and only if T is quasinormal. The study proceeds to prove the result: Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator.

Suppose that $\widehat{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$, and X is a closed proper subset of $\mathbb{C}$ such that X contains a neighborhood of $\sigma(S)$. Further, assume that $f(\widehat{T}^{(n)}) \to f(S)$ for all rational functions f having poles off $\check{X}$. Then, X is a spectral set for T if and only if X is a spectral set for S.

In section 2.4, the contractivity and positivity of the map $f(T) \to f(\widetilde{T})$ between the Riesz Dunford algebras determined by T and $\widetilde{T}$, and of the map $f(T) \to f(\widehat{T})$ between the Riesz Dunford algebras determined by T and $\widehat{T}$, are used to prove some of the consequences including the following theorem. If T is a a bounded linear operator on a Hilbert space $\mathcal{H}$ such that the Riesz Dunford algebra determined by T is closed in $\mathcal{L}(\mathcal{H})$ and f, g are holomorphic functions on neighborhoods of $\sigma(T)$ satisfying $(f(T))^* = g(T)$, then $(f(\widehat{T}))^* = g(\widehat{T}),\ \ (f(\widetilde{T}))^* = g(\widetilde{T})$.

The polar decomposition of Aluthge transformation and Duggal transformation of operators is the focus of study of chapter 3. In [20], Masatoshi Ito, Takeaki Yamazaki, and Masahiro Yanagida obtained several results on the polar decomposition of Aluthge transformation. In [21], Ito, Yamazaki, and Yanagida showed results on the polar decomposition of the product of two operators and of Aluthge transformation. They also showed properties and characterizations of binormal and centered operators from the viewpoint of the polar decomposition and Aluthge transformation. In [20], Ito, Yamazaki, and Yanagida gave an example of a binormal, invertible operator T such that the Aluthge transformation $\widetilde{T}$ is not binormal. In chapter 3, it is shown that if T is a binormal, invertible operator, then the Duggal transformation $\widehat{T}$ is binormal. Some of the consequences of applying Aluthge transformation and Duggal transformation successively on an invertible operator T are discussed. As a result, theorem 3.2.10 shows that if T is invertible and binormal, then $\widehat{(\widetilde{T})} = \widetilde{(\widehat{T})}$. The thesis further extends this result to iterated Aluthge transformations and Duggal transformations. The study proceeds to show that if T is an invertible operator with polar decomposition

$T = U\,|T|$, then the polar decomposition of $\widehat{T}$ is $\widehat{T} = U\,|\widehat{T}|$.

Let $T = U|T|$ be the polar decomposition of an operator T. A theorem in [21] says that T is binormal if and only if $\widetilde{T} = \widetilde{U}\,|\widetilde{T}|$ is the polar decomposition of the Aluthge transformation $\widetilde{T}$. The thesis discusses a similar situation for Duggal transformations. Necessary and sufficient condition for $\widehat{T}$ to have the polar decomposition $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is obtained. The following theorem is an important consequence. If T is binormal, then $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$. In section 3.2.3, the discussion on polar decomposition of Aluthge transformations and Duggal transformations is concluded by giving a modification of the proof of a theorem in [21], using the semigroup properties of factors in the polar decomposition of a bounded linear operator on a Hilbert space.

In [29], M. Schreiber characterized by means of normal dilations, those operators, the closure of whose numerical range is a spectral set. He obtained results on the equality of the convex hull of the spectrum with the closure of the numerical range in relation to the spectrality of the numerical range. In section 3.3, Aluthge and Duggal transformations are discussed in the context of these results.

Minimal spectral sets and $n-$level spectral sets are the subjects of study of chapter 4. If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, then a closed subset X of $\mathbb{C}$ is called a minimal spectral set for T, if X is a spectral set for T such that X contains no other spectral set for T. As observed by J. P. Williams in [34], for every operator T on a Hilbert space $\mathcal{H}$, there is a minimal spectral set, and in fact, every spectral set contains a minimal spectral set. A theorem in [7] says that $\sigma(T)$ is a spectral set for T if and only if $f(T)$ is normaloid for every rational function f having poles off the spectrum $\sigma(T)$. In chapter 4, it is observed that these statements are equivalent to the uniqueness of the minimal spectral set. Situations when the minimal spectral set of an operator is unique are discussed.

If X is a closed proper subset of the complex plane, $\mathcal{R}(X)$ the subalgebra of the C^*-algebra $C(\partial\check{X})$ consisting of the rational functions having poles off $\check{X}$, $\overline{\mathcal{R}(X)}$ the set of all complex conjugates of members of $\mathcal{R}(X)$, and X happens to be an $n-$level spectral set for the operator T on a Hilbert space $\mathcal{H}$, then there is a map $\tilde{\rho} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \to \mathcal{L}(\mathcal{H})$ that extends the natural functional calculus map $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$. In section 4.3, results on the contractivity and positivity of the map $\tilde{\rho}$ are obtained.

Further research that is possible beyond the thesis, and some of the problems and possibilities that are left open, are briefly outlined in the *Epilogue.*

Chapter 1

Preliminaries

1.1 Introduction

This chapter is devoted to the basic definitions and results that are necessary for the study, and covers some topics on bounded linear operators on Hilbert spaces, rational and holomorphic functional calculi, C^*-algebras, and spectral sets.

Notation: In what follows, $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{R}_+$ the set of all non-negative real numbers. We denote by $\mathcal{H}$ a Hilbert space, $\langle \, , \, \rangle$ the inner product, and $\mathcal{L}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we denote the spectrum of T by $\sigma(T)$, and the adjoint of T by T^*. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then we denote by $P_{\mathcal{M}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. All the projections we consider are orthogonal projections. If $T \in \mathcal{L}(\mathcal{H})$, we denote by $ranT$, the range $\{Tx : x \in \mathcal{H}\}$, and by $kerT$ the set $\{x \in \mathcal{H} : Tx = 0\}$. T always denotes a bounded linear operator on a Hilbert space $\mathcal{H}$, unless specified otherwise. If X is a subset of a normed linear space, or of $\mathbb{C}$, we denote by $\overline{X}$, the closure of X in the respective space, except in section 4.3. In section 4.3, the overbar is used to denote complex conjugation. The term 'operator' is freely used to mean 'bounded linear operator on the Hilbert space $\mathcal{H}$ '.

1.2 Bounded operators on Hilbert spaces

Definition 1.2.1. Let $\mathcal{H}$ be a complex Hilbert space, and $T \in \mathcal{L}(\mathcal{H})$. The set of $x \in \mathcal{H}$ such that $Tx = 0$ is a closed subspace of $\mathcal{H}$. Let $\mathcal{Y} = \{x \in \mathcal{H} : Tx = 0\}$. (ie., $\mathcal{Y} = kerT$). Let $\mathcal{X} = \mathcal{Y}^{\perp}$. The projection $E = P_{\mathcal{X}}$ is called the *support* of T.

If E is the support of T, then $TE = T$. Also E is the smallest of the projections E_1 of $\mathcal{L}(\mathcal{H})$ such that $TE_1 = T$. (If E_1 and E_2 are two projections, we say that $E_1 \leq E_2$ if $ranE_1 \subset ranE_2$).

The closure of $T(\mathcal{H})$ is a closed linear subspace $\mathcal{Y}$ of $\mathcal{H}$. Let $F = P_{\mathcal{Y}}$. Then F is the smallest of the projections F_1 of $\mathcal{L}(\mathcal{H})$ such that $T^*F_1 = T^*$. Also, F is the support of T^*.

Definition 1.2.2 (Partial isometry)**.** Let $U \in \mathcal{L}(\mathcal{H})$, and E its support. We say that U is a *partial isometry* if U is an isometry on $\mathcal{X} = E(\mathcal{H})$.

If $U \in \mathcal{L}(\mathcal{H})$ is a partial isometry, then $U(\mathcal{H}) = U(\mathcal{X})$ is a closed linear subspace $\mathcal{Y}$ of $\mathcal{H}$; and U maps $\mathcal{X}$ isometrically onto $\mathcal{Y}$. Let $F = P_{\mathcal{Y}}$. We say that E is the *initial projection* of U and that F is the *final projection* of U. We say that $\mathcal{X}$ is the *initial space* of U and that $\mathcal{Y}$ is the *final space* of U.

Let $x \in \mathcal{X}$, $y = Ux \in \mathcal{Y}$. For every $z \in \mathcal{H}$, we have $\langle x, z\rangle = \langle x, Ez\rangle = \langle Ux, UEz\rangle = \langle y, UEz\rangle = \langle y, Uz\rangle = \langle U^*y, z\rangle$. Hence $x = U^*y$. Thus the mapping $x \to Ux$ of $\mathcal{X}$ onto $\mathcal{Y}$ has for its inverse (isometric) mapping the mapping $y \to U^*y$ of $\mathcal{Y}$ onto $\mathcal{X}$.

Since, furthermore, the support of U^* is F, we see that U^* is a partial isometry with initial projection F and final projection E. We also see that $U^*U = E$

and $UU^* = F$. Thus U^*U and UU^* are the initial and final projections of U. Conversely, if $V \in \mathcal{L}(\mathcal{H})$ such that V^*V is a projection, then V is a partial isometry. Similarly, if $W \in \mathcal{L}(\mathcal{H})$ and WW^* is a projection, then W^* is a partial isometry, and hence W is a partial isometry.

Remark 1.2.3. If U is a nonzero partial isometry, then $\| U \|= 1$. (Proof: U is a nonzero partial isometry $\Longrightarrow U^*U$ is a nonzero projection $\Longrightarrow 1 \in \sigma(U^*U) \subset \{0, 1\} \Longrightarrow \| U \|^2= r(U^*U) = 1$, where $r(U^*U)$ denotes the spectral radius of the operator U^*U.)

Definition 1.2.4 (Polar decomposition of an operator)**.** Let $T \in \mathcal{L}(\mathcal{H})$, E the support of T, F the support of T^*, $\mathcal{X} = E(\mathcal{H})$ and $\mathcal{Y} = F(\mathcal{H})$. We put $|T| = (T^*T)^{1/2}$. We have, for every $x \in \mathcal{H}$, $\| Tx \|^2= \langle T^*Tx, x\rangle = \langle |T|^2x, x\rangle = \langle |T|x, |T|x\rangle =\| |T|x \|^2$. Therefore, $\| Tx \|=\| |T|x \|$ for all $x \in \mathcal{H}$. Hence $|T|$ has support E, and consequently, $\overline{|T|(\mathcal{H})} = \mathcal{X}$.

Furthermore, the mapping $|T|x \to Tx$ is a linear isometry of $|T|(\mathcal{H})$ onto $T(\mathcal{H})$, and therefore extends to a linear isometry V of $\mathcal{X} = \overline{|T|(\mathcal{H})}$ onto $\mathcal{Y} = \overline{T(\mathcal{H})}$. Let U be the partial isometry with support E and which coincides with V on $\mathcal{X}$; this partial isometry has E as initial projection and F as final projection.

We have $T = U|T|$ an equality called the *polar decomposition* of T.

Thus any $T \in \mathcal{L}(\mathcal{H})$ can be written as

$$T = U|T|$$

where U is a partial isometry and $|T|$ is a positive operator such that the initial projection of U is E which is the support of T, the final projection of U is F which is the support of T^*, and such that the support of $|T|$ is also E. Note that $TE = T$, $UE = U$ and $|T|E = |T|$.

On the other hand, if we have an equality $T = U_1T_1$ where T_1 is positive hermitian and where U_1 is a partial isometry whose initial projection is support of T_1, then we have $T_1 = |T|$ and $U_1 = U$.

The equality

$$T^* = U^*(U|T|U^*)$$

is the polar decomposition of T^*.

The following theorem in [27] deals with polar decomposition.

Theorem 1.2.5. [27] *Let A be a bounded operator on a Hilbert space $\mathcal{H}$. Then there is a partial isometry U such that $A = U|A|$. The partial isometry U is uniquely determined by the condition that $kerU = kerA$. Moreover, $ranU = \overline{ranA}$.*

Remark 1.2.6. [12] Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. If $T \in \mathcal{A}$ and $T = U|T|$ the polar decomposition of T, then $U \in \mathcal{A}$ and $|T| \in \mathcal{A}$. If $\mathcal{M}$ is a left ideal in $\mathcal{A}$ and $T \in \mathcal{A}$, then $T \in \mathcal{M}$ if and only if $|T| \in \mathcal{M}$.

Definition 1.2.7 (Shifts)**.** An operator S_+ on a Hilbert space $\mathcal{H}$ is a *unilateral shift* if there exists an infinite sequence $\{\mathcal{H}_k\}_{k=0}^{\infty}$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \oplus_{k=0}^{\infty}\mathcal{H}_k$ and S_+ maps each $\mathcal{H}_k$ isometrically onto $\mathcal{H}_{k+1}$.

Since $S_{+|\mathcal{H}_k} : \mathcal{H}_k \to \mathcal{H}_{k+1}$ is unitary (ie., a surjective isometry), it follows that $dim\mathcal{H}_{k+1} = dim\mathcal{H}_k$, for every $k \geq 0$. This constant dimension is the multiplicity of S_+. The adjoint S_+^* of S_+ lies in $\mathcal{L}(\mathcal{H})$, and is referred to as the *backward unilateral shift,* also denoted by S_-. Writing $\oplus_{k=0}^{\infty}x_k$ for $\{x_k\}_{k=0}^{\infty}$ in $\mathcal{H} = \oplus_{k=0}^{\infty}\mathcal{H}_k$, it follows that S_+ and S_- are given by the formulas

$$S_+x = 0 \oplus \oplus_{k=1}^{\infty}U_k x_{k-1} \quad \text{and} \quad S_+^*x = \oplus_{k=0}^{\infty}U_{k+1}^* x_{k+1}$$

for every $x = \oplus_{k=0}^{\infty}x_k$ in $\mathcal{H} = \oplus_{k=0}^{\infty}\mathcal{H}_k$, where 0 is the origin of $\mathcal{H}_0$ and U_{k+1} is any unitary transformation of the space $\mathcal{H}_k$ onto the space $\mathcal{H}_{k+1}$ so that $S_{+|\mathcal{H}_k} = U_{k+1}$, for each $k \geq 0$. These are identified with the infinite matrices

$$S_+ = \begin{bmatrix} 0 & \cdots & & & \\ U_1 & 0 & \cdots & & \\ & U_2 & 0 & \cdots & \\ & & U_3 & 0 & \cdots \\ & & & \ddots & \ddots \end{bmatrix} \text{ and } S_+^* = \begin{bmatrix} 0 & U_1^* & \cdots & & \\ & 0 & U_2^* & \cdots & \\ & & 0 & U_3^* & \cdots \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}$$

of transformations.

In particular, if $\mathcal{H}$ is an infinite dimensional separable Hilbert space; that is, if the Hilbert space $\mathcal{H}$ has a countably infinite orthonormal basis, say $\{e_k\}_{k=0}^{\infty}$, then set $\mathcal{H}_k = span\,\{e_k\}$ for $k \geq 0$ so that $dim\mathcal{H}_k = 1$ for all $k \geq 0$. It can be verified that S_+ is a unilateral shift of multiplicity 1 on $\mathcal{H}$ if it shifts the orhonormal basis $\{e_k\}_{k=0}^{\infty}$ for $\mathcal{H}$; that is, if $S_+ e_k = e_{k+1}$ for every $k \geq 0$. Conversely, an operator S_+ is a unilateral shift of multiplicity 1 on $\mathcal{H}$ only if it shifts some orhonormal basis $\{u_k\}_{k=0}^{\infty}$ for $\mathcal{H}$; that is, only if $S_+ u_k = u_{k+1}$ for every $k \geq 0$.

If $x = \oplus_{k=0}^{\infty} x_k$ is any vector in the Hilbert space $\mathcal{H} = \oplus_{k=0}^{\infty}\mathcal{H}_k$, then $\|S_+ x\|^2 = \|0 \oplus \oplus_{k=1}^{\infty} U_k x_{k-1}\|^2 = \sum_{k=1}^{\infty} \|U_k x_{k-1}\|^2 = \sum_{k=1}^{\infty} \|x_{k-1}\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2 = \|x\|^2$. Thus the unilateral shift S_+ is an isometry. This can also be verified observing that $S_+^* S_+ = I$.

An operator S on a Hilbert space $\mathcal{H}$ is a *bilateral shift* if there exists an infinite sequence $\{\mathcal{H}_k\}_{k=-\infty}^{\infty}$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \oplus_{k=-\infty}^{\infty}\mathcal{H}_k$ and S maps each $\mathcal{H}_k$ isometrically onto $\mathcal{H}_{k+1}$.

Definition 1.2.8 (Coisometry)**.** Let $\mathcal{H}$ be a Hilbert space, and $U \in \mathcal{L}(\mathcal{H})$. We say that U is a *coisometry* if $UU^* = I$, where I is the identity operator in $\mathcal{L}(\mathcal{H})$.

The operator U is a coisometry if and only if U^* is an isometry. Obviously every unitary operator is a coisometry. Also an invertible coisometry is always unitary. The backward unilateral shift is an example of a coisometry.

Definition 1.2.9 (Quasinomal operator)**.** An operator T is called quasinormal if T commutes with T^*T.

Every normal operator is quasinormal. Converse is not true. For instance, if A is an isometry, then $A^*A = I$. So A commutes with A^*A. If A is not unitary, then A is not normal. For example, the unilateral shift is quasinormal, but not normal.

Definition 1.2.10 (Dilations)**.** Suppose that $\mathcal{H}$ is a subspace of a Hilbert space $\mathcal{K}$ and let P be the projection from $\mathcal{K}$ onto $\mathcal{H}$. If $B \in \mathcal{L}(\mathcal{K})$, then B induces an operator A on $\mathcal{H}$, defined by

$$Ax = PBx, \;\; x \in \mathcal{H}.$$

We have $AP = PB$.

The operator A is called the *compression* of B to $\mathcal{H}$, and B is called a *dilation* of A to $\mathcal{K}$.

If $\mathcal{H}$ is invariant under B then A is the restriction of B to $\mathcal{H}$ and B is an extension of A to $\mathcal{K}$. The following are some known facts.

- Every operator has a normal dilation.
- If $\| A \| \leq 1$, then A has a unitary dilation.
- If $0 \leq A \leq 1$, then A has a dilation that is a projection.

An operator B is called a *power dilation* (or *strong dilation*) of an operator A if B^n is a dilation of A^n for $n = 1, 2, \ldots$ (ie., $A^n x = PB^n x, \; x \in \mathcal{H}, \; n = 1, 2, \ldots$).

Definition 1.2.11 (Subnormal operator)**.** An operator is *subnormal* if it has a normal extension. More precisely, an operator A on a Hilbert space $\mathcal{H}$ is subnormal if there exists a normal operator B on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace o $\mathcal{K}$, $\mathcal{H}$ is invariant under the operator B, and the restriction of B to $\mathcal{H}$ coincides with A.

Normal $\Rightarrow$ subnormal.

On finite dimensional Hilbert spaces, the concepts normal and subnormal are the same. Unilateral shift is subnormal, the bilateral shift is the normal extension.

Normal $\Rightarrow$ quasinormal $\Rightarrow$ subnormal.

On finite dimensional Hilbert spaces the three concepts, normal, quasinormal and subnormal, are the same.

Subnormal $\not\Rightarrow$ quasinormal.

For example, let U be the unilateral shift and $0 \neq c$ be a scalar. Let B be the normal extension of U. Then B is the bilateral shift on a Hilbert space $\mathcal{K}$. Let $I_{\mathcal{K}}$ be the identity operator on $\mathcal{K}$. We know that if S and T are normal operators such that S commutes with T^* and T commutes with S^*, then $S+T$ is normal. Therefore, the operator $B + cI_{\mathcal{K}}$ is normal. Also, $B + cI_{\mathcal{K}}$ is an extension of $U + cI$. This shows that $U + cI$ is subnormal. But $U + cI$ is not quasinormal, since if $U+cI$ were quasinormal, $U+cI$ would commute with $(U+cI)^*(U+cI)$, ie., $(U+cI)[(U+cI)^*(U+cI)] = [(U+cI)^*(U+cI)](U+cI)$ which would imply that $UU^* = U^*U$ and would contradict the fact that U is not normal.

Definition 1.2.12 (Spectral radius)**.** If $T \in \mathcal{L}(\mathcal{H})$, the *spectral radius* $r(T)$ of T is defined as

$$r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

We see that

$$r(T) = \lim_{k\to\infty} \parallel T^k \parallel^{1/k} .$$

Definition 1.2.13 (Normaloid)**.** An operator T is said to be *normaloid* if $r(T) = \| T \|$.

Definition 1.2.14 (Numerical range)**.** If $T \in \mathcal{L}(\mathcal{H})$, then the set $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \| x \| = 1\}$ is called the *numerical range* of T.

Theorem 1.2.15 (Toeplitz-Hausdorff theorem)**.** *The numerical range of an operator is always convex.*

In 1918, Toeplitz proved that for any operator A, the boundary of $W(A)$ is a convex curve, but left open the possibility that it had interior holes. In 1919, Hausdorff proved that it did not.

Theorem 1.2.16. *The closure of the numerical range includes the spectrum.*

Theorem 1.2.17. *If T is normal, then $\overline{W(T)}$ is the closed convex hull $\mathcal{C}(\sigma(T))$ of the spectrum $\sigma(T)$ of T.*

If T is not normal, then it can happen that $\mathcal{C}(\sigma(T)) \neq \overline{W(T)}$. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\sigma(A) = \{0\}$ and $W(A) = \{z : |z| \leq 1/2\}$.

Definition 1.2.18 (Convexoid operator)**.** An operator T is said to be *convexoid* if $\mathcal{C}(\sigma(T)) = \overline{W(T)}$.

Definition 1.2.19 (Unitarily equivalent operators)**.** Let $\mathcal{H}$ be a Hilbert space, and $S, T \in \mathcal{L}(\mathcal{H})$. We say that S and T are *unitarily equivalent* if there exists a unitary operator U in $\mathcal{L}(\mathcal{H})$ such that $S = U^*TU$.

Unitarily equivalent operators have the same numerical range. (Proof: Suppose that S and T are unitarily equivalent. Assume that $S = U^*TU$, where U is unitary. For every $x \in \mathcal{H}$, $\langle Sx, x\rangle = \langle U^*TUx, x\rangle = \langle TUx, Ux\rangle = \langle Ty, y\rangle$, where $y = Ux$, and we have $\| y \| = \| Ux \| = \| x \|$. On the other hand, for every $x \in \mathcal{H}$, $\langle Tx, x\rangle = \langle TUU^*x, UU^*x\rangle = \langle SU^*x, U^*x\rangle = \langle Sz, z\rangle$, where $z = U^*x$, and we have $\| z \| = \| U^*x \| = \| x \|$. Thus $W(S) = W(T)$).

1.3 Rational functional calculus and holomorphic functional calculus

Let $\mathcal{U}$ be an associative algebra with unity. We shall define $\mathbb{C}[t]$ as the integral domain of all polynomials over $\mathbb{C}$ in the variable t.

Theorem 1.3.1. [7] *Let $x \in \mathcal{U}$. Then there is a unique algebra homomorphism $\phi : \mathbb{C}[t] \to \mathcal{U}$ such that $\phi(1) = 1$ and $\phi(t) = x$.*
Notation: If $p \in \mathbb{C}[t]$ we define $p(x) = \phi(p)$; thus if $p(t) = \sum_{k=0}^{n} a_k t^k$, then $p(x) = \sum_{k=0}^{n} a_k x^k$.

We shall denote by $\mathbb{C}(t)$ the field of fractions of the integral domain $\mathbb{C}[t]$. Thus $\mathbb{C}(t)$ is the set of all rational forms $f = p/q$ where $p, q \in \mathbb{C}[t]$ and $q \neq 0$. The rational form $f = p/q$ is said to be in the *reduced form* if p an q are relatively prime in the integral domain $\mathbb{C}[t]$.

Definition 1.3.2 (The algebra $\mathcal{R}(X)$)**.** [7] Let X be a closed proper subset of $\mathbb{C}$, and let $\check{X}$ denote the closure of X, when we regard X as a subset of the Riemann sphere $\mathfrak{S}$. That is, $\check{X} = X$, when X is compact, and otherwise $\check{X}$ is X together with the point at ∞. We let $\mathcal{R}(X)$ denote the quotients of polynomials with poles off $\check{X}$, that is, the bounded, rational functions on X with a limit at ∞. Clearly, $\mathcal{R}(X)$ is a subalgebra of $\mathbb{C}(t)$.

Theorem 1.3.3. [7] *Let $\mathcal{U}$ be an associative algebra with unity and $A \in \mathcal{U}$. There is a unique algebra homomorphism $\phi : \mathcal{R}(\sigma(A)) \to \mathcal{U}$ such that $\phi(1) = 1$ and $\phi(t) = A$.*
Notation: If $f \in \mathcal{R}(\sigma(A))$ we define $f(A) = \phi(f)$; writing $f = p/q$ where $p, q \in \mathbb{C}[t]$ and q has no zeros in $\sigma(A)$, we have $f(A) = q(A)^{-1}p(A) = p(A)q(A)^{-1}$.

Note that if $f = p/q$ belong to $\mathcal{R}(\sigma(A))$, then q has no zeros in $\sigma(A)$. This means $0 \notin q(\sigma(A))$. But by the spectral mapping theorem, $q(\sigma(A)) = \sigma(q(A))$.

Therefore, $0 \notin \sigma(q(A))$, and hence $q(A)$ is invertible. Thus the definition of $f(A)$ makes sense in theorem 1.3.3.

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Suppose that X is a closed proper subset of the complex plane $\mathbb{C}$ such that $X \supset \sigma(T)$. Obviously, $\mathcal{R}(X) \subset \mathcal{R}(\sigma(T))$. Therefore, by applying the functional calculus of theorem 1.3.3, $f(T)$ is defined and exists in $\mathcal{L}(\mathcal{H})$, for every $f \in \mathcal{R}(X)$. The mapping $f \to f(T) : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$ is an algebra homomorphism by theorem 1.3.3.

Definition 1.3.4 (Integrals of Banach space-valued functions)**.** [23] Let $\mathcal{U}$ be a Banach space and f be a continuous function of the complex variable z, with $f(z) \in \mathcal{U}$. Let C be a smooth closed curve in the plane $\mathbb{C}$, defined by $t \to z(t) : [a, b] \to \mathbb{C}$ (which is continuously differentiable on $[a, b]$). We define

$$\int_C f(z)dz \left(= \int_a^b f(z(t))z'(t)dt\right)$$

as the norm limit of the "Riemann sums" of the form

$$\sum_{j=1}^{n} f(z(t'_j))[z(t_j) - z(t_{j-1})]$$

where $a = t_0 < t_1 < \ldots < t_n = b$, $t_{j-1} \leq t'_j \leq t_j$, the limit being taken as $\max\{|t_j - t_{j-1}| : j = 1, \ldots n\} \to 0$.

Definition 1.3.5 (Holomorphic functional calculus)**.** [23] Suppose that $\mathcal{U}$ is a Banach algebra and let $A \in \mathcal{U}$. Let $\sigma(A)$ be the spectrum of A, let C be a smooth closed curve whose interior contains $\sigma(A)$, and let n be a positive integer. It can be seen that

$$A^n = \frac{1}{2\pi i} \int_C z^n (zI - A)^{-1} dz \tag{1.1}$$

Hence it follows that

$$p(A) = \frac{1}{2\pi i} \int_C p(z)(zI - A)^{-1} dz \tag{1.2}$$

for each polynomial p, where C is as described above. If f is a holomorphic function (classical complex valued of complex variable), holomorphic in an open set containing $\sigma(A)$, we define

$$f(A) = \frac{1}{2\pi i}\int_C f(z)(zI - A)^{-1}dz \tag{1.3}$$

for each $A \in \mathcal{U}$, where C is a smooth closed curve whose interior contains $\sigma(A)$. By 1.3.4, the integral on the right hand side of (1.3) converges in the norm. So it represents an element $f(A)$ in $\mathcal{U}$.

We shall denote by $Hol(\sigma(A))$ the set of functions holomorphic in some open set containing $\sigma(A)$ (the open set may vary with the function). It can be seen that $Hol(\sigma(A))$ is an algebra.

Theorem 1.3.6. [23] *The mapping $f \to f(A)$ is a homomorphism from the algebra $Hol(\sigma(A))$ into $\mathcal{U}$ for each A in the Banach algebra $\mathcal{U}$. If f is represented by the power series $\sum_{n=0}^{\infty} a_n z^n$ throughout an open set containing $\sigma(A)$, then*

$$f(A) = \sum_{n=0}^{\infty} a_n A^n.$$

Remark 1.3.7. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$. Let X be a closed proper subset of $\mathbb{C}$ such that $X \supset \sigma(T)$. If f is a rational function, then we see that $f \in \mathcal{R}(X) \Longrightarrow f = p/q$, where $p, q \in \mathbb{C}(t)$ such that q has no zeros in $X \Longrightarrow f$ is holomorphic on an open set containing $X \Longrightarrow f \in Hol(\sigma(T))$. Thus $\mathcal{R}(X)$ is a subalgebra of $Hol(\sigma(T))$.

1.4 Some topics in C^*-algebras

Definition 1.4.1 (Operator system). If $\mathcal{S}$ is a subset of a C^*-algebra $\mathcal{A}$, we set $\mathcal{S}^* = \{a : a^* \in \mathcal{S}\}$, and we call $\mathcal{S}$ self-adjoint when $\mathcal{S}^* = \mathcal{S}$. If $\mathcal{A}$ has a unit 1 and $\mathcal{S}$ is a self-adjoint subspace of $\mathcal{A}$ containing 1, then we call $\mathcal{S}$ an *operator system.*

Definition 1.4.2 (The C^*-algebra $\mathcal{M}_n(\mathcal{A})$ and the norm in $\mathcal{M}_n(\mathcal{A})$)**.** Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{M}_n(\mathcal{A})$ denote the set of all $n \times n$ matrices with entries from $\mathcal{A}$. We will denote a typical element of $\mathcal{M}_n(\mathcal{A})$ by (a_{ij}). There is a natural way to make $\mathcal{M}_n(\mathcal{A})$ into a $*$-algebra. For (a_{ij}) and (b_{ij}) in $\mathcal{M}_n(\mathcal{A})$, set

$$\begin{aligned} (a_{ij}) + (b_{ij}) &= (a_{ij} + b_{ij}), \\ c(a_{ij}) &= (ca_{ij}) \text{ if } c \in \mathbb{C}, \\ (a_{ij}) \cdot (b_{ij}) &= \left(\Sigma_{k=1}^{n} a_{ik} b_{kj} \right), \end{aligned}$$

and

$$(a_{ij})^* = (a_{ji}^*).$$

Also, there is a unique way to introduce a norm such that $\mathcal{M}_n(\mathcal{A})$ becomes a C^*-algebra.

One way that $\mathcal{M}_n(\mathcal{A})$ can be viewed as a C^*-algebra is to first choose a one-to-one *-representation of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, and then let $\mathcal{M}_n(\mathcal{A})$ act on the direct sum of n copies of $\mathcal{H}$ in the obvious way. It can be easily verified that this defines a one-to-one representation of $\mathcal{M}_n(\mathcal{A})$ for which the above multiplication and * operation become operator composition and operator adjoint. It is straight-forward to verify that the image of $\mathcal{M}_n(\mathcal{A})$ under this representation is closed and hence a C^*-algebra.

Thus we have a way to turn $\mathcal{M}_n(\mathcal{A})$ into a C^*-algebra. But since the norm is unique on a C^*-algebra we see that the norm on $\mathcal{M}_n(\mathcal{A})$ defined in this fashion is independent of the particular representation of $\mathcal{A}$ we choose.

We shall use the notation $\mathcal{M}_n$ for the C^*-algebra of all $n \times n$ complex matrices.

In other words, $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$.

Definition 1.4.3. Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{S} \subset \mathcal{A}$ be an operator system. If $\mathcal{B}$ is a unital C^*-algebra and $\phi : \mathcal{S} \to \mathcal{B}$ is a linear map, we say that ϕ is a *unital map* if $\phi(1) = 1$. We say that ϕ is *self-adjoint* if $\phi(a^*) = (\phi(a))^*$ for all $a \in \mathcal{S}$.

Definition 1.4.4 (Operator space)**.** Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{M}$ be a subspace, then we call $\mathcal{M}$ an *operator space.*

Clearly $\mathcal{M}_n(\mathcal{M})$ can be regarded as a subspace of $\mathcal{M}_n(\mathcal{A})$, and we let $\mathcal{M}_n(\mathcal{M})$ have the norm structure that it inherits from the unique norm structure on the C^*-algebra $\mathcal{M}_n(\mathcal{A})$. Similarly if $\mathcal{S} \subset \mathcal{A}$ is an operator system, then we endow $\mathcal{M}_n(\mathcal{S})$ with the norm and the order structure that it inherits as a subspace of the C^*-algebra $\mathcal{M}_n(\mathcal{A})$.

Definition 1.4.5. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ be a subspace, and let n be a positive integer. If $\mathcal{B}$ is a C^*-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ is a linear map, we define $\phi_n : \mathcal{M}_n(\mathcal{M}) \to \mathcal{M}_n(\mathcal{B})$ by

$$\phi_n((a_{ij})) = (\phi(a_{ij})),$$

and call ϕ_n the n^{th} *amplification* of ϕ.

Definition 1.4.6. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ a subspace of $\mathcal{A}$, and let n be a positive integer. If $\mathcal{B}$ is a C^*-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ is a linear map, we say that

i. ϕ is *positive* if ϕ maps positive elements of $\mathcal{M}$ to positive elements of $\mathcal{B}$.

ii. ϕ is *contractive* if $\|\phi\| \leq 1$.

iii. ϕ is *n-positive* if ϕ_n is positive.

iv. ϕ is *completely positive* if ϕ is n-positive for all n.

v. ϕ *is completely bounded* if $\sup_n \|\phi_n\|$ is finite, and, in this case, we set $\|\phi\|_{cb} = \sup_n \|\phi_n\|$.

vi. ϕ is *completely isometric* if each ϕ_n is isometric.

vii. ϕ is *completely contractive* if $\|\phi\|_{cb} \leq 1$.

viii. ϕ is *n-contractive* if $\|\phi_n\| \leq 1$.

A positive map need not be completely positive; and a bounded map need not be completely bounded. A contractive map need not be $2-$contractive. In general, $\|\phi_n\| \neq \|\phi\|$.

For example, let $\{E_{ij}\}_{i,j=1}^2$ denote the system of matrix units for $\mathcal{M}_2$, that is E_{ij} is the 2×2 matrix with 1 in the ij^{th} entry and 0 elsewhere. Let $\phi : \mathcal{M}_2 \to \mathcal{M}_2$ be the transpose map. It can be easily seen that $\phi(A)$ is positive if A is positive, and that $\|\phi(A)\| = \|A\|$ for all $A \in \mathcal{M}_2$. Thus ϕ is positive and $\|\phi\| = 1$.

Now, consider $\phi_2 : \mathcal{M}_2(\mathcal{M}_2) \to \mathcal{M}_2(\mathcal{M}_2)$. The matrix of matrix units

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

can be written as B^*B where

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

and hence is positive. But

$$\phi_2\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\right) = \begin{bmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive. For instance, if T is the operator on $\mathbb{C}^4$ that is represented by the above matrix, and if $x = (0, -1, 1, 0)$, then $x \in \mathbb{C}^4$ and $\langle Tx, x\rangle = -2 < 0$.

Also, if

$$D = \begin{bmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{bmatrix},$$

then $\|D\| = 1$, but $\|\phi_2(D)\| = 2$. Therefore, $\|\phi_2\| \geq 2$. Note that ϕ is contractive, but not $2-$contractive.

1.5 Spectral sets

Definition 1.5.1 (Spectral set, K-spectral set)**.** Let X be a closed proper subset of $\mathbb{C}$, and let $\check{X}$ denote the closure of X when we regard X as a subset of the Riemann sphere $\mathfrak{S}$. That is, $\check{X} = X$, when X is compact, and otherwise $\check{X}$ is X together with the point at ∞. We let $\mathcal{R}(X)$ denote the quotients of polynomials with poles off $\check{X}$, that is, the bounded, rational functions on X with a limit at infinity. We regard $\mathcal{R}(X)$ as a subalgebra of the C^*-algebra $C(\partial\check{X})$, which defines norms on $\mathcal{R}(X)$ and on each $\mathcal{M}_n(\mathcal{R}(X))$.

If X is a closed, proper subset of $\mathbb{C}$, and $T \in \mathcal{L}(\mathcal{H})$, with $\sigma(T) \subset X$, then there is still a functional calculus, ie., a homomorphism $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$, given by $\rho(f) = f(T)$, where $f(T) = p(T)q(T)^{-1}$ if $f = p/q$ (see the remarks after 1.3.3).

If $\| \rho \| \leq 1$, then X is called a *spectral set* for T.

If $\| \rho \| \leq K$, then X is called a *K-spectral set* for T.

If $\| \rho \|_{cb} \leq 1$, then X is called a *complete spectral set* for T.

If $\| \rho \|_{cb} \leq K$, then X is called a *complete K-spectral set* for T.

Of course, $\| \rho \|$ is defined as follows.

$$\begin{aligned} \| \rho \| &= \sup\{\| \rho(f) \| : f \in \mathcal{R}(X), \| f \|_\infty = 1\} \\ &= \sup\{\| f(T) \| : f \in \mathcal{R}(X), \| f \|_\infty = 1\}, \end{aligned}$$

where $\| f \|_\infty$ is the norm of f in the C^*-algebra $C(\partial \check{X})$.

A well known theorem known as von Neumann's inequality, says that an operator T is a contraction if and only if the closed unit disk is a spectral set for T. Thus for every $T \in \mathcal{L}(\mathcal{H})$, the set $\{z \in C : |z| \leq ||T||\}$ is always a spectral set for T.

By replacing the algebra $\mathcal{R}(X)$ in the above definition by the algebra $Hol(X)$ and by defining $f(T)$ according to the Riesz-Dunford functional calculus we obtain more general definitions of spectral sets and K-spectral sets.

Theorem 1.5.2 (Berger-Foias-Lebow)**.** *If S is a compact, convex spectral set for the operator T on a Hilbert space $\mathcal{H}$, and if ∂S denotes the boundary of S, then* *there exists a normal operator N defined on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that*

i. $\sigma(N) \subset \partial S$

ii. $T^n x = PN^n x, \ x \in \mathcal{H}, n = 1, 2, \dots$

where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

The theorem says that if S is a compact spectral set for T, then there exists a strong normal dilation N of T such that $\sigma(N) \subset \partial S$.

Chapter 2

Aluthge and Duggal transformations

2.1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with $2 \leq dim\mathcal{H} \leq \aleph_0$ and $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$. Let $T = U|T|$ be the unique polar decomposition of T, where U is a partial isometry such that $kerU = kerT = ker|T|$ and $|T| = (T^*T)^{1/2}$. Obviously $|T|$ is a positive operator. Also $\| Tx \| = \| |T|x \|$ for all $x \in \mathcal{H}$, and if $E = U^*U$ then E is the initial projection of U (ie., $E = P_{\mathcal{X}}$ where $\mathcal{X} = (kerU)^{\perp} = (kerT)^{\perp}$) and E is the support of T as well as the support of $|T|$. The following definition is due to Aluthge [1].

Definition 2.1.1 (Aluthge transformation [1]). If $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ is the polar decomposition of T, then

$$\widetilde{T} = |T|^{1/2}U|T|^{1/2}$$

is called the *Aluthge transformation* of T.

Definition 2.1.2 (Duggal transformation [16]). If $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ is the polar decomposition of T, then

$$\widehat{T} = |T|U$$

is called the *Duggal transformation* of T.

Definition 2.1.3 ($\lambda-$Aluthge transformation)**.** If $T \in \mathcal{L}(\mathcal{H})$, $T = U|T|$ the polar decomposition of T, and $0 < \lambda < 1$, then $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ is called the $\lambda-$*Aluthge transformation* of T. When $\lambda = 1/2$, the $\lambda-$Aluthge transformation is the Aluthge transformation.

The notion of Aluthge transformation was first studied in [1] in relation with the $p-$hyponormal and $\log-$hyponormal operators. Roughly speaking, the Aluthge transformation of an operator is closer to being normal. Aluthge transformation has received much attention in recent years. One reason is the connection of Aluthge transformation with the invariant subspace problem. Jung, Ko and Pearcy proved in [22] that T has a nontrivial invariant subspace if and only if $\widetilde{T}$ does. Another reason is related with the iterated Aluthge transformation.

In [16], Foias, Jung, Ko and Pearcy introduced the the concept of Duggal transformations, and proved several analogous results for Aluthge transformations and Duggal transformations. Yamazaki in [35] proved that for every $T \in \mathcal{L}(\mathcal{H})$, the sequence of the norms of the Aluthge iterates of T converges to the spectral radius $r(T)$. Derming Wang in [33] gave another proof of this result. We started studying Aluthge and Duggal transformations hoping to prove, the analogue of the result of Yamazaki, that the sequence of the norms of the Duggal iterates of T converges to the spectral radius $r(T)$, for every $T \in \mathcal{L}(\mathcal{H})$. Several finite dimensional examples suggested that the result is true for Duggal transformations. We succeeded in proving that the sequence of the norms of the Duggal iterates converges to the spectral radius, for certain classes of operators. We give this result as theorem 2.2.2. Further investigation led to an example of a finite dimensional operator showing that there exist operators such that the sequence of the norms of the Duggal iterates does not converge to the spectral radius. We exhibit this example in section 2.2.3.

Definition 2.1.4. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$.

i. T is called *hyponormal* if $T^*T \geq TT^*$.

ii. For $p > 0$, T is $p-$*hyponormal* if $(T^*T)^p \geq (TT^*)^p$. (Thus $1-$hyponormal means simply hyponormal).

iii. If T is invertible, T is called $\log -$*hyponormal* if $\log T^*T \geq \log TT^*$.

Theorem 2.1.5. [1] *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$.*

i. For $0 < p < 1/2$, if T is $p-$hyponormal, then $\widetilde{T}$ is $p + 1/2-$hyponormal.

ii. For $1/2 \leq p \leq 1$, if T is $p-$hyponormal, then $\widetilde{T}$ is $1-$hyponormal.

Theorem 2.1.6. [31] *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$, and T be invertible. If T is $\log -$hyponormal, then $\widetilde{T}$ is $1/2-$hyponormal.*

It is well known that $\sigma(T) = \sigma(\widetilde{T}) = \sigma(\widehat{T})$ ([22], [16]). The following theorem shows some known results.

Theorem 2.1.7. *Let $T \in \mathcal{L}(\mathcal{H})$.*

i. $\| \widetilde{T} \| \leq \| T \|, \;\; \| \widehat{T} \| \leq \| T \|$.

ii. T is quasinormal if and only if $T = \widetilde{T}$ if and only if $T = \widehat{T}$.

Proof. Let $T = U|T|$ be the polar decomposition of T. One can note that (if $U \neq 0$) $\| U \| = 1$ (see the remark 1.2.3 on page 12). Further, one can see that $\| T \| = \| \, |T|^2 \, \|^{1/2} = \| \, |T| \, \| = \| \, |T|^{1/2} \, \|^2$ and hence $\| \, |T|^{1/2} \, \| = \|T\|^{1/2}$ [24].

Now

$$\begin{aligned}
\| \widetilde{T} \| &= \| |T|^{1/2}U|T|^{1/2} \| \\
&\leq \| |T|^{1/2} \| \cdot \| U \| \cdot \| |T|^{1/2} \| \\
&= \| |T|^{1/2} \|^2 \\
&= \| T \| \qquad (2.1) \\
\| \widehat{T} \| &= \| |T|U \| \\
&\leq \| |T| \| \cdot \| U \| \\
&= \| |T| \| \\
&= \| T \| \qquad (2.2)
\end{aligned}$$

Further, $T = \widehat{T} \implies T = |T|U \implies U|T| = |T|U \implies |T|$ commutes with $U \implies |T|^{1/2}$ commutes with $U \implies |T|^{1/2}U = U|T|^{1/2} \implies |T|^{1/2}U|T|^{1/2} = U|T| \implies \widetilde{T} = T$. On the other hand, $T = \widetilde{T} \implies T = |T|^{1/2}U|T|^{1/2} \implies T|T|^{1/2} = |T|^{1/2}U|T| \implies T|T|^{1/2} = |T|^{1/2}T \implies |T|^{1/2}$ commutes with $T \implies |T|$ commutes with $T \implies T^*T$ commutes with $T \implies T$ is quasinormal $\implies U$ and $|T|$ commute (see [18]) $\implies |T|U = U|T| \implies \widehat{T} = T$. Also, T is quasinormal $\iff U$ and $|T|$ commute $\iff \widehat{T} = T$. Thus T is quasinormal $\iff T = \widehat{T} \iff T = \widetilde{T}$. □

Definition 2.1.8. For $T \in \mathcal{L}(\mathcal{H})$, denote by $Hol(\sigma(T))$ the algebra of all complex-valued functions which are analytic on some neighborhood of $\sigma(T)$, where linear combinations and products in $Hol(\sigma(T))$ are defined (with varying domains) in the obvious way. The (Riesz-Dunford) algebra $\mathcal{A}_T \subseteq \mathcal{L}(\mathcal{H})$ is defined as

$$\mathcal{A}_T = \{f(T) : f \in Hol(\sigma(T))\},$$

where the operator $f(T) \in \mathcal{L}(\mathcal{H})$ is defined by the Riesz-Dunford functional calculus as in 1.3.5.

The following theorem in [16] gives useful information about $\widetilde{T}$ and $\widehat{T}$ by studying maps between the algebras $\mathcal{A}_T, \mathcal{A}_{\widetilde{T}}$ and $\mathcal{A}_{\widehat{T}}$.

Theorem 2.1.9. [16] *For every $T \in \mathcal{L}(\mathcal{H})$, with $\widetilde{T}, \widehat{T}$, and $Hol(\sigma(T))$ as defined above:*

(a) The maps $\widetilde{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widetilde{T}}$ and $\widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widehat{T}}$ defined by

$$\widetilde{\Phi}(f(T)) = f(\widetilde{T}), \;\; \widehat{\Phi}(f(T)) = f(\widehat{T}), \;\; f \in Hol(\sigma(T))$$

are well defined contractive algebra homomorphisms. Thus

$$max\{\| f(\widetilde{T}) \|, \| f(\widehat{T}) \|\} \leq \| f(T) \|, \;\; f \in Hol(\sigma(T)).$$

(b) More generally, the maps $\widetilde{\Phi}$ and $\widehat{\Phi}$ are completely contractive, meaning that for every $n \in \mathbb{N}$ and every $n \times n$ matrix (f_{ij}) with entries from $Hol(\sigma(T))$,

$$max\{\| (f_{ij}(\widehat{T})) \|, \| (f_{ij}(\widetilde{T})) \|\} \leq \| (f_{ij}(T)) \| .$$

(The norm here is the natural norm in the C^*-algebra $\mathcal{M}_n(\mathcal{L}(\mathcal{H}))$).

(c) Every spectral set for T is a spectral set for both $\widetilde{T}$ and $\widehat{T}$. For fixed $K > 1$, every K-spectral set for T is a K-spectral set for both $\widetilde{T}$ and $\widehat{T}$.

(d) If $W(S)$ denotes the numerical range of an operator S in $\mathcal{L}(\mathcal{H})$, then

$$\overline{W(f((\widehat{T}))} \cup \overline{W(f((\widetilde{T}))} \subset \overline{W(f((T))}, \;\; f \in Hol(\sigma(T))$$

(the overbar denoting the closure).

Remark 2.1.10. Aluthge transformations and Duggal transformations enjoy several analogous properties. The following are some.

i. $\| \widetilde{T} \| \leq \| T \|, \quad \| \widehat{T} \| \leq \| T \|$.

ii. $\sigma(\widetilde{T}) = \sigma(T), \quad \sigma(\widehat{T}) = \sigma(T)$

iii. T is quasinormal $\Longleftrightarrow T = \widetilde{T} \Longleftrightarrow T = \widehat{T}$.

iv. $r(T) = r(\widetilde{T}) = r(\widehat{T})$.

v. $\| f(\widetilde{T}) \| \leq \| f(T) \|, \quad \| f(\widehat{T}) \|\} \leq \| f(T) \|, \quad f \in Hol(\sigma(T))$.

vi. Every spectral set for T is a spectral set for $\widetilde{T}$, every spectral set for T is a spectral set for $\widehat{T}$.

vii. Every K-spectral set for T is a K-spectral set for $\widetilde{T}$, every K-spectral set for T is a K-spectral set for $\widehat{T}$.

viii. $\| (f_{ij}(\widetilde{T})) \| \leq \| (f_{ij}(T)) \|, \quad \| (f_{ij}(\widehat{T})) \| \leq \| (f_{ij}(T)) \|$ for every positive integer n and every $n \times n$ matrix (f_{ij}) with entries in $Hol(\sigma(T))$.

ix. $\overline{W(\widetilde{T})} \subset \overline{W(T)}, \quad \overline{W(\widehat{T})} \subset \overline{W(T)}$, where $\overline{W(T)}$ denotes the closure of the numerical range $W(T)$ of T.

2.2 Aluthge and Duggal iterates

2.2.1 Aluthge and Duggal iterates

Definition 2.2.1 (Iterated Aluthge transformations)**.** Denote $\widetilde{T}^{(0)} = T$, $\widetilde{T}^{(1)} = \widetilde{T}$, $\widetilde{T}^{(2)} = \widetilde{(\widetilde{T}^{(1)})}, \ \ldots, \ \widetilde{T}^{(n)} = \widetilde{(\widetilde{T}^{(n-1)})}, \ \ldots$.

For every $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\| (\widetilde{T}^{(n)}) \|\}_{n=0}^{\infty}$ is decreasing such that $r(T) \leq \left\|\widetilde{T}^{(n)}\right\| \leq \|T\|$. (Proof: Since $\sigma(T) = \sigma(\widetilde{T}) = \sigma(\widetilde{T}^{(n)})$ for all $n \in \mathbb{N}$, we have, $r(T) = r(\widetilde{T}^{(n)}) \leq \| \widetilde{T}^{(n)} \|$ for all $n \in \mathbb{N}$. The fact $\| \widetilde{T}^{(n)} \| \leq \| T \|$ follows easily from an application of the inequality (2.1) on page 30). Hence

$\{\| (\widetilde{T}^{(n)}) \|\}_{n=0}^{\infty}$ is a convergent sequence. In 2002, Yamazaki in the excellent paper [35] proved that for every $T \in \mathcal{L}(\mathcal{H})$, the sequence of the norms of the Aluthge iterates of T converges to the spectral radius $r(T)$.

Theorem 2.2.2. [35] *For every* $T \in \mathcal{L}(\mathcal{H})$, *the sequence* $\{||\widetilde{T}^{(n)}||\}$ *converges to* $r(T)$.

In 2003, Derming Wang in [33] used Mc Intosh inequality and Heinz inequality to give another proof of the above theorem.

Definition 2.2.3 (Iterated Duggal transformations)**.** Denote $\widehat{T}^{(0)} = T$, $\widehat{T}^{(1)} = \widehat{T}$, $\widehat{T}^{(2)} = \widehat{(\widehat{T}^{(1)})}$, $\ldots$, $\widehat{T}^{(n)} = \widehat{(\widehat{T}^{(n-1)})}$, $\ldots$.

In the coming sections, we investigate the convergence of the norms of the Duggal iterates of a bounded linear operator on a Hilbert space.

2.2.2 Convergence of the norms of Duggal iterates

We shall prove that $\lim_{n\to\infty} \| \widehat{T}^{(n)} \| = r(T)$ for operators T belonging to certain classes of operators in $\mathcal{L}(H)$. By the inequality (2.2), $\| \widehat{T}^{(n+1)} \| \leq \| \widehat{T}^{(n)} \|$ for all $n \in \mathbb{N}$. Moreover $\sigma(\widehat{T}^{(n)}) = \sigma(T)$, and hence $r(\widehat{T}^{(n)}) = r(T)$ for all $n \geq 0$. Thus $\{ ||\widehat{T}^{(n)}|| \}_{n=0}^{\infty}$ is a decreasing sequence which is bounded below by $r(T)$. The following lemma is an easy consequence.

Lemma 2.2.4. *There is an* $s \geq r(T)$ *for which* $\lim_{n\to\infty} \| \widehat{T}^{(n)} \| = s$.

Remark 2.2.5. We notice one more analogy between Aluthge and Duggal transformations.

> The sequence $\{\| (\widetilde{T}^{(n)}) \|\}_{n=0}^{\infty}$ is decreasing such that $r(T) \leq \| \widetilde{T}^{(n)} \| \leq \|T\|$, and $r(\widetilde{T}^{(n)}) = r(T)$ for all n. The sequence $\{\| (\widehat{T}^{(n)}) \|\}_{n=0}^{\infty}$ is decreasing such that $r(T) \leq \| \widehat{T}^{(n)} \| \leq \|T\|$, and $r(\widehat{T}^{(n)}) = r(T)$ for all n.

Theorem 2.2.6 (Mc Intosh inequality). *For bounded linear operators A,B and X,*

$$\| A^*XB \| \leq \| AA^*X \|^{1/2} \| XBB^* \|^{1/2} .$$

Theorem 2.2.7 (Heinz inequality). *For positive linear operators A and B, and bounded linear operator X,*

$$\| A^\alpha XB^\alpha \| \leq \| AXB \|^\alpha \| X \|^{1-\alpha}$$

for all $0 \leq \alpha \leq 1$.

Using these inequalities we prove the following results.

Lemma 2.2.8. *For any positive integer k,*

$$\| (\widehat{T}^{(n+1)})^k \| \leq \| (\widehat{T}^{(n)})^k \|$$

for all $n \geq 0$. *Consequently, the decreasing sequence* $\{\| (\widehat{T}^{(n)})^k \|\}_{n=0}^\infty$ *is convergent.*

Proof. Let $f(t) = t^k, t \in$ a neighborhood of $\sigma(T)$, and note that $\sigma(T) = \sigma(\widehat{T}^{(n)})$. We have $f \in Hol(\sigma(T))$. Applying theorem 2.1.9 (a) , the proof is complete. □

Lemma 2.2.9. *If* $\widehat{T}^{(n)} = U_n|\widehat{T}^{(n)}|$ *is the polar decomposition of* $\widehat{T}^{(n)}$, *then for any positive integer k,*

$$\| (\widehat{T}^{(n+1)})^k \| \leq \| |\widehat{T}^{(n)}|^2 (\widehat{T}^{(n)})^{k-1} \|^{1/2} \| (\widehat{T}^{(n)})^{k-1}U_nU_n^* \|^{1/2}$$

Proof. We have $\widehat{T}^{(n+1)} = |\widehat{T}^{(n)}|U_n$ and therefore $(\widehat{T}^{(n+1)})^k = |\widehat{T}^{(n)}|(\widehat{T}^{(n)})^{k-1}U_n$. Hence by theorem 2.2.6,

$$\begin{aligned} \| (\widehat{T}^{(n+1)})^k \| &\leq \| |\widehat{T}^{(n)}|(\widehat{T}^{(n)})^{k-1}U_n \| \\ &\leq \| |\widehat{T}^{(n)}|^2(\widehat{T}^{(n)})^{k-1} \|^{1/2} \| (\widehat{T}^{(n)})^{k-1}U_nU_n^* \|^{1/2} \end{aligned}$$ □

Lemma 2.2.10. *Let n be a positive integer and $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition* $\| \, |\widehat{T}^{(n)}|^2 (\widehat{T}^{(n)})^{k-1} \, \| \leq \| \, (\widehat{T}^{(n)})^{k+1} \, \|$. *Then*

$$\| \, (\widehat{T}^{(n+1)})^k \, \| \leq \| \, (\widehat{T}^{(n)})^{k+1} \, \|^{1/2} \, \| \, (\widehat{T}^{(n)})^{k-1} \, \|^{1/2}$$

Proof. By lemma 2.2.9,

$$\begin{aligned} \| \, (\widehat{T}^{(n+1)})^k \, \| &\leq \| \, |\widehat{T}^{(n)}|^2 (\widehat{T}^{(n)})^{k-1} \, \|^{1/2} \, \| \, (\widehat{T}^{(n)})^{k-1} U_n U_n^* \, \|^{1/2} \\ &\leq \| \, (\widehat{T}^{(n)})^{k+1} \, \|^{1/2} \, \| \, (\widehat{T}^{(n)})^{k-1} U_n U_n^* \, \|^{1/2} \\ &\leq \| \, (\widehat{T}^{(n)})^{k+1} \, \|^{1/2} \, \| \, (\widehat{T}^{(n)})^{k-1} \, \|^{1/2} \end{aligned}$$

since $U_n U_n^*$ is a projection. □

Lemma 2.2.11. *Let n be a positive integer and $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition* $|\widehat{T}^{(n)}| \, \widehat{T}^{(n)} = \widehat{T}^{(n)} \, |\widehat{T}^{(n)}|$. *Then*

$$\| \, (\widehat{T}^{(n+1)})^k \, \| \leq \| \, (\widehat{T}^{(n)})^{k+1} \, \|^{1/2} \, \| \, (\widehat{T}^{(n)})^{k-1} \, \|^{1/2}$$

Proof.

$$\begin{aligned} \| \; |\widehat{T}^{(n)}|^2 \; (\widehat{T}^{(n)})^{k-1} \; \| &= \| \, |\widehat{T}^{(n)}| \cdot |\widehat{T}^{(n)}| \; (\widehat{T}^{(n)})^{k-1} \, \| \\ &= \| \, \widehat{T}^{(n)} \; |\widehat{T}^{(n)}| \; (\widehat{T}^{(n)})^{k-1} \, \| \\ &= \| \; |\widehat{T}^{(n)}| \; (\widehat{T}^{(n)})^k \; \| \\ &= \| \, (\widehat{T}^{(n)})^{k+1} \, \| \, . \end{aligned}$$

By lemma 2.2.10, the result follows. □

Lemma 2.2.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition*

$$\| \; |\widehat{T}^{(n)}|^2 (\widehat{T}^{(n)})^{k-1} \; \| \leq \| \, (\widehat{T}^{(n)})^{k+1} \, \|$$

for all $k = 1, 2, \ldots,$ *and for all large positive integers* n. *Then* $\lim_{n\to\infty} \| (\widehat{T}^{(n)})^k \| = s^k$ *for any positive integer* k.

Proof. We prove the lemma by induction on k. By lemma 2.2.4, the result is true for $k = 1$. Suppose that the result is true for $1 \leq k \leq m$. By lemma 2.2.10, for large n,

$$\begin{aligned} \| (\widehat{T}^{(n+1)})^m \| &\leq \| (\widehat{T}^{(n)})^{m+1} \|^{1/2} \| (\widehat{T}^{(n)})^{m-1} \|^{1/2} \\ &\leq \| (\widehat{T}^{(n)})^m \|^{1/2} \| \widehat{T}^{(n)} \|^{1/2} \| (\widehat{T}^{(n)})^{m-1} \|^{1/2} \end{aligned} \tag{2.3}$$

Put $\lim_{n\to\infty} \| (\widehat{T}^{(n)})^{m+1} \| = t$ (the limit exists by lemma 2.2.8 .) Now, taking limits as $n \to \infty$ in (2.3), the induction hypothesis shows that

$$s^m \leq t^{1/2} s^{(m-1)/2} \leq s^{m/2} s^{1/2} s^{(m-1)/2} = s^m.$$

Therefore,

$$t^{1/2} s^{(m-1)/2} = s^m.$$

Hence

$$t = s^{m+1}.$$

The lemma follows by induction. □

Theorem 2.2.13. *If* $T \in \mathcal{L}(\mathcal{H})$ *is such that* $\| \ |\widehat{T}^{(n)}|^2 \ (\widehat{T}^{(n)})^{k-1} \ \| \leq \| (\widehat{T}^{(n)})^{k+1} \|$ *for all* $k = 1, 2, \ldots,$ *and for all large positive integers* n, *then*

$$\lim_{n\to\infty} \| \widehat{T}^{(n)} \| = r(T).$$

Proof. By lemma 2.2.8, we see that for each fixed positive integer k, the sequence $\{\| (\widehat{T}^{(n)})^k \|^{1/k}\}_{n=0}^{\infty}$ is convergent, and by lemma 2.2.12, it converges to s.

Therefore,

$$s \leq \| (\widehat{T}^{(n)})^k \|^{1/k}$$

for all n and k. By lemma 2.2.4,

$$r(T) \leq s.$$

Suppose, if possible, $r(T) < s$. For every fixed k, the sequence $\{\| (\widehat{T}^{(n)})^k \|\}_{n=0}^{\infty}$ is decreasing. Now fix an n. We have

$$\| (\widehat{T}^{(n)})^k \| \leq \| (\widehat{T}^{(0)})^k \| = \| T^k \|$$

for all k. Therefore,

$$\| (\widehat{T}^{(n)})^k \|^{1/k} \leq \| T^k \|^{1/k}$$

for all k. Since $r(T) < s$, and $\lim_{k\to\infty} \| T^k \|^{1/k} = r(T)$, we see that

$$\| (\widehat{T}^{(n)})^k \|^{1/k} < s$$

for sufficiently large k. This is a contradiction. Hence

$$s = r(T).$$

$$\text{ie.,} \quad \lim_{n\to\infty} \| \widehat{T}^{(n)} \| = r(T).$$

□

Theorem 2.2.14. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition that $\widehat{T}^{(n)}\ |\widehat{T}^{(n)}| = |\widehat{T}^{(n)}|\ \widehat{T}^{(n)}$ for all large positive integers n. Then*

$$\lim_{n\to\infty} \| \widehat{T}^{(n)} \| = r(T).$$

Proof. (See proof of lemma 2.2.11).

$$\| \ |\widehat{T}^{(n)}|^2 \ (\widehat{T}^{(n)})^{k-1} \ \| = \| \ (\widehat{T}^{(n)})^{k+1} \ \|$$

for all $k = 1, 2, \ldots,$ and for all large n. Hence by theorem 2.2.13, the proof is complete. □

Remark 2.2.15. An operator S is quasinormal if and only if $S|S| = |S|S$. (For S is quasinormal $\iff S$ commutes with $S^*S \iff S$ commutes with $(S^*S)^{1/2} = |S|$). Thus theorem 2.2.14 says that if $\widehat{T}^{(n)}$ is quasinormal for large positive integers n, then $\lim_{n\to\infty} \| \ \widehat{T}^{(n)} \ \| = r(T)$. But this is obvious since if $\widehat{T}^{(n)}$ is quasinormal for some n, then $\widehat{(\widehat{T}^{(n)})} = \widehat{T}^{(n)}$. ie., $\widehat{T}^{(n+1)} = \widehat{T}^{(n)}$ and hence $\widehat{T}^{(m)} = \widehat{T}^{(n)}$ for all $m \geq n$. Being a quasinormal operator, $\widehat{T}^{(n)}$ is normaloid, and therefore, $\| \ \widehat{T}^{(m)} \ \| = \| \ \widehat{T}^{(n)} \ \| = r(\widehat{T}^{(n)}) = r(T)$ for all $m \geq n$. Thus $\lim_{n\to\infty} \| \ \widehat{T}^{(n)} \ \| = r(T)$.

Corollary 2.2.16. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying $|(\widehat{T}^{(n)})^2| = |\widehat{T}^{(n)}|^2$ for all large positive integers n. Then*

$$\lim_{n\to\infty} \| \ \widehat{T}^{(n)} \ \| = r(T).$$

Proof. $\| \ |\widehat{T}^{(n)}|^2 \ (\widehat{T}^{(n)})^{k-1} \ \| = \| \ |(\widehat{T}^{(n)})^2| \ (\widehat{T}^{(n)})^{k-1} \ \| = \| \ (\widehat{T}^{(n)})^{k+1} \ \|$ for all large n, and for all $k = 1, 2, \ldots$. By theorem 2.2.13, the proof follows. □

Remark 2.2.17. An operator S is quasinormal if and only if S is hyponormal and $|S^2| = |S|^2$ [15]. Thus theorem 2.2.14 can be deduced as a consequence of corollary 2.2.16.

If $\widehat{T}^{(n)}$ is quasinormal for some n, then obviously, $\| \ \widehat{T}^{(n)} \ \| \to r(T)$ as $n \to \infty$ (see remark 2.2.15). Similarly, if $\widehat{T}^{(n)}$ is normaloid for some n, then for all $m \geq n$, we have $r(T) = r(\widehat{T}^{(m)}) \leq \| \ \widehat{T}^{(m)} \ \| \leq \| \ \widehat{T}^{(n)} \ \| = r(\widehat{T}^{(n)}) = r(T)$, hence $\| \ \widehat{T}^{(m)} \ \| = r(T)$, and therefore $\lim_{n\to\infty} \| \ \widehat{T}^{(n)} \ \| = r(T)$. As a special case, if T itself is normaloid, then $\| \ \widehat{T}^{(n)} \ \| = \| \ T \ \| = r(T)$ for all n.

Remark 2.2.18. Now we pose the crucial question. Is it true that for every $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\| \widehat{T}^{(n)} \|\}_{n=0}^{\infty}$ converge to the spectral radius $r(T)$?

2.2.3 The norms of the Duggal iterates of T need not converge to $r(T)$

In 2002, Yamazaki [35], proved that for every $T \in \mathcal{L}(\mathcal{H})$, the Aluthge norm sequence $\{\| \widetilde{T}^{(n)} \|\}_{n=0}^{\infty}$ converges to the spectral radius $r(T)$. In 2003, T. Ando and T. Yamazaki [3], proved that in the case of a 2×2 matrix the sequence of the iterated Aluthge transformations itself converges. In 2006, Jorge Antezana, Enrique R. Pujals and Demetrio Stojanoff [4], proved the convergence of iterated Aluthge transformation sequence for diagonalizable matrices.

We construct below an example showing that the analogues of these three results fail in the case of Duggal transformations. Note that we thus answer the question in remark 2.2.18 in the negative.

Example 2.2.19. Let $\mathcal{H}$ be the Hilbert space $\mathbb{C}^2$ and consider $A \in \mathcal{L}(\mathcal{H})$ given by

$$A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Then the polar decomposition of A is $A = U|A|$, where $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $|A| = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. We see that $\sigma(A) = \{-\sqrt{3}i, \sqrt{3}i\}, \| A \| = 3, r(A) = \sqrt{3}$. The Duggal transformation of A is

$$\widehat{A} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

So $\| \widehat{A} \| = 3$. The polar decomposition of $\widehat{A}$ is $\widehat{A} = U_1|\widehat{A}|$, where $U_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $|\widehat{A}| = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Hence the second Duggal iterate $\widehat{A}^{(2)}$ of A is

$$\widehat{A}^{(2)} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} = A.$$

This shows that

$$\widehat{A}^{(n)} = \begin{cases} A & \text{if } n \text{ is even} \\ A_1 & \text{if } n \text{ is odd} \end{cases}$$

where $A_1 = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$. Hence $\| \widehat{A}^{(n)} \| = 3$ for all n. Thus $\{ \| \widehat{A}^{(n)} \| \}_{n=0}^{\infty}$ does not converge to $r(A)$. More obviously, $\{ \widehat{A}^{(n)} \}_{n=0}^{\infty}$ does not converge. Also note that A is a 2×2 diagonalizable matrix. (A is diagonalizable because the eigen values of A are distinct).

Remark 2.2.20. In 2007, Huajun Huang and Tin-Yau Tam, proved in [19] that the iterated λ-Aluthge sequence converges for an $n \times n$ matrix if the nonzero eigenvalues of the matrix have distinct moduli. Earlier in [4], Jorge Antezana, Enrique R. Pujals and Demetrio Stojanoff proved the convergence of iterated Aluthge transformation sequence for diagonalizable matrices. If A is a normal $n \times n$ matrix over $\mathbb{C}$, then it is always possible to choose an orthonormal basis of $\mathbb{C}^n$ such that the corresponding matrix is diagonal [13]. Conversely if A is an $n \times n$ matrix over $\mathbb{C}$ and if it is possible to choose an orthonormal basis of $\mathbb{C}^n$ such that the corresponding matrix is diagonal, then obviously A is normal. If A is normal, then $\widetilde{A}^{(n)} = A$ for all $n \in \mathbb{N}$, and hence the sequence $\{\widetilde{A}^{(n)}\}$ converges trivially. Thus the result of Antezana, Pujals and Stojanoff is trivial in the case of matrices which are diagonalizable with respect to an orthonormal basis.

The question of whether for every $T \in \mathcal{L}(\mathcal{H})$ the sequence of iterated Aluthge transformation sequence converge remained unanswered for some time. Recently

M. Chō, I. B. Jung, and W. Y. Lee in [10] constructed a hyponormal bilateral weighted shift $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ such that $\{\widetilde{T}^{(n)}\}_{n=0}^{\infty}$ does not converge in the norm topology. However, the convergence of iterated Aluthge transformation sequence for $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a finite dimensional Hilbert space remains as an open problem.

Remark 2.2.21. Even though, in general, $\{\| \widehat{T}^{(n)} \|\}_{n=0}^{\infty}$ does not converge to $r(T)$ (as shown in the example), there are operators T for which $\| \widehat{T}^{(n)} \| \longrightarrow r(T)$. For instance, if T is normaloid, (in particular, if T is quasinormal, subnormal, or hyponormal), then $\| \widehat{T}^{(n)} \| \longrightarrow r(T)$ (see remark 2.2.17). We proved in theorem 2.2.13, for certain class of operators $\| \widehat{T}^{(n)} \| \longrightarrow r(T)$.

2.3 More on Aluthge and Duggal transformations

2.3.1 Invertible operators and Duggal transformations

In 2004, T. Ando in [2] proved the remarkable result that if A is an $n \times n$ matrix over $\mathbb{C}$, then the convex hull of $\sigma(A)$ equals the numerical range $W(A)$ if and only if A and the Aluthge transformation $\widetilde{A}$ have the same numerical range. In this section we show that in the analogous case of Duggal transformations, the implication in one direction holds, and the converse fails. We give an example to show that the converse fails even for 2×2 matrices. Also, we prove that if S and T are unitarily equivalent, then so are $\widehat{S}$ and $\widehat{T}$.

Lemma 2.3.1. *If $T \in \mathcal{L}(\mathcal{H})$ is invertible, then $\widehat{T}$ is invertible.*

Proof. If T is invertible, then T has the polar decomposition $T = U|T|$, where U is unitary [28]. Also $|T| = U^{-1}T$. Therefore, $|T|$ is invertible, and hence, $\widehat{T} = |T|U$ is invertible. □

Lemma 2.3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be any operator. If $T = U|T|$ is the polar decomposition of T, then $\widehat{T} = U^*TU$.*

Proof. Let $E = U^*U$. Then E is a projection and $E|T| = |T|$. Therefore, $U^*TU = U^*U|T|U = E|T|U = |T|U = \widehat{T}$. □

Theorem 2.3.3. *Let $T \in \mathcal{L}(\mathcal{H})$. If V is unitary and $S = V^*TV$, then $\widehat{S} = V^*\widehat{T}V$.*

Proof. We have

$$\begin{aligned} S^*S &= (V^*TV)^*(V^*TV) \\ &= V^*T^*VV^*TV \\ &= V^*T^*TV \\ &= V^*|T|^2V \\ &= (V^*|T|V)(V^*|T|V) \end{aligned}$$

and $V^*|T|V$ is positive (note that $\langle V^*|T|Vx, x\rangle = \langle |T|Vx, Vx\rangle \geq 0 \quad \forall x \in \mathcal{H}$). Therefore, $|S| = V^*|T|V$.

Let $T = U|T|$ be the polar decomposition of T. Then $kerU = kerT$. Let $U_1 = V^*UV$. Now $U_1^*U_1 = V^*U^*VV^*UV = V^*U^*UV$ and since U^*U is a projection

$$\begin{aligned} V^*U^*UV &= V^*(U^*U)^2V \\ &= (V^*U^*UV)^2 \end{aligned}$$

and hence $V^*U^*UV = U_1^*U_1$ is a projection. Thus U_1 is a partial isometry.

Let $x \in \mathcal{H}$. Then $x \in kerU_1 \iff V^*UVx = 0 \iff UVx = 0 \iff Vx \in kerU \iff Vx \in kerT \iff TVx = 0 \iff V^*TVx = 0 \iff x \in kerS$. Thus $kerU_1 = kerS$.

Hence $S = U_1|S|$ is the polar decomposition of S. Therefore, $\widehat{S} = |S|U_1 = V^*|T|VV^*UV = V^*|T|UV = V^*\widehat{T}V$ □

Theorem 2.3.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. If $T = U|T|$ is the polar decomposition of T, then for all $n \in \mathbb{N}$, the n^{th} Duggal iterate $\widehat{T}^{(n)} = (U^*)^n TU^n$.*

Proof. We prove the result by induction. Since T is invertible, U is unitary. By lemma 2.3.2, $\widehat{T} = U^*TU$. Thus the result is true for $n = 1$. (The case $n = 0$ is trivial).

Suppose that $n \geq 2$ and assume that the result is true for all $m \leq n-1$. Then

$$\begin{aligned} \widehat{T}^{(n-1)} &= (U^*)^{n-1}TU^{n-1} \\ &= U^*[(U^*)^{n-2}TU^{n-2}]U \\ &= U^*\widehat{T}^{(n-2)}U \text{ by the induction hypothesis.} \end{aligned}$$

Therefore, by theorem 2.3.3,

$$\begin{aligned} \widehat{T}^{(n)} &= U^*\widehat{T}^{(n-1)}U \\ &= U^*[(U^*)^{n-1}TU^{n-1}]U \\ &= (U^*)^n TU^n. \end{aligned}$$

□

Remark 2.3.5. If T is invertible, by theorem 2.3.4, every Duggal iterate of T is unitarily equivalent to T. So if T is invertible, every Duggal iterate of T has the same numerical range as that of T.

Remark 2.3.6. In [2], T. Ando proved that if A is an $n \times n$ matrix over $\mathbb{C}$, then the convex hull of $\sigma(A)$ equals $W(A)$ if and only if A and $\widetilde{A}$ have the same numerical range .

Consider the analogous case of Duggal transformations. Let A be an $n \times n$ matrix over $\mathbb{C}$, and let $\mathcal{C}(\sigma(A))$ denote the convex hull of $\sigma(A)$. Suppose that $\mathcal{C}(\sigma(A)) = W(A)$. By theorem 2.1.9, $\overline{W(\widehat{A})} \subset \overline{W(A)}$. Since this is a finite dimensional case, numerical ranges are compact, and hence closed. Thus $W(\widehat{A}) \subset W(A) = \mathcal{C}(\sigma(A)) = \mathcal{C}(\sigma(\widehat{A})) \subset W(\widehat{A})$. Therefore, $W(\widehat{A}) = W(A)$. Thus if the convex hull of $\sigma(A)$ equals the numerical range $W(A)$, then A and the Duggal transformation $\widehat{A}$ have the same numerical range.

The converse fails in the case of Duggal transformations. For example if A is invertible, then A and $\widehat{A}$ are unitarily equivalent, and therefore, A and $\widehat{A}$ have the same numerical range. But in this case, the convex hull of $\sigma(A)$ need not be equal to $W(A)$, as the following example shows.

Example 2.3.7. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then A is an invertible matrix and $\sigma(A) = \{-1, 1\}$.

It is fairly standard that if A is a 2×2 matrix with distinct eigen values α and β, and corresponding eigen vectors f and g, so normalized that $\| f \| = \| g \| = 1$, then $W(A)$ is a closed elliptical disc with foci at α and β; if $\gamma = |\langle f, g \rangle|$ and $\delta = \sqrt{1-\gamma^2}$, then the minor axis is $\gamma|\alpha - \beta|/\delta$ and the major axis is $|\alpha - \beta|/\delta$. Also, if A has only one eigen value α, then $W(A)$ is the circular disc with center α and radius $\frac{1}{2} \| A - \alpha \|$. The results given in this paragraph can be seen in [18].

If $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, then $\alpha = 1$ and $\beta = -1$ are the distinct eigen values of A with corresponding eigen vectors $f = (1, 0)$ and $g = (-1/\sqrt{5}, 2/\sqrt{5})$. We have $\| f \| = \| g \| = 1$. Let $\gamma = |\langle f, g \rangle|$ and $\delta = \sqrt{1-\gamma^2}$. Then $\gamma|\alpha - \beta|/\delta = 1$ and $|\alpha - \beta|/\delta = \sqrt{5}$.

Therefore, the numerical range $W(A)$ is the closed elliptical disc with foci at

1 and -1; the minor axis is 1 and the major axis is $\sqrt{5}$.

The convex hull of $\sigma(A)$ is the straight line segment with end points $(-1, 0)$ and $(1, 0)$. Thus the convex hull of $\sigma(A)$ does not equal $W(A)$. Notice that since A is invertible and by lemma 2.3.2, $\widehat{A}$ is unitarily equivalent to A. So A and $\widehat{A}$ have the same numerical range.

By the remark 2.3.6 and the example 2.3.7, we have discussed the complete Duggal transformation analogue of Ando's result.

2.3.2 When the partial isometry in the polar decomposition is a coisometry

Let $T \in \mathcal{L}(\mathcal{H})$ and let $T = U|T|$ be the polar decomposition of T. In this section we study the Aluthge and the Duggal transformations of T when the partial isometry U in the polar decomposition of T happens to be a coisometry. A particular case is when U is actually unitary. We know that when T is invertible, the partial isometry U in the polar decomposition of T is a unitary operator.

We introduce in this section the concept of $n-$level spectral sets. We show that if the partial isometry U in the polar decomposition of T is a coisometry, then the obvious algebra homomorphism between the Riez Dunford algebras $\mathcal{A}_T$ and $\mathcal{A}_{\widehat{T}}$ is a complete isometry. As a consequence, we prove that in such cases, the operators T and $\widehat{T}$ have the same collection of complete spectral sets. Also we show that for any invertible non-normaloid T, the sequence of the norms of Duggal iterates of T cannot converge to the spectral radius of T; and this result is an improvement of the results of section 2.2.3.

Lemma 2.3.8. *If $T = U|T|$ is the polar decomposition of T and if U is a coisometry, then $\| \widehat{T} \| = \| T \|$.*

Proof. We have $\widehat{T} = U^*TU$. Therefore, $U\widehat{T}U^* = UU^*TUU^* = T$, and therefore, $\| T \| = \| U\widehat{T}U^* \| \leq \| U \| \cdot \| \widehat{T} \| \cdot \| U^* \| \leq \| \widehat{T} \|$. But by the inequality (2.2) on page 30, $\| \widehat{T} \| \leq \| T \|$. Hence $\| \widehat{T} \| = \| T \|$. $\square$

If T is invertible, we can prove the following stronger result.

Theorem 2.3.9. *If $T \in \mathcal{L}(H)$ is invertible, then $\| \widehat{T}^{(n)} \| = \| T \|$ for all $n \in \mathbb{N}$.*

Proof. If T is invertible, and $T = U|T|$ is the polar decomposition of T, then U is unitary. Also $\widehat{T} = U^*\, T\, U$. By theorem 2.3.4, $\widehat{T}^{(n)} = (U^*)^n\ T\ U^n$ for all $n \in \mathbb{N}$. Therefore, $U^n\ \widehat{T}^{(n)}\ (U^*)^n = T$. So, $\| T \| \leq \| \widehat{T}^{(n)} \|$. But $\| \widehat{T}^{(n)} \| \leq \|T\|$. $\square$

If we apply the following lemma from [16], we can prove theorem 2.3.11 which is much more general than lemma 2.3.8.

Lemma 2.3.10 ([16]). *If $T = U|T|$ is the polar decomposition of T, then for every $f \in Hol(\sigma(T))$, we have $f(T)U = U f(\widehat{T})$.*

Theorem 2.3.11. *If $T = U|T|$ is the polar decomposition of T and if U is a coisometry, then for every $f \in Hol(\sigma(T))$, we have $\| f(\widehat{T}) \| = \| f(T) \|$.*

Proof. By theorem 2.1.9, $\| f(\widehat{T}) \| \leq \| f(T) \|$. On the other hand, we have by lemma 2.3.10, $f(T)U = U f(\widehat{T})$. Therefore, $U\, f(\widehat{T})\, U^* = f(T)\, U\, U^* = f(T)$. So, $\| f(T) \| = \| U\, f(\widehat{T})\, U^* \| \leq \| f(\widehat{T}) \|$. $\square$

Corollary 2.3.12. *If $T \in \mathcal{L}(H)$ is invertible, then $\| f(\widehat{T}) \| = \| f(T) \|$ for all $f \in Hol(\sigma(T))$.*

Theorem 2.3.13. *If $T = U|T|$ is the polar decomposition of T, and U is coisometry, then the map $\widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widehat{T}}$ defined by $\widehat{\Phi}(f(T)) = f(\widehat{T}),\ \ f \in Hol(\sigma(T))$ is an isometry.*

Proof. By theorem 2.3.11,

$$\| \widehat{\Phi}(f(T)) \|=\| f(\widehat{T}) \|=\| f(T) \|,$$

for all $f \in Hol(\sigma(T))$. □

Corollary 2.3.14. *If $T \in \mathcal{L}(H)$ is invertible, then the map $\widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widehat{T}}$ defined by $\widehat{\Phi}(f(T)) = f(\widehat{T}),\ \ f \in Hol(\sigma(T))$ is an isometry.*

Let $T \in \mathcal{L}(H)$ be invertible. By an application of lemma 2.3.1, we see that $\widehat{T}^{(n)}$ is invertible for all $n \in \mathbb{N}$. Also, $\sigma(T) = \sigma\left(\widehat{T}^{(n)}\right)$ for all $n \in \mathbb{N}$. So by applying 2.3.12 inductively, we can prove the following result.

Theorem 2.3.15. *If $T \in \mathcal{L}(H)$ is invertible, then $\| f(\widehat{T}^{(n)}) \|=\| f(T) \|$ for all $n \in \mathbb{N}$ and for all $f \in Hol(\sigma(T))$.*

Remark 2.3.16. If T is invertible, then by theorem 2.3.9, $\| \widehat{T}^{(n)} \|=\| T \|$ for all $n \in \mathbb{N}$, and hence $\{\| \widehat{T}^{(n)} \|\}$ is a constant sequence converging to $\| T \|$. Thus if T is any invertible non-normaloid, then the sequence $\{\| \widehat{T}^{(n)} \|\}$ cannot converge to the spectral radius $r(T)$. Referring back to the section 2.2.3, notice that the operator considered in example 2.2.19 was invertible and non-normaloid. For the operator in the example, we proved constructively that the norms of the Duggal iterates do not converge to the spectral radius. Now we realize that it was not accidental, and it is the case with every invertible non-normaloid.

Remark 2.3.17. Theorem 2.3.9 says that if T is invertible, then $\| \widehat{T}^{(n)} \|=\| T \|$ for all $n \in \mathbb{N}$. But the condition that $\| \widehat{T}^{(n)} \|=\| T \|$ for all n, does not imply T is invertible. It does not even imply that U is a coisometry. Consider the following example.

Example 2.3.18. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then the matrix A is not invertible. Since A is self-adjoint, $\widehat{A}^{(n)} = A$ for all n, and therefore, $\| \widehat{A}^{(n)} \|=\| A \|$ for all n.

The polar decomposition of A is $A = U|A|$, where the partial isometry $U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $|A| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Notice that U is not even a coisometry.

Lemma 2.3.19. *Let $T = U|T|$ be the polar decomposition of T. If U is a coisometry, then for every $n \times n$ matrix (f_{ij}) with $f_{ij} \in Hol(\sigma(T))$, we have* $\| (f_{ij}(\widehat{T})) \| = \| (f_{ij}(T)) \|$.

Proof. Let (f_{ij}) be an $n \times n$ matrix with $f_{ij} \in Hol(\sigma(T))$. By lemma 2.3.10, $f_{ij}(T)U = Uf_{ij}(\widehat{T})$ for all i, j. Therefore, $f_{ij}(T) = Uf_{ij}(\widehat{T})U^*$ for all i, j. Thus $(f_{ij}(T)) = (Uf_{ij}(\widehat{T})U^*)$. Therefore,

$$\begin{aligned}
\| (f_{ij}(T)) \| &= \| (Uf_{ij}(\widehat{T})U^*) \| \\
&= \left\| \begin{bmatrix} U & 0 & \cdots & 0 \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U \end{bmatrix} (f_{ij}(\widehat{T})) \begin{bmatrix} U^* & 0 & \cdots & 0 \\ 0 & U^* & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U^* \end{bmatrix} \right\| \\
&\leq \left\| \begin{bmatrix} U & 0 & \cdots & 0 \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U \end{bmatrix} \right\| . \left\| (f_{ij}(\widehat{T})) \right\| . \left\| \begin{bmatrix} U^* & 0 & \cdots & 0 \\ 0 & U^* & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U^* \end{bmatrix} \right\|
\end{aligned}$$

But the above diagonal matrices have norm less than or equal to 1. (For example, let $R = \begin{bmatrix} U & 0 & \cdots & 0 \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U \end{bmatrix}$. Then $R \in \mathcal{M}_n(\mathcal{L}(\mathcal{H}))$, $RR^* = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

Therefore, $\| RR^* \| = 1$. So, $\| R^* \|^2 = \| R \|^2 = 1$. Therefore, $\| R \| = 1$.) Thus

$$\| (f_{ij}(T)) \| \leq \| (f_{ij}(\widehat{T})) \| .$$

On the other hand, by theorem 2.1.9,

$$\| (f_{ij}(\widehat{T})) \| \leq \| (f_{ij}(T)) \| .$$

Thus $\| (f_{ij}(\widehat{T})) \| = \| (f_{ij}(T)) \|$. □

Theorem 2.3.20. *If $T = U|T|$ is the polar decomposition of T and U a coisometry, then the map $\widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widehat{T}}$ defined by $\widehat{\Phi}(f(T)) = f(\widehat{T}),\ \ f \in Hol(\sigma(T))$ is a complete isometry.*

Proof. By lemma 2.3.19,

$$\| (f_{ij}(\widehat{T})) \| = \| (f_{ij}(T)) \| .$$

for every $n \times n$ matrix (f_{ij}) where $f_{ij} \in Hol(\sigma(T))$. In other words, the equality $\| \widehat{\Phi}_n(f_{ij}) \| = \| (f_{ij}) \|$ holds for every positive integer n and for every $n \times n$ matrix (f_{ij}) where $f_{ij} \in Hol(\sigma(T))$. Thus for all positive integers n, $\| \widehat{\Phi}_n \| = 1$, where $\widehat{\Phi}_n$ denotes the n^{th} amplification of $\widehat{\Phi}$. Hence, $\widehat{\Phi}$ is a complete isometry. □

Corollary 2.3.21. *If T is invertible, then the map $\widehat{\Phi}$ defined as above is a complete isometry.*

Recall the definitions of spectral set and complete spectral set, given in section 1.5. Let X be a closed proper subset of $\mathbb{C}$, and let $\check{X}$ denote the closure of X, when we regard X as a subset of the Riemann sphere $\mathfrak{S}$. We let $\mathcal{R}(X)$ denote the quotients of polynomials with poles off $\check{X}$, that is, the bounded, rational functions on X with a limit at ∞. We regard $\mathcal{R}(X)$ as a subalgebra of the

C^*-algebra $C(\partial\check{X})$, which defines norms on $\mathcal{R}(X)$ and each $\mathcal{M}_n(\mathcal{R}(X))$.

If X is a closed, proper subset of $\mathbb{C}$, and $T \in \mathcal{L}(\mathcal{H})$, with $\sigma(T) \subset X$, then there is a functional calculus, ie., a homomorphism $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$, given by $\rho(f) = f(T)$, where $f(T) = p(T)q(T)^{-1}$ if $f = p/q$. If $\| \rho \| \leq 1$, then X is called a spectral set for T. If $\| \rho \|_{cb} \leq 1$, then X is called a complete spectral set for T.

Now let us introduce the concept of n-level spectral sets. They are discussed in more detail in section 4.3 on page 89.

Definition 2.3.22 (n-level spectral set)**.** Let n be a positive integer. Let X, ρ, and T be as discussed above. If $\|\rho_n\| \leq 1$, where ρ_n denotes the n^{th} amplification of ρ, then we say that X is an *n-level spectral set* for T.

Lemma 2.3.23. *If $T = U|T|$ is the polar decomposition of T, and U is a coisometry, then T and $\widehat{T}$ have the same collection of spectral sets.*

Proof. By theorem 2.3.11,

$$\| f(\widehat{T}) \| = \| f(T) \|,$$

for all $f \in Hol(\sigma(T))$. If X is a closed set in the complex plane such that $\sigma(T) \supset X$, then $\mathcal{R}(X)$ is a subalgebra of $Hol(\sigma(T))$. Hence

$$\| f(\widehat{T}) \| = \| f(T) \|,$$

for all $f \in \mathcal{R}(X)$. Therefore, T and $\widehat{T}$ have the same collection of spectral sets. □

Corollary 2.3.24. *If T is invertible, then T and $\widehat{T}$ have the same collection of spectral sets.*

Let $T = U|T|$ be the polar decomposition of T. If U is a coisometry, then lemma 2.3.19 says that for every $n \times n$ matrix (f_{ij}) with $f_{ij} \in Hol(\sigma(T))$, we have $\| (f_{ij}(\widehat{T})) \| = \| (f_{ij}(T)) \|$. Applying this we get the following result.

Lemma 2.3.25. *If $T = U|T|$ is the polar decomposition of T, and U is coisometry, then T and $\widehat{T}$ have the same collection of complete spectral sets. Also, for every fixed positive integer n, the operators T and $\widehat{T}$ have the same collection of $n-$level spectral sets.*

Theorem 2.3.26. *Let $T \in \mathcal{L}(H)$ be an invertible operator. If for some n, the n^{th} Duggal iterate $\widehat{T}^{(n)}$ is normal, then T is normaloid. In fact, $f(T)$ is normaloid for every $f \in \mathcal{R}(\sigma(T))$.*

Proof. By theorem 2.3.15, $\| f(\widehat{T}^{(n)}) \| = \| f(T) \|$ for all $f \in Hol(\sigma(T))$. If X is a closed set in the complex plane such that $\sigma(T) \supset X$, then $\mathcal{R}(X)$ is a subalgebra of $Hol(\sigma(T))$. It follows that $\widehat{T}^{(n)}$ and T have the same collection of spectral sets. Since $\widehat{T}^{(n)}$ is normal, $\sigma(\widehat{T}^{(n)})$ is a spectral set for $\widehat{T}^{(n)}$. But $\sigma(\widehat{T}^{(n)}) = \sigma(T)$. Thus $\sigma(T)$ is a spectral set for T. By a theorem in [7], $\sigma(T)$ is a spectral set for T if and only if $f(T)$ is normaloid for every $f \in \mathcal{R}(\sigma(T))$. In particular, T is normaloid. □

2.3.3 Continuity of the maps $T \to \widetilde{T}$ and $T \to \widehat{T}$

Ken Dykema and Hanne Schultz proved in [14] that the Aluthge transformation map $T \to \widetilde{T}$ is continuous on $\mathcal{L}(\mathcal{H})$. This result can be seen in [30] also. In this section we examine the continuity of the Duggal transformation map $T \to \widehat{T}$. The method of proof in [14] to prove the Aluthge transformation map $T \to \widetilde{T}$ is continuous (which uses continuous functional calculus), does not readily translate to the context of Duggal transformations. So we examine the continuity of the Duggal transformation map $T \to \widehat{T}$ on the set of invertible operators in $\mathcal{L}(\mathcal{H})$ and prove that the map is continuous on the set of invertible operators. As a consequence we show that the sequence of the Duggal iterates of an invertible operator T converges to an invertible operator if and only if T is quasinormal.

Further we obtain some results regarding the relation between the spectral sets of an operator and the spectral sets of the limit of the sequence of the Duggal iterates.

Theorem 2.3.27. [14]

i. Given $R \geq 1$ and $\epsilon > 0$, there are real polynomials p and q such that for every $T \in \mathcal{L}(\mathcal{H})$ with $\| T \| \geq R$, we have $\| \widetilde{T} - p(T^*T)Tq(T^*T) \| < \epsilon$.

ii. For every $T \in \mathcal{L}(\mathcal{H})$, the Aluthge transformation $\widetilde{T}$ of T belongs to the C^*-algebra generated by T and the identity.

Theorem 2.3.28. [14] *The Aluthge transformation map $T \to \widetilde{T}$ is $(\| . \| , \| . \|)$ continuous on $\mathcal{L}(\mathcal{H})$.*

Remark 2.3.29. Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that the Aluthge transformation sequence $\{\widetilde{T}^{(n)}\}$ is convergent and that $\widetilde{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$. Then S is quasinormal. (The fact that S is quasinormal can be proved as follows. Define $\Delta(T) = \widetilde{T}$ for all $T \in \mathcal{L}(\mathcal{H})$. By theorem 2.3.28, the map $\Delta : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is continuous. Therefore, $\Delta(\widetilde{T}^{(n)}) \to \Delta(S)$ as $n \to \infty$. But $\Delta(\widetilde{T}^{(n)}) = \widetilde{T}^{(n+1)}$, and $\widetilde{T}^{(n+1)} \to S$ as $n \to \infty$. Hence $\Delta(S) = S$. ie., $\widetilde{S} = S$. Thus S is quasinormal).

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Every quasinormal operator on $\mathcal{H}$ is normal (see remarks after 1.2.11). Hence if $T \in \mathcal{L}(\mathcal{H})$, and $\widetilde{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$, then S is normal.

Definition 2.3.30. Let $\mathcal{H}$ be a Hilbert space. Let $\Delta : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ and $\Gamma : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be defined by

$$\Delta(T) = \widetilde{T}, \quad \Gamma(T) = \widehat{T} \text{ for } T \in \mathcal{L}(\mathcal{H}).$$

Let $\mathcal{D}$ be a subset of $\mathcal{L}(\mathcal{H})$. We say that $\mathcal{D}$ is a *Duggal continuity family* if the map $\Gamma_{|\mathcal{D}} : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is continuous.

Notice that by theorem 2.3.28, the Aluthge transformation map Δ is continuous on all of $\mathcal{L}(\mathcal{H})$.

Our next aim is to prove that the set of all invertible operators in $\mathcal{L}(\mathcal{H})$ is a Duggal continuity family, or in other words, our aim is to prove that the map $T \to \widehat{T}$ is continuous on the set of invertible operators.

Theorem 2.3.31. [11] *Let $\mathcal{A}$ be a unital complex C^*-algebra. Let A, B be subsets of $\mathbb{C}$ and $f : A \to B$ a homeomorphism. Put $\mathcal{A}_0 = \{x \in \mathcal{N}_\mathcal{A} : \sigma(x) \subset A\}$, $\mathcal{B}_0 = \{x \in \mathcal{N}_\mathcal{A} : \sigma(x) \subset B\}$ where $N_\mathcal{A}$ denotes the set of all normal elements of the C^*-algebra $\mathcal{A}$. Then $\{f(x) : x \in \mathcal{A}_0\} = \mathcal{B}_0$, and the map $x \to f(x) : \mathcal{A}_0 \to \mathcal{B}_0$ is a homeomorphism.*

Notice that $f \in C(\sigma(x))$ for every $x \in \mathcal{A}_0$, and hence by the continuous functional calculus on C^*-algebras, $f(x)$ is defined as an element in $\mathcal{A}$.

If $A = B = \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+ \backslash \{0\}$, then $f : A \to B$ defined by $f(t) = t^\alpha$ is a homeomorphism. Here, $\mathcal{A}_0 = \mathcal{B}_0 = \{x \in \mathcal{N}_\mathcal{A} : \sigma(x) \subset \mathbb{R}_+\} = \mathcal{A}_+$, the set of all positive elements in $\mathcal{A}$. Hence by the above theorem, the map $x \to x^\alpha : \mathcal{A}_+ \to \mathcal{A}_+$ is a homeomorphism.

If $x \in \mathcal{A}$, then $|x| = (x^*x)^{1/2}$. Since the maps $x \to x^*x : \mathcal{A} \to \mathcal{A}_+$ and $x \to x^{1/2} : \mathcal{A}_+ \to \mathcal{A}_+$ are continuous, the theorem below follows.

Theorem 2.3.32. [11] *Let $\mathcal{A}$ be a unital complex C^*-algebra and $\mathcal{A}_+$ be the set of all positive elements in $\mathcal{A}$. The map*

$$x \to |x| : \mathcal{A} \to \mathcal{A}_+$$

is continuous.

Theorem 2.3.33. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}$ be the set of all invertible operators in $\mathcal{L}(\mathcal{H})$. Then $\mathcal{S}$ is a Duggal continuity family. In other words, the map $T \to \widehat{T}$ is continuous on $\mathcal{S}$.*

Proof. If $T \in \mathcal{L}(\mathcal{H})$, let $T = \theta(T)\mu(T)$ be the polar decomposition of T. We have, $\mu(T) = |T| = (T^*T)^{1/2}$. If $T \in \mathcal{S}$, then $\mu(T) \in \mathcal{S}$, and in this case, $\theta(T) = T\mu(T)^{-1}$.

By theorem 2.3.32, the map $T \to \mu(T)$ is continuous on $\mathcal{L}(\mathcal{H})$. Also, the inversion map $S \to S^{-1} : \mathcal{S} \to \mathcal{L}(\mathcal{H})$ is continuous. Hence the map $\theta : \mathcal{S} \to \mathcal{L}(\mathcal{H})$ is continuous. If $T \in \mathcal{L}(\mathcal{H})$, then $\widehat{T} = \mu(T)\theta(T)$.

Suppose that $\{S_n\}$ is a sequence in $\mathcal{S}$ such that $S_n \to S$ in $\mathcal{S}$ as $n \to \infty$. Since $\theta : \mathcal{S} \to \mathcal{L}(\mathcal{H})$ is continuous, and the map $T \to \mu(T)$ is continuous on $\mathcal{L}(\mathcal{H})$, we see that $\theta(S_n) \to \theta(S)$ and $\mu(S_n) \to \mu(S)$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$. Therefore, $\widehat{S_n} \to \widehat{S}$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$. Thus the map $T \to \widehat{T}$ is continuous on $\mathcal{S}$. In other words, $\Gamma_{|\mathcal{S}} : \mathcal{S} \to \mathcal{L}(\mathcal{H})$ is continuous. Hence $\mathcal{S}$ is a Duggal continuity family. □

Theorem 2.3.34. *Let $\mathcal{H}$ be a Hilbert space. Suppose that $\mathcal{D}$ is a Duggal continuity family in $\mathcal{L}(\mathcal{H})$. Let $T \in \mathcal{D}$ be such that*

i. $\widehat{T}^{(n)} \in \mathcal{D}$ *for all* $n \in \mathbb{N}$.

ii. $\widehat{T}^{(n)} \to S$ *in* $\mathcal{L}(\mathcal{H})$ *as* $n \to \infty$.

iii. $S \in \mathcal{D}$.

Then S is quasinormal.

Proof. Since $\mathcal{D}$ is a Duggal continuity family, the Duggal transformation map

$\Gamma_{|\mathcal{D}} : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is continuous. Therefore,

$$\begin{aligned}\Gamma(S) &= \Gamma\left(lim_{n\to\infty}\widehat{T}^{(n)}\right)\\ &= lim_{n\to\infty}\Gamma\left(\widehat{T}^{(n)}\right)\\ &= lim_{n\to\infty}\widehat{T}^{(n+1)}\\ &= S.\end{aligned}$$

Hence, S is quasinormal. □

Corollary 2.3.35. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Suppose that $\mathcal{D}$ is a Duggal continuity family in $\mathcal{L}(\mathcal{H})$. Let $T \in \mathcal{D}$ be such that*

i. $\widehat{T}^{(n)} \in \mathcal{D}$ *for all* $n \in \mathbb{N}$.

ii. $\widehat{T}^{(n)} \to S$ *in* $\mathcal{L}(\mathcal{H})$ *as* $n \to \infty$.

iii. $S \in \mathcal{D}$.

Then S is normal.

Proof. On finite dimensional Hilbert spaces every quasinormal operator is normal (see remarks after 1.2.11 on page 15). Therefore, by theorem 2.3.34, the proof follows. □

Theorem 2.3.36. *Let $T \in \mathcal{L}(\mathcal{H})$ be such that*

(i) T is invertible

(ii) $\{\widehat{T}^{(n)}\}$ converges to $S \in \mathcal{L}(\mathcal{H})$ such that S is invertible.

Then S is quasinormal.

Proof. Since T is invertible, $\widehat{T}^{(n)}$ is invertible for every n. By theorem 2.3.33, the set $\mathcal{S}$ of invertible operators in $\mathcal{L}(\mathcal{H})$ is a Duggal continuity family. By theorem 2.3.34, S is quasinormal. $\square$

Corollary 2.3.37. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let $T \in \mathcal{L}(\mathcal{H})$ be such that*

(i) T is invertible

(ii) $\{\widehat{T}^{(n)}\}$ converges to $S \in \mathcal{L}(\mathcal{H})$ such that S is invertible.

Then S is normal.

Lemma 2.3.38. *Let $U, S \in \mathcal{L}(\mathcal{H})$ and assume that U is unitary. Then*

i. S is normal if and only if U^*SU is normal.

ii. S is quasinormal if and only if U^*SU is quasinormal.

Proof. S normal $\Rightarrow$ $S^*S = SS^*$ $\Rightarrow$ $(U^*SU)^*(U^*SU) = U^*S^*UU^*SU = U^*S^*SU = U^*SS^*U = U^*SUU^*S^*U = (U^*SU)(U^*SU)^* \Rightarrow$ U^*SU normal.

S quasinormal $\Rightarrow$ $S(S^*S) = (S^*S)S \Rightarrow$ $(U^*SU)[(U^*SU)^*(U^*SU)] = U^*SUU^*S^*UU^*SU = U^*S(S^*S)U = U^*(S^*S)SU = U^*S^*UU^*SUU^*SU = [(U^*SU)^*(U^*SU)](U^*SU) \Rightarrow$ U^*SU quasinormal.

On the other hand, since U^* is unitary, U^*SU normal (quasinormal) $\Rightarrow$ $(U^*)^*(U^*SU)U^*$ normal (quasinormal) $\Rightarrow$ S normal (quasinormal). $\square$

Theorem 2.3.39. *Let $\mathcal{N}$ be the set of all normal operators and $\mathcal{Q}$ be the set of all quasinormal operators in $\mathcal{L}(\mathcal{H})$. Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then*

i. T and $\widehat{T}$ are at the same distance from $\mathcal{N}$.

ii. T and $\widehat{T}$ are at the same distance from $\mathcal{Q}$.

Proof. Let $T = U|T|$ be the polar decomposition of T. By lemma 2.3.2, $\widehat{T} = U^*TU$. Since T is invertible, U is unitary. Therefore, $\| U^*RU \| = \| R \|$ for every $R \in \mathcal{L}(\mathcal{H})$. By lemma 2.3.38, $\mathcal{N} = \{U^*SU : S \in \mathcal{N}\}$ and $\mathcal{Q} = \{U^*SU : S \in \mathcal{Q}\}$. Therefore,

$$\begin{aligned}
\operatorname{dist}(\widehat{T}, \mathcal{N}) &= \inf \{\| \widehat{T} - S \| : S \in \mathcal{N}\} \\
&= \inf \{\| \widehat{T} - U^*SU \| : S \in \mathcal{N}\} \\
&= \inf \{\| U^*TU - U^*SU \| : S \in \mathcal{N}\} \\
&= \inf \{\| U^*(T - S)U \| : S \in \mathcal{N}\} \\
&= \inf \{\| T - S \| : S \in \mathcal{N}\} \\
&= \operatorname{dist}(T, \mathcal{N}).
\end{aligned}$$

Similarly, $\operatorname{dist}(\widehat{T}, \mathcal{Q}) = \operatorname{dist}(T, \mathcal{Q})$. □

Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then $\{\widehat{T}^{(n)}\}_{n=0}^{\infty}$ is a sequence of invertible operators. The following theorem shows that this sequence can converge to an invertible operator in $\mathcal{L}(\mathcal{H})$ only when T is quasinormal. Notice that if T is quasinormal then $\widehat{T}^{(n)} = T$ for all n.

Theorem 2.3.40. *Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then $\{\widehat{T}^{(n)}\}_{n=0}^{\infty}$ converges to an invertible operator in $\mathcal{L}(\mathcal{H})$ if and only if T is quasinormal.*

Proof. If T is quasinormal then $\widehat{T}^{(n)} = T$ for all n, and therefore one part of the proof is trivial.

Conversely, suppose that $\{\widehat{T}^{(n)}\}_{n=0}^{\infty}$ converges to S in $\mathcal{L}(\mathcal{H})$, and assume that S is invertible. Let $\mathcal{Q}$ be the set of all quasinormal operators in $\mathcal{L}(\mathcal{H})$. Then $\mathcal{Q}$ is a closed subset of $\mathcal{L}(\mathcal{H})$. By theorem 2.3.36, $S \in \mathcal{Q}$. By repeated application of theorem 2.3.39, we see that dist $(T, \mathcal{Q}) =$ dist $(\widehat{T}^{(n)}, \mathcal{Q})$ for all n. Therefore,

$$\begin{aligned} \text{dist } (T, \mathcal{Q}) &= \text{dist } (\widehat{T}^{(n)}, \mathcal{Q}) \\ &\leq \parallel \widehat{T}^{(n)} - S \parallel \end{aligned}$$

for every n. Since $\widehat{T}^{(n)} \to S$ as $n \to \infty$, it follows that dist $(T, \mathcal{Q}) = 0$. Since $\mathcal{Q}$ is closed, $T \in \mathcal{Q}$. □

Corollary 2.3.41. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then $\{\widehat{T}^{(n)}\}_{n=0}^{\infty}$ converges to an invertible operator in $\mathcal{L}(\mathcal{H})$ if and only if T is normal.*

Remark 2.3.42. The following is another form of the corollary 2.3.41 and it appeared in [5]. If $\mathcal{G}l_r(\mathbb{C})$ is the general linear group of $r \times r$ invertible complex matrices and $T \in \mathcal{G}l_r(\mathbb{C})$, then the sequence $\{\widehat{T}^{(n)}\}$ can not converge (in $\mathcal{G}l_r(\mathbb{C})$), unless T is normal.

Definition 2.3.43. Let $T \in \mathcal{L}(\mathcal{H})$. Let X be a closed proper subset of $\mathbb{C}$ with $\sigma(T) \subset X$. Let f be a rational function with poles off $\check{X}$. We shall say that $f \in \mathcal{R}_{(X,T)}$ if it satisfies the condition that $f(\widehat{T}^{(n)}) \to f(S)$ in $\mathcal{L}(\mathcal{H})$ whenever $\widehat{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$.

If p is any polynomial, then the map $A \to p(A)$ is continuous on $\mathcal{L}(\mathcal{H})$. Therefore, if $T \in \mathcal{L}(\mathcal{H})$ is any operator, and if X is any closed proper subset of $\mathbb{C}$ with $\sigma(T) \subset X$, then $p \in \mathcal{R}_{(X,T)}$.

Theorem 2.3.44. *Let $T \in \mathcal{L}(\mathcal{H})$ be invertible, and X be any closed proper subset of $\mathbb{C}$ such that $\sigma(T) \subset X$ and $\mathcal{R}_{(X,T)} = \mathcal{R}(X)$. Suppose that $\widehat{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$, and that $\sigma(S) \subset X$. If X is a spectral set for T, then X is a spectral set for S.*

Proof. Let $f \in \mathcal{R}(X)$. Then $f \in Hol(\sigma(T))$. Since T is invertible, by theorem 2.3.15, $\parallel f(\widehat{T}^{(n)}) \parallel = \parallel f(T) \parallel$ for all $n \in \mathbb{N}$. Since $f \in \mathcal{R}_{(X,T)}$, we have

$f(\widehat{T}^{(n)}) \to f(S)$ in $\mathcal{L}(\mathcal{H})$ as $n \to \infty$. Therefore, $\| f(\widehat{T}^{(n)}) \| \to \| f(S) \|$ as $n \to \infty$. Hence $\| f(S) \| = \| f(T) \|$. This proves the theorem. □

The following theorem talks about the upper semi-continuity of the spectrum. We use this theorem to prove the useful result in theorem 2.3.46

Theorem 2.3.45. [28] *Suppose $\mathcal{A}$ is a Banach algebra, $x \in \mathcal{A}$, Ω is an open set in $\mathbb{C}$, and $\sigma(x) \subset \Omega$. Then there exists $\delta > 0$ such that $\sigma(x+y) \subset \Omega$ for every $y \in \mathcal{A}$ with $\| y \| < \delta$.*

Theorem 2.3.46. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. If $\{\widehat{T}^{(n)}\}$ converges to $S \in \mathcal{L}(\mathcal{H})$, then $Hol(\sigma(S)) \subset Hol(\sigma(T))$.*

Proof. Let Ω be an open set in $\mathbb{C}$ with $\Omega \supset \sigma(S)$. By theorem 2.3.45, there exists $\delta > 0$ such that $\sigma(S+R) \subset \Omega$ for every $R \in \mathcal{L}(\mathcal{H})$ with $\| R \| < \delta$. Since $\widehat{T}^{(n)} \to S$ as $n \to \infty$, there exists a positive integer n_0 such that $\| \widehat{T}^{(n)} - S \| < \delta$ for all $n \geq n_0$. Hence $\sigma(S + \widehat{T}^{(n_0)} - S) \subset \Omega$. ie., $\sigma(\widehat{T}^{(n_0)}) \subset \Omega$. But $\sigma(\widehat{T}^{(n_0)}) = \sigma(T)$. So $\sigma(T) \subset \Omega$. Thus if Ω is an open set in $\mathbb{C}$ with $\Omega \supset \sigma(S)$, then $\Omega \supset \sigma(T)$.

Now, $f \in Hol(\sigma(S)) \Rightarrow f$ is holomorphic on an open set Ω that contains $\sigma(S) \Rightarrow f$ is holomorphic on an open set Ω that contains $\sigma(T) \Rightarrow f \in Hol(\sigma(T))$. □

Remark 2.3.47. The analogue of theorem 2.3.46 is true in the case of Aluthge transformations. The proof uses the fact that $\sigma(\widetilde{T}^{(n)}) = \sigma(T)$ for all n. We state the result in the following theorem.

Theorem 2.3.48. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. If $\{\widetilde{T}^{(n)}\}$ converges to $S \in \mathcal{L}(\mathcal{H})$, then $Hol(\sigma(S)) \subset Hol(\sigma(T))$.*

Theorem 2.3.49. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator. Suppose that $\widehat{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$, and X is a closed proper subset of $\mathbb{C}$ such that X contains a neighborhood of the spectrum $\sigma(S)$. Further, assume that*

$f(\widehat{T}^{(n)}) \to f(S)$ for all $f \in \mathcal{R}(X)$. Then X is a spectral set for T if and only if X is a spectral set for S.

Proof. As in the first paragraph of the proof of theorem 2.3.46, we see that X contains a neighborhood of $\sigma(T)$. Now, let $f \in \mathcal{R}(X)$. Then $f \in Hol(\sigma(S)) \subset Hol(\sigma(T))$. Since T is invertible, by theorem 2.3.15, $\| f(\widehat{T}^{(n)}) \| = \| f(T) \|$ for all $n \in \mathbb{N}$. It follows that $\| f(T) \| = \| f(S) \|$. Hence, X is a spectral set for T if and only if X is a spectral set for S. □

2.4 Contractivity and positivity of the maps $f(T) \to f(\widetilde{T})$ and $f(T) \to f(\widehat{T})$

Let $\mathcal{H}$ be an arbitrary Hilbert space whose dimension satisfies $2 \leq dim\mathcal{H} \leq \aleph_0$. If $T \in \mathcal{L}(\mathcal{H})$, let

$$\mathcal{A}_T = \{f(T) : f \in Hol(\sigma(T))\}.$$

Then $\mathcal{A}_T$ is a subalgebra of the C^*-algebra $\mathcal{L}(\mathcal{H})$. In [16], Foias, Jung, Ko, and Pearcy proved that the maps $f(T) \to f(\widetilde{T})$ and $f(T) \to f(\widehat{T})$ are completely contractive algebra homomorphisms from $\mathcal{A}_T$ onto $\mathcal{A}_{\widetilde{T}}$ and from $\mathcal{A}_T$ onto $\mathcal{A}_{\widehat{T}}$ respectively. Also, these maps are unital.

Let $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. In this case $\mathcal{A}_T$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$ and therefore is a subspace, that is, a closed linear manifold. In such cases the set $\mathcal{A}_T + (\mathcal{A}_T)^*$ is an operator system in the C^*-algebra $\mathcal{L}(\mathcal{H})$.

Notice that there are operators in $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. For example, if $\mathcal{H}$ is finite dimensional, then $\mathcal{A}_T$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$ for every $T \in \mathcal{L}(\mathcal{H})$.

If $T = I$, the identity operator on any Hilbert space $\mathcal{H}$, then for every function

$f \in Hol(\sigma(T))$,

$$\begin{aligned} f(T) &= f(I) \\ &= \frac{1}{2\pi i}\int_C f(z)(zI - I)^{-1}dz \quad \text{where } C \text{ is a smooth closed curve whose} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{interior contains } \sigma(I) \\ &= \frac{1}{2\pi i}\int_C f(z)(z-1)^{-1}\ I\ dz \\ &= I\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-1)}dz \\ &= I\ \frac{1}{2\pi i}\ f(1)\ 2\pi i \quad \text{since } 1 \in \sigma(I) \\ &= f(1)\, I, \end{aligned}$$

which shows that $\mathcal{A}_T = \mathcal{C}_{\mathcal{H}}$, where $\mathcal{C}_{\mathcal{H}}$ denotes the set of all scalar operators in $\mathcal{L}(\mathcal{H})$. Obviously, $\mathcal{C}_{\mathcal{H}}$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$.

We use the following two results from [26] to prove some consequences.

Theorem 2.4.1 ([26]). *Let A be a unital C^*-algebra and let M be a subspace of A containing 1. If B is a unital C^*-algebra and $\phi : M \to B$ is a unital contraction, then ϕ extends uniquely to a positive map $\tilde{\phi} : M + M^* \to B$ with $\tilde{\phi}$ given by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$.*

Theorem 2.4.2 ([26]). *If S is an operator system in a unital C^*-algebra A, B a unital C^*-algebra and if $\phi : S \to B$ is a unital positive map, then ϕ is self-adjoint.*

Theorem 2.4.3. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. If $f, g \in Hol(\sigma(T))$ such that $(f(T))^* = g(T)$, then $(f(\widehat{T}))^* = g(\widehat{T})$, $\quad (f(\widetilde{T}))^* = g(\widetilde{T})$.*

Proof. Being a closed subalgebra of $\mathcal{L}(\mathcal{H})$, the set $\mathcal{A}_T$ is a subspace of the unital C^*-algebra $\mathcal{L}(\mathcal{H})$ and $I \in \mathcal{A}_T$. By theorem 2.1.9, the maps $\widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widehat{T}}$ and

$\widetilde{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\widetilde{T}}$ defined by

$$\widehat{\Phi}(h(T)) = h(\widehat{T}),\ \widetilde{\Phi}(h(T)) = h(\widetilde{T}),\ h \in Hol(\sigma(T))$$

are well-defined and contractive. Also $\widehat{\Phi}$ and $\widetilde{\Phi}$ are unital (Let h be the constant polynomial $h(z) = 1$. Then $h \in Hol(\sigma(T)) = Hol(\sigma(\widetilde{T})) = Hol(\sigma(\widehat{T}))$ and $h(T) = h(\widehat{T}) = h(\widetilde{T}) = I$). Since $\widehat{\Phi}$ is a unital contraction of the algebra $\mathcal{A}_T$ into the algebra $\mathcal{A}_{\widehat{T}} \subset \mathcal{L}(\mathcal{H})$, by theorem 2.4.1, $\widehat{\Phi}$ extends uniquely to a positive map $\widehat{\Psi} : \mathcal{A}_T + (\mathcal{A}_T)^* \to \mathcal{L}(\mathcal{H})$ defined by $\widehat{\Psi}(f(T) + g(T)^*) = \widehat{\Phi}(f(T)) + (\widehat{\Phi}(g(T)))^*$. Since $\mathcal{A}_T + (\mathcal{A}_T)^*$ is an operator system, by theorem 2.4.2, $\widehat{\Psi}$ is self-adjoint. Therefore,

$$\begin{aligned} (f(\widehat{T}))^* &= (\widehat{\Phi}(f(T)))^* \\ &= (\widehat{\Psi}(f(T)))^* \\ &= \widehat{\Psi}((f(T))^*) \\ &= \widehat{\Psi}(g(T)) \\ &= \widehat{\Phi}(g(T)) \\ &= g(\widehat{T}). \end{aligned}$$

The proof goes similar in the case of Aluthge transformations. □

Corollary 2.4.4. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. If $T^* = g(T)$ for some $g \in Hol(\sigma(T))$, then $(\widehat{T})^* = g(\widehat{T})$ and $(\widetilde{T})^* = g(\widetilde{T})$.*

Proof. Apply theorem 2.4.3 taking $f \in Hol(\sigma(T))$ defined by $f(z) = z$. □

Remark 2.4.5. Note that $\mathcal{A}_T$ is a commutative algebra. The fact that $\mathcal{A}_T$ is a commutative algebra can be proved as follows: By the definition of the holomorphic functional calculus in 1.3.6, the mapping $f \to f(T) : Hol(\sigma(T)) \to \mathcal{L}(\mathcal{H})$

is an algebra homomorphism, and the range of this homomorphism is $\mathcal{A}_T$. The function algebra $Hol(\sigma(T))$ is commutative. Therefore, the algebra $\mathcal{A}_T$ is commutative. Thus if $T^* = g(T)$ for some $g \in Hol(\sigma(T))$, then since both T and T^* belong to $\mathcal{A}_T$, we see that T and T^* commute, or in other words, T is normal. So in this case $\widehat{T} = \widetilde{T} = T$. Thus the corollary 2.4.4, which we proved using theorem 2.4.3, is true even otherwise.

Chapter 3

Polar decomposition of Aluthge and Duggal transformations

3.1 Introduction

Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathcal{L}(\mathcal{H})$. Then one can think about the polar decomposition of Aluthge transformation $\widetilde{T}$, and of Duggal transformation $\widehat{T}$. In [20], Masatoshi Ito, Takeaki Yamazaki, and Masahiro Yanagida obtained results on the polar decomposition of Aluthge transformation. In [21], Ito, Yamazaki, and Yanagida showed results on the polar decomposition of the product of two operators and of Aluthge transformation. They also showed properties and characterizations of binormal and centered operators from the viewpoint of the polar decomposition and Aluthge transformation.

In [20], Ito, Yamazaki, Yanagida gave an example of a binormal, invertible operator T such that the Aluthge transformation $\widetilde{T}$ is not binormal. In this chapter, first we show that if T is a binormal, invertible operator, then the Duggal transformation $\widehat{T}$ is binormal. We discuss some consequences of applying Aluthge transformation and Duggal transformation successively on an invertible operator T. In theorem 3.2.10, we show that if T is invertible and binormal, then $\widehat{(\widetilde{T})} = \widetilde{(\widehat{T})}$. Further we extend this result to iterated Aluthge transformations and Duggal transformations. We proceed to show that if T is an invertible operator with polar decomposition $T = U\,|T|$, then the polar decomposition of $\widehat{T}$ is $\widehat{T} = U\,|\widehat{T}|$.

Let $T = U|T|$ be the polar decomposition of an operator T. A theorem in [21] says that T is binormal if and only if $\widetilde{T} = \widetilde{U}\,|\widetilde{T}|$ is the polar decomposition of the Aluthge transformation $\widetilde{T}$. We discuss the similar situation of Duggal transformations. We give necessary and sufficient condition for $\widehat{T}$ to have the polar decomposition $\widehat{T} = \widehat{U}\,|\widehat{T}|$. As a consequence, we prove that if T is binormal, then $\widehat{T} = \widehat{U}\,|\widehat{T}|$ is the polar decomposition of $\widehat{T}$. Characterization of operators $T = U|T|$, having the property that $\widetilde{T}^{(n)} = \widetilde{U}^{(n)}\,|\widetilde{T}^{(n)}|$ is the polar decomposition of $\widetilde{T}^{(n)}$ for all $n = 1, 2, \ldots$, exists. In theorem 3.2.27, we prove that the final space of U is invariant under every $|\widehat{T}^{(n)}|$, if $\widehat{T}^{(n)} = \widehat{U}^{(n)}\,|\widehat{T}^{(n)}|$ is the polar decomposition of $\widehat{T}^{(n)}$ for all $n = 1, 2, \ldots$.

In [9], Ximena Catepillan and Waclaw Szymanski proved the semigroup properties of factors in the polar decomposition of operators. In section 3.2.3, we use them to give a modification of the proof of a theorem in [21].

In [29], M. Schreiber discussed operators for which the closure of the numerical range is a spectral set. He characterized such operators by means of normal dilations. He obtained relations between the spectrality of the numerical range and the equality of the convex hull of the spectrum with the closure of the numerical range. In section 3.3, we discuss some consequences of these results on Aluthge and Duggal transformations. In theorem 3.3.6, we prove a general version of one part of Ando's theorem. We proceed to prove that if A is an $n \times n$ matrix over $\mathbb{C}$, and if the convex hull of the spectrum of A is a spectral set for the matrix A, then the matrices A, $\widetilde{A}$, and $\widehat{A}$ have the same numerical range.

3.2 Aluthge and Duggal transformations of binormal and centered operators

3.2.1 Aluthge and Duggal transformations of binormal or centered invertible operators

The following lemma shows results about Aluthge and Duggal transformation of invertible operators.

Lemma 3.2.1. *If $T \in \mathcal{L}(\mathcal{H})$ is invertible, then $\widetilde{T}$ and $\widehat{T}$ are invertible. In this case,*

$$\begin{aligned} \widetilde{T} &= |T|^{1/2} T |T|^{-1/2} \\ \widehat{T} &= |T| \, T \, |T|^{-1} \end{aligned}$$

(see lemma 2.3.1).

Definition 3.2.2. An operator T is said to be *binormal* if $[\ |T|, |T^*|\ ] = 0$, where $[\ A, B\ ] = AB - BA$. The operator T is said to be *centered* if the following sequence

$$\ldots,\ T^3(T^3)^*,\ T^2(T^2)^*,\ TT^*,\ T^*T,\ (T^2)^*T^2,\ (T^3)^*T^3,\ \ldots$$

is commutative.

Binormal and centered operators were defined by S. L. Campbell in [8] and B. B. Morrel and P. S. Muhly in [25], respectively. Relations among these classes and that of quasinormal operators are easily obtained as follows.

$$\text{quasinormal} \subset \text{centered} \subset \text{binormal}\ .$$

Theorem 3.2.3. [20] *Let $T \in \mathcal{L}(H)$, and suppose that $T = U|T|$ is the polar decomposition of T. Then $\widetilde{T} = U|\widetilde{T}|$ if and only if T is binormal. (Note that, this assertion does not mean that $\widetilde{T} = U|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$, when T is binormal).*

Theorem 3.2.4. [20] *Let $T \in \mathcal{L}(H)$, and suppose that $T = U|T|$ be the polar decomposition of T. If T is binormal, then $\widetilde{T} = U^*UU|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.*

Theorem 3.2.5. *Let $T \in \mathcal{L}(H)$ be invertible. Suppose that $T = U|T|$ is the polar decomposition of T. If T is binormal, then $\widetilde{T} = U|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.*

Proof. Since T is invertible, U is unitary. By theorem 3.2.4, the proof follows. □

If S, T and V are operators with $S = V^*TV$, and V unitary, then it can be easily verified that $|S| = V^*\,|T|\,V$ and $|S|^{1/2} = V^*\,|T|^{1/2}\,V$. Further, if the polar decomposition of T is $T = U|T|$, then the polar decomposition of S is $S = (V^*UV)\,|S|$. Hence $\widetilde{S} = V^*\widetilde{T}V$.

Remark 3.2.6. The binormality of T does not imply the binormality of the Aluthge transformation $\widetilde{T}$. The following example is from the paper of Ito, Yamazaki, and Yanagida [20]. It gives a binormal operator T such that the Aluthge transformation $\widetilde{T}$ is not binormal. We give the example here not only for the sake of completeness but also for the reason that it gives a simple example of an invertible operator which is not binormal. Notice that the operator in the example is invertible. We shall show that the analogous case of the Duggal transformations is different. We prove in theorem 3.2.9 that, if T is an invertible operator, then the binormality of the operator T implies the binormality of the Duggal transformation $\widehat{T}$.

Example 3.2.7. [20] There exists a binormal operator T such that the Aluthge transformation $\widetilde{T}$ is not binormal. Let

$$T = \begin{bmatrix} 0 & 0 & 5 \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$$

and $T = U|T|$ be the polar decomposition of T. Then T is binormal since

$$T^*T.TT^* = TT^*.T^*T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

and also

$$|T| = (T^*T)^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

so that

$$U = T|T|^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{bmatrix}.$$

Therefore,

$$\widetilde{T} = |T|^{1/2}U|T|^{1/2} = \begin{bmatrix} 0 & 0 & \sqrt{5} \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{15}/2 & -\sqrt{5}/2 & 0 \end{bmatrix}.$$

We get that

$$(\widetilde{T})^*\widetilde{T}.\widetilde{T}(\widetilde{T})^* = \begin{bmatrix} 20 & -\sqrt{3} & 0 \\ -5\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

and

$$\widetilde{T}(\widetilde{T})^*.(\widetilde{T})^*\widetilde{T} = \begin{bmatrix} 20 & -5\sqrt{3} & 0 \\ -\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{bmatrix}.$$

Hence $\widetilde{T}$ is not binormal.

Theorem 3.2.8. *Let $T = U\,|T|$ is the polar decomposition of the operator T, and U a coisometry. If T is binormal, then $\widehat{T}$ is binormal.*

Proof. We have $\widehat{T} = U^*TU$. Therefore, $(\widehat{T})^*\widehat{T} = U^*\ |T|^2\ U \geq 0$, and $|\widehat{T}| = U^*\ |T|\ U$. On the other hand, $\widehat{T}(\widehat{T})^* = U^*\ |T^*|^2\ U \geq 0$, and $|(\widehat{T})^*| = U^*\ |T^*|\ U$. Hence, $|\widehat{T}|\ |(\widehat{T})^*| = |(\widehat{T})^*|\ |\widehat{T}|$, and $\widehat{T}$ is binormal. □

If S and T are unitarily equivalent operators, then S is binormal if and only if T is binormal.

Theorem 3.2.9. *Let T be invertible. Then T is binormal if and only if $\widehat{T}$ is binormal.*

Proof. Since T is invertible, the polar decomposition of T has the form $T = U|T|$ where U is unitary. Also, $\widehat{T} = U^*TU$. Thus T and $\widehat{T}$ are unitarily equivalent. □

Theorem 3.2.10. *Let $T \in \mathcal{L}(H)$ be invertible. If T is binormal, then $\widehat{(\widetilde{T})} = \widetilde{(\widehat{T})}$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since T is invertible, U is unitary. By lemma 2.3.2, $\widehat{T} = U^*TU$. ie., $\widehat{T}$ is unitarily equivalent to T. Therefore, $\widetilde{(\widehat{T})} = U^*\widetilde{T}U$.

By theorem 3.2.5, $\widetilde{T} = U|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$, and therefore, again by lemma 2.3.2, $\widehat{(\widetilde{T})} = U^*\widetilde{T}U$. □

For convenience of notation we make the following definitions. We use these notations till the end of the present section.

Definition 3.2.11. Let T be an operator. Define $\Delta(T) = \widetilde{T}$, $\Gamma(T) = \widehat{T}$. For every non-negative integer n, define $\Delta^n(T) = \widetilde{T}^{(n)}$, $\Gamma^n(T) = \widehat{T}^{(n)}$.

Theorem 3.2.10 says that if T is invertible and binormal, then $\Gamma(\Delta(T)) = \Delta(\Gamma(T))$.

The following characterization of centered operators from the viewpoint of the polar decomposition and the Aluthge transformation can be seen in [20].

Theorem 3.2.12. [20] *Let T be an operator. Then $\Delta^n(T)$ is binormal for all $n \geq 0$ if and only if T is a centered operator.*

Theorem 3.2.13. *Let T be invertible and centered. Then $\Gamma(\Delta^n(T)) = \Delta^n(\Gamma(T))$ for all $n \geq 0$.*

Proof. Since T is centered, T is binormal. Therefore, the result is true for the case $n = 1$, by theorem 3.2.10. Suppose that the result is true for $n = m - 1$. Then $\Gamma(\Delta^{m-1}(T)) = \Delta^{m-1}(\Gamma(T))$.

Now, $\Delta^{m-1}(T)$ is invertible since T is invertible. By theorem 3.2.12, $\Delta^{m-1}(T)$ is binormal. Therefore, by theorem 3.2.10, $\Gamma[\Delta(\Delta^{m-1}(T))] = \Delta[\Gamma(\Delta^{m-1}(T))]$. Hence,

$$\begin{aligned}
\Gamma(\Delta^m(T)) &= \Gamma[\Delta(\Delta^{m-1}(T))] \\
&= \Delta[\Gamma(\Delta^{m-1}(T))] \\
&= \Delta[\Delta^{m-1}(\Gamma(T))] \\
&= \Delta^m(\Gamma(T))
\end{aligned}$$

The theorem follows by induction. □

Theorem 3.2.14. *If T is invertible and binormal, then $\Delta(\Gamma^n(T)) = \Gamma^n(\Delta(T))$ for all $n \geq 0$.*

Proof. The result is true for $n = 1$, by theorem 3.2.10.

Suppose that the result is true for $n = m-1$. ie., $\Delta(\Gamma^{m-1}(T)) = \Gamma^{m-1}(\Delta(T))$.

Since T is invertible and binormal, by lemma 3.2.1 and theorem 3.2.9, $\Gamma^n(T)$ is invertible and binormal for all $n \geq 0$. Since $\Gamma^{m-1}(T)$ is binormal and invertible, by theorem 3.2.10, $\Delta[\Gamma(\Gamma^{m-1}(T))] = \Gamma[\Delta(\Gamma^{m-1}(T))]$. Hence,

$$\begin{aligned}\Delta(\Gamma^m(T)) &= \Delta[\Gamma(\Gamma^{m-1}(T))] \\ &= \Gamma[\Delta(\Gamma^{m-1}(T))] \\ &= \Gamma[\Gamma^{m-1}(\Delta(T))] \\ &= \Gamma^m(\Delta(T))\end{aligned}$$

The theorem follows by induction. □

Every centered operator is binormal. So combining theorem 3.2.13 and theorem 3.2.14 we have the following result.

Theorem 3.2.15. *Let T be invertible and centered. Then*

$$\Gamma(\Delta^n(T)) = \Delta^n(\Gamma(T)), \ \Delta(\Gamma^n(T)) = \Gamma^n(\Delta(T))$$

for all $n \geq 0$.

Corollary 3.2.16. *Let T be invertible and centered. Then*

$$\Gamma^m(\Delta^n(T)) = \Delta^n(\Gamma^m(T))$$

for all $m, n \geq 0$.

Proof. The proof follows by theorem 3.2.12, theorem 3.2.14, and theorem 3.2.15. □

3.2.2 Polar decomposition of Aluthge and Duggal transformations

The following result gives the polar decomposition of the product of two operators.

Theorem 3.2.17. [17] *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. If T and S are doubly commutative (ie., $[T\ ,\ S] = [T\ ,\ S^*] = 0$), then*

$$TS = UV|TS|$$

is the polar decomposition of TS.

The following is a generalization of this result.

Theorem 3.2.18. [21] *Let $T = U|T|$, $S = V|S|$ and $|T|\ |S^*| = W|\ |T|\ |S^*|\ |$ be the polar decompositions. Then*

$$TS = UWV|TS|$$

is also the polar decomposition.

This theorem can be used to obtain the polar decomposition of the Duggal transformation of an invertible operator as follows.

Theorem 3.2.19. *Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. If $T = U|T|$ is the polar decomposition of T, then*

$$\widehat{T} = U|\widehat{T}|$$

is the polar decomposition of $\widehat{T}$.

Proof. Since T is invertible, U is unitary and $|T|$ is invertible. We have, $|U| = |U^*| = I$ the identity operator, because $UU^* = U^*U = I = I^2$ and $I \geq 0$. Since

$kerI = kerU = kerU^* = \{0\}$, we see that $U = UI$ is the polar decomposition of U. We have $|T|^* \; |T| = |T|^2$ and $|T| \geq 0$. Therefore, $| \; |T| \; | = |T|$. Since $|T|$ is invertible, $ker|T| = kerI = \{0\}$, and hence $|T| = I| \; |T| \; |$ is the polar decomposition of $|T|$. Also $|T| \; |U^*| = |T|$. Replacing T and S by $|T|$ and U respectively in 3.2.18, we obtain $|T|U = IIU| \; |T|U \; |$ is the polar decomposition of $|T|U$, or in other words, $\widehat{T} = U|\widehat{T}|$ is the polar decomposition of $\widehat{T}$. □

Remark 3.2.20. Theorem 3.2.19 shows that the Duggal analogue of theorem 3.2.3 is false. By theorem 3.2.19, whenever $T = U\,|T|$ is invertible, $\widehat{T} = U|\widehat{T}|$. An invertible operator need not always be binormal. For an example of invertible non-binormal operator, consider the matrix $\widetilde{T}$ in example 3.2.7.

Lemma 3.2.21. *If U is a partial isometry, then $|U| = U^*U$ and $U = U\;|U|$ is the polar decomposition of U. Also, $\widehat{U} = \widetilde{U} = U^*UU$.*

Proof. Since U^*U is a projection, $(U^*U)^2 = U^*U$ and $U^*U \geq 0$. Therefore, $|U| = (U^*U)^{1/2} = U^*U$. Since U^*U is the support of U, we have $UU^*U = U$. ie., $U\;|U| = U$. The kernel condition for the polar decomposition is satisfied automatically. Hence $U = U\;|U|$ is the polar decomposition of U.

Since $|U| = U^*U = (U^*U)^2$ and $U^*U \geq 0$, we have $|U|^{1/2} = U^*U$. Therefore, $\widehat{U} = |U|\;U = U^*UU$ and $\widetilde{U} = |U|^{1/2}\;U\;|U|^{1/2} = (U^*U)U(U^*U) = U^*UU$. □

Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. The following theorem gives an equivalent condition so that $TS = UV\,|TS|$ becomes the polar decomposition.

Theorem 3.2.22. [21] *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. Then $|T|\;|S^*| = |S^*|\;|T|$ if and only if*

$$TS = UV|TS|$$

is the polar decomposition.

We use this theorem to prove the following result on the polar decomposition of the Duggal transformation of an operator.

Theorem 3.2.23. *Let $T = U|T|$ be the polar decomposition of T. Then $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$ if and only if $|T|\ |U^*| = |U^*|\ |T|$.*

Proof. Since U is a partial isometry, $(kerU)^{\perp}$ is the initial space of U. Since U^*U is the support of both T and U, we see that $ran\,(U^*U) = (kerU)^{\perp} = (kerT)^{\perp}$.

Also, U^*U is self-adjoint. Therefore, $ker\,|T| = kerT = [ran\,(U^*U)]^{\perp} = ker\,(U^*U)$. Every projection is a partial isometry, in particular, U^*U is a partial isometry. Further, $(U^*U)\ |T| = |T|\ (U^*U) = |T|$. Hence $|T| = (U^*U)\ |T|$ is the polar decomposition of $|T|$, recalling $|\ |T|\ | = |T|$.

By lemma 3.2.21, $U = U\ |U|$ is the polar decomposition of U and $\widehat{U} = U^*UU$.

Replacing T and S in theorem 3.2.22 by $|T|$ and U respectively, we see that

$$|\ |T|\ | \cdot |U^*| = |U^*| \cdot |\ |T|\ |$$

if and only if

$$|T|U = U^*UU\ \mid |T|U\ \mid$$

is the polar decomposition of $|T|U$.
ie.,

$$|T| \cdot |U^*| = |U^*| \cdot |T|$$

if and only if

$$\widehat{T} = U^*UU\ |\widehat{T}|$$

is the polar decomposition of $\widehat{T}$.

ie.,

$$|T| \cdot |U^*| = |U^*| \cdot |T|$$

if and only if

$$\widehat{T} = \widehat{U}\ |\widehat{T}|$$

is the polar decomposition of $\widehat{T}$. □

Let $T = U|T|$ be the polar decomposition of T. A theorem in [21] says that T is binormal if and only if $\widetilde{T} = \widetilde{U}\,|\widetilde{T}|$ is the polar decomposition of the Aluthge transformation $\widetilde{T}$. In the following two theorems we discuss the similar situation of Duggal transformations.

Theorem 3.2.24. *Let $T = U|T|$ be the polar decomposition of T. If T is binormal, then $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$.*

Proof. Let $F = UU^*$. Then F is the support of T^*. If T is binormal, then $ker\,|T^*|$ is invariant under $|T|$. Therefore, $(ker\,|T^*|)^\perp$ is invariant under $|T|$. But $(ker\,|T^*|)^\perp = (ker\,T^*)^\perp = ran\,F$. Therefore, $F\,|T|\,F = |T|\,F$, and hence $|T|\,F = F\,|T|$. It follows that $|T|\,|U^*| = |U^*|\,|T|$. By theorem 3.2.23, $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$. □

Theorem 3.2.25. *Let $T = U|T|$ be the polar decomposition of T, and E, F the initial and final projections, respectively, of the partial isometry U. If $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$, then $EF = FE$, or equivalently, U is binormal.*

Proof. We have, $E = U^*U$ and $F = UU^*$. If $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$, then by theorem 3.2.23, $|T|\ |U^*| = |U^*|\ |T|$. Thus, $|T|\ F = F\ |T|$, and therefore, $ran\,|T|$ is invariant under F. Hence $\overline{(ran\,|T|)}$ is invariant under F. But $\overline{(ran\,|T|)} = (kerT)^\perp = ran\,E$. Hence $EF = FE$.

Next, U is binormal, if and only if, $|U|\,|U^*| = |U^*|\,|U|$, if and only if, $EF = FE$. □

Remark 3.2.26. Let T be an operator, and $T = U|T|$ the polar decomposition of T. Let E, F be the initial and final projections, respectively, of the partial isometry U. By lemma 3.2.21, $\widehat{U} = \widetilde{U} = U^*UU$. Theorem 3.1 in [21] says that if U is a partial isometry, then U is binormal, if and only if, $\widetilde{U}$ is a partial isometry, if and only if, U^2 is a partial isometry. Thus if $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$, then the following hold.

i. $|T|\ |U^*| = |U^*|\ |T|$.

ii. $EF = FE$.

iii. U is binormal.

iv. $\widehat{U}$ is a partial isometry.

v. $\widetilde{U}$ is a partial isometry.

vi. U^2 is a partial isometry.

On the other hand, if T is binormal, then $\widehat{T} = \widehat{U}\ |\widehat{T}|$ is the polar decomposition of $\widehat{T}$, and this in turn implies each of the above statements.

Theorem 3.2.27. *Let $T = U\,|T|$ be the polar decomposition of T. If $\widehat{T}^{(n)} = \widehat{U}^{(n)}\,|\widehat{T}^{(n)}|$ is the polar decomposition of $\widehat{T}^{(n)}$ for all $n = 1, 2, \ldots,$ then $|U^*|$ commutes with every $|\widehat{T}^{(n)}|$, or in other words, $ran\, U$ is invariant under every $|\widehat{T}^{(n)}|$.*

Proof. Let $E = U^*U$ and $F = UU^*$. First note that $ran\, U = ran\, F$.

By theorem 3.2.23 and theorem 3.2.25, we have

$$|T|\,|U^*| = |U^*|\,|T| \tag{3.1}$$

and $EF = FE$.

Since $(\widehat{U})^*$ is a partial isometry,

$$|(\widehat{U})^*| = \widehat{U}(\widehat{U})^* = U^*UUU^*U^*U = EFE = EF.$$

Also,

$$\begin{aligned}(\widehat{T})^*\widehat{T} &= (|T|\,U)^*\,|T|\,U = U^*\,|T|^2\,U \\ &= U^*\,|T|^2\,UU^*U \\ &= U^*\,|T|\,UU^*\,|T|\,U \text{ by (3.1)} \\ &= (U^*\,|T|\,U)^2\end{aligned}$$

and $U^*\,|T|\,U \geq 0$. Therefore, $|\widehat{T}| = U^*\,|T|\,U$.

Since $\widehat{T}^{(2)} = \widehat{U}^{(2)}\,|\widehat{T}^{(2)}|$ and $\widehat{T} = \widehat{U}\,|\widehat{T}|$ are polar decompositions, by theorem 3.2.23,

$$|\widehat{T}|\,|(\widehat{U})^*| = |(\widehat{U})^*|\,|\widehat{T}|.$$

But

$$|\widehat{T}|\,|(\widehat{U})^*| = U^*\,|T|\,UEF = U^*\,|T|\,UF = |\widehat{T}|\,F$$

and

$$|(\widehat{U})^*|\,|\widehat{T}| = EFU^*\,|T|\,U = FEU^*\,|T|\,U = FU^*\,|T|\,U = F\,|\widehat{T}|.$$

Thus $|\widehat{T}|\,F = F\,|\widehat{T}|$. Therefore, $|\widehat{T}|\,|U^*| = |U^*|\,|\widehat{T}|$.

Next, since $\widehat{T}^{(n)} = \widehat{U}^{(n)}\,|\widehat{T}^{(n)}|$ is the polar decomposition of $\widehat{T}^{(n)}$ for $n \leq 3$, by the same argument as above, we see that

$$|\widehat{T}^{(2)}| = (\widehat{U})^*\,|\widehat{T}|\,(\widehat{U}).$$

Also,

$$|\widehat{T}^{(2)}|\,|(\widehat{U})^*| = |(\widehat{U})^*|\,|\widehat{T}^{(2)}|.$$

ie.,

$$|\widehat{T}^{(2)}|\,EF = EF\,|\widehat{T}^{(2)}|.$$

Now, $\widehat{U}E = U^*UUU^*U = U^*UU = \widehat{U}$ and $E(\widehat{U})^* = (\widehat{U})^*$. Therefore,

$$|\widehat{T}^{(2)}|\,EF = |\widehat{T}^{(2)}|\,F$$

and

$$EF\,|\widehat{T}^{(2)}| = FE\,|\widehat{T}^{(2)}| = F\,|\widehat{T}^{(2)}|.$$

Thus

$$|\widehat{T}^{(2)}|\,F = F\,|\widehat{T}^{(2)}|.$$

Therefore, $|\widehat{T}^{(2)}|\,|U^*| = |U^*|\,|\widehat{T}^{(2)}|$.

Proceeding like this, we obtain $|\widehat{T}^{(n)}|\,|U^*| = |U^*|\,|\widehat{T}^{(n)}|$ for all n. □

3.2.3 Semigroup properties of factors in the polar decomposition and some applications

Masatoshi Ito, Takeaki Yamazaki and Masahiro Yanagida in [21] proved the following theorem 3.2.28 using properties of the polar decomposition of the product of operators. In this section we give a modification of the proof of this theorem using semigroup properties of factors in the polar decomposition.

Theorem 3.2.28. [21] *Let $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ be the polar decomposition of T. Then the following are equivalent.*

i. T is centered.

ii. $\widetilde{T}^{(n)} = \widetilde{U}^{(n)}\,|\widetilde{T}^{(n)}|$ is the polar decomposition for all nonnegative integer n.

iii. $T^n = U^n\,|T^n|$ is the polar decomposition for all natural number n.

Theorem 3.2.29. [21] *Let $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ be the polar decomposition of T. Then T is binormal if and only if $\widetilde{T} = \widetilde{U}\ |\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.*

Definition 3.2.30. Let S be a commutative semigroup with unit. A mapping $\pi : S \to \mathcal{L}(\mathcal{H})$ is called a

i. *semigroup homomorphism* if $\pi(s+t) = \pi(s)\pi(t)$, $s, t \in S$ and $\pi(0) = I$.

ii. *normal homomorphism* if the set $\{\pi(s), \pi(t)^*,\ s, t \in S\}$ is commutative.

iii. *quasinormal homomorphism* if the set $\{\pi(s)^*\pi(s), \pi(t),\ s, t \in S\}$ is commutative.

iv. *subnormal homomorphism* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal homomorphism $\tau : S \to \mathcal{L}(\mathcal{H})$ such that $\mathcal{H}$ is invariant for each $\tau(s)$ and $\tau(s)_{|\mathcal{H}} = \pi(s)$, $s \in S$.

v. *centered homomorphism* if the set $\{\pi(s)^*\pi(s), \pi(t)\pi(t)^*,\ s, t \in S\}$ is commutative.

All these special kinds of homomorphisms are, clearly, assumed to be semigroup homomorphisms. Notice that $\mathcal{L}(\mathcal{H})$ is considered here as a semigroup under the operation of multiplication of operators.

Let $\pi : S \to \mathcal{L}(\mathcal{H})$ be a semigroup homomorphism. Let

$$\pi(s) = \theta(s)\mu(s)$$

be the polar decomposition of the operator $\pi(s)$, $s \in S$.

The following two theorems in [9] discuss the semigroup properties of the factors θ and μ in the polar decomposition of π.

Theorem 3.2.31. [9]

i. If π is centered, then θ is a semigroup homomorphism.

ii. Assume additionally that for each $s, t \in S$ there exists $r \in S$ such that $s = t + r$ or $t = s + r$. If θ is a semigroup homomorphism, then π is centered.

The semigroup $\mathbb{N}$ satisfies the additional condition in (ii) of theorem 3.2.31, but the semigroup $\mathbb{N} \times \mathbb{N}$ does not.

Theorem 3.2.32. [9] *Let S be a commutative semigroup with unit and let $\pi : S \to \mathcal{L}(\mathcal{H})$ be a semigroup homomorphism. π is a quasinormal homomorphism if and only if μ is a semigroup homomorphism.*

Proof of theorem 3.2.28. $\mathbb{N}$ is a semigroup that satisfies the additional condition in (ii) of theorem 3.2.31. Let $\pi : \mathbb{N} \to \mathcal{L}(\mathcal{H})$ be defined by

$$\pi(n) = T^n,\ n \in \mathbb{N}.$$

Then π is a semigroup homomorphism. Let

$$\pi(n) = \theta(n)\mu(n)$$

be the polar decomposition of the operator $\pi(n)$, $n \in \mathbb{N}$. Now, $\theta(1) = U$ and $\mu(1) = |T|$.

Assume that T is centered. Then the set

$$\{\pi(n)^*\pi(n), \pi(m)\pi(m)^*,\ n, m \in \mathbb{N}\} = \{(T^n)^*T^n, T^m(T^m)^*,\ n, m \in \mathbb{N}\}$$

is commutative, and therefore, we see that π is a centered homomorphism. By theorem 3.2.31, $\theta : \mathbb{N} \to \mathcal{L}(\mathcal{H})$ is a semigroup homomorphism. Therefore, for every $n \in \mathbb{N}$, $U^n = (\theta(1))^n = \theta(n)$.

Hence the polar decomposition of $T^n = \pi(n)$ is

$$T^n = \theta(n)\mu(n) = \theta(n)|\pi(n)| = U^n\,|T^n|$$

for every $n \in \mathbb{N}$. This proves (i) $\Rightarrow$ (iii).

Assume that $T^n = U^n\,|T^n|$ is the polar decomposition for all natural number n. Then $\theta(n) = U^n$ for all $n \in \mathbb{N}$. Therefore, for $n, m \in \mathbb{N}$,

$$\theta(n+m) = U^{n+m} = U^nU^m = \theta(n)\theta(m),$$

and hence θ is a semigroup homomorphism. By theorem 3.2.31, π is a centered homomorphism. It follows that T is centered. This proves (iii) $\Rightarrow$ (i).

The rest of the proof is essentially the same as that in [21]. We give it here for completeness.

To prove (i) $\Rightarrow$ (ii). Assume that T is centered. By theorem 3.2.12, $\widetilde{T}^{(n)}$ is binormal for all $n \in \mathbb{N}$. By theorem 3.2.29,

$$\widetilde{T} = \widetilde{U}\ |\widetilde{T}| \text{ is the polar decomposition of } \widetilde{T}. \tag{3.2}$$

Next since $\widetilde{T}$ is binormal and since (3.2) holds, we have by theorem 3.2.29,

$$\widetilde{T}^{(2)} = \widetilde{U}^{(2)}\ |\widetilde{T}^{(2)}| \text{ is the polar decomposition of } \widetilde{T}^{(2)}.$$

Repeating this method, we have $\widetilde{T}^{(n)} = \widetilde{U}^{(n)}\,|\widetilde{T}^{(n)}|$ is the polar decomposition for all nonnegative integer n.

To prove (ii) $\Rightarrow$ (i). Assume that $\widetilde{T}^{(n)} = \widetilde{U}^{(n)}\,|\widetilde{T}^{(n)}|$ is the polar decomposition for all nonnegative integer n. By theorem 3.2.29, $\widetilde{T}^{(n)}$ is binormal for all $n \in \mathbb{N}$. Hence T is centered by theorem 3.2.12. $\square$

3.3 Spectral sets and numerical range of Aluthge and Duggal transformations

In [29], M. Schreiber characterized by means of normal dilations those operators the closure of whose numerical range is a spectral set. He obtained results on the equality of the convex hull of the spectrum with the closure of the numerical range, in relation to the spectrality of the numerical range. In this section we discuss Aluthge and Duggal transformations in the context of these results.

The Toeplitz - Hausdorff theorem states that the numerical range $W(T)$ of an operator $T \in \mathcal{L}(\mathcal{H})$ is always convex. If T is normal, then $\overline{W(T)}$ is the closed convex hull $\mathcal{C}(\sigma(T))$ of the spectrum $\sigma(T)$ of T. These facts can be seen in [18].

We defined dilations in 1.2.10 as follows. Let $T \in \mathcal{L}(\mathcal{H})$. If there exists a normal operator N on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $Tx = PNx$ for all $x \in \mathcal{H}$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then N is called a normal dilation of T. If, in addition, $T^n x = PN^n x$ for all $x \in \mathcal{H}$ and all $n = 0, 1, 2, \ldots$, then N is called a strong normal dilation of T.

Theorem 3.3.1. [16] *Every spectral set for T is a spectral set for $\widetilde{T}$ and a spectral set for $\widehat{T}$. Also $\overline{W(\widetilde{T})} \subset \overline{W(T)}$, and $\overline{W(\widehat{T})} \subset \overline{W(T)}$*

Theorem 3.3.2. [29] *Let $\mathcal{C}(X)$ denote the convex hull of X. If $\mathcal{C}(\sigma(T))$ is a spectral set for T, then there exists a strong normal dilation N of T such that $\mathcal{C}(\sigma(T)) = \overline{W(T)} = \overline{W(N)}$.*

Theorem 3.3.3. [29] *If there exists a strong normal dilation N of T such that $\overline{W(N)} = \overline{W(T)}$, then $\overline{W(T)}$ is a spectral set for T.*

Theorem 3.3.4. [2] *Let A be an $n \times n$ matrix over C. The convex hull of the eigen values of A equals $W(A)$ if and only if A and $\widetilde{A}$ have the same numerical range*

Theorem 3.3.5. [18] *If $\mathcal{H}$ is a finite dimensional Hilbert space and $T \in \mathcal{L}(H)$, then the numerical range $W(T)$ is compact.*

Theorem 3.3.6. *Let $T \in \mathcal{L}(H)$ be such that $\mathcal{C}(\sigma(T)) = \overline{W(T)}$. Then $\overline{W(\widetilde{T})} = \overline{W(T)}$. Also $\overline{W(\widehat{T})} = \overline{W(T)}$. (In other words, if T is convexoid, then so are $\widetilde{T}$ and $\widehat{T}$).*

Proof. By theorem 3.3.1, $\overline{W(\widetilde{T})} \subset \overline{W(T)} = \mathcal{C}(\sigma(T)) = \mathcal{C}(\sigma(\widetilde{T})) \subset \overline{W(\widetilde{T})}$. The proof of $\overline{W(\widehat{T})} = \overline{W(T)}$ is similar. □

Remark 3.3.7. This theorem generalizes one part of Ando's theorem 3.3.4. The finite dimensional case was discussed in remark 2.3.6.

Theorem 3.3.8. *Let $T \in \mathcal{L}(H)$ be such that $\mathcal{C}(\sigma(T)) = \overline{W(T)}$. Suppose there exists a strong normal dilation N of T such that $\overline{W(N)} = \overline{W(T)}$. Then*

(a) there exists a strong normal dilation N_1 of $\widetilde{T}$ such that $\overline{W(N_1)} = \overline{W(\widetilde{T})}$,

(b) there exists a strong normal dilation N_2 of $\widehat{T}$ such that $\overline{W(N_2)} = \overline{W(\widehat{T})}$.

(Remember $\overline{W(\widetilde{T})} = \overline{W(\widehat{T})} = \overline{W(T)}$, in this case by theorem 3.3.6).

Proof. By theorem 3.3.3, $\overline{W(T)}$ is a spectral set for T. By theorem 3.3.1, $\overline{W(T)}$ is a spectral set for $\widetilde{T}$. But $\mathcal{C}(\sigma(\widetilde{T})) = \mathcal{C}(\sigma(T)) = \overline{W(T)}$. Thus $\mathcal{C}(\sigma(\widetilde{T}))$ is a spectral set for $\widetilde{T}$. By theorem 3.3.2, there exists a strong normal dilation N_1 of $\widetilde{T}$ such that $\overline{W(N_1)} = \overline{W(\widetilde{T})}$, proving (a).

The proof of (b) is similar. □

Theorem 3.3.9. *Let $T \in \mathcal{L}(H)$. If $\mathcal{C}(\sigma(T))$ is a spectral set for T, then $\overline{W(T)} = \overline{W(\widetilde{T})} = \overline{W(\widehat{T})}$.*

Proof. If $\mathcal{C}(\sigma(T))$ is a spectral set for T, by theorem 3.3.2, $\mathcal{C}(\sigma(T)) = \overline{W(T)}$. Therefore, by theorem 3.3.6, $\overline{W(T)} = \overline{W(\widetilde{T})} = \overline{W(\widehat{T})}$. □

Corollary 3.3.10. *Let A be an $n \times n$ matrix over $\mathbb{C}$. If $\mathcal{C}(\sigma(A))$ is a spectral set for A, then $A, \widetilde{A}$, and $\widehat{A}$ have the same numerical range.*

Proof. Obvious by theorem 3.3.5 and theorem 3.3.9. □

Chapter 4

Minimal spectral sets and $n-$level spectral sets

4.1 Introduction

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. A closed subset X of $\mathbb{C}$ is called a *minimal spectral set* for T, if X is a spectral set for T such that X contains no other spectral set for T.

It is known that that for every operator T on a Hilbert space $\mathcal{H}$, there is a minimal spectral set; in fact, every spectral set contains a minimal spectral set [34]. A theorem in [7] says that $\sigma(T)$ is a spectral set for T if and only if $f(T)$ is normaloid for every $f \in \mathcal{R}(\sigma(T))$. In this chapter we observe that these statements are equivalent to the uniqueness of the minimal spectral set. We discuss situations when the minimal spectral set of an operator is unique.

Earlier in section 2.3.2, we introduced $n-$level spectral sets. If X is a closed set in the complex plane and if X is an $n-$level spectral set for an operator T in $\mathcal{L}(\mathcal{H})$, then there is a map $\tilde{\rho} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \rightarrow \mathcal{L}(\mathcal{H})$ that extends the natural functional calculus map $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$, where $\overline{\mathcal{R}(X)}$ is the set of all complex conjugates of members of $\mathcal{R}(X)$. In this chapter, we obtain some results on the contractivity and positivity of the map $\tilde{\rho}$.

4.2 Uniqueness of minimal spectral sets

Lemma 4.2.1. *Let $T \in \mathcal{L}(\mathcal{H})$, and let $\mathcal{F}$ be a collection of spectral sets for T such that $\mathcal{F}$ is totally ordered under set inclusion. Then $\cap_{M\in\mathcal{F}} M$ is a spectral set for T.*

Proof. Let $N = \cap_{M\in\mathcal{F}} M$. Clearly N is closed. Being a subset of a second countable space we see that N^c is Lindelöf. We have $N^c = \cup_{M\in\mathcal{F}} M^c$. Thus $\{M^c : M \in \mathcal{F}\}$ is an open covering of N^c. There is a countable subcollection $\{M_i^c : i = 1, 2, \ldots\}$ such that $N^c = \cup_i M_i^c$. Therefore, $N = \cap_i M_i$. Since $\mathcal{F}$ is totally ordered under set inclusion, the collection $\{M_i : i = 1, 2, \ldots\}$ is totally ordered under set inclusion. Hence by the von Neumann's result [32], N is a spectral set for T. □

Theorem 4.2.2 ([6])**.** *If $T \in \mathcal{L}(\mathcal{H})$, then there exists a minimal spectral set for T.*

Proof. Let $\mathcal{A}$ be the collection of all spectral sets of T. If $M_1, M_2 \in \mathcal{A}$, define $M_1 \leq M_2$ if $M_2 \subset M_1$. Then $\leq$ is a partial order on $\mathcal{A}$. By lemma 4.2.1, every totally ordered subset of $\mathcal{A}$ has an upper bound in $\mathcal{A}$. By Zorn's lemma, $\mathcal{A}$ has a maximal element M. Then M is a spectral set for T and M contains no other spectral set for T. □

Theorem 4.2.3 ([34])**.** *Let $T \in \mathcal{L}(\mathcal{H})$. If X is any spectral set for T, then there is a minimal spectral set M for T such that $M \subset X$.*

Proof. Let $\mathcal{A}$ be the collection of all spectral sets for T which are contained in X. Clearly $\mathcal{A}$ is nonempty since $X \in \mathcal{A}$. By lemma 4.2.1 and by Zorn's lemma, there exists $M \in \mathcal{A}$ such that M contains no other element of $\mathcal{A}$. Then M is a minimal spectral set for T such that $M \subset X$. □

Corollary 4.2.4. *If $T \in \mathcal{L}(\mathcal{H})$, there is a minimal spectral set M for T such that M is contained in $\{z : |z| \leq\| T \|\}$.*

Theorem 4.2.5 ([32])**.** *For every $T \in \mathcal{L}(\mathcal{H})$, one has $\sigma(T) = \cap\tau$, where τ varies over all spectral sets of T.*

Theorem 4.2.6. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\mathcal{F}$ is the collection of all minimal spectral sets for T, then $\cap_{M\in\mathcal{F}} M = \sigma(T)$.*

Proof. If $\mathcal{A}$ is the collection of all spectral sets of T, by theorem 4.2.5, $\cap_{X\in\mathcal{A}} X = \sigma(T)$. Given $X \in \mathcal{A}$, by theorem 4.2.3, there is a minimal spectral set M_X such that $M_X \subset X$. Hence $\cap_{X\in\mathcal{A}} M_X \subset \cap_{X\in\mathcal{A}} X = \sigma(T)$. Since each M_X is in $\mathcal{F}$ we see that $\cap_{M\in\mathcal{F}} M \subset \cap_{X\in\mathcal{A}} M_X$. Thus $\cap_{M\in\mathcal{F}} M \subset \sigma(T)$.

On the other hand, $\sigma(T) \subset M$ for every $M \in \mathcal{F}$ since M is a spectral set and hence $\sigma(T) \subset \cap_{M\in\mathcal{F}} M$. This proves the theorem. □

If $\sigma(T)$ itself is a spectral set for T, then the minimal spectral set for T is unique, namely the spectrum $\sigma(T)$. In particular, if T is normal, then $\sigma(T)$ is the unique minimal spectral set for T. In general, the minimal spectral set for an operator is not unique. The following example 4.2.8 gives one such operator.

Theorem 4.2.7 ([7])**.** *Suppose $T \in \mathcal{L}(\mathcal{H}), \alpha \in C$ and $\beta > 0$.*

(i) The closed disc $\{\lambda \in C : |\lambda - \alpha| \leq \beta\}$ is a spectral set for T if and only if $\| T - \alpha I \|\leq \beta$.

(ii) The set $\{\lambda \in C : |\lambda - \alpha| \geq \beta\}$ is a spectral set for T if and only if $\alpha \notin \sigma(T)$ and $\| (T - \alpha I)^{-1} \|\leq 1/\beta$.

Example 4.2.8. Let

$$A = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}.$$

Then A is invertible, $\sigma(A) = \{1\}$, and

$$A^{-1} = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}.$$

We have,

$$A^* = \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix},$$

and

$$A^*A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 13/4 \end{bmatrix},$$

hence $\sigma(A^*A) = \{4, 1/4\}, r(A^*A) = 4 =\| A \|^2$. Therefore, $\| A \|=\| A^{-1} \|= 2$.

Hence $X_1 = \{z : |z| \leq 2\}$ and $X_2 = \{z : |z| \geq 1/2\}$ are spectral sets for A. (X_2 is a spectral set for A follows from (ii) of theorem 4.2.7. Note that $0 \notin \sigma(A)$, and $\| A \|\leq 2$).

By theorem 4.2.3, there are minimal spectral sets M_1 and M_2 for T such that $M_1 \subset X_1$ and $M_2 \subset X_2$. Then $M_1 \neq M_2$. To see this, suppose if possible, $M_1 = M_2 = M$ (say). Then $M \subset X_1 \cap X_2$. Since M is a spectral set for T, it follows that $X_1 \cap X_2$ is a spectral set for T. But $X_1 \cap X_2 = \{z : 1/2 \leq |z| \leq 2\}$ is not a spectral set for T (page 145 in [26] (since $R = 2 > \sqrt{3}$)). This contradiction proves that $M_1 \neq M_2$. Thus A has more than one minimal spectral set.

Theorem 4.2.9. *Let $T \in \mathcal{L}(\mathcal{H})$. The following are equivalent.*

(a) T has a unique minimal spectral set.

(b) If X_1 and X_2 are spectral sets for T, then $X_1 \cap X_2$ is a spectral set for T.

(c) $\sigma(T)$ is a spectral set for T.

(d) For every $f \in \mathcal{R}(\sigma(T)), r(f(T)) = \| f(T) \|$.

Proof. (a) $\Rightarrow$ (b). Suppose T has a unique minimal spectral set. Let X_1 and X_2 be spectral sets for T. By theorem 4.2.3, there exist minimal spectral sets M_1 and M_2 for T such that $M_1 \subset X_1$ and $M_2 \subset X_2$. By the uniqueness, $M_1 = M_2$. Thus $M_1 \subset X_1 \cap X_2$. Since M_1 is a spectral set, $X_1 \cap X_2$ is a spectral set.

(b) $\Rightarrow$ (a). Assume (b). Suppose M_1 and M_2 are minimal spectral sets for T. By hypothesis $M_1 \cap M_2$ is a spectral set for T. But $M_1 \cap M_2 \subset M_1$ and $M_1 \cap M_2 \subset M_2$. By the minimality $M_1 \cap M_2 = M_1$ and $M_1 \cap M_2 = M_2$. Hence $M_1 = M_2$.

(c) and (d) are equivalent [7] . Now we prove (a) and (c) are equivalent.

(a) $\Rightarrow$(c). Suppose T has a unique minimal spectral set. By theorem 4.2.6, this minimal spectral set must be $\sigma(T)$.

(c) $\Rightarrow$(a). If $\sigma(T)$ is a spectral set for T, then T has a unique minimal spectral set, namely $\sigma(T)$. □

Remark 4.2.10. The equivalence of (b) and (c) are known facts by [34]. However we have given a proof here for completeness.

4.3 n-level spectral sets

Recall the definition of n-level spectral set on page 50. If X is a closed, proper subset of $\mathbb{C}$, and $T \in \mathcal{L}(\mathcal{H})$ with $\sigma(T) \subset X$, then there is a functional calculus, ie., a homomorphism $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$, given by $\rho(f) = f(T)$, where $f(T) = p(T)q(T)^{-1}$ if $f = p/q$, a quotient of polynomials. We regard $\mathcal{R}(X)$ as a subalgebra of the C^*-algebra $C(\partial\check{X})$, which defines norms on $\mathcal{R}(X)$ and on each $\mathcal{M}_n(\mathcal{R}(X))$. If $\| \rho \| \leq 1$, then X is called a spectral set for T. If $\| \rho \|_{cb} \leq 1$, then X is called a complete spectral set for T. If n is a positive integer, and if $\|\rho_n\| \leq 1$, where ρ_n denotes the n^{th} amplification of ρ, then we say that X is an n-level spectral set for T.

Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{M}$ be a subspace (then we call $\mathcal{M}$ an operator space). Let $\mathcal{B}$ be a C^*-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ a linear map.

Lemma 4.3.1 ([26]). *If ϕ is n-contractive and $k \leq n$, then ϕ is k-contractive. If ϕ is n-positive and $k \leq n$, then ϕ is k-positive.*

So we have the following theorem.

Theorem 4.3.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If k and n are positive integers such that $k \leq n$ and if X is an n-level spectral set for T, then X is a k-level spectral set for T. In particular, every n-level spectral set for T is a spectral set for T, and every complete spectral set for T is a spectral set for T.*

Theorem 4.3.3 ([26]). *Let $\mathcal{B}, \mathcal{C}$ be C^*-algebras with units, let $\mathcal{A}$ be a subalgebra of $\mathcal{B}$, $1 \in \mathcal{A}$ and let $\mathcal{S} = \mathcal{A} + \mathcal{A}^*$. If $\phi : \mathcal{S} \to \mathcal{C}$ is positive, then $\|\phi(a)\| \leq \|\phi(1)\| \, \|a\|$ for all $a \in \mathcal{A}$.*

Theorem 4.3.4 ([26]). *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{M}$ be a subspace of $\mathcal{A}$ containing 1. If $\mathcal{B}$ is a unital C^*-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ is a unital contraction, then ϕ extends uniquely to a positive map $\tilde{\phi} : \mathcal{M} + \mathcal{M}^* \to B$ with $\tilde{\phi}$ given by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$.*

Let X be a closed set in the complex plane and $\mathcal{M} = \mathcal{R}(X)$. Let $\overline{\mathcal{R}(X)}$ denote the set of all complex conjugates of members of $\mathcal{R}(X)$. Put $\mathcal{S} = \mathcal{R}(X) + \overline{\mathcal{R}(X)}$. Let $T \in \mathcal{L}(\mathcal{H})$. If X is a spectral set for T, by theorem 4.3.4, there is a well defined positive map $\tilde{\rho} : \mathcal{S} \to \mathcal{L}(\mathcal{H})$ given by $\tilde{\rho}(f + \overline{g}) = f(T) + g(T)^*$. Conversely, by theorem 4.3.3, if the above map $\tilde{\rho}$ is well-defined and positive, then X is a spectral set for T (noticing the fact that $\|\tilde{\rho}(1)\| = \|I\| = 1$).

Theorem 4.3.5 ([26]). *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with units, let $\mathcal{M}$ be a subspace of $\mathcal{A}, 1 \in \mathcal{M}$ and $\mathcal{S} = \mathcal{M} + \mathcal{M}^*$. If $\phi : \mathcal{M} \to \mathcal{B}$ is unital and 2-contractive, then the map $\tilde{\phi} : \mathcal{S} \to \mathcal{B}$ given by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ is 2-positive and contractive.*

Theorem 4.3.6. *Let X be closed set in the complex plane. If X is a 2-level spectral set for T, then the map $\tilde{\rho} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \to \mathcal{L}(\mathcal{H})$ given by $\tilde{\rho}(f + \overline{g}) = f(T) + g(T)^*$ is 2-positive and contractive.*

Proof. $\mathcal{R}(X)$ is a subspace of the unital C^*-algebra $C(X)$. If X is a 2-level spectral set for T, then the map $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$ defined by $\rho(f) = f(T)$ is unital and 2-contractive. By theorem 4.3.5, the map $\tilde{\rho} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \to \mathcal{L}(\mathcal{H})$ given by $\tilde{\rho}(f + \overline{g}) = f(T) + g(T)^*$ is 2-positive and contractive. □

Theorem 4.3.7. *Let X be closed set in the complex plane. If X is a $2n$-level spectral set for T, then the map $\tilde{\rho} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \to \mathcal{L}(\mathcal{H})$ given by $\tilde{\rho}(f + \overline{g}) = f(T) + g(T)^*$ is $2n$-positive and n-contractive.*

Proof. The proof is only a simple generalization of the proof of theorem 4.3.6. □

Corollary 4.3.8. *Let X be closed set in the complex plane. If X is a complete spectral set for T, then the map $\tilde{\rho} : \mathcal{R}(X) + \overline{\mathcal{R}(X)} \to \mathcal{L}(\mathcal{H})$ given by $\tilde{\rho}(f + \overline{g}) = f(T) + g(T)^*$ is completely positive and completely contractive.*

Proof. If X is a complete spectral set for T, then X is a $2n$-level spectral set for all n. By theorem 4.3.7, $\tilde{\rho}$ is $2n$-positive and n-contractive for all n. Then $\tilde{\rho}$ is completely contractive. Also by lemma 4.3.1, $\tilde{\rho}$ is completely positive. □

Epilogue

Some of the problems that were thought about and where further research is possible, are discussed below briefly.

It is proved in chapter 2 that if T is a bounded linear operator on a Hilbert space, then the sequence of the norms of Duggal iterates of T converges to the spectral radius $r(T)$, if T satisfies certain conditions. In general, unlike Aluthge transformations, the sequence need not converge to $r(T)$. One can attempt to characterize operators T on a Hilbert space $\mathcal{H}$, having the property that the sequence of the norms of Duggal iterates of T converges to the spectral radius $r(T)$.

The example 2.2.19 furnishes a 2×2 matrix A such that $\{ \| \widehat{A}^{(n)} \| \}_{n=0}^{\infty}$ does not converge to $r(A)$, and such that $\{ \widehat{A}^{(n)} \}_{n=0}^{\infty}$ does not converge. In this case, the sequence $\{ \widehat{A}^{(n)} \}_{n=0}^{\infty}$ does not converge because it is an alternating sequence. By the results of Ando and Yamazaki [3], and of Antezana, Pujals and Stojanoff [4], the Aluthge sequence $\{ \widetilde{A}^{(n)} \}_{n=0}^{\infty}$ cannot be alternating (at least for a 2×2 matrix A or for a diagonalizable matrix A). The study of the class of operators (matrices) for which the sequence of Duggal iterates is alternating remains to be explored.

The continuity of the Duggal transformation map $T \rightarrow \widehat{T}$ on the set of invertible operators is established. The continuity of the Duggal transformation map $T \rightarrow \widehat{T}$ on all of $\mathcal{L}(\mathcal{H})$ is a problem that remains to be answered.

In theorem 2.3.46, it is proved that if $T \in \mathcal{L}(\mathcal{H})$, and if $\{\widehat{T}^{(n)}\}$ converges to $S \in \mathcal{L}(\mathcal{H})$, then $Hol(\sigma(S)) \subset Hol(\sigma(T))$. The possibility of operators for which the inclusion is proper needs to be analyzed further.

Theorem 2.3.49 is a result about the relation between spectral sets of an invertible operator and spectral sets of the limit of the sequence of Duggal iterates of the operator. More exploration is needed to resolve completely the similar problems that arise in the case of arbitrary operators.

Theorem 3.2.10 shows that if $T \in \mathcal{L}(H)$ is invertible and binormal, then $\widehat{(\widetilde{T})} = \widetilde{(\widehat{T})}$; and corollary 3.2.16 proves that if $T \in \mathcal{L}(H)$ is invertible and centered, then $\Gamma^m(\Delta^n(T)) = \Delta^n(\Gamma^m(T))$ for all $m, n \geq 0$. One can work out more general classes of operators T having the property $\widehat{(\widetilde{T})} = \widetilde{(\widehat{T})}$, and operators T having the property $\Gamma^m(\Delta^n(T)) = \Delta^n(\Gamma^m(T))$ for all $m, n \geq 0$.

For an operator T with polar decomposition $T = U|T|$, theorem 3.2.23 gives the necessary and sufficient condition for $\widehat{T}$ to have the the polar decomposition $\widehat{T} = \widehat{U}\ |\widehat{T}|$. Characterization of operators $T = U|T|$, having the property that $\widetilde{T}^{(n)} = \widetilde{U}^{(n)}\,|\widetilde{T}^{(n)}|$ is the polar decomposition of $\widetilde{T}^{(n)}$ for all $n = 1, 2, \ldots,$ exists. Theorem 3.2.27 gives some partial results in the similar case of Duggal transformations, and demands further research.

If $\mathcal{A}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$, and $T \in \mathcal{A}$, then obviously, the Aluthge transformation $\widetilde{T}$ and the Duggal transformation $\widehat{T}$ are also in $\mathcal{A}$. So, some operator algebraic study of Aluthge and Duggal transformations is possible, although it was not within the scope of the thesis.

Identification of some or all minimal spectral sets of an arbitrary operator is an unsettled problem, except in the case of some special classes of operators. There exist some results of this nature in the case of compact and completely non-normal operators.

By theorem 4.3.2, every n-level spectral set for $T \in \mathcal{L}(\mathcal{H})$ is a spectral set for T. So far, one does not know an example of a spectral set which is not an n-level spectral set. Further research on n-level spectral sets may help the study of spectral sets and complete spectral sets.

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