MODELLING AND ANALYSIS OF RELIABILITY DATA USING BATHTUB SHAPED FAILURE RATE DISTRIBUTIONS

Thesis

submitted in partial fulfillment of

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in

STATISTICS

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by

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under the Guidance of

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DECLARATION

I hereby declare that the matter embodied in this thesis entitled '**Modelling and Analysis of Reliability Data using Bathtub shaped Failure rate Distributions**', submitted to the St. Thomas' College (Autonomous), Thrissur in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy in Statistics**, is the result of my studies and this has been composed by me under the guidance and supervision of Dr. Chacko V.M., Assistant Professor & Head of the Department, Department of Statistics, St. Thomas' College (Autonomous), Thrissur during 2017-2020.

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Thrissur,

14th December 2020 **Deepthi K.S.**

I dedicate this work

to my parents who paved the way for me during their lifetime and

to my beloved husband Mr. Praveen R., and my child for their unconditional love and support.

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14th December 2020 **Deepthi K.S.**

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Reliability theory deals with the interdisciplinary use of probability, statistics and stochastic modeling, combined with engineering insights into the system design and the scientific understanding of the failure mechanisms. The study of reliability characteristics and performances requires a comprehensive understanding of many different concepts of the system. System reliability describes the probability of completing the mission with in a pre-specified time interval. An engineer need to find the probability of successful functioning of many engineering systems such as Airplanes, linear accelerators, power generation system etc. The computation of reliability or expected system performance is a problem for engineers and manufacturers. Traditional reliability theory is built on a statistical framework in which the system and its components can be in one of the two states such as functioning state or failed state. As a result, the system structure function is a binary function of binary variables. Reliability calculation is an important task for increasingly sophisticated technological systems.

The reliability of a unit (a system or a component) is also defined as the probability that a unit can perform satisfactorily for a specified period of time without failure. Proper modeling of lifetime with appropriate statistical distributions makes reliability computation easy. Moreover reliability and maintenance activities can be planned with the help of distribution of lifetime of the system. Identification of failure rate model is crucial in reliability analysis to select appropriate distribution for the given data. Many of the distributions available in literature is not sufficient for explain distributional properties and reliability analysis for the given data. So searching for more appropriate distributions for the given data is an open challenge among researchers. Most of the systems are subjected to certain type of stresses, so reliability computation of stress-strength models using different distributions is an important research problem. Moreover identification of failure rate model for the given data or transformed data makes the selection of distribution for the given data easy. While considering lifetime data, study on burn-in process is unavoidable to understand the length of time to burn-in.

1.2 Binary state system

In the binary state system, the components in the system are assumed to be in one of the two states, functioning or failed, see Barlow and Proschan (1975). For a system with n components, let x_i indicate the state of the i^{th} component, $i = 1, 2, \ldots, n$. That is,

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed.} \end{cases}$$

for $i = 1, 2, \ldots, n$. Similarly, let Φ be a binary random variable indicating the state of the system

$$\Phi = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed.} \end{cases}$$

We assume that the state of the system is a function of the states of the components, so that we may write $\Phi = \Phi(\underline{x})$, structure function or structure, where $\underline{x} = (x_1, x_2, \ldots, x_n)$. For a series system,

$$\Phi(\underline{x}) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, \ldots, x_n)$$

$$= \begin{cases} 1, & \text{if each component is functioning} \\ 0, & \text{if atleast one of the components is failed.} \end{cases}$$

For a parallel system,

$$\Phi(\underline{x}) = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, \ldots, x_n)$$

$$= \begin{cases} 1, & \text{if atleast one component is functioning} \\ 0, & \text{if all components are failed} \end{cases}$$

where $\coprod_{i=1}^{n} = 1 - \prod_{i=1}^{n}(1 - x_i)$.

A system of components is coherent if its structure function $\Phi(\underline{x})$ is increasing in each component and each component is relevant.

The reliability of a system is given by $P(\Phi(\underline{x}) = 1) = h = E\Phi(\underline{x})$. Under the assumption of independence of components, we may represent system reliability as a function of component reliabilities, p_i, $i = 1, 2, \ldots, n$. That is $h = h(\underline{p})$, $\underline{p} = (p_1, p_2, \ldots, p_n)$. Accordingly the reliability function of series structure is $h(\underline{p}) = \prod_{i=1}^{n} p_i$ and the reliability function of parallel structure is $h(\underline{p}) = 1 - \prod_{i=1}^{n}(1 - p_i)$.

In Barlow and Proschan (1975), we find that "reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered". Generally the period of time intended is $[0, t]$. In many problems, we consider the life lengths of the components of the system, for the reliability analysis. By lifetime we mean the maximum period for which the unit can work satisfactorily whereas age of the unit is the time which it requires for the completion of a particular mission without failure. In general, life lengths are random variables and therefore lead us to a study of life distributions. The reliability of a fresh unit corresponding to a mission of duration t is, by definition, $\bar{F}(t) = P(T > t) = 1 - F(t)$, where F is the cumulative distribution function (cdf) of lifetime of the unit.

There is a reason to believe that in many applications, the lifetime of the system will be reduced beyond the specified age. That is, survival decreases as the system ages. If units have this behavior, the corresponding life distributions are called positive ageing distributions. To understand which real life distributions are appropriate in reliability data analysis, we need to consider a notion of ageing. Ageing can be conveniently defined in terms of the failure rate function.

1.2.1 Notion of ageing

The notion of ageing plays a significant role in reliability theory. Several classes of life distributions based on the notion of ageing have been studied and explored during the past several years. Most common ageing properties like increasing failure rate (IFR), decreasing failure rate (DFR), increasing failure rate average (IFRA), decreasing failure rate average (DFRA), new better than used (NBU), new worse than used (NWU), new better than used in expectation (NBUE), new worse than used in expectation (NWUE), increasing mean residual life (IMRL), decreasing mean residual life (DMRL) etc. are discussed by Barlow and Proschan (1975) and Deshpande et al. (1986). The discrete and continuous versions of these classes have become very common in literature. Concepts of ageing describes how a component or system improves or deteriorates with age. The most popular lifetime distributions such as Exponential, Weibull, Gamma, Rayleigh, Pareto and Gompertz have monotonic failure rate functions, see Lawless (1982). If we observe a constant failure rate pattern for a data, then Exponential distribution serves as very useful model for reliability analysis. The Poisson process has many direct and indirect applications in reliability, especially in formulating shock models. 'No ageing' means that the age of a component does not influence the distribution of the remaining life of the component or system. Positive ageing means adverse effect of age exist on the random residual life of the component or system whereas negative ageing means beneficial effect of age exist on the random residual life of the component or system. Positive ageing describes the situation where the residual lifetime tends to decrease, in some probabilistic sense, with increasing age of the component. This situation is common in reliability engineering, as increased wear and tear may worsen over time. Negative ageing, on the other hand, has the opposite effect during the rest of life. Negative ageing is also known as beneficial ageing. In other words, the residual lifetime is monotonic with respect to age. However, in many practical applications, the effect of age is initially beneficial but after a certain period of time, adverse indicating a 'ware-out' phase where age is positive. Certain lifetime data, for example, human mortality, machine life cycles and data from some biological and medical studies require non-monotonic shapes like bathtub shape or upside-down bathtub shape. Initially, the failure rate (death rate) of the newborn babies is very high especially in the first six months after birth, caused by deformities, heart dysfunctions or other infant diseases. Then, the risk of death decreases rapidly until it reaches its lowest level and remains approximately constant for a long period. At some point, during the ages between 50 and 80 the death risk increases over time. This kind of non-monotonic ageing phenomenon is often modeled using life distributions that display bathtub shaped failure rate (BFR). Model the lifetime data using a distribution with BFR is important in reliability analysis. An upside-down bathtub shape is analyzed by Efron (1988) in the context of head and neck cancer data, in which the failure rate initially increased, reached a maximum and then decreased before it finally stabilized because of a therapy.

Let X be a continuous non-negative random variable (r.v) representing the lifetime of a unit which is in operation. This unit may be a living organism, a mechanical component, a system of components etc. Now we describe the failure rate function or hazard function, see Barlow and Proschan (1975). Let $F(x)$ be the cdf of X, then the survival function of a fresh unit is $\bar{F}(x) = 1 - F(x) = P(X > x)$. Also let $F(x|t) = P(X > t + x|X > t)$ be the survival probability or reliability of a unit which has attained the age t. It can be seen that $\bar{F}(x|t) = \bar{F}(x+t)/\bar{F}(t)$. Note that this represents the survival function of a unit of age t, *i.e.*, the conditional probability that a unit of age t will survive for an extra x units of time. When $t = 0$, $\bar{F}_0(x) = \bar{F}(x)$ is the survival function of a new unit.

When the derivative $F'(t) = f(t)$ exists, where $f(t)$ be the probability density function (pdf), failure rate (hazard rate) of a component is defined as $r(t) = f(t)/\bar{F}(t)$, $\bar{F}(t) > 0$. This can also be written as $r(t) = \lim_{\Delta \to 0} \frac{Pr(t \le X < t+\Delta | t \le X)}{\Delta}$. Thus for small Δ, $r(t)\Delta$ is approximately the probability of a failure occurring in $(t, t+\Delta]$ given no failure has occurred in $(0, t]$.

It follows that, if $r(t)$ exists, then $-\log\ \bar{F}(t) = \int_0^t r(x)\ dx$ represents the cumulative failure rate (cumulative hazard rate) which may be denoted by $H(t)$. Hence $\bar{F}(t) = \exp\left\{-\int_0^t r(x)\ dx\right\} = \exp\{-H(t)\}$.

Now we consider a unit which does not age stochastically, that is, probability distribution of the residual lifetime at age t of the unit does not depend on t. Hence $\bar{F}_t(x) = \bar{F}_0(x)\ \forall\ t,\ x > 0$. This is equivalent to $\bar{F}(t+x) = \bar{F}(t)\bar{F}(x)\ \forall\ t,\ x > 0$. It is well known that among the continuous survival functions only the exponential survival function $\bar{F}(x) = e^{-\lambda x},\ x > 0,\ \lambda > 0$ satisfies the above equation, and this property is known as lack of memory property or no-ageing property of Exponential distribution in reliability theory.

We recall the definition of failure rate behaviors IFR, DFR, IFRA, DFRA, NBU, NWU, NBUE, NWUE, IMRL and DMRL (Barlow and Proschan (1975)).

Definition 1.2.1. *When the density exists, IFR (DFR) is equivalent to* $r(t) = f(t)/\bar{F}(t)$ *is increasing (decreasing) in* $t \geq 0$. *When* F *is not absolutely continuous,* F *is said to be IFR (DFR) distribution if* $\bar{F}(x|t)$ *is decreasing (increasing) in* $t,\ 0 \leq t < \infty$ *for each* $x > 0$. F *is IFR (DFR) iff* $-\log \bar{F}(t)$ *is convex (concave).*

Definition 1.2.2. F *is said to be IFRA (DFRA) if and only if* $\int_0^t r(x)\ dx/t$ *increasing (decreasing) in* $t \geq 0$ *equivalently* $-(1/t)\log \bar{F}(t)$ *is increasing (decreasing) in* $t \geq 0$ *(This is equivalent to* $-\log \bar{F}(t)$ *being a star-shaped function) equivalently* $\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t),\ 0 < \alpha < 1,\ t \geq 0$.

Definition 1.2.3. F *is said to be NBU (NWU) if* $\bar{F}(x|t) \leq (\geq) \bar{F}(x)$ *equivalently* $\bar{F}(x+t) \leq (\geq)\ \bar{F}(x)\bar{F}(t)$ *for* $x,\ t \geq 0$ *equivalently* $\log \bar{F}(x+t) \leq (\geq)\ \log \bar{F}(x) + \log \bar{F}(t)$ *equivalently* $\int_0^t r(u)\ du \leq (\geq) \int_x^{x+t} r(u)\ du$.

Definition 1.2.4. F *is said to be NBUE (NWUE) if* $\int_0^\infty \bar{F}(x|t)\ dx \leq (\geq)\ \mu$ *for* $t \geq 0$.

DEFINITION 1.2.5. *F is said to be IMRL (DMRL) if* $\mu(t) = \int_0^\infty \bar{F}(x|t)\, dx$ *is increasing (decreasing) in* t, *i.e.,* $\mu(s) \geq \mu(t)$ *for* $0 \leq s \leq t$.

We assume that the failure rate function $r(t)$ is a real-valued differentiable function $r(t) : R^+ \rightarrow R^+$. As usual, by increasing we mean non-decreasing and by decreasing, we mean non-increasing. $r(t)$ is said to be

1. strictly increasing if $r'(t) > 0$ for all t;
2. strictly decreasing if $r'(t) < 0$ for all t;
3. bathtub shaped if $r'(t) < 0$ for $t \in (0, t_o)$, $r'(t_o) = 0$, $r'(t) > 0$ for $t > t_o$;
4. upside-down bathtub shaped if $r'(t) > 0$ for $t \in (0, t_o)$, $r'(t_o) = 0$, $r'(t) < 0$ for $t > t_o$;
5. modified bathtub shaped if $r(t)$ is first increasing and then bathtub shaped;
6. roller-coaster shaped if there exist n consecutive change points $0 < t_1 < t_2 < \ldots < t_n < \infty$ such that in each interval $[t_{j-1}, t_j]$, $1 \leq j \leq n+1$, where $t_o = 0, t_{n+1} = \infty$, $r(t)$ is strictly monotone and it has opposite monotonicity in any two adjacent such intervals.

A class of life distributions that has received considerable attention is the class of BFR life distributions, see Rajarshi and Rajarshi (1988) for a systematic review. We say that F is BFR model, if failure rate decreases first, then remains constant for a period, and eventually increases over time. In other words, the failure rate function has bathtub shape. This corresponds to the three distinct phases of a unit: early life, useful life and wear out as shown in Figure 1.1. In the initial region that begins at time zero, product is characterized by a high but rapidly decreasing failure rate. This region is known as the early failure period (also referred to as infant mortality period). Next, the failure rate levels off and remain roughly constant for the majority of the life of the product. This long period of a constant failure rate is known as the useful period. Many systems spend most of their lifetimes operating in this flat portion of the bathtub curve. Finally, if unit remain in use long enough, the failure rate increase as materials wear out and degradation failures occur at an ever increasing rate. This is called the wear-out period (Rajarshi and Rajarshi (1988)).

Another important family of life distributions is known as the upside-down bathtub-shaped failure rate (UBFR) class. Chang (2000) proposed a UBFR model. We say that F is its upside-down bathtub shaped failure rate, its failure rate increases first, then remains constant for a period, and eventually decreases over time.

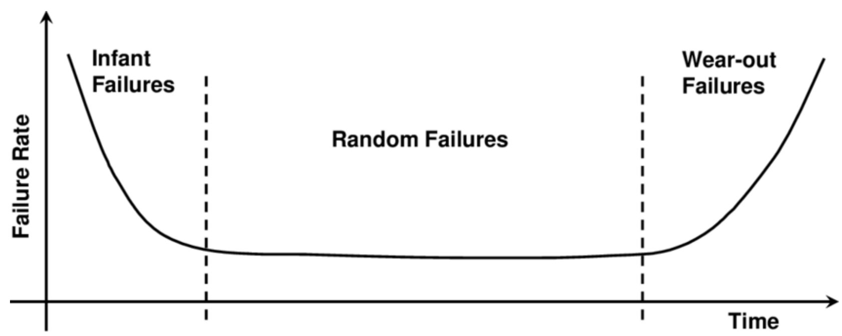


Figure 1.1: Bathtub failure rate curve

1.3 Failure Rates of Mixtures of Distributions

Mixture of distributions arise naturally in a number of reliability situations. For example, suppose a manufacturer produces $p100$ percent of a certain product in production line 1 and $(1-p)100$ percent in production line 2, $0 \leq p \leq 1$. Suppose, the life length of a unit produced in production line 1 has distribution F_1, where as the life length of a unit produced in production line 2 has distribution $F_2(\neq F_1)$. After production, units from both production lines will be allowed to campaign together, in such a way that outgoing lots consist of a random mixture of the output of the two production lines. Then, a unit selected at random from a lot would have life distribution $F = pF_1 + (1-p)F_2$, a mixture of the two underlying distributions. More generally, the distribution being mixed may be uncountably infinite in number. See Barlow and Proschan (1975) for more details.

Mixtures are important in burn-in procedures. The pdf of a mixture of two sub-populations with density functions f_1 and f_2 is $f(t) = pf_1(t) + (1-p)f_2(t),\ t \geq 0,\ 0 \leq p \leq 1$. Survival function of a mixture is also a mixture of the two survival functions, i.e., $\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t)$. The mixture failure rate is $r(t) = \frac{pf_1(t)+(1-p)f_2(t)}{p\bar{F}_1(t)+(1-p)\bar{F}_2(t)}$ where $f_i(t),\ \bar{F}_i(t)$ are the pdf and survival function of the distribution having failure rate $r_i(t), i = 1, 2$, see Lai and Xie (2006) for more details.

Below Lai and Xie (2006) has given some examples in mixture failure rates.

Example 1.3.1. Consider two IFR Weibull distributions with pdfs $f_1(t) = 2t\exp\{-t^2\}$, $t > 0$ and $f_2(t) = 3t^2\exp\{-t^3\}$, $t > 0$. If $p = 0.5$, $r(t)$ is IFR.

The next example shows that a mixture of two IFR distributions results in a DFR distribution.

Example 1.3.2. Let $r_1(t) = 1 - \exp\{-5t\}$, $t > 0$, $r_2(t) = 6 - \exp\{-5t\}$, $t > 0$. We note that $r_1(t)$ strictly increases to 1 and $r_2(t)$ strictly increases to 6. However, if $p = 0.5$, $r(t)$ is DFR and strictly decreases to 1.

Example 1.3.3. Take $f_1(t) = \exp\{-t\}$, $t > 0$, pdf of exponential distribution, $f_2(t) = 16t \exp\{-4t\}$, $t > 0$, pdf of Gamma distribution with IFR property. Let $p = 0.5$. In this case, $r(t)$ is UBFR.

Example 1.3.4. Let $f_1(t) = 4 \exp\{-4t\}$, $t > 0$, pdf of exponential distribution, $f_2(t) = t \exp\{-t\}$, $t > 0$, pdf of Gamma distribution with IFR property. Let $p = 0.5$. Then $r(t)$ is BFR.

Example 1.3.5. Consider two Weibull distributions, $f_1(t) = 2t \exp\{-t^2\}$, $t > 0$ and $f_2(t) = 4t^3 \exp\{-t^4\}$, $t > 0$. Let $p = 0.5$; both $r_1(t)$ and $r_2(t)$ increases to ∞. The mixture failure rate $r(t)$ is BFR, see Jiang and Murthy (1998).

Example 1.3.6. Consider the mixture of two Gamma probability densities: $f(t) = pf_1(t) + (1-p)f_2(t)$ where $f_i(t) = \frac{\lambda^{\alpha_i} t^{\alpha_i}}{\Gamma(\alpha_i)} e^{-\lambda t}$, $t > 0$, α_i, $\lambda > 0$, $i = 1, 2$. Assuming $\alpha_1 < \alpha_2$, Glaser (1980) was able to determine the shape of the failure rate of the distribution in all cases except for one case: $\alpha_1 > 1$, $\alpha_2 - \alpha_1 > 0$ with $\alpha_1 - 1 < (\alpha_2 - \alpha_1 - 1)^2/4$. For this case, he conjectured that the mixture density is IFR.

Jiang and Murthy (1998) categorized the possible shapes of failure rate function for a mixture of two Weibull distributions. The mixture failure rate of two strictly IFR Weibull distributions with the same shape parameter can be either BFR or IFR. The asymptotic behavior of mixtures of exponentials has been studied by Clarotti and Spizzichino (1990). Al-Hussaini and Sultan (2001) has given a comprehensive review on reliability and failure rates of mixture models. Finkelstein and Esaulova (2001) considered several types of continuous mixtures of IFR distributions.

1.3.1 Mean Residual Life

Let $\bar{F}$ be the survival function of an item with a finite first moment μ and X be the r.v that corresponds to $\bar{F}$ assuming $F(0) = 0$. The residual life r.v at age t is same as the remaining lifetime after the time of inspection. The mean residual life (MRL) (also known as the mean remaining life) is defined as $\mu(t) = E(X - t | X > t)$ which can be given as

$$\mu(t) = E(X - t | X > t) = \left[\frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)\ dx \right]. \tag{1.3.1}$$

Clearly, $\mu(0) = \mu = E(X)$. If F has a density f, then we can write

$$\mu(t) - t = \left(\int_t^\infty x\ f(x)\ dx \right) / \bar{F}(t). \tag{1.3.2}$$

Park (1985) found that, the time at which a bathtub failure rate is a minimum does not maximize the mean residual life. The mean residual life function $\mu(t)$ in the constant failure rate region of a bathtub shaped failure curve is not constant but decreasing.

1.3.2 Decreasing Percentile Residual Life Function

The α-percentile residual life function (α-percentile RLF) was first defined by Haines and Singpurwalla (1974). Joe and Proschan (1984) showed that this function may be expressed as

$$q_{\alpha,F}(t) = F^{-1}(1 - (1-\alpha)\bar{F}(t)). \tag{1.3.3}$$

A distribution is DFRL-α, if and only if for some α, $0 < \alpha < 1$, $q_{\alpha,F}(t)$ decreases in t.

Launer (1993) has shown that a BFR distribution is DPRL-α for all $\alpha_o < \alpha < 1$ for some $\alpha_o > 0$, provided there exists a t_o with $r(t_o) \geq r(0)$.

1.4 Stress-strength Reliability

The stress-strength reliability model has attracted a great deal of attention in the fields of reliability engineering, medicine and psychology. In manufacturing process, the information about the mechanical reliability of design through stress-strength model prior to production can significantly decrease the cost of production. The concept of stress and strength in engineering devices have been become the deciding factors of failure of the devices. It has been customary to define safety factors for longer lives of systems in terms of the inherent strength that they have and the external stress being experienced by the systems. If x_0 is the fixed strength and y_0 is the fixed stress that a system is experiencing, then the ratio $\frac{x_0}{y_0}$ is called safety factor and the difference $x_0 - y_0$ is called safety margin. Thus in the deterministic stress-strength situation the system survives only if the safety factor is greater than 1 or equivalently safety margin is positive, see Pratapa (2012).

In the traditional approach to design a system, the safety factor or the safety margin is constructed to resolve uncertainties in the values of stress and strength. Uncertainties in the stress and strength of a system therefore tend to cause the system life to be viewed as random variables. However, the probabilistic analysis demands the use of random variables for the concepts of stress and strength for the evaluation of survival probabilities of such systems. This analysis is particularly useful in situations in which no fixed bound can be put on the stress. For example, with earthquakes, floods and other natural phenomena, stress can lead to failures of systems with unusually small strengths. Similarly when economics is the primary criterion rather than safety, it is best to compare survival performance by understanding the increase in the likelihood of failure when stress and strength are close to each other.

1.5 Total Time on Test Transform

Total time of test (TTT) transform is widely accepted as a statistical tool with applications in various fields such as reliability analysis, econometrics, cryptocurrency modeling, tail ordering, order of delivery, etc. An important part of the literature on TTT transformation deals with reliability issues, including the nature of aging features, model identification, testing of assumptions, age replacement policies, adjusting life distributions, and defining new types of life distributions. TTT transform of a lifetime distribution F is defined as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(x))\ dx \quad t \in [0,1]$$

where $F^{-1}(t) = \inf\{x : F(t) \geq t\}$. The scaled TTT transform is defined as $\phi(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)}$. A detailed description on TTT can be seen in Barlow and Campo (1975).

1.6 Burn-in

Burn-in is a widely used engineering screening technique to eliminate vulnerable units. Systems can be electronic systems such as circuit boards having different types of chips and printed circuits. An air conditioner having a condenser, fan and circuits is an example for a typical mechanical system. A population of components may include both strong components with long lifetimes and weak components with very short lifetimes. To ensure that only strong components are given to the customer, a manufacturer can subject all components to tests in normal or harsh use conditions so that the weak components will fail during the test, leaving only the strong components. This type of test can be performed on systems to determine weak or strong components or to detect defects during assembly. These tests are usually called burn-in tests in reliability.

Let the lifetime T of a component have a continuous bathtub shaped failure rate $r(t)$. This component is required to accomplish a mission which lasts for time τ . The reliability of completing the mission is thus $\bar{F}(\tau)$. If we burn-in the component for a time b and if the component survives the burn-in, then the conditional reliability of accomplishing the mission is given by

$$\frac{\bar{F}(b+\tau)}{\bar{F}(b)} = \exp\left\{-\int_b^{b+\tau} r(t)\ dt\right\}.$$

1.7 Goodness of Fit Tests

There are different methods that can be used for testing whether a given random sample $x_1, x_2, \ldots, x_n$, of n observations, are coming from a population with specific distribution or for comparing the underlying distribution with other distributions for fitting a given data set. Some of the test for the confirmation of distributions are given below.

1.7.1 Kolmogorov-Smirnov Test

Kolmogorov (1933) proposed the Kolmogorov-Smirnov test (K-S test) for testing whether a given random sample $x_1, x_2, \ldots, x_n$ belongs to a population with a specific distribution or not. The K-S test calculates the distance between the empirical distribution function of the given sample and the estimated cdf of the distribution. The null and alternative hypotheses are H_0 : sample follow the specific distribution versus $H_1 : H_0$ is false.

Let $F(x_i)$ denote the value of the cumulative distribution function of the candidate distribution at x_i and $\hat{F}(x_i)$ denote the value of the empirical distribution function at x_i. The value of the K-S test statistic is defined by

$$\text{K-S test statistic} = \max\left\{|F(x_i) - \hat{F}(x_i)|, |F(x_i) - \hat{F}(x_{i-1})|\right\},$$

where $\hat{F}(x_i) = \frac{\{x_j : x_j \leq x_i\}}{n}$.

The computed K-S statistic is then compared with the tabulated K-S value at a pre-specified significance level to decide whether a distribution is appropriate or not. Moreover, if there are more than one distributions to be compared, the distribution with smaller K-S value will be more appropriate to fit the given sample.

1.7.2 Anderson-Darling Test

The Anderson-Darling (A-D) test is used to test if a sample of data is coming from a population with a specific distribution. It is a modification of the K-S test and gives more weight to the tails than does the K-S test. The A-D test makes use of the specific distribution in calculating critical values. This has the advantage of allowing a more sensitive test and the disadvantage that critical values must be calculated for each distribution, see Stephens (1974). The A-D test is defined as: H_0 : The data follow a specified distribution. H_1 : The data do not follow the specified distribution. The test statistic is

$$A^2 = -N - S$$

where

$$S = \sum_{i=1}^{N} \frac{(2i-1)}{N}[\log F(Y_i) + \log 1 - F(Y_{N+1-i})].$$

F is the cdf of the specified distribution. Note that the Y_i are the ordered data.

1.7.3 Akaike's information criterion

Akaike's information criterion (AIC) compares the quality of a set of statistical models to each other. AIC will take each model and rank it from the best to the worst. The "best" model may be inappropriate or overly compatible. AIC is usually calculated with software. The basic formula is defined as:

$$\text{AIC} = -2l + 2K,$$

where:

- K is the number of model parameters.
- l denotes the log-likelihood function. Log-likelihood is a measure of model fit. The higher the number, the better the fit.

1.7.4 Bayesian information criterion

Bayesian information criterion (BIC) is a criterion for model selection among a finite set of models. It is based, in part, on the likelihood function, and it is closely related to AIC. Mathematically BIC can be defined as

$$\text{BIC} = -2l + K \log n,$$

- K is the number of model parameters.
- l denotes the log-likelihood function. Log-likelihood is a measure of model fit.
- n is the sample size.

The theory of AIC requires that the log-likelihood has been maximized. When comparing models fitted by maximum likelihood to the same data, the smaller the BIC, the better the fit. Standard Normal (SN) distribution is uses for obtaining asymptotic distribution of estimators.

1.8 Objectives of the Study

The objectives of the study are listed below.

1. To study on bathtub shaped failure rate distributions and its applications for modeling life time data.
2. To propose new bathtub shaped failure rate models.
3. To compare existing bathtub shaped failure rate distributions.
4. To study on the stress-strength reliability models and its estimation process.

5. To enhance the application of bathtub shaped failure rate distribution in system engineering and other scientific area.

6. To develop the theory and application of TTT transformation in identification of bathtub shaped failure rate model.

7. To explore the applications of bathtub shaped failure rate models in Burn in process.

1.9 An Outline of the Present Work

The thesis is arranged into eight chapters. Two new lifetime distributions for modeling bathtub shaped failure rate distributions and one lifetime distribution for modeling upside down bathtub shaped failure rate distribution are proposed. Stress-strength reliability estimation in the context of multi-component reliability data has been done using Three-Parameter Generalized Lindley (TPGL) distribution and Power Lindley (PL) distribution. Identification procedure of failure rate distribution of increasing convex (concave) transformation of lifetime data is given.

The chapters of thesis are organized as below.

In *chapter 1*, basic concepts and definitions used in this thesis are given.

In *chapter 2*, extensive reviews of some of these bathtub (or upside down bathtub) shaped failure rate distributions have been presented. This review includes the existing bathtub life distributions that have been proposed in the last several years. In order to attain the results of proposed research work, a review study has been conducted on increasing, decreasing, bathtub shaped, upside down bathtub shaped and constant failure rate distributions. The importance of bathtub shaped failure rate distribution and its practical relevance are studied.

In *chapter 3*, two new bathtub shaped failure rate distributions, Generalized X-Exponential Distribution and Weibull-Lindley distribution are proposed and studied in detail. The new distributions provided a better fit than other well known distributions. Some of the mathematical properties, moments, moment generating function, characteristics function and order statistics, etc., are studied. The estimation of parameters by maximum likelihood is discussed. The proposed distributions are applied to several real data sets and compared with some other bathtub shaped life distributions.

In *chapter 4*, a new upside down bathtub shaped failure rate distribution, based on DUS transformation using Lomax distribution as baseline, is proposed. A very few study on upside down bathtub shaped failure rate models are available in literature. The shapes of its probability density and failure rate functions are investigated. Some of the properties including moments, moment generating function, characteristic function, quantiles, entropy of DUS Lomax distribution are studied. Distributions of minimum and maximum are obtained. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of stress-strength models is derived. Using a simulation study the performance of the maximum likelihood estimators (MLE) with respect to biases and mean squared errors are studied. The proposed distribution is applied to three real data sets and compared with other lifetime distributions.

In manufacturing, if we have any information about the mechanical reliability of design through stress-strength model prior to production, a manufacturer can significantly decrease the cost of production. The inherent strength and external stress being experienced by the systems are customary to define safety factors for their long lives. In *chapter 5*, stress-strength reliability in two different cases using three parameter Generalized Lindley distribution and Power Lindley distribution are discussed. The procedure of estimating reliability of single component and multi-component stress-strength models are considered. Performance of the MLEs are presented by the way of a simulation study. Two applications are provided to show how the distribution work in practice using real data sets.

The total time on test transforms is a widely accepted statistical tool, which has applications in different fields such as reliability analysis, econometrics, stochastic modeling, tail ordering, ordering of distributions, etc. TTT transform technique is discussed in *chapter 6* for the problem of identification of failure rate behavior of increasing convex (concave) function of random variable based on distributional properties of the baseline lifetime variable. In this chapter, various properties of TTT transform of increasing convex (concave) function of random variable are studied. Some results about the ageing patterns are investigated.

Burn-in is a technology that used to improve the quality of components and systems which delivered to a customer using the item under normal or accelerated environmental conditions prior to export. If the burn-in procedure is effective, the items delivered to the user are better than those delivered without burn-in. In *chapter 7*, expression of long run average cost function per unit time for obtaining optimal burn-in time and optimal age using Weibull Lindley and and Generalized

X-Exponential distributions are given.

In *Chapter 8*, the conclusion of the thesis is given and presented possible future work. The references are appended at the end of the thesis.

CHAPTER 2

REVIEW OF BATHTUB-SHAPED FAILURE RATE DISTRIBUTIONS

2.1 Introduction

Attempts in modeling or summarizing survival data are mainly based on three types of distributions: lifetime distributions with constant failure rate, IFR and DFR. However, there seems to be an increased interest in non-monotone failure rate distributions, especially BFR distributions and UBFR distributions. These distributions serve as adequate models for the survival time of many industrial products. Such failure rate curves are also known as the U-shaped or J-shaped curves. Many parametric families of BFR distributions have been introduced in literature during past several years. The BFR distributions are widely used in reliability engineering and survival analysis. Monotonic ageing concepts are popular among many reliability engineers. However, in many practical applications, the effect of age is initially beneficial, but after a certain period of time, it is effecting adversely. Many products, especially electronic, electro-mechanical and mechanical items have BFR distribution, see Barlow and Proshan (1975).

Glaser (1980) and Lawless (1982) have been given many examples of BFR life distributions. Hjorth (1980) described BFR distributions by mixtures of a set of IFR distribution for competing risk model. Lai et al. (2001) discussed the BFR distributions. Xie et al. (2002) studied modified Weibull extension models with BFR function useful in reliability related decision making and cost analysis. Xie et al. (2003) investigated some models extending the traditional two-parameter Weibull distribution. Navarro and Hernandez (2004) studied the shape of reliability functions by using the s-equilibrium distribution of a renewal process and also studied how to obtain distribution with BFR using mixture of two positive truncated Normal distributions. Kundu (2004) proposed two parameter exponentiated Exponential distribution and discussed several properties and different estimation procedures. Wondmagegnehu et al. (2005) studied the failure rate of the mixture of an Exponential distribution and a Weibull distribution. Block et al. (2008) discussed the continuous mixture of whole families of distribution having a BFR functions. Sarhan and Kundu (2009) derived the generalized linear failure rate distribution and its properties.

Extension of Weibull distributions to make it compatible with BFR data are introduced by Mudholkar and Srivastava (1993), Xie and Lai (1995), and Xie et al. (2002). Chen (2000) also introduced a two parameter BFR model for survival data analysis. Wang (2000) studied an additive model based on the Burr XII distribution for lifetime data with BFR. Wang et al. (2014) derived Weibull extension with BFR function based on type-II censored samples.

Recently, Lemonte (2013) proposed a new exponential type distribution with constant failure rate, IFR, DFR, UBFR and BFR function which can be used in modeling survival data in reliability problems and fatigue life studies. Zhang et al. (2013) investigated the parameter estimation of 3-parameter Weibull related model with decreasing, increasing, bathtub and upside-down bathtub shaped failure rates. Parsa et al. (2014) investigated the difference between the change points of failure rate and mean residual life functions of some generalized Gamma type distribution due to the capability of these distribution in modeling various BFR functions. Wang et al. (2015) discussed new finite interval lifetime distribution model for fitting BFR curve. Shehla and Ali khan (2016) studied reliability analysis using an exponential power model with BFR function. Zeng et al. (2016) derived two lifetime distributions, one with 4 parameters and the other with 5 parameters, for the modeling of BFR data.

Shafiq and Viertl (2017) proposed generalized estimators for the parameters and failure rates of BFR distributions used to model fuzzy lifetime data. Gauss et al. (2018) introduced new Lindley Weibull distribution which accommodates unimodal and bathtub shaped failure rates. Dey et al. (2019) introduced a new distribution alpha-power transformed Lomax distribution with decreasing and UBFR distribution. Al-abbasi et al. (2019) proposed a three parameter generalized Weibull uniform distribution that extends the Weibull distribution to have BFR or DFR property. Shoaee (2019) investigated two bivariate models, viz., bivariate

Chen distribution and bivariate Chen-Geometric distribution, that has BFR or IFR functions. Ahsan et al. (2019) studied the reliability analysis of gas-turbine engine with BFR distribution. Chen and Gui (2020) discussed the estimation problem of two parameters of a lifetime distributions with a BFR functions based on adaptive progressive type-II censored data.

The aim of this chapter is to provide a review of BFR and UBFR models. Basic definitions and results are given in section 2.2. Construction of BFR or UBFR model is recalled in section 2.3.

2.2 Definitions and Results

In this section, some definitions of BFR distributions are presented.

DEFINITION 2.2.1. *(Glaser, 1980). Let F be a cdf with a failure rate function $r(t)$ which is continuous. Then F is BFR distribution if there exists a t_o such that:*

(a) *$r(t)$ is decreasing for $t < t_o$,*

(b) *$r(t)$ is increasing for $t > t_o$. i.e., $r'(t) < 0$ for $t < t_o$, $r'(t_o) = 0$ and $r'(t) > 0$ for $t > t_o$.*

Here, when $r(t)$ is increasing (decreasing), it is strictly increasing (decreasing). The bathtub curves given in this definition would probably represent some U-shaped tubs. There is no interval for which $r(t)$ is a constant.

DEFINITION 2.2.2. *(Deshpande and Suresh, 1990). A life distribution F having support on $[0, \infty)$ is said to be a BFR distribution if there exists a point t_o such* *that* $-\log \bar{F}(t)$ *is concave in* $[0, t_o)$ *and convex in* $[t_0, \infty)$.

DEFINITION 2.2.3. *(Mitra and Basu, 1995). An absolutely continuous life distribution* F *having support* $[0, \infty)$ *is said to be a BFR distribution if there exists a,* $t_o \geq 0$ *such that* $r(t)$ *is non-increasing for* $[0, t_o)$ *and non-decreasing on* $[t_o, \infty)$.

DEFINITION 2.2.4. *(Mi, 1995). A lifetime distribution* F *is said to be a BFR distribution if there exists* $0 \leq t_1 \leq t_2 < \infty$ *such that:*

(a) $r(t)$ *is strictly decreasing if* $0 < t < t_1$;

(b) $r(t)$ *is a constant if* $t_1 \leq t \leq t_2$; *and*

(c) strictly increasing if $t > t_2$.

The points t_1 and t_2 are the change points of $r(t)$. If $t_1 = t_2 = 0$, then $r(t)$ becomes IFR, and if $t_1 = t_2 \to \infty$, then $r(t)$ becomes DFR. In general, if $t_1 = t_2$ then the interval for which $r(t)$ is constant degenerates to a single point. The points in the interval (t_1, t_2) are not change points according to Mi (1995).

If F is not absolutely continuous, then BFR property can be explained through the conditional reliability function

$$\bar{F}(x|t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}, \quad \bar{F}(t) = 1 - F(t) > 0,\, t > 0,\, x > 0. \tag{2.2.1}$$

DEFINITION 2.2.5. *(Haupt and Scabe, 1997).* F *is said to be BFR if there exists a* t_o *such that*

- $\bar{F}(x|t)$ *is increasing in* t *for* $0 \leq t < t_o,\ 0 \leq x \leq (t_o - t)$,
- $\bar{F}(x|t)$ *is decreasing in* t *for* $t_o < t < \infty,\ x \geq 0$.

2.2.1 Some Basic Properties

Mitra and Basu (1996a) presented some basic properties of the survival functions and moments of a BFR distribution. Let T represent lifetime r.v with cdf F.

- Suppose F is BFR, then $\bar{F}(t) \leq \bar{G}(t)$ where G is exponential with mean $r(t_o)^{-1}$. Here t_o is a change point at which $r(t)$ is minimum.
- $E(T^k) \leq \frac{\Gamma(k+1)}{\{r(t_o)\}^k}, \quad k > 0.$
- A BFR life distribution F with $E(T^k) = \frac{\Gamma(k+1)}{\{r(t_o)\}^k}$ is necessarily an exponential.
- Convolution of BFR distributions is not necessarily BFR.
- The mixture of BFR distributions need not be BFR.
- Suppose we have a competing risk model: $\bar{F}(t) = \bar{F}_1(t)\bar{F}_2(t)$ where the lifetime of each component is BFR with a common turning point t_o. Then the lifetime of the system is again has a BFR distribution with t_o as one of its turning points.
- A parallel system of two independent BFR components need not be BFR.

In many distributions, survival functions and failure rate functions do not have an analytically tractable form. In such cases, Glaser's technique can be applied, Glaser (1980).

Assume that the pdf $f(t)$ of T is positive on $(0, \infty)$ and that it is twice differentiable on $(0, \infty)$. Let $\eta(t) = -\frac{f'(t)}{f(t)}$, $g(t) = \frac{1}{r(t)}$, then, we have following the results. Clearly $g'(t) = \int_t^\infty [f(y)/f(t)] [\eta(t) - \eta(y)] \, dy$.

Theorem 2.2.1. *(Glaser, 1980)*

(a) *If* $\eta'(t) > 0$ *for all* $t > 0$, *then F is IFR.*

(b) *If* $\eta'(t) < 0$ *for all* $t > 0$, *then F is DFR.*

(c) $\exists\, t_o > 0$ *such that* $\eta'(t_o) = 0$, $\eta'(t) < 0 \;\; \forall \;\; t \in (0, t_o)$ *and* $\eta'(t) > 0 \;\; \forall \;\; t > t_o$.

  (i) *If* $\exists\, y_o > 0$ *such that* $g'(y_o) = 0$, *then F is BFR.*

  (ii) *If there does not exists* $y_o > 0$ *such that* $g'(y_o) = 0$, *then F is IFR.*

(d) $\exists\, t_o > 0$ *such that* $\eta'(t_o) = 0$, $\eta'(t) > 0 \;\; \forall \;\; t \in (0, t_o)$, *and* $\eta'(t) < 0 \;\; \forall \;\; t > t_o$.

  (i) *If* $\exists\, y_o > 0$ *such that* $g'(y_o) = 0$, *then F is UBFR.*

  (ii) *If there does not exists* $y_o > 0$ *such that* $g'(y_o) = 0$, *then F is DFR.*

We can use $\eta'(t)$ when the failure rate function is very complicated or not determined.

Following two results from Glaser further ease the computations by avoiding the complications which usually arise because of the function $g(t)$.

Lemma 2.2.2. (a) *Let* $\epsilon = \lim_{t \downarrow 0} f(t)$. *If the condition (c) of Theorem 2.2.1 hold and if* $\epsilon = \infty$, *then the corresponding distribution is BFR. If the condition (d) of Theorem 2.2.1 hold and if* $\epsilon = 0$, *then the corresponding distribution is UBFR.*

(b) *Let* $\delta = \lim_{t \downarrow 0} g(t)\eta(t)$. *If the condition (c) of Theorem 2.2.1 hold and if* $\delta > 1$, *then the corresponding distribution is BFR. If the condition (d) of Theorem 2.2.1 hold and if* $\delta < 1$, *then the corresponding distribution is UBFR.*

The conditions on ϵ and δ in the above results are reflections of the high infant mortality rate, a characteristic of the bathtub distributions.

2.3 Construction Techniques for Bathtub Distributions

There are lots of ways available for the construction of BFR distribution. Schabe (1994a) has constructed BFR distributions from DFR distributions by truncations. Techniques for construction of a BFR model are given below.

1. **Convex function:** Define a BFR by choosing a positive convex function $r(t)$ over $(0, \infty)$ such that $\int r(t)\ dt = \infty$ (Rajarshi and Rajarshi, 1988). The distribution having a failure rate function $r(t) = \exp\{\alpha + \beta t + \gamma t^2\}$, a strictly increasing function of BFR function, is BFR.

2. **Glaser's technique:** Glaser's theorem (Theorem 2.2.1 above) can be applied to derive new bathtub distributions. i.e., we can choose a function $\eta(t)$ that satisfies the conditions of the theorem.

3. **Function of random variables:** This procedure is due to Griffith (1982). Let T be a continuous r.v and let $g(u)$ be a strictly increasing function which is differentiable on $[0, \infty)$ with $g(0) = 0$. Let g^{-1} be the inverse function of g. Then the failure rate function of $g(T)$ is given by

$$r_{g(T)}(t) = r_T(g^{-1}(t))[g^{-1}(t)]' \tag{2.3.1}$$

Let T be an exponential r.v and let $g(u)$ be a strictly increasing differentiable function on $[0, \infty)$ with $g(0) = 0$. If g is a convex on $(0, \tau]$ and concave on $[\tau, \infty)$ for a positive τ, then the distribution of $g(T)$ is a bathtub distribution.

4. **Series system (competing risk model):** Suppose we have a series system of two independent components. Everyone knows that the system failure rate is the sum of the two component failure rates. If one of them has IFR distribution and the other has DFR distribution, the system distribution may be BFR. This type of models are obtained by Murthy et al. (1973).

5. **Mixtures:** Mixtures of distributions often give rise to bathtub distributions. For example, Glaser (1980) showed that Gamma mixture has BFR distribution. The mixture of the two increasing linear failure rate distributions provided a BFR distribution, Block et al. (2008).

6. **Sectional models:** Shooman (1968) reported BFR with piecewise linear shape in three areas. Other sectional models that give rise to bathtub distributions were given in Murthy and Jiang (1997).

7. **Polynomial of finite order:** Jaisingh et al. (1987) suggested a polynomial of finite order failure model: $r(t) = a_0 + a_1 t + \ldots + a_n t_n$. As the constants $a_i,\ i = 0, \ldots, n$ may be positive or negative, bathtub shapes for $r(t)$ can be achieved.

8. **TTT Transform:** In Kunitz (1989) and Haupt and Schabe (1997), the TTT transform was used to construct parametric bathtub life distributions.

9. **Truncation of DFR distribution:** Schabe (1994a) has constructed BFR distributions from DFR distributions by truncations.

2.4 Some BFR and UBFR Distributions

Several parametric families of BFR and UBFR life distributions have been constructed in various contexts over the past two decades. Ideally, we should classify them into groups according to some common characteristics. (1) Lifetime distributions with explicit expressions for failure rates and (2) Lifetime distributions with inefficient or unknown failure rate functions.

Quadratic model and its generalization

Bain (1978) considered a quadratic failure rate model,

$$r(x) = \alpha + \beta x + \gamma x^2, \quad \alpha \geq 0, \quad -2(\alpha\gamma)^{1/2} \leq \beta < 0, \; \gamma > 0, \, x > 0, \qquad (2.4.1)$$

which has a bathtub shape. Here, $r(0) = \alpha, \; r(x) \to \infty$ as $x \to \infty$. It is easy to verify that $\hat{r}(x) = e^{r(x)}$ also has a bathtub shape if $r(x)$ has a bathtub shape.

A flexible family

Gaver and Acar (1979) have proposed a BFR model with $r(x) = \lambda + g(x) + k(x)$ where $g(x)$ is a non-negative decreasing function of x with $\lim_{x\to\infty} g(x) \to 0$ whereas $k(x)$ is an increasing function of x such that $k(0) = 0, \; \lim_{x\to\infty} k(x) \to \infty$ and λ is any real number such that $r(x) \to 0$. This is a popular method for making BFR functions. Several special cases of this family are presented below:

- $r(x) = \lambda + \frac{\theta}{x+\phi} + \alpha x^p, \quad x > 0, \phi > 0, \alpha > 0, \theta > 0, p > 0$; the model is an extension of Murthy et al. (1973). If both $g(x)$ and $k(x)$ are failure rate functions, then this model is simply a competing risks model involving three distributions.

- $r(x) = \theta_1 \alpha_1 x^{\alpha_1 - 1} + \theta_2 + \theta_3 \alpha_3 x^{\alpha_3 - 1},\ \alpha_3 > 2,\ 0 < \alpha_1 < 1$. $r(x) \to \infty$ as $x \to 0$ or ∞, the model is studied by Canfield and Borgman (1975).

Additive Weibull Distribution

Xie and Lai (1995) considered a competing risk model involving two Weibull distributions. For $\alpha > 0$, $\theta > 0$, $\beta > 0$, $\gamma < 1$, pdf, cdf and failure rate function are

$$
\begin{aligned}
f(x) &= \left(\alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1}\right) e^{-\alpha x^{\theta} - \beta x^{\gamma}},\ x > 0, \\
F(x) &= 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}},\ x > 0 \\
\text{and} \qquad r(x) &= \alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1},\ x > 0. \qquad (2.4.2)
\end{aligned}
$$

The function $r(x)$ has a bathtub shape when $\alpha < 1$ and $\beta > 1$. Also $r(0) = r(\infty) = \infty$. The turning point x_o is given by

$$x_o = \left[\frac{\alpha(1-\alpha)\theta^{\alpha}}{\beta(1-\beta)\gamma^{\beta}}\right]^{\frac{1}{\alpha-\beta}}.$$

Additive Burr XII Distribution

Wang (2000) considered an additive Burr XII model that combines two Burr XII distributions, one with DFR property and another with IFR property. The failure rate function of the additive Burr XII is given by

$$r(x) = \frac{k_1 c_1 (x/s_1)^{c_1 - 1}}{s_1[1 + (x/s_1)^{c_1}]} + \frac{k_2 c_2 (x/s_2)^{c_2 - 1}}{s_2[1 + (x/s_2)^{c_2}]}, \quad x > 0, \tag{2.4.3}$$

where $k_1 \geq 0$, $k_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, $0 < c_1 < 1 \geq 0$, $c_2 > 2$. It was shown that $r(x)$ has bathtub shape.

Modified Weibull Distribution

Lai et al. (2003) proposed a modified Weibull (MW) distribution with cdf

$$F(x) = 1 - e^{-\beta x^{\gamma} e^{\lambda x}}, \quad \beta > 0,\, \gamma,\, \lambda \geq 0,\, x > 0,$$

where at most one of γ, λ is equal to zero. The pdf of the MW distribution is

$$f(x) = \beta(\gamma + \lambda x)\, x^{\gamma - 1} e^{\lambda x} e^{-\beta x^{\gamma} e^{\lambda x}}, \qquad \beta > 0,\, \gamma,\, \lambda \geq 0,\, x > 0.$$

The corresponding failure rate function is

$$r(x) = \beta(\gamma + \lambda x)\, x^{\gamma - 1} e^{\lambda x}, \quad \beta > 0,\, \gamma,\, \lambda \geq 0,\, x > 0. \tag{2.4.4}$$

The pdf of the MW distribution can be unimodal or decreasing. The failure rate function can be increasing or bathtub shaped.

Sectional model with two Weibull Distributions

Murthy and Jiang (1997) have considered two sectional models involving two Weibull distributions having failure rate function

$$r(x) = \begin{cases} (\alpha_1/\beta_1)(x/\beta_1)^{\alpha_1-1}; & 0 \leq x \leq x_o, \alpha_1 > 0,\ \beta_1 > 0 \\ (\alpha_2/\beta_2)\left(\frac{x-\gamma}{\beta_2}\right)^{\alpha_2-1}; & x_o < x < \infty, \alpha_2 > 0,\ \beta_2 > 0 \end{cases}$$

with change point $x_o = [\beta_1^{\alpha_1}(\alpha/\beta_2)^{\alpha_2}]^{1/(\alpha_1-\alpha_2)},\ \gamma = (1-\alpha)x_o$ where $\alpha = \frac{\alpha_2}{\alpha_1}$, and $r(x)$ is continuous at x_o. i.e.,

For $\alpha_1 < \alpha_2,\ r(x)$ have a bathtub shape if $\alpha_1 < 1$ and $\alpha_2 > 1$.

Exponential power Distribution

Smith and Bain (1975) studied the exponential power model having density function

$$f(x) = \lambda\alpha(\lambda x)^{\alpha-1}\exp\{-(e^{(\lambda x)^\alpha} - (\lambda x)^\alpha - 1)\},\ x > 0,\ \alpha > 0,\ \lambda > 0.$$

The survival function is

$$\bar{F}(x) = \exp\{-(e^{(\lambda x)^\alpha} - 1)\},\ x > 0,\ \alpha > 0,\ \lambda > 0. \qquad (2.4.5)$$

The failure function is given by

$$r(x) = \lambda\alpha(\lambda x)^{\alpha-1}e^{(\lambda x)^{\alpha}},\ x > 0,\ \alpha > 0,\ \lambda > 0. \tag{2.4.6}$$

For $\alpha < 1,\ r(x) \to \infty$ when $x \to 0$ or $x \to \infty$, $r(x)$ has bathtub shape.

Weibull Extension Distribution

Consider the case $\lambda = 1$ in the exponential power model. Then (2.4.5) becomes

$$\bar{F}(x) = \exp\{-(e^{x^{\alpha}} - 1)\}, \alpha > 0,\ x > 0. \tag{2.4.7}$$

Chen (2000) introduced another parameter λ to the distribution specified in (2.4.7), so that the new cdf becomes

$$\bar{F}(x) = \exp\{-\lambda(e^{x^{\alpha}} - 1)\},\ \alpha > 0,\ \lambda > 0,\ x > 0$$

with failure rate function

$$r(x) = \lambda\alpha x^{\alpha-1}e^{x^{\alpha}}. \tag{2.4.8}$$

The parameter λ here does not alter the shape of the failure rate function so (2.4.8) behaves similarly to the function given in (2.4.6). In particular, $r(x)$ is increasing for $\alpha \geq 1$ and $r(x)$ is bathtub shaped for $\alpha < 1$.

Double Exponential power Distribution

Paranjpe et al. (1985) considered the following model having failure rate function

$$r(x) = \beta\alpha x^{\alpha-1} e^{(\beta x^{\alpha})} \exp\{e^{(\beta x^{\alpha})} - 1\},\ \alpha < 1,\ \beta > 0,\ x > 0. \tag{2.4.9}$$

The above expression is obviously quite complex. Clearly, $r(x) \to \infty$ as $x \to 0$ or $x \to \infty$, so a bathtub shape is obtained.

Power-function Distribution

Mukherjee and Islam (1983) proposed a finite range distribution with a bathtub failure rate:

$$r(x) = \frac{px^{p-1}}{\theta^p - x^p},\quad 0 \le x < \theta,\ p < 1,\ \theta > 0 \tag{2.4.10}$$

and $r(x) \to \infty$ when $x \to 0$ or θ, thus bathtub is obtained.

Beta failure rate Distribution

Moore and Lai (1994) proposed another finite range distribution, an extension of beta function, with BFR function,

$$r(x) = c(x+p)^{a-1}(q-x)^{b-1},\quad 0 < a < 1,\ b < -1,\ 0 \le x < q,\ c > 0,\ p \ge 0. \tag{2.4.11}$$

Clearly $r(0) = cp^{a-1}q^{b-1},\ r(x) \to \infty$ as $x \to q$.

Integrated beta failure rate Distribution

Lai et al. (1998) considered a lifetime distribution with failure rate function

$$r(x) = x^{a-1}(1-x)^{b-1}\{a-(a+b)x\},\ 0<x<1,\ a>0,\ b>0. \tag{2.4.12}$$

It is obvious that $r(x) \to \infty$ as $x \to 0$ or 1 and hence $r(x)$ is bathtub shaped.

J-shaped Distribution

Topp and Leone (1955) proposed J-shaped distribution. Nadarajah and Kotz (2003) discussed moments of J-shaped distribution. For $0 < v < 1$, $b > 0$, J-shaped distribution has pdf and cdf

$$f(x) = \frac{2v}{b}\left(\frac{x}{b}\right)^{v-1}\left(1-\frac{x}{b}\right)\left(2-\frac{x}{b}\right)^{v-1},\ 0<x<b,$$

and

$$F(x) = \begin{cases} \left(\frac{x}{b}\right)^{v}\left(2-\frac{x}{b}\right)^{v}; & \text{if } 0 \le x \le b < \infty \\ 0; & \text{if } x < 0 \\ 1; & \text{if } x > b. \end{cases}$$

The failure rate function $r(x)$ is

$$r(x) = \frac{2v}{b}\frac{y(1-y^2)^{v-1}}{1-(1-y^2)^{v}}, \tag{2.4.13}$$

where $y = 1-(x/b)$. $r(x) \to \infty$ as $x \to 0$ and $x \to b$ for all $v \in (0,1)$ and $r(x)$ attains a minimum at $x = x_0$, where $y_0 = 1-(x_0/b)$ is the root of the equation

$(1-y)^v = 1 - 2vy/(1+y)$.

Beta Distribution

For $p > 0$, $q > 0$, $0 < x < 1$, the standard beta distribution has pdf and cdf

$$f(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}$$

and

$$F(x) = \frac{B_x(p,q)}{B(p,q)}$$

respectively, where $B_x(.,.)$ is the incomplete beta function defined by

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt,\ 0 < x < 1,\ p > 0,\ q > 0.$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q) - B_x(p,q)},\ 0 < x < 1,\ p > 0,\ q > 0. \tag{2.4.14}$$

Ghitany (2004) showed that $r(x)$ is bathtub-shaped if $p < 1$.

Mukherjee and Roy's Distribution

Mukherjee and Roy (1993) defined a distribution having pdf

$$f(x) = (\delta(|x-a| + |x-b|)/(b-a)) \exp\left[-\delta a^2/(b-a) - \delta x + (-1)^{k(x-a,|x-a|)}\right]$$
$$\exp\left[((b-a)\delta/4)((|x-a| + |x-b|)/(b-a) - 1)^2\right],\ x > 0,\ 0 < a < b < \infty,\ \delta > 0,$$

and cdf

$$\begin{aligned} F(x) =& 1 - \exp\big[-\delta a^2/(b-a) - \delta x \\ & + (-1)^{k(x-a,|x-a|)}((b-a)\delta/4)((|x-a|+|x-b|)/(b-a)-1)^2\big], \end{aligned}$$

where $k(x, y)$ is Kronecker's function taking the value 1 when $x = y$ and 0 whenever $x \neq y$. The failure rate function $r(x)$ can be expressed as

$$r(x) = \delta(|x-a|+|x-b|)/(b-a). \tag{2.4.15}$$

It is clear that $r(x)$ take bathtub shape for all values of δ, a and b.

Haupt and Schabe's Distribution

Haupt and Schabe (1992) developed a distribution having pdf

$$f(x) = \begin{cases} \frac{1+2\beta}{2T\sqrt{\beta^2+(1+2\beta)x/T}}; & \text{if } 0 \le x \le T,\ \infty < \beta < \infty,\ T > 0, \\ 0; & \text{otherwise} \end{cases}$$

and cdf

$$F(x) = \begin{cases} 1; & \text{if } x \ge T, \\ -\beta + \sqrt{\beta^2 + (1+2\beta)x/T}; & \text{if } 0 \le x \le T, \\ 0; & \text{otherwise.} \end{cases}$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \begin{cases} \frac{1+2\beta}{2T\sqrt{\beta^2+(1+2\beta)x/T}(1+\beta-\sqrt{\beta^2+(1+2\beta)x/T})}; & \text{if } 0 \leq x \leq T, \\ 0; & \text{otherwise} \end{cases} \quad (2.4.16)$$

$r(x)$ is bathtub-shaped if $-1/3 < \beta < 1$ and $r(x)$ attaining the minimum at $x_0 = T(1+2\beta-3\beta^2)/[4(1+2\beta)]$.

Schabe's Distribution

Schabe (1994a) considered a simple distribution having pdf

$$f(x) = \frac{2\gamma + (1-\gamma)x/\theta}{\theta(\gamma + x/\theta)^2},\ x \leq \theta,\ \theta > 0,\ -\infty < \gamma < \infty$$

and cdf

$$F(x) = \frac{(1+\gamma)x}{\theta\gamma + x}.$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{1}{\theta(\gamma + x/\theta)} + \frac{1}{\theta(1 - x/\theta)}. \quad (2.4.17)$$

$r(x)$ is bathtub-shaped if $\gamma < 1$ and minimum of $r(x)$ occurring at $x_0 = (\theta/2)(1-\gamma)$. $r(x) \to \infty$ as $x \to \infty$.

Hjorth's Distribution

For $\delta > 0,\ \beta > 0,\ \theta > 0$, Hjorth (1980) proposed a distribution with pdf and cdf

$$f(x) = \frac{[(1+\beta x)\delta x + \theta]exp(-\delta x^2/2)}{(1+\beta x)^{\frac{\theta}{\beta+1}}},\ x > 0$$

and

$$F(x) = 1 - \frac{\exp(-\delta x^2/2)}{(1+\beta x)^{\frac{\theta}{\beta}}},\ x > 0$$

respectively. The failure rate function $r(x)$ can be expressed as

$$r(x) = \delta x + \frac{\theta}{1+\beta x}. \tag{2.4.18}$$

It is easily seen that $r(x)$ is bathtub-shaped if $0 < \delta < \theta\beta$.

Gamma mixture Distribution

Gupta and Warren (2001) examined the mixture of Gamma distributions having pdf

$$f(x) = \frac{px^{\alpha_1 - 1}e^{-x/\beta}}{\beta^{\alpha_1}\Gamma(\alpha_1)} + \frac{(1-p)x^{\alpha_2 - 1}e^{-x/\beta}}{\beta^{\alpha_2}\Gamma(\alpha_2)},$$

for $x > 0,\ \alpha_1 > 0,\ \alpha_2 > 0,\ \beta > 0$ and $0 < p < 1$. Corresponding failure rate function is bathtub-shaped if either $\beta_1 = \beta_2,\ \alpha_1 > 1$ and $\alpha_2 < 1$ or $\alpha_1 > 2$.

Normal mixture Distribution

Navarro and Hernandez (2004) examined the mixture of truncated Normal distributions with pdf

$$f(x) = \frac{p}{\sqrt{2\Pi}\sigma_0\Phi(\mu_0/\sigma_0)}\exp\left[-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right] + \frac{1-p}{\sqrt{2\Pi}\sigma_1\Phi(\mu_1/\sigma_1)}\exp\left[-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right],$$

for $x > 0,\ -\infty < \mu_0 < \infty,\ -\infty < \mu_1 < \infty,\ \sigma_0 > 0,\ \sigma_1 > 0$ and $0 < p < 1$, in which the corresponding failure rate function exhibiting a bathtub shape.

Exponentiated Weibull Distribution

Mudholkar et al. (1995) introduced the exponentiated Weibull distribution with pdf

$$f(x) = a\alpha\lambda^\alpha x^{\alpha-1}e^{-(\lambda x)^\alpha}(1-e^{-(\lambda x)^\alpha})^{a-1},\ x > 0,\ a > 0,\ \alpha > 0,\ \lambda > 0$$

and cdf

$$F(x) = (1-e^{-(\lambda x)^\alpha})^a,\ x > 0,\ a > 0,\ \alpha > 0,\ \lambda > 0$$

respectively. The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{a\alpha(\lambda x)^{\alpha-1}e^{-(\lambda x)^\alpha}(1-e^{-(\lambda x)^\alpha})^{a-1}}{1-(1-e^{-(\lambda x)^\alpha})^a} \tag{2.4.19}$$

in which $r(x)$ is bathtub-shaped if $\alpha > 1$ and $a\alpha < 1$.

Stacy's Weibull Distribution

Stacy (1962) proposed a distribution with pdf

$$f(x) = c\beta^{-c\alpha}(\Gamma\alpha)^{-1}x^{c\alpha-1}e^{-(x/\beta)^c}$$

and cdf

$$F(x) = (\Gamma\alpha)^{-1}\gamma\left(\alpha, \left(\frac{x}{\beta}\right)^c\right),$$

where $x > 0,\ c > 0,\ \alpha > 0,\ \beta > 0,\ \gamma(\cdot,\cdot)$ denotes the incomplete Gamma function defined by

$$\gamma(a,\ x) = \int_0^x t^{a-1}e^{-t}dt,\ x > 0,\ a > 0.$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{c\beta^{-c\alpha}x^{c\alpha-1}e^{-(x/\beta)^c}}{(\Gamma\alpha)^{-1}\gamma\left(\alpha, \left(\frac{x}{\beta}\right)^c\right)}. \tag{2.4.20}$$

$r(x)$ is bathtub-shaped if $c > 1$.

Truncated Weibull Distribution

McEwen and Parresol (1991) proposed doubly truncated Weibull distribution having pdf

$$f(x) = \frac{p(x)}{P(b) - P(a)}$$

and cdf

$$F(x) = \frac{P(x) - P(a)}{P(b) - P(a)},$$

where $0 \leq a < x < b < \infty$, $p(x)$ and $P(x)$ are the pdf and cdf, respectively, of the traditional Weibull distribution, $p(x) = \alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}}$, and $P(x) = 1 - e^{-(\lambda x)^{\alpha}}$, for $x > 0,\ \alpha > 0$ and $\lambda > 0$. Corresponding failure rate function has bathtub shape.

Xie et al.'s Weibull Distribution

Xie et al. (2002) proposed a modification of the Weibull distribution having pdf

$$f(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left[\left(\frac{x}{\alpha}\right)^{\beta} + \lambda\alpha\left\{1 - \exp\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right]$$

and cdf

$$F(x) = 1 - \exp\left[\lambda\alpha\left\{1 - \exp\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right],$$

where $x > 0,\ \lambda > 0,\ \alpha > 0$ and $\beta > 0$. The failure rate function $r(x)$ can be expressed as

$$r(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(\frac{x}{\alpha}\right)^{\beta}. \qquad (2.4.21)$$

It is easily seen that $r(x)$ is bathtub-shaped if $0 < \beta < 1$ with $r(x)$ attaining the minimum at $x_o = \alpha(1/\beta - 1)^{1/\beta}$ and $r(x)$ increases to ∞ as $x \to 0$ and ∞.

Generalized Lindley Distribution

Nadarajah et al. (2011) proposed Generalized Lindley (GL) distribution whose cdf is

$$F(x) = \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda}e^{-\lambda x}\right]^{\alpha}$$

for $x > 0$, $\lambda > 0$, and $\alpha > 0$. The failure rate function is given by

$$r(x) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{\alpha-1} e^{-\lambda x}[1 - V^{\alpha}(x)]^{-1} \qquad (2.4.22)$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$, where $V(x) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]$. The shape of the failure rate function appears monotonically decreasing or to initially decrease and then increase, a bathtub shape if $\alpha < 1$, the shape appears monotonically increasing if $\alpha \geq 1$.

Burr XII Distribution

The Burr XII distribution was first introduced by Burr (1942). Burr XII distribution having reliability function

$$\bar{F}(x) = \frac{1}{(1+x^c)^k},\ k,\ c > 0,\ x > 0.$$

The failure rate function is

$$r(x) = \frac{kcx^{c-1}}{(1+x^c)} \qquad (2.4.23)$$

For $c \leq 1$, the slope is always negative, for $c > 1$ the slope is positive for $x^c < c-1$ and negative for $x^c > c-1$. Thus $r(x)$ is decreasing for $c \leq 1$ and UBFR if $c > 2$.

Birnbaum and Saunders Distribution

Birnbaum and Saunders (1969a,b) introduced a lifetime distribution

$$\begin{aligned} F(x) &= \Phi\left\{\frac{1}{\alpha}.\left[\left(\frac{x}{\beta}\right)^{1/2} - \left(\frac{x}{\beta}\right)^{-1/2}\right]\right\} \\ &= \Phi\left\{\frac{1}{\alpha}\xi\left(\frac{x}{\beta}\right)\right\},\ x > 0, \end{aligned}$$

where $\xi(x) = x^{1/2} - x^{-1/2},\ \alpha,\ \beta > 0$ and $\Phi(.)$ denotes the cdf of the standard normal. While the failure rate of Birnbaum and Saunders is zero at $x = 0$, then increases to a maximum for some x_0 and finally decreases to a finite positive value (i.e., $r(x)$ is UBFR) when $\beta = 1$ and $\alpha > 0.8$, the failure rate of the log-Normal also has a UBFR function but decreases to zero.

Inverse Gaussian Distribution

The name 'inverse Gaussian' was first applied to a certain class of distributions by Tweedie (1947). The density function of the inverse Gaussian is

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right],\ \lambda > 0,\ x \geq 0.$$

The corresponding distribution function is

$$F(x) = \Phi\left\{\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right\}$$

and the expression for $r(x)$ is quite complicated. By Glaser's theorem, $r(x)$ is UBFR.

Inverse Weibull Distribution

The cdf of two-parameter inverse Weibull distribution is given by

$$F(x) = \exp\left\{-\left(\frac{\alpha}{x}\right)^{\beta}\right\}, \ \alpha, \beta > 0, \ x > 0.$$

The failure rate function is

$$r(x) = \beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-1}. \tag{2.4.24}$$

It has been shown that $\lim_{x\to 0} r(x) = \lim_{x\to\infty} r(x) = 0$ and $r(x) \in$ UBFR.

Log-Logistic Distribution

The pdf, cdf and failure rate of Log-Logistic distribution are given by

$$f(x) = \frac{k\rho(x\rho)^{k-1}}{[1+(\rho x)^k]^2}, \ x > 0, \ \rho > 0, \ \ k > 0$$
$$F(x) = 1 - \frac{1}{1+(\rho x)^k}$$

and

$$r(x) = \frac{k\rho(\rho x)^{k-1}}{1+(\rho x)^k}. \tag{2.4.25}$$

It can be shown easily that $r(x)$ is DFR when $k \leq 1$; $r(x)$ is UBFR when $k > 1$.

Log-Normal Distribution

The distribution function and failure rate function of log-Normal distribution are

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

and

$$r(x) = \frac{(1/2\pi x\sigma)\exp\{-(\log ax)^2/2\sigma^2\}}{1 - \Phi\{\log ax/\sigma\}}, \tag{2.4.26}$$

where $a = e^{-\alpha}$. $r(0) = 0$ and $r(x) \to 0$ when $x \to \infty$.

2.5 Optimal burn-in time

Burn-in plays an important role in reliability engineering, Jensen and Petersen (1982). In this section, we discuss the concepts of burn-in.

2.5.1 Concepts of burn-in

Due to the high failure rate (most importantly, silicon and integrated circuits) in the early stages of module life, burn-in is widely accepted as a method of screening failures before sending or delivering these components to the field operations. That is, before delivery to the customers, the components are tested in approximate electrical or thermal conditions that approximate the working conditions in field operation. The components that fail in the burn-in process are removed or repaired, and only those that survive the burn-in process are considered to be of high quality. These are how users send them to the field or to the field function.

With adequate burn-in, a high initial failure rate results in high maintenance costs. A general background on burn-in can be found in Kuo and Kuo (1983), and Kuo (1984).

Estimating change points is particularly relevant for BFR in the context of maintenance policies, naturally, do not consider replacing a component of such a life distribution before the 'threshold' unknown age t_o is reached. Kulasekera and Saxena (1991) have constructed kernel density estimators and empirical cdf to estimate f, F, r and its change point.

Mitra and Basu (1995) considered the problem of estimating change points in different monotonic aging models. Suresh (1992) also obtained two estimates of the change point; one by using the definition of BFR distribution, another by a characterization of BFR distribution in terms of TTT transform. Nguyen et al. (1984) considered the estimation of the turning point of a two-step piecewise linear failure rate function. Pham and Nguyen (1993) considered estimating the turning point of a truncated BFR function. Gupta et al. (1999) considered estimation of the turning point of the failure rate function in the case of the log-Logistic model. Details about optimal burn-in time can be seen in Myung and Young (2002). The expressions of long run average cost function per unit time for obtaining optimal burn-in time and optimal age for various distributions is an open problem.

This chapter gave a comprehensive review of known BFR and UBFR distributions. Some tools and methods that will be used for data analysis are also reviewed.

CHAPTER 3

NEW BATHTUB SHAPED FAILURE RATE DISTRIBUTIONS

3.1 Introduction

[1] In many applied sciences such as medicine, engineering, bio statistics, survival analysis etc modeling and analysis of lifetime data are crucial, Deshpande and Suresh (1990). In analyzing lifetime data one often uses the Exponential, Gamma, Weibull and Generalized Lindley distributions. It is well known that Exponential distribution has constant hazard function, Generalized Lindley distribution has a BFR function whereas Weibull and Gamma distributions have constant or monotone (increasing/decreasing) failure rate functions. In this chapter we present two new simple distributions which have BFR function. The proposed distributions

[1] Some contents of this chapter are based on Chacko and Deepthi (2018 & 2019).

are capable of modeling the real problems.

In this chapter, Generalized X-Exponential distribution and Weibull-Lindley Distribution are discussed in section 3.2 and 3.3. Summary are given in section 3.4.

3.2 Generalized X-Exponential distribution

Here we consider a new distribution named as Generalized X-Exponential distribution, having BFR function, which generalizes the distribution having df $F(x) = 1 - (1 + \lambda x^2)e^{-\lambda x}$, $x > 0$, $\lambda > 0$, Chacko (2016). The failure rate function of X-Exponential distribution appears monotonically decreasing and bathtub shape. The generalization considered is the distribution of a series system having distribution function $F(x) = 1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}$, $x > 0$, $\lambda > 0$, for its components.

In section 3.2.1, the distribution function of the Generalized X-Exponential distribution (GXE) is given. In section 3.2.2 discussed the statistical properties of the distribution. In section 3.2.3 discussed the distribution of maximum and minimum to address the reliability problems of parallel system and series system, respectively. The maximum likelihood estimation of the parameters is explained in section 3.2.4. In section 3.2.5 discussed the asymptotic confidence bounds of MLEs of the distribution. In section 3.2.6 a simulation study is given. Two real data sets are analyzed in section 3.2.7 and the results are compared with some existing distributions.

3.2.1 Generalized X-Exponential Distribution

Let X be a life time r.v having cdf with parameter α and λ,

$$F(x) = \left(1 - (1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}, \quad x > 0,\ \alpha > 0,\ \lambda > 0. \qquad (3.2.1)$$

Clearly $F(0) = 0$, $F(\infty) = 1$, F is non-decreasing and right continuous. More over F is absolutely continuous. Then the pdf of the r.v X is given by

$$f(x) = \alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1) - 2\lambda x)\left(1 - (1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}, \quad x > 0,\ \alpha, \lambda > 0. \qquad (3.2.2)$$

Here α and λ are the shape and scale parameters. It is clear that F is a positively skewed distribution. The distribution with pdf of the form (3.2.2) is named as GXE distribution with parameters α and λ and denoted by GXE(α, λ). Failure rate function of GXE distribution is

$$r(x) = \frac{\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1) - 2\lambda x)\left(1 - (1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}}{1 - (1-(1+\lambda x^2)e^{-\lambda(x^2+x)})^{\alpha}}, \quad x > 0,\ \alpha, \lambda > 0. \qquad (3.2.3)$$

Considering the behavior near the change point $x_0, x_0 > 0$ and if $\frac{d}{dx}h(x_0) = 0$.

(i) If $0 < \alpha < 1/2$, and $0 < \lambda < 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$ and $\frac{d}{dx}h(x) > 0$ when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(ii) If $0 < \alpha < 1/2$, and $\lambda > 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$, $\frac{d}{dx}h(x) > 0$

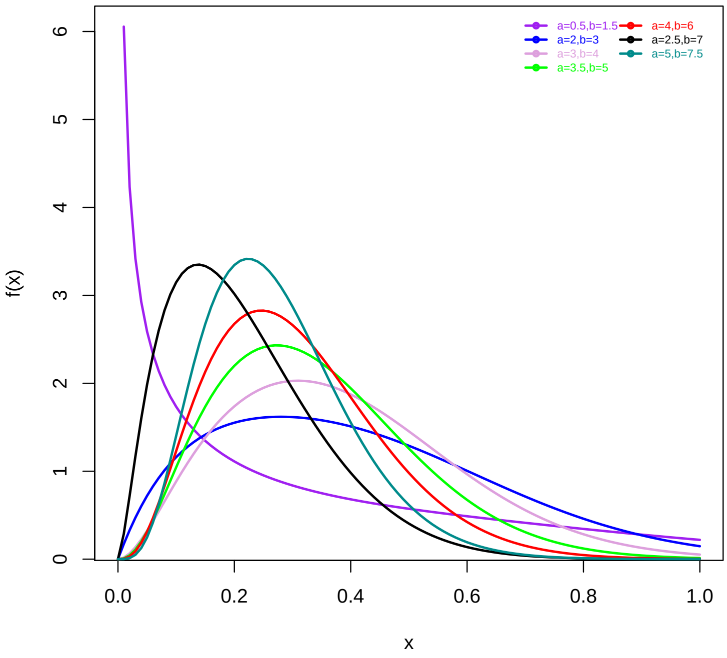


Figure 3.1: PDF of GXE distribution for values of parameters $\alpha = 0.5, 2, 3, 3.5, 4, 2.5, 5$ and $\lambda = 1.5, 3, 4, 5, 6, 7, 7.5$ with color shapes purple, blue, plum, green, red, black and dark cyan, respectively.

when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(iii) If $1/2 < \alpha < 1$, and $0 < \lambda < 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$ and $\frac{d}{dx}h(x) > 0$ when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(iv) If $1/2 < \alpha < 1$, and $\lambda > 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$ and $\frac{d}{dx}h(x) > 0$ when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(v) If $\alpha > 1$, and $\lambda > 1$, then $\frac{d}{dx}h(x) > 0$ for $x > 0$.

The shape of (3.2.3) appears monotonically decreasing or to initially decrease and then increase, a bathtub shape if $\alpha < 1$. GXE(α, λ) allows for monotonically decreasing, monotonically increasing and bathtub shapes for its failure rate function. As α decreases from 1 to 0, the graph shift above whereas if λ increases from 1 to ∞ the shape of the graph concentrate near to 0, see Figures. 3.1, 3.2, 3.3.

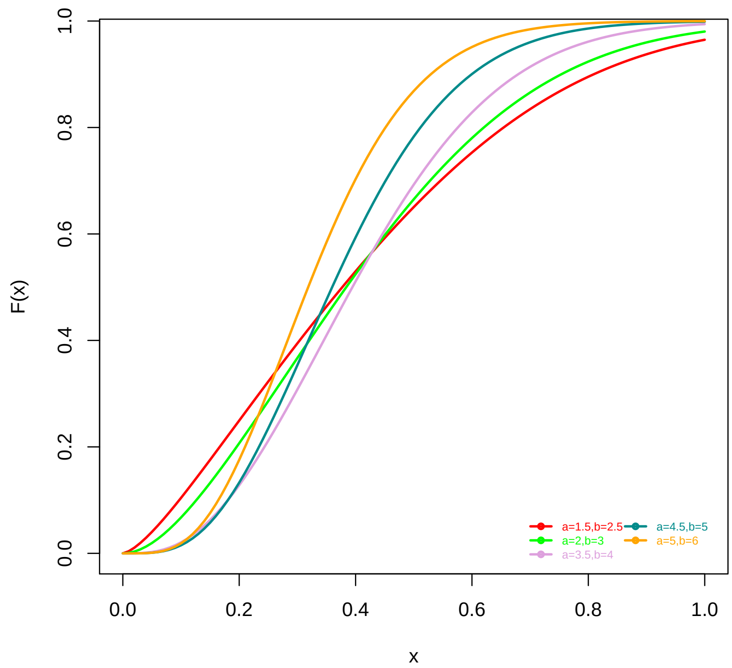


Figure 3.2: CDF of GXE distribution for values of parameters $\alpha = 1.5, 2, 3.5, 4.5, 5$ and $\lambda = 2.5, 3, 4, 5, 6$ with color shapes red, green, plum, dark cyan and orange respectively.

3.2.2 Moments

In order to calculate moments of X, we require the following lemma.

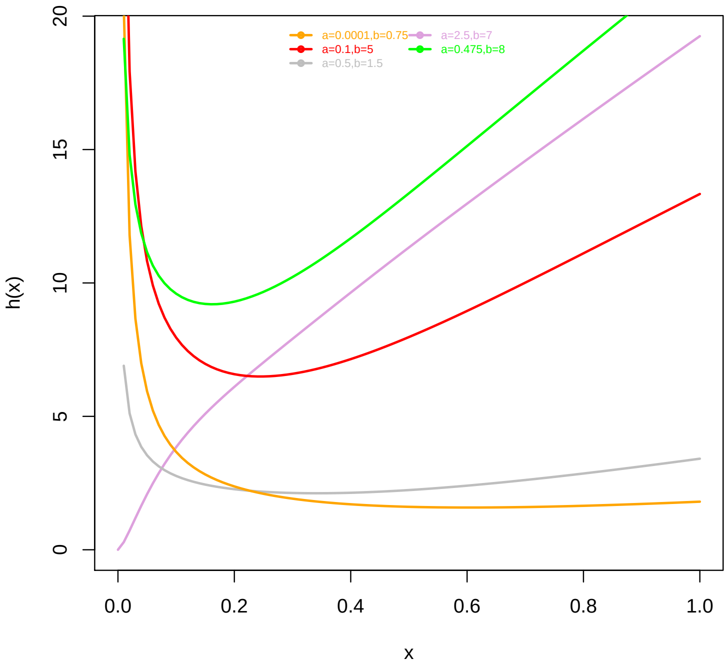


Figure 3.3: Failure rate function GXE distribution for values of parameters $\alpha = 0.0001, 0.1, 0.5, 2.5, 0.475$ and $\lambda = 0.75, 5, 1.5, 7, 8$ with color shapes orange, red, grey, plum and green, respectively.

Lemma 3.2.1. *For* $\alpha, \lambda > 0$, $x > 0$,

$$K(\alpha, \lambda, c) = \int_0^\infty x^c \left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)}\, dx.$$

Then,

$$K(\alpha, \lambda, c) = \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \lambda^j \int_0^\infty x^{2j+c} e^{-\lambda(x^2+x)}\, dx.$$

Proof. Using Binomial expansion, $(1-z)^{\alpha-1} = \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i}(-1)^i z^i$, we have,

$$\begin{aligned} K(\alpha, \lambda, c) &= \int_0^\infty x^c \left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)}\, dx \\ &= \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i}(-1)^i \int_0^\infty x^c \left[(1+\lambda x^2)e^{-\lambda(x^2+x)}\right]^i e^{-\lambda(x^2+x)}\, dx \\ &= \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i}(-1)^i \int_0^\infty x^c \sum_{j=0}^{i} \binom{i}{j} (\lambda x^2)^j e^{-(i+1)\lambda(x^2+x)}\, dx \\ &= \sum_{i=0}^{\alpha-1}\sum_{j=0}^{i} \binom{\alpha-1}{i}\binom{i}{j}(-1)^i \lambda^j \int_0^\infty x^{2j+c} e^{-\lambda(x^2+x)}\, dx. \end{aligned}$$

The result of the lemma follows by the definition of the Gamma function. The first raw moment is

$$E(X) = \alpha\lambda K(\alpha,\lambda,1) + 2\alpha\lambda^2 K(\alpha,\lambda,4) + \alpha\lambda^2 K(\alpha,\lambda,3).$$

The n^{th} raw moment is

$$E(X^n) = \alpha\lambda K(\alpha,\lambda,n) + 2\alpha\lambda^2 K(\alpha,\lambda,n+3) + \alpha\lambda^2 K(\alpha,\lambda,n+2).$$

□

Moment Generating Function

Moment generating function can be obtained from following formula

$$M_X(t) = \int_0^\infty e^{tx}\alpha e^{-\lambda(x^2+x)}\left(\lambda+\lambda x^2+2\lambda^2x^3\right)\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}\, dx$$

$$= \int_0^{\infty} \alpha \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1 - (1 + \lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)+tx} \; dx.$$

Characteristic Function

Characteristic function can be obtained from following formula

$$\begin{aligned}\phi_X(t) &= \int_0^{\infty} e^{itx} \alpha e^{-\lambda(x^2+x)} \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1 - (1 + \lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} \; dx \\ &= \int_0^{\infty} \alpha \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1 - (1 + \lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)+tx} \; dx.\end{aligned}$$

Mean Deviation About Mean

The scatter in a population is measured by using Mean deviation about the mean μ is defined by

$$\begin{aligned}\text{MD(mean)} &= 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} x f(x) \; dx \\ &= 2\mu F(\mu) - 2\mu + 2\left(\alpha\lambda L(\alpha, \lambda, 1, \mu) + 2\alpha\lambda^2 L(\alpha, \lambda, 4, \mu)\right) \\ &+ 2\alpha\lambda^2 L(\alpha, \lambda, 3, \mu)\end{aligned}$$

where

$$\begin{aligned}L(\alpha, \lambda, c, \mu) &= \int_{\mu}^{\infty} x^c \left(1 - (1 + \lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)} \; dx \\ &= \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \lambda^j \left(\int_{\mu}^{\infty} x^{2j+c+1} e^{-(j+1)\lambda(x^2+x)} dx\right).\end{aligned}$$

Similarly, Mean deviation about the Median M is defined by

$$\begin{aligned}\text{MD(Median)} &= -M + 2\int_M^\infty xf(x)\,dx \\ &= -M + 2\left(\alpha\lambda L(\alpha,\lambda,1,M) + 2\alpha\lambda^2 L(\alpha,\lambda,4,M)\right) \\ &+ 2\alpha\lambda^2 L(\alpha,\lambda,3,M).\end{aligned}$$

3.2.3 Distribution of Maximum and Minimum

Series, Parallel, Series-Parallel and Parallel-Series systems are general system structure of many engineering systems. The theory of order statistics provides a useful tool for analysing life time data of such systems. Let $X_1, X_2, \ldots, X_n$ be a random sample from GXE distribution with cdf and pdf as in (3.2.1) and (3.2.2), respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(r)}$ is given by,

$$\begin{aligned}f_{(r:n)}(x) = &\frac{1}{B(r,n-r+1)}\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{r-1} \\ &\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-r}\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1)-2\lambda x) \\ &\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}, \quad x>0,\ \alpha,\lambda>0. \qquad (3.2.4)\end{aligned}$$

The cdf of $X_{(r)}$ is given by

$$\begin{aligned}F_{r:n}(x) &= \sum_{j=r}^{n}\binom{n}{j}F^j(x)[1-F(x)]^{n-j} \\ &= \sum_{j=r}^{n}\binom{n}{j}\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{j}\end{aligned}$$

$$\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-j}, \quad x>0,\ \alpha,\lambda>0. \tag{3.2.5}$$

The pdf of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$ are respectively given by:

$$f_1(x)=\frac{1}{B(1,n)}\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-1}\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}$$
$$\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1)-2\lambda x),\ x>0,\ \alpha,\lambda>0, \tag{3.2.6}$$

and

$$f_n(x)=\frac{1}{B(n,1)}\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-1}$$
$$\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1)-2\lambda x)\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1},$$
$$x>0,\ \alpha,\lambda>0. \tag{3.2.7}$$

The cdf of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$ are respectively given by

$$F_1(x)=1-\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n}, \quad x>0,\ \alpha,\lambda>0$$

and

$$F_n(x)=\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n}, \quad x>0,\ \alpha,\lambda>0.$$

Reliability of a series system having n components with GXE(α, λ) is

$$R(x) = \left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^n.$$

Reliability of a parallel system having n components having GXE(α, λ) is

$$R(x) = 1-\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^n.$$

Both the reliability functions can be used in various reliability calculations.

3.2.4 Parameter Estimation

In this section, estimation of the unknown parameters of the GXE by using the method of moments and method of maximum likelihood is explained.

Let $X_1, X_2, \ldots, X_n$ are random sample taken from GXE. Let $m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Equating sample moments to population moments we get moment estimators for parameters.

$$m_1 = \alpha\lambda K(\alpha,\lambda,1) + 2\alpha\lambda^2 K(\alpha,\lambda,4) + \alpha\lambda^2 K(\alpha,\lambda,3)$$
$$m_2 = \alpha\lambda K(\alpha,\lambda,2) + 2\alpha\lambda^2 K(\alpha,\lambda,5) + \alpha\lambda^2 K(\alpha,\lambda,4)$$

where $K(\alpha,\lambda,1) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{i}\binom{\alpha-1}{i}\binom{i}{j}(-1)^i\lambda^j\int_0^{\infty} x^{2j+1}e^{-\lambda(x^2+x)}\,dx$. The solution of these equations are moment estimators.

To find MLE, consider likelihood function as,

$$
\begin{aligned}
L(x;\alpha,\lambda) &= \prod_{i=1}^{n} f(x_i) \\
&= \alpha^n e^{-\lambda\sum_{i=1}^{n}(x_i^2+x_i)} \prod_{i=1}^{n}(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i) \\
&\prod_{i=1}^{n}\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)^{\alpha-1}.
\end{aligned}
$$

The log-likelihood function is,

$$
\begin{aligned}
l = \log L(x;\alpha,\lambda) = n\log\alpha &+ \sum_{i=1}^{n}\log\big(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i\big) \\
&- \lambda\sum_{i=1}^{n}(x_i^2+x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right).
\end{aligned}
$$

The first partial derivatives of the log-likelihood function with respect to the two-parameters are

$$
\begin{aligned}
\frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right) \\
\frac{\partial l}{\partial \alpha} &= 0 \\
\implies \hat{\alpha} &= -\frac{1}{n}\log\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right) \qquad (3.2.8) \\
\text{and } \frac{\partial l}{\partial \lambda} &= -\sum_{i=1}^{n}(x_i^2+x_i) + \sum_{i=1}^{n}\frac{((2x_i+1)(1+2\lambda x_i^2)-2x_i)}{(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i)} \\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{((1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)-e^{-\lambda(x_i^2+x_i)}x_i^2)}{\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)} \\
\frac{\partial l}{\partial \lambda} &= 0
\end{aligned}
$$

$$\implies \sum_{i=1}^{n}(x_i^2+x_i)=\sum_{i=1}^{n}\frac{((2x_i+1)(1+2\lambda x_i^2)-2x_i)}{(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i)}$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{((1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)-e^{-\lambda(x_i^2+x_i)}x_i^2)}{\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)}. \tag{3.2.9}$$

Solving this system of equations (3.2.8) and (3.2.9) in α and λ gives the MLE of α and λ. Estimates can be obtained by using 'nlm' package in R software with arbitrarily initial values.

3.2.5 Asymptotic Confidence bounds

In this section, since the MLEs of the unknown parameters $\alpha > 0$ and $\lambda > 0$ cannot be obtained in closed forms, we derive the asymptotic confidence intervals of these parameters when $\alpha > 0$ and $\lambda > 0$, by using variance covariance matrix I^{-1}, where I^{-1} is the inverse of the observed information matrix which is defined as follows

$$I=\begin{pmatrix} E(-\frac{\partial^2 l}{\partial\alpha^2}) & E(-\frac{\partial^2 l}{\partial\alpha\lambda}) \\ E(-\frac{\partial^2 l}{\partial\lambda\alpha}) & E(-\frac{\partial^2 l}{\partial\lambda^2}) \end{pmatrix}$$
$$=\begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{Cov}(\hat{\lambda},\hat{\alpha}) & \mathrm{Var}(\hat{\lambda}) \end{pmatrix}.$$

The second partial derivatives are

$$\frac{\partial^2 l}{\partial\alpha^2}=-\frac{n}{\alpha^2},$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \sum_{i=0}^{n} \frac{\Big(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i\Big)[2x_i^2(2x_i+1)]}{\Big(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i\Big)^2}$$
$$- \sum_{i=0}^{n} \frac{((2x_i+1)(1+2\lambda x_i^2) - 2x_i)^2}{\Big(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i\Big)^2}$$
$$+ \sum_{i=1}^{n} \frac{\Big(x_i^2 e^{-\lambda(x_i^2+x_i)} - (1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)\Big) + e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)}{\big(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\big)}$$
$$- \sum_{i=1}^{n} \frac{\Big((1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i) - x_i^2 e^{-\lambda(x_i^2+x_i)}\Big)^2}{\big(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\big)^2}$$

and $$\frac{\partial^2 l}{\partial \alpha \lambda} = \sum_{i=1}^{n} \frac{(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i) - x_i^2 e^{-\lambda(x_i^2+x_i)}}{\big(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\big)}.$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters α and λ by using variance matrix as in the form

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{\mathrm{Var}(\hat{\alpha})},\ \hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{\mathrm{Var}(\hat{\lambda})}$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})^{\text{th}}$ percentile of the SN distribution.

3.2.6 Simulation

To understand the performance of the MLEs given by (3.2.8) and (3.2.9) with respect to sample size n, a simulation study for assessment is considered:

(i) Generate five thousand samples from (3.2.2). Using Newton Raphson method, values of the GXE random variable are generated using

$$(1 + \lambda x^2)e^{-\lambda(x^2+x)} = 1 - u^{\frac{1}{\alpha}}$$

where $u \sim U(0, 1)$.

(ii) Compute the MLEs for the five thousand samples, say (α_i, λ_i) for $i = 1, 2, \ldots, 5000$.

(iii) Compute the biases of the estimator and mean squared errors using $\text{bias}_h(n) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{h}_i - h)$ and $\text{MSE}_h(n) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{h}_i - h)^2$ for $h = (\alpha, \lambda)$.

We repeated these steps for $n = 10, 20, \ldots, 100$ with different values of parameters, for computing $\text{bias}_h(n)$ and $\text{MSE}_h(n)$ for $n = 10, 20, \ldots, 100$.

3.2.7 Data Analysis

In this section, we present the analysis of a real data for using the GXE(α, λ) model and compare it with Generalized Lindley (GL) distribution using AIC, BIC and K-S statistic. We considered the survival data for psychiatric inpatients (Klein and Moesch Berger (1997)) to estimate the parameter values. The data are given in Table 3.2. Table 3.3 provides the parameter estimates, standard errors obtained by inverting the observed information matrix and log-likelihood values. Table 3.4 provides values of AIC, BIC, and p-values based on the K-S statistic. The corresponding probability plots and histogram are shown in Figure 3.4 and 3.5. We can see that the GXE distribution provides the smallest AIC

Table 3.1: Simulation study on different choices of parameter values

	$\alpha = 2.5$		$\lambda = 0.75$		$\alpha = 0.5$		$\lambda = 0.001$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.37515	1.40735	0.03116	0.00971	0.00256	6.537×10^{-5}	-7.531×10^{-6}	5.6723×10^{-10}
20	0.00777	0.00121	0.00967	0.00187	-0.001132	2.5621×10^{-5}	6.3116×10^{-6}	7.9673×10^{-10}
30	-0.00347	0.000361	-0.00133	5.2976×10^{-5}	0.00992	0.001967	3.9965×10^{-6}	3.1944×10^{-10}
40	0.002158	0.0001863	-0.001515	9.1865×10^{-5}	0.001568	9.8350×10^{-5}	-1.1745×10^{-6}	5.5179×10^{-11}
50	-0.008035	0.003228	0.001646	0.0001354	0.000202	2.0495×10^{-6}	5.5596×10^{-6}	1.5455×10^{-9}
60	0.00111	7.4300×10^{-5}	-7.695×10^{-6}	3.5528×10^{-9}	-0.000341	6.9802×10^{-6}	-2.9708×10^{-6}	5.2953×10^{-10}
70	0.002488	0.0004332	-0.0001722	2.0761×10^{-6}	-0.000331	7.6762×10^{-6}	-3.4171×10^{-6}	8.1738×10^{-10}
80	0.001709	0.0002337	5.6173×10^{-5}	2.5243×10^{-7}	2.4323×10^{-5}	4.7331×10^{-8}	7.1859×10^{-7}	4.1309×10^{-11}
90	0.001912	0.0003292	0.0001403	1.7714×10^{-6}	0.000429	1.6546×10^{-5}	-4.3202×10^{-7}	1.6798×10^{-11}
100	0.001929	0.0003720	0.000216	4.6694×10^{-6}	0.001706	0.0002910	2.5672×10^{-6}	6.5907×10^{-10}
	$\alpha = 1$		$\lambda = 0.5$		$\alpha = 1.5$		$\lambda = 1$	
10	0.00465	0.000216	-0.00458	0.00021	-0.05683	0.03229	-0.04672	0.02183
20	-0.00292	0.000171	0.02258	0.0102	-0.020347	0.008281	-0.00697	0.000973
30	-9.052×10^{-5}	2.4583×10^{-7}	-0.00184	0.000102	0.001420	6.0464×10^{-5}	0.005746	0.000991
40	-0.000876	3.0695×10^{-5}	-0.000901	3.2446×10^{-5}	-0.003607	0.0005203	0.006072	0.001475
50	0.00128	8.1932×10^{-5}	-0.000622	1.9383×10^{-5}	0.00479	0.001149	0.002649	0.000351
60	0.000594	2.117×10^{-5}	0.00118	8.3813×10^{-5}	0.001702	0.000174	-0.000463	1.288×10^{-5}
70	-0.00155	0.000168	-0.000892	5.5728×10^{-5}	0.003831	0.001027	0.000398	1.1106×10^{-5}
80	-0.000983	7.7277×10^{-5}	-0.000934	6.9806×10^{-5}	0.00285	0.000649	0.000584	2.7314×10^{-5}
90	0.001544	0.000214	-5.4667×10^{-5}	2.6896×10^{-7}	0.000841	6.359×10^{-5}	0.000968	8.4369×10^{-5}
100	0.001396	0.000195	0.000373	1.3918×10^{-5}	0.000537	2.8865×10^{-5}	0.000275	7.5845×10^{-6}
	$\alpha = 2.5$		$\lambda = 1.25$		$\alpha = 3$		$\lambda = 1.75$	
10	0.008654	0.000749	0.01351	0.001825	0.03068	0.009415	0.02618	0.006855
20	-0.005014	0.000503	-0.00401	0.000321	0.017103	0.00585	0.009213	0.001698
30	0.00994	0.002963	0.002236	0.000149	-0.00845	0.002143	0.00851	0.002172
40	-0.004481	0.000803	-0.000823	2.7113×10^{-5}	-0.00231	0.000213	0.003504	0.000491
50	0.00666	0.002220	-0.000140	9.8505×10^{-7}	0.005896	0.001738	0.000748	2.801×10^{-5}
60	0.000206	2.5387×10^{-6}	0.001005	6.0642×10^{-5}	0.000311	5.7859×10^{-6}	0.002391	0.000343
70	-4.04×10^{-5}	1.1425×10^{-7}	0.000722	3.6513×10^{-5}	0.001375	0.0001324	-0.000786	4.3222×10^{-5}
80	0.000882	6.2230×10^{-5}	0.001149	0.000106	0.001103	9.7265×10^{-5}	0.000192	2.9403×10^{-6}
90	0.000732	4.8281×10^{-5}	-0.000923	7.6744×10^{-5}	-0.000874	6.8733×10^{-5}	-0.000445	1.7813×10^{-5}
100	-0.000957	9.1617×10^{-5}	0.000349	1.2245×10^{-5}	8.322×10^{-5}	6.9256×10^{-7}	-0.000947	8.9755×10^{-5}

Table 3.2: Survival data for psychiatric inpatients

1	1	2	22	30	28	32	11	14	36	31	33	33
37	35	25	31	22	26	24	35	34	30	35	40	39

value and BIC value, the largest p-value based on the K-S statistic. Hence, the GXE distribution provides the better fit. The variance covariance matrix I^{-1} of

Table 3.3: MLEs of parameters, SE and Log-likelihood

Model	ML estimates	Standard error	Log L
GXE	$\hat{\alpha} = 0.6967$ $\hat{\lambda} = 0.00184$	0.03313 7.852×10^5	-99.360
GL	$\hat{\alpha} = 1.0691$ $\hat{\lambda} = 0.07547$	0.05637 0.00274	-107.657

Table 3.4: AIC, BIC, K-S Statistic and p-value of the model

Model	AIC	BIC	K-S Statistic	p value
GXE	202.721	205.237	0.2113	0.1962
GL	219.315	221.831	0.3011	0.0179

the MLEs of GXE distribution for the data set 1 is computed as

$$= \begin{pmatrix} 2.8532 \times 10^{-2} & 3.9810 \times 10^{-5} \\ 3.9810 \times 10^{-5} & 1.6031 \times 10^{-7} \end{pmatrix}.$$

Thus, the variances of the MLE of α and λ are $\text{Var}(\hat{\alpha}) = 2.853 \times 10^{-2}$ and $\text{Var}(\hat{\lambda}) = 1.603 \times 10^{-7}$, respectively. Therefore, 95% confidence intervals for α and λ are [0.6417, 0.7507] and [0.0017, 0.00197], respectively. The data set 2 consist of the lifetimes of 50 devices (Aarset (1987)) and it is provided in Table 3.5. The parameter estimates, standard errors of the estimators and the various

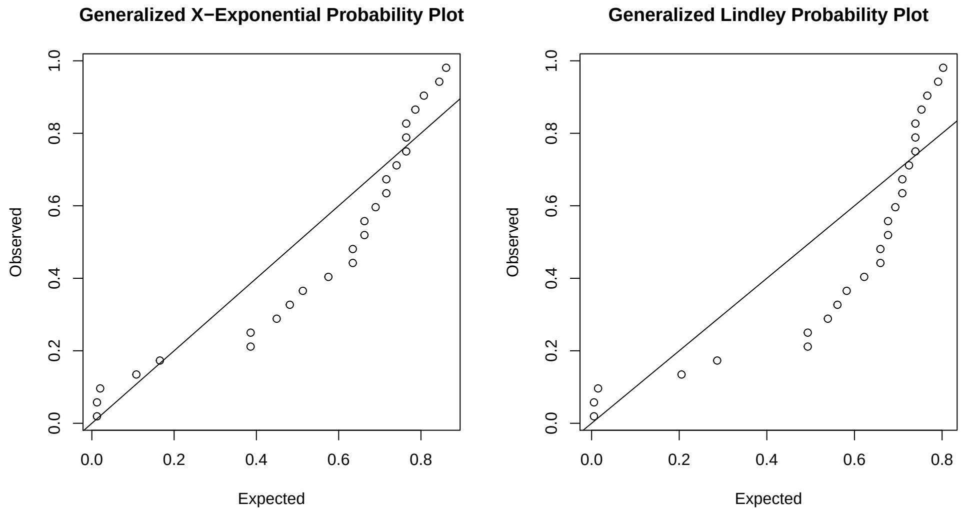


Figure 3.4: Probability plots of data set 1.

Table 3.5: Lifetimes of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18
18	18	18	21	32	36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83	84	84	84	85	85
85	85	85	86	86										

measures are given in Table 3.6. The corresponding histogram and probability plots are shown in Figure 3.6 and 3.7.

We can see again that the GXE distribution gives the smallest AIC value, the smallest BIC value, and largest p-value based on the K-S statistic, see Table 3.7. Hence, the GXE distribution again provides the better fit. The variance covariance matrix I^{-1} of the MLEs under the GXE distribution for the data set

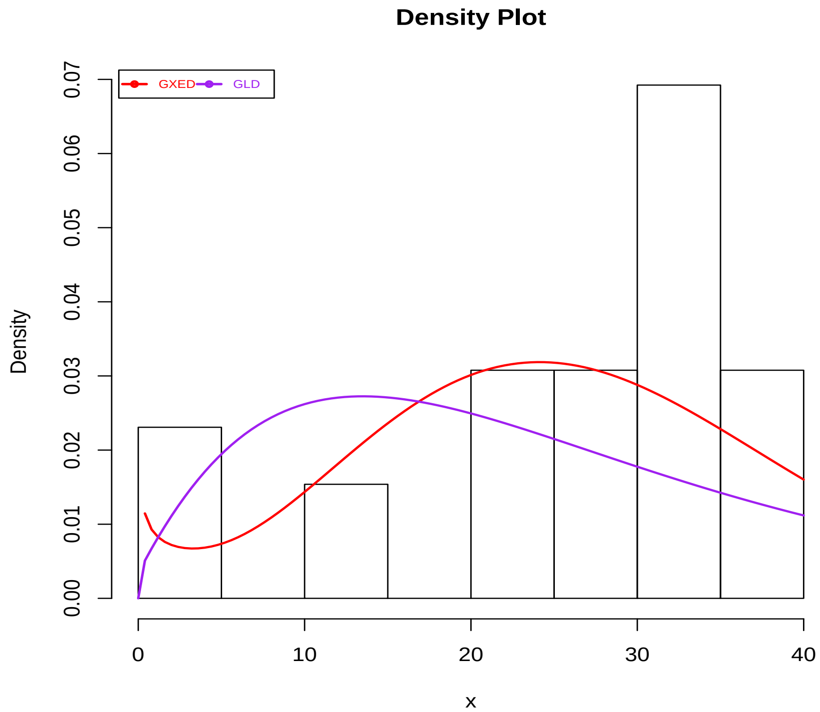


Figure 3.5: Histogram with fitted Pdfs for the data set 1.

Table 3.6: MLEs of parameters, SE and Log-likelihood

Model	ML estimates	Standard error	Log L
GXE	$\hat{\alpha} = 0.3181$ $\hat{\lambda} = 0.000302$	0.00715 1.0452×10^5	-231.609
GLD	$\hat{\alpha} = 0.4547$ $\hat{\lambda} = 0.0278$	0.01123 0.000691	-238.9909

2 is computed as

$$= \begin{pmatrix} 2.5596 \times 10^{-3} & 1.7080 \times 10^{-6} \\ 1.7080 \times 10^{-6} & 5.4625 \times 10^{-9} \end{pmatrix}.$$

Table 3.7: AIC, BIC, K-S Statistic and p-value of the model

Model	AIC	BIC	K-S Statistic	p value
GXE	467.218	471.042	0.1555	0.1783
GLD	481.982	485.806	0.1936	0.0472

Thus, the variances of the MLEs of α and λ are $\text{Var}(\hat{\alpha}) = 2.56 \times 10^{-3}$ and $\text{Var}(\hat{\lambda}) = 5.463 \times 10^{-9}$, respectively. Therefore, 95% confidence intervals for α and λ are [0.3064, 0.3299] and [0.00029, 0.00032], respectively.

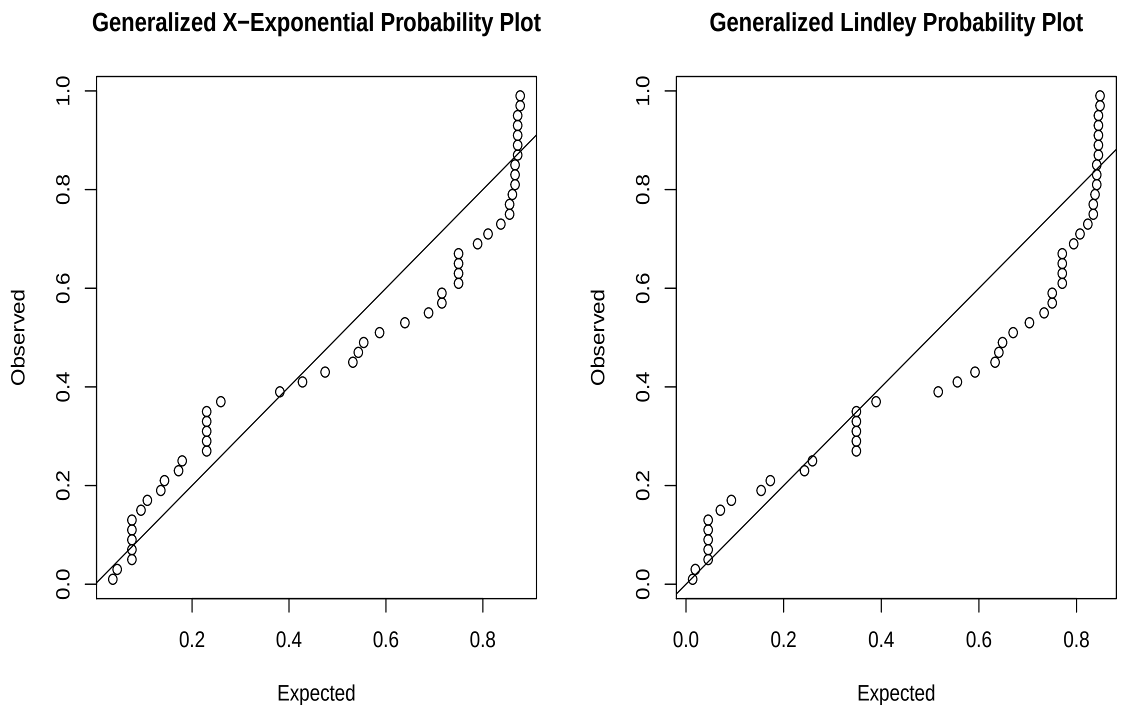


Figure 3.6: Probability plots of data set 2.

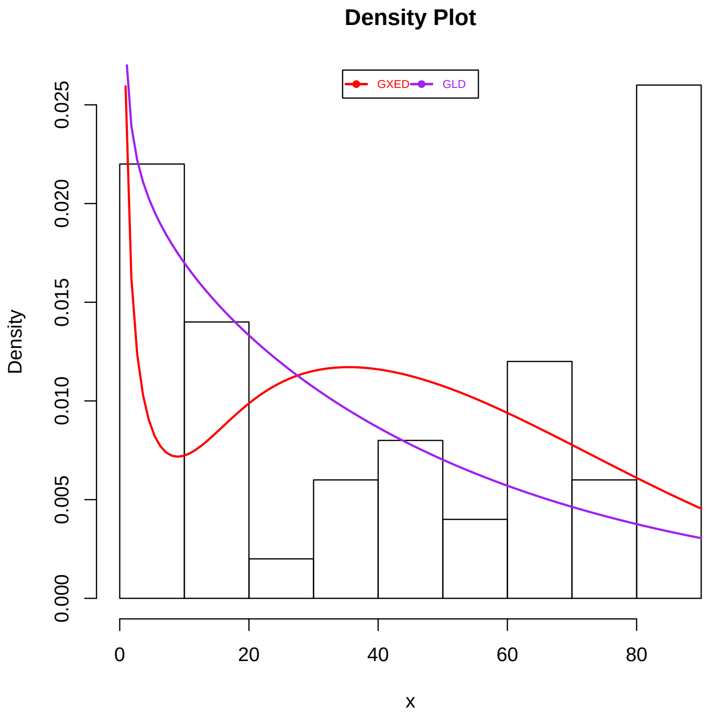


Figure 3.7: Histogram with fitted Pdfs for the data set 2.

3.3 Weibull-Lindley Distribution

Let X be a random variable with the cdf,

$$F(x;\alpha,\beta) = 1 - e^{-\alpha\left((1+x)e^{(x)^\beta}-1\right)}, \quad x > 0,\ \alpha, \beta > 0. \tag{3.3.1}$$

The r.v X is said to have Weibull-Lindly (WL) distribution if its distribution function is in the form (3.3.1). It will be denoted by WL(α,β). Then, the pdf corresponding to (3.3.1) is given by

$$f(x;\alpha,\beta) = \alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta} + e^{x^\beta}\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)},$$

$$x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.2)$$

Here β is shape parameter. The pdf of WL distribution can be rewritten as

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1+x)e^{x^\beta}e^{-\alpha\left((1+x)e^{x^\beta}-1\right)} + \alpha e^{x^\beta}e^{-\alpha\left((1+x)e^{x^\beta}-1\right)},$$

$$x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.3)$$

By using the power series expansion for the exponential function, we obtain

$$e^{-\alpha\left((1+x)e^{x^\beta}-1\right)} = \sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left((1+x)e^{x^\beta}-1\right)^i. \quad (3.3.4)$$

Substituting (3.3.4) in (3.3.3), we get

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1+x)e^{x^\beta}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left((1+x)e^{x^\beta}-1\right)^i$$
$$+ \alpha e^{x^\beta}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left((1+x)e^{x^\beta}-1\right)^i,\quad x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.5)$$

Using the generalized binomial theorem, we have

$$\left((1+x)e^{x^\beta}-1\right)^i = \sum_{j=o}^{i}\frac{i!}{j!(i-j)!}\left((1+x)e^{x^\beta}\right)^{i-j}(-1)^j. \quad (3.3.6)$$

Inserting (3.3.6) in (3.3.5), we get

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1+x)e^{x^\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i\alpha^i}{j!(i-j)!}\left((1+x)e^{x^\beta}\right)^{i-j}$$
$$+\alpha e^{x^\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i\alpha^i}{j!(i-j)!}\left((1+x)e^{x^\beta}\right)^{i-j}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.7}$$

The pdf can be further simplified as

$$f(x;\alpha,\beta) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{(-1)^{i+j}\alpha^{i+1}\beta(i-j+1)}{j!k!(i-j-k+1)!}e^{x^\beta(i-j+1)}x^{\beta+k-1}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{i-j}\frac{(-1)^{i+j}\alpha^{i+1}}{j!m!(i-j-m)!}e^{x^\beta(i-j+1)}x^m, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.8}$$

Figure 3.8 provide the pdfs of WL(α,β) for different parameter values. From the below figures it is immediate that the pdfs are unimodal. The survival function $S(x)$, failure rate function $r(x)$, reversed failure rate function $h(x)$ and cumulative failure rate function $H(x)$ of X are

$$S(x;\alpha,\beta) = e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.9}$$

$$r(x;\alpha,\beta) = \alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta}+e^{x^\beta}\right), \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.10}$$

$$h(x;\alpha,\beta) = \frac{\alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta}+e^{x^\beta}\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}}{1-e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.11}$$

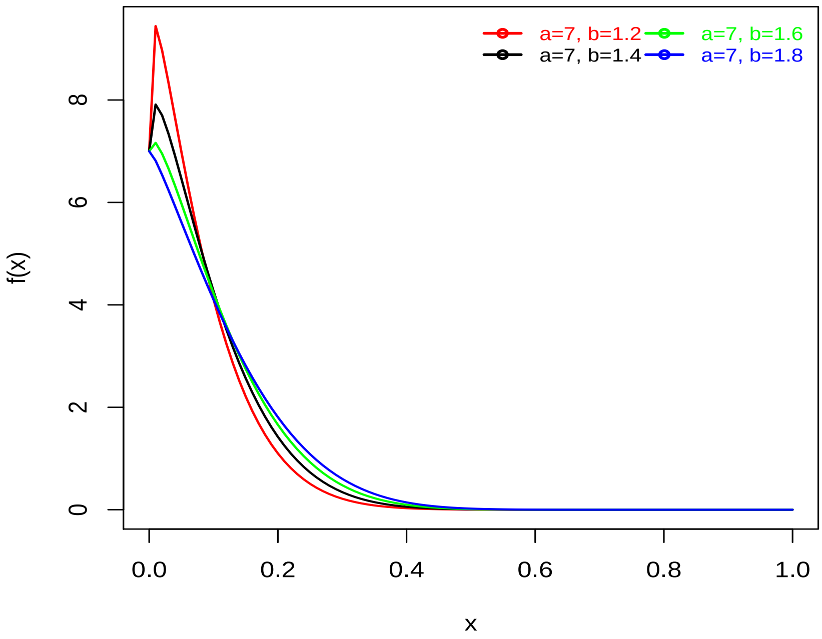


Figure 3.8: PDF of the WL(α, β).

and

$$H(x;\alpha,\beta) = \int_0^x r(t;\alpha,\beta)\, dt = \alpha(1+x)e^{x^{\beta}}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \qquad (3.3.12)$$

The failure rate function of the WL(α, β) exhibit increasing, decreasing and bathtub shapes. We can see from that

$$\lim_{x \to 0} r(x) = \begin{cases} \infty, & \beta < 1 \\ 2\alpha, & \beta = 1 \\ \alpha, & \beta > 1. \end{cases}$$

Figure 3.9 provide the failure rate functions of WL(α, β) for different parameter values. From the below figures it is immediate that the failure rate function can be increasing, decreasing or bathtub shaped. It is clear that the pdf and the

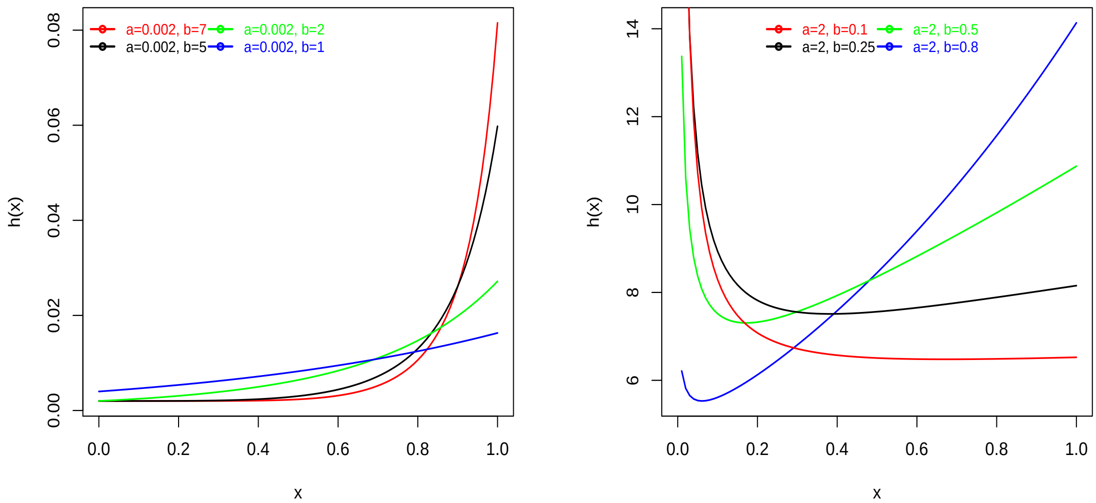


Figure 3.9: Failure rate function of the WL(α, β).

failure rate function have many different shapes, which allows this distribution to fit different types of lifetime data. For fixed α, the failure rate function is non-decreasing function if $\beta > 1$ (left) and non-increasing and bathtub function if $\beta < 1$ (right).

3.3.1 Statistical Properties

In this section, we study the statistical properties for the WL(α, β), specially Quantile function, Median, Mode, Moments etc.

Quantile and Median

We obtain the $100p^{\text{th}}$ percentile,

$$(1+x)e^{x^\beta} = -\frac{1}{\alpha}\log(1-p)+1. \tag{3.3.13}$$

Setting in (3.3.13), we get the median of WL(α,β) from

$$(1+x)e^{x^\beta} = \frac{1}{\alpha}\log\left(\frac{1}{1-0.5}\right)+1.$$

x_p is the solution of above monotone increasing function. Software can be used to obtain the Quantiles/Percentiles.

Mode

Mode can be obtained as solution of

$$[h'(x;\alpha,\beta)-(h(x;\alpha,\beta))^2].S(x;\alpha,\beta)=0. \tag{3.3.14}$$

It is not possible to get an analytic solution in x for (3.3.14). It can be obtained numerically by using methods such as fixed-point or bisection method.

Moments

We obtain the r^{th} moment of WL(α, β) in the form

$$\begin{aligned}\mu'_r &= \int_0^\infty x^r f(x;\alpha,\beta)\ dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j}\alpha^{i+1}}{j!k!(i-j-k+1)!}\int_0^\infty x^{r+\beta+k-1}e^{x^\beta(i-j+1)}\ dx \\ &+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}\alpha^{i+1}}{j!m!(i-j-m)!}\int_0^\infty x^{r+m}e^{x^\beta(i-j+1)}\ dx.\end{aligned}$$

By using the definition of Gamma function, we get

$$\begin{aligned}\mu'_r &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j} \frac{(-1)^{i+j+\frac{\beta+r+k}{\beta}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+r+k}{\beta})}{\beta(i-j+1)^{\frac{\beta+r+k}{\beta}}} \\ &+ \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j} \frac{(-1)^{i+j+\frac{m+r+1}{\beta}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma(\frac{m+r+1}{\beta})}{\beta(i-j+1)^{\frac{m+r+1}{\beta}}}. \qquad (3.3.15)\end{aligned}$$

If (3.3.15) is a convergent series for any $r \geq 0$, therefore all the moments exist and for integer values of α and β, it can be represented as a finite series representation. Therefore putting $r = 1$, we obtain the mean as

$$\begin{aligned}E(X) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j} \frac{(-1)^{i+j+\frac{\beta+k+1}{\beta}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+k+1}{\beta})}{\beta(i-j+1)^{\frac{\beta+k+1}{\beta}}} \\ &+ \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j} \frac{(-1)^{i+j+\frac{m+2}{\beta}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma(\frac{m+2}{\beta})}{\beta(i-j+1)^{\frac{m+2}{\beta}}}\end{aligned}$$

and putting $r=2$, we obtain the second moment as

$$E(X^2)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{(-1)^{i+j+\frac{\beta+k+2}{\beta}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+k+2}{\beta})}{\beta(i-j+1)^{\frac{\beta+k+2}{\beta}}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j}\frac{(-1)^{i+j+\frac{m+3}{\beta}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma(\frac{m+3}{\beta})}{\beta(i-j+1)^{\frac{m+3}{\beta}}}.$$

It can be used to obtain the higher central moments and variance.

Moment Generating Function and Characteristic Function

The moment generating function, $M_X(t)$, of WL(α,β) is obtained as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{t^r(-1)^{i+j+\frac{\beta+r+k}{\beta}}\alpha^{i+1}}{r!j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+r+k}{\beta})}{\beta(i-j+1)^{\frac{\beta+r+k}{\beta}}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j}\frac{t^r(-1)^{i+j+\frac{m+r+1}{\beta}}\alpha^{i+1}}{r!j!m!(i-j-m)!}\frac{\Gamma(\frac{m+r+1}{\beta})}{\beta(i-j+1)^{\frac{m+r+1}{\beta}}}.$$

The characteristic function, $\phi_X(t)$, of WL(α,β) is obtained as

$$\phi_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{(it)^r(-1)^{i+j+\frac{\beta+r+k}{\beta}}\alpha^{i+1}}{r!j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+r+k}{\beta})}{\beta(i-j+1)^{\frac{\beta+r+k}{\beta}}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j}\frac{(it)^r(-1)^{i+j+\frac{m+r+1}{\beta}}\alpha^{i+1}}{r!j!m!(i-j-m)!}\frac{\Gamma(\frac{m+r+1}{\beta})}{\beta(i-j+1)^{\frac{m+r+1}{\beta}}}.$$

3.3.2 Distribution of Maximum and Minimum

Let $X_1, X_2, \ldots, X_n$ be a random sample from WL(α, β) with cdf and pdf as in (3.3.1) and (3.3.2), respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(r)}$ is given by,

$$f_{(r:n)}(x) = \frac{1}{B(r, n-r+1)} \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{r-1} \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n-r}$$
$$\alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta} + e^{x^\beta}\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.16)$$

The cdf of $X_{(r)}$ is given by,

$$F_{r:n}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{j} \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n-j},$$
$$x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.17)$$

The cdf of $X_{(1)}$ is

$F_{X_{(1)}}(x; \alpha, \beta) = P(X_{(1)} \leq x) = 1 - \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$

The cdf of $X_{(n)}$ is

$$F_{X_{(n)}}(x; \alpha, \beta) = P(X_{(n)} \leq x) = \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

Reliability of a series system having n components with WL(α, β) is

$$R(x) = \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n}.$$

Reliability of a parallel system having n components with WL(α, β) is

$$R(x) = 1 - \left[e^{-\alpha\left((1+x)e^{x^{\beta}} - 1\right)} \right]^n.$$

3.3.3 Parameter Estimation

In this section, point estimation of the unknown parameters of the WL(α, β) are conducted using MLE. First partial derivatives of the log-likelihood function with respect to the two-parameters are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} (1 + x_i) e^{x_i^{\beta}} \tag{3.3.18}$$

and

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} x_i^{\beta} \log x_i - \alpha \sum_{i=1}^{n} \left((1 + x_i) e^{x_i^{\beta}} x_i^{\beta} \log x_i\right) + \sum_{i=1}^{n} \frac{(1 + x_i)\left[x_i^{\beta-1} + \beta x_i^{\beta-1} \log x_i\right]}{\beta x_i^{\beta-1}(1 + x_i) + 1}. \tag{3.3.19}$$

Setting the left side of the above two equations to zero, we get the likelihood equations as a system of two non-linear equations in α and β. Solving this system in α and β gives the MLE of α and β. It is very easy to obtain estimates using R software by numerical methods.

3.3.4 Asymptotic Confidence bounds

In this section, we derive the asymptotic confidence intervals of the parameters α and β, since the MLEs of the unknown parameters α and β cannot be obtained in closed forms. Let the variance covariance matrix be denoted by I^{-1}, where I^{-1} is the inverse of the observed information matrix which defined as follows

$$I^{-1} = \begin{pmatrix} E(-\frac{\partial^2 l}{\partial \alpha^2}) & E(-\frac{\partial^2 l}{\partial \alpha \beta}) \\ E(-\frac{\partial^2 l}{\partial \beta \alpha}) & E(-\frac{\partial^2 l}{\partial \beta^2}) \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\beta}, \hat{\alpha}) & \mathrm{Var}(\hat{\beta}) \end{pmatrix}$$

where the second partial derivatives of log-likelihood function are

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha}, \quad \frac{\partial^2 l}{\partial \alpha \beta} = \sum_{i=1}^{n} (1 + x_i) e^{x_i^\beta} x_i^\beta \log x_i$$

and

$$\begin{aligned} \frac{\partial^2 l}{\partial \beta^2} &= \sum_{i=1}^{n} x_i^\beta (\log x_i)^2 - \alpha \sum_{i=1}^{n} (1 + x_i) \left[e^{x_i^\beta} (x_i^\beta \log x_i)^2 + e^{x_i^\beta} x_i^\beta (\log x_i)^2 \right] \\ &+ \sum_{i=1}^{n} \frac{\left(\beta x_i^{\beta-1} (1 + x_i) + 1 \right) \left(\beta x_i^{\beta-1} (1 + x_i) (\log x_i)^2 + 2(1 + x_i) x_i^{\beta-1} \right)}{\left(\beta x_i^{\beta-1} (1 + x_i) + 1 \right)^2} \\ &- \sum_{i=1}^{n} \frac{\left((1 + x_i) \left[x_i^{\beta-1} + \beta x_i^{\beta-1} \log x_i \right] \right)^2}{\left(\beta x_i^{\beta-1} (1 + x_i) + 1 \right)^2}. \end{aligned}$$

We can derive the $(1 - \xi)100\%$ confidence intervals of the parameters α and β as

$$\hat{\alpha} \pm Z_{\frac{\xi}{2}} \sqrt{\mathrm{Var}(\hat{\alpha})}, \; \hat{\beta} \pm Z_{\frac{\xi}{2}} \sqrt{\mathrm{Var}(\hat{\beta})}$$

where $Z_{\frac{\xi}{2}}$ is the upper $(\frac{\xi}{2})^{\text{th}}$ percentile of the standard Normal distribution.

3.3.5 Three parameter Weibull-Lindley Distribution

A random variable X is said to have three parameter Weibull-Lindley (3WL) distribution if its cdf is of the form,

$$F(x;\alpha,\beta,\lambda) = 1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}}-1\right)}, \quad x>0,\ \alpha>0, \beta>0, \lambda>0. \qquad (3.3.20)$$

The pdf corresponding to Eq.(3.3.20) is given by

$$f(x;\alpha,\beta,\lambda) = \alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^{\beta}} + \lambda e^{(\lambda x)^{\beta}}\right)e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}}-1\right)},$$
$$x>0,\ \alpha,\ \beta,\ \lambda>0. \qquad (3.3.21)$$

Here β is shape parameter and λ is scale parameter. The distribution of this form with parameters α, β, and λ will be denoted by 3WL(α,β,λ). The survival function $S(x;\alpha,\beta,\lambda)$, failure rate function $r(x;\alpha,\beta,\lambda)$, reversed failure rate function $h(x;\alpha,\beta,\lambda)$ and cumulative failure rate function $H(x;\alpha,\beta,\lambda)$ of X are

$$S(x;\alpha,\beta,\lambda) = 1 - F(x;\alpha,\beta,\lambda) = e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}}-1\right)}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0, \qquad (3.3.22)$$

$$r(x;\alpha,\beta,\lambda) = \alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^{\beta}} + \lambda e^{(\lambda x)^{\beta}}\right), \quad x>0,\ \alpha,\ \beta,\ \lambda>0, \qquad (3.3.23)$$

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^{\beta}} + \lambda e^{(\lambda x)^{\beta}}\right)e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}}-1\right)}}{1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^{\beta}}-1\right)}},$$
$$x>0,\ \alpha,\ \beta,\ \lambda>0 \qquad (3.3.24)$$

and

$$H(x;\alpha,\beta,\lambda) = \int_0^x r(t;\alpha,\beta,\lambda)dt = \alpha(1+\lambda x)e^{(\lambda x)^\beta}, \;\; x > 0, \; \alpha, \; \beta, \; \lambda > 0 \quad (3.3.25)$$

respectively. Figure 3.10 and Figure 3.11 provide the pdfs and the failure rate functions of 3WL(α, β, λ) for different parameter values. From the below figures it is immediate that the pdfs can be unimodal and the failure rate function can be increasing, decreasing or bathtub shaped. It is clear that the pdf and the

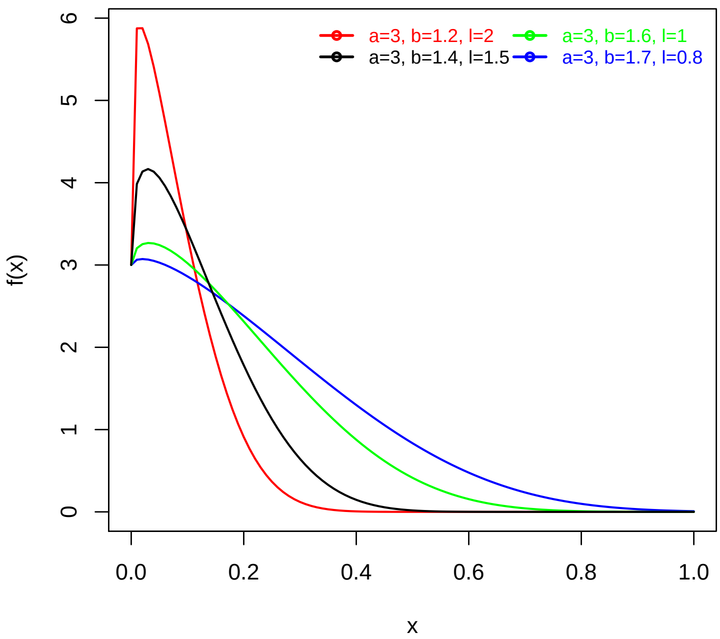


Figure 3.10: PDF of the 3WL(α, β, λ).

failure rate function have many different shapes, which allows this distribution to fit different types of lifetime data. For fixed α, F is IFR if $\beta > 1$ and $\lambda > 1$, (left)

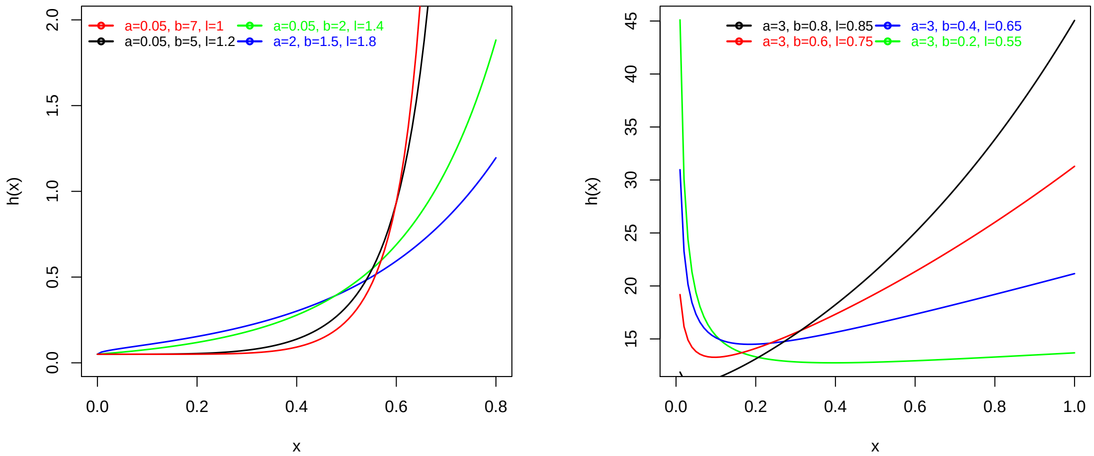


Figure 3.11: Failure rate function of the 3WLD(α, β, λ).

and DFR and BFR if $\beta < 1$ and $\lambda < 1$ (right).

Parameter Estimation

In this section, point estimation of the unknown parameters of the 3WL(α, β, λ) are done by method of maximum likelihood. The first partial derivatives of the log-likelihood function with respect to the three-parameters are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n} \big(1 + \lambda x_i\big) e^{(\lambda x_i)^{\beta}} + n, \tag{3.3.26}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} (\lambda x_i)^{\beta} \log(\lambda x_i) - \alpha \sum_{i=1}^{n} \left((1 + \lambda x_i) e^{(\lambda x_i)^{\beta}} (\lambda x_i)^{\beta} \log(\lambda x_i) \right) + \sum_{i=1}^{n} \frac{(1 + \lambda x_i)\big((\lambda x_i)^{\beta - 1} + \beta (\lambda x_i)^{\beta - 1} \log(\lambda x_i)\big)}{\beta (\lambda x_i)^{\beta - 1} (1 + \lambda x_i) + 1} \tag{3.3.27}$$

and

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{\beta} \beta \lambda^{\beta-1} - \alpha \sum_{i=1}^{n} \left((1+\lambda x_i) e^{(\lambda x_i)^{\beta}} x_i \beta (\lambda x_i)^{\beta} + x_i e^{(\lambda x_i)^{\beta}} \right) + \sum_{i=1}^{n} \frac{\beta x_i^{\beta} \left((\beta - 1)\lambda^{(\beta-2)} + x_i \beta \lambda^{\beta-1} \right)}{\left(\beta (\lambda x_i)^{\beta-1} (1+\lambda x_i) + 1 \right)}. \quad (3.3.28)$$

Setting the left side of the above three equations to zero, we get the likelihood equations as a system of three non-linear equations in α, β and λ. Solving this system in α, β and λ gives the MLEs of α, β and λ. It is very easy to obtain estimates using R software by numerical methods.

3.3.6 Application

In this section, we present the analysis of a real data set using the WL(α, β) and 3WL(α, β, λ) model and compare it with the other bathtub models such as Generalized Lindley distribution (GL), Nadarajah et al. (2011) and Exponentiated Weibull distribution (EW), Pal et al. (2006), using K-S statistic. We considered two sets of data, which are strengths of 1.5 cm glass fibers data, Smith and Naylor (1987) and infection for AIDS data, Klein and Moesch Berger (1997).

Data Set 1: The data are the strengths of 1.5 cm glass fibers, Smith and Naylor (1987), measured at the National Physical Laboratory, England. The data set 1 is given in Table 3.8. Table 3.9 gives MLEs of parameters of the WL(α, β), GL, EW and 3WL(α, β, λ) and goodness of fit statistics.

3WL(α, β, λ) gives the smallest K-S value and largest p-value. The second

Table 3.8: Strengths of 1.5 cm glass fibres

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68
1.73	1.81	2	0.74	1.04	1.27	1.39	1.49	1.53	1.59
1.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42
1.5	1.54	1.6	1.62	1.66	1.69	1.76	1.84	2.24	0.81
1.13	1.29	1.48	1.5	1.55	1.61	1.62	1.66	1.7	1.77
1.84	0.84	1.24	1.3	1.48	1.51	1.55	1.61	1.63	1.67
1.7	1.78	1.89							

Table 3.9: MLEs of parameters, Log-likelihood

Model	**MLEs of Parameters**	**log L**	**K-S**	***p*-value**
WL	$\hat{\alpha}$=0.0285 $\hat{\beta}$=1.893	-16.639	0.1368	0.189
GL	$\hat{\alpha}$=26.172 $\hat{\lambda}$=2.9901	-30.6199	0.2264	0.00314
EW	$\hat{\alpha}$=7.285 $\hat{\beta}$=0.67122 $\hat{\lambda}$=0.582	-14.676	0.146	0.135
3WL	$\hat{\alpha}$=0.000212 $\hat{\beta}$=0.8378 $\hat{\lambda}$=5.3257	-14.4228	0.1256	0.273

smallest K-S value and largest p-value are obtained for the WL distribution. The second largest log-likelihood value is given by the EW distribution. Fitted pdfs and probability plots of the three best fitting distributions for data set 1 are given in Figure 3.12 and 3.13.

Data Set 2: Consider times to infection for AIDS for two hundred and ninety five

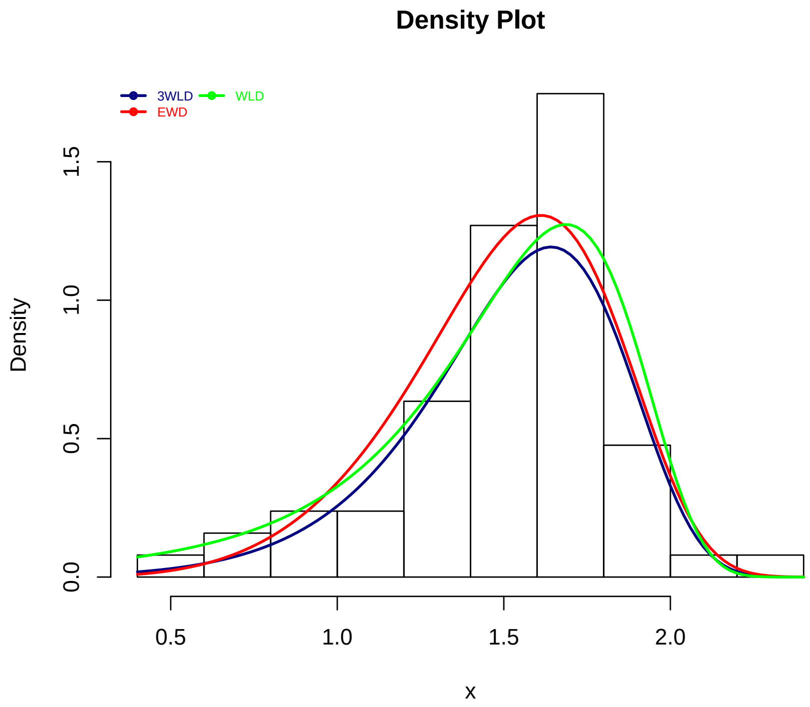


Figure 3.12: Fitted Pdfs of the three best fitting distributions for data set 1.

patients, Klein and Moesch Berger (1997). Table 3.10 gives MLEs of parameters of the WL(α, β), GL, EW and 3WL(α, β, λ) and goodness of fit statistics.

Here, the smallest K-S value and largest p-value are obtained for 3WL(α, β, λ). EW gives the largest log-likelihood value and second largest p-value. The third largest log-likelihood value and p-value based are obtained for WL(α, β). It is observed that 3WL(α, β, λ) fits as the best in the first data set whereas EW fits as the best in the second data in terms of likelihood and in terms of KS Statistic. Therefore, it is not guaranteed the 3WL(α, β, λ) will behave always better than WL(α, β) or EW but at least it can be said in certain circumstances 3WL(α, β, λ) might work better than WL(α, β) or EW. Fitted pdfs and probability plots of the

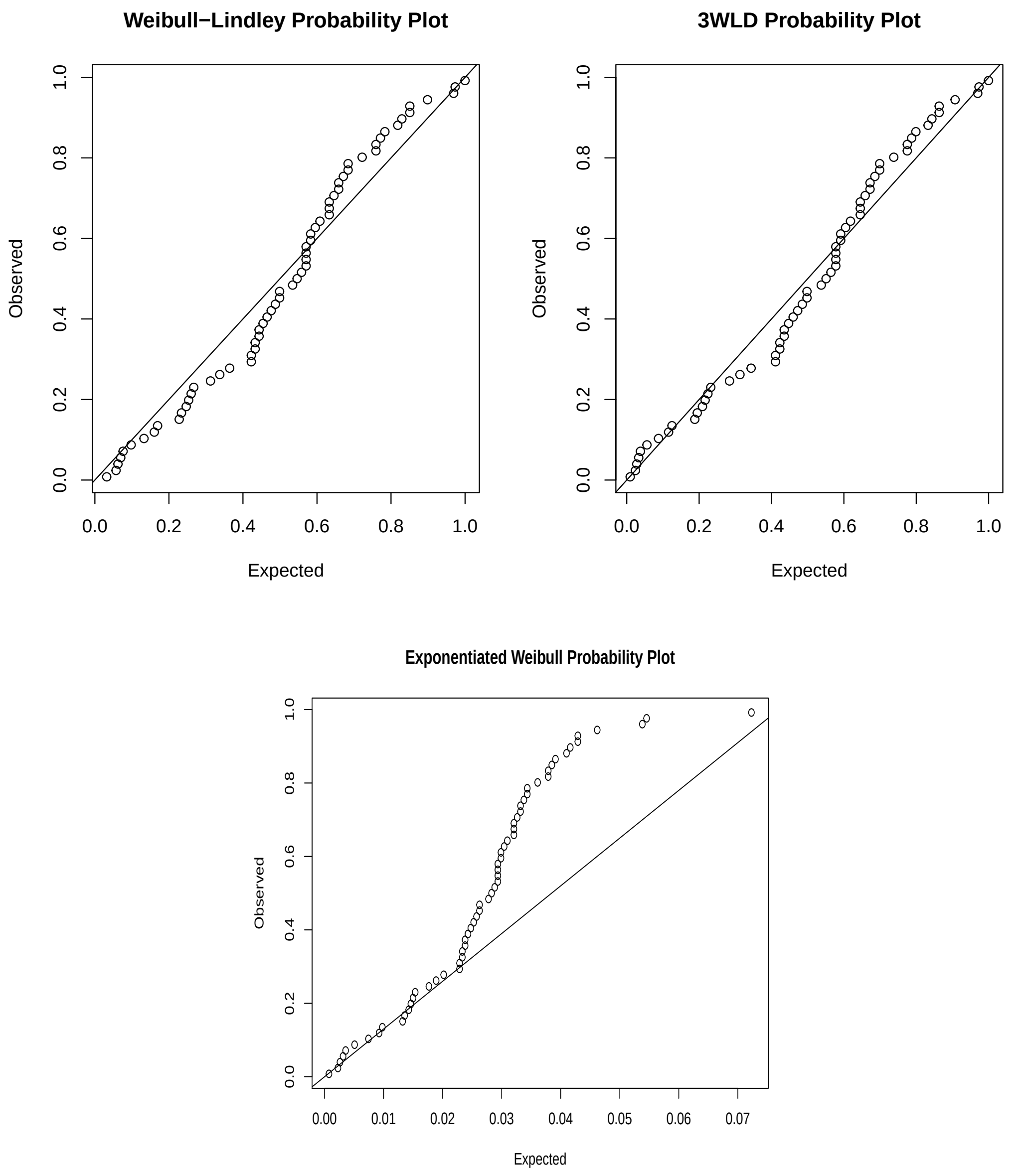


Figure 3.13: Probability plots of the three best fitting distributions for data set 1.

three best fitting distributions for data set 2 are given in Figure 3.14 and 3.15.

Table 3.10: MLEs of parameters, Log-likelihood

Model	MLEs of Parameters	log L	K-S	p-value
WL	$\hat{\alpha}$=0.0355 $\hat{\beta}$=0.5712	-457.302	0.0776	0.0893
GL	$\hat{\alpha}$=2.414 $\hat{\lambda}$=0.8929	-453.523	0.717	2.22×10^{-16}
EW	$\hat{\alpha}$=1.9566 $\hat{\beta}$=0.9598 $\hat{\lambda}$=0.3213	-450.131	0.064	0.2426
3WL	$\hat{\alpha}=8.752\times10^{-04}$ $\hat{\beta}$=0.2994 $\hat{\lambda}$=15.0999	-451.875	0.0619	0.2755

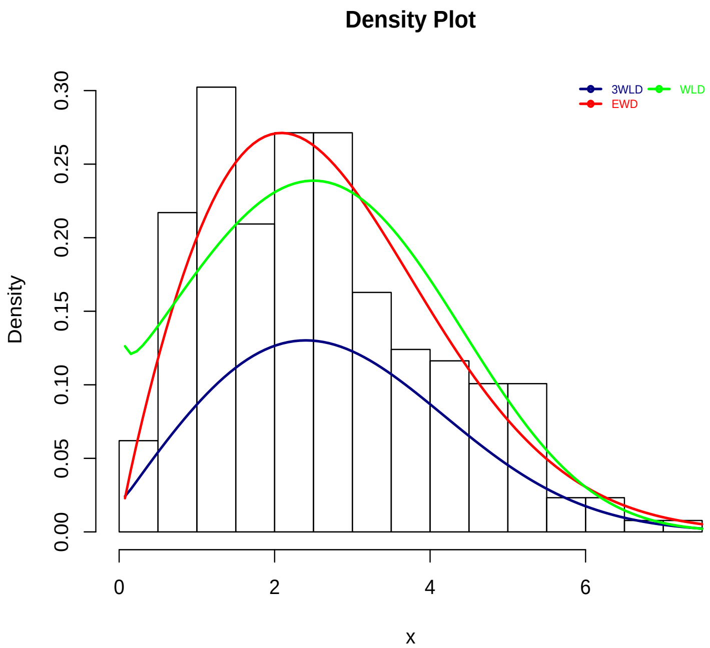


Figure 3.14: Fitted Pdfs of the three best fitting distributions for data set 2.

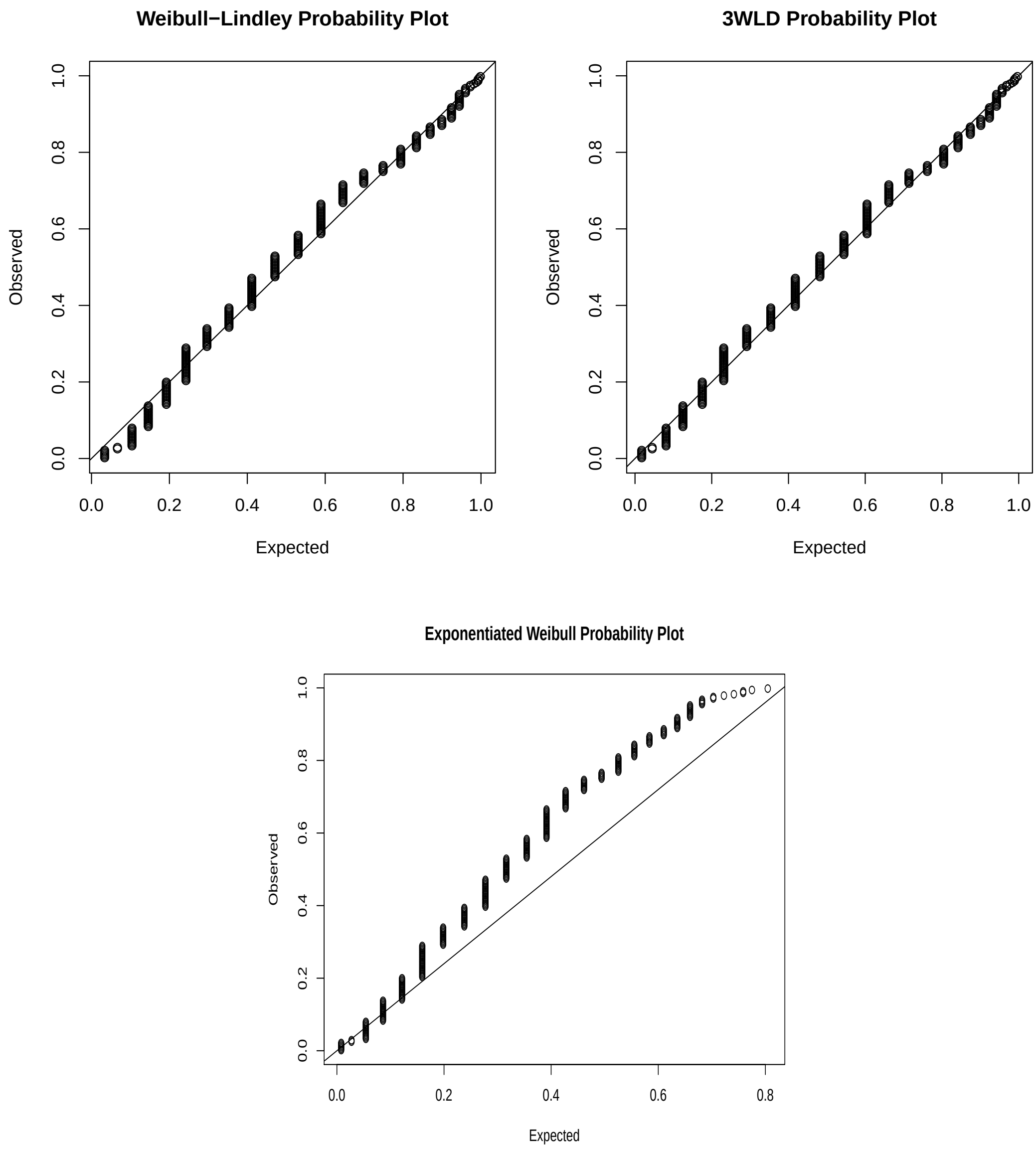


Figure 3.15: Probability plots of the three best fitting distributions for data set 2.

3.4 Summary

GXE is generalizes the X-Exponential distribution. Several properties of the distribution, hazard rate function, moments, moment generating function etc are derived. Also we presented the maximum likelihood estimation of this distribution. A simulation study is performed for validate MLE. Two real data sets are analyzed. The first data set provided smallest AIC and BIC value, the largest p-value for GXE distribution than GLD distribution. And also the second data set also provided smallest AIC and BIC value, the largest p-value for GXE distribution than GLD distribution. It shows that the proposed distribution is a better alternative among BFR models.

We proposed Weibull-Lindley distribution which exhibits bathtub shaped failure rate function, with high initial failure rate, which decreases rapidly and then slowly increases. Three parameter Weibull-Lindley distribution (3WLD) is introduced for avoid scale problem. We have studied maximum likelihood estimators and the parameters estimation is carried out in the presence of real data. We present two real life data sets, where in one data set it is observed that 3WLD has a better fit compare to EW or WLD but in the other the EW has a better fit than 3WLD or WLD.

CHAPTER 4

DUS TRANSFORMATION OF LOMAX DISTRIBUTION: AN UPSIDE-DOWN BATHTUB SHAPED FAILURE RATE MODEL

4.1 Introduction

[1] A little research works have been discussed on the upside-down bathtub (UBT) shaped failure rate distributions. Efron (1988) discussed head and neck cancer data having UBT shape for failure rate because of a therapy. Inverse Lindley (IL) distribution is used to model UBT data, Sharma et al. (2014 & 2015). In reliability literature, the stress-strength model describes the life of a component that has a random strength X and is subjected to random stress Y. The system fails if and

[1] Some contents of this chapter are based on Deepthi and Chacko (2020).

only if the stress is greater than strength. The estimation of a stress-strength model when X and Y are having a specified distribution, has been discussed by many researchers, Al-Mutairi et al. (2013).

Use of heavy tailed distribution required for many of the lifetime data analysis. Pareto distribution is one of the heavy tailed distributions which usually models nonnegative data. It was introduced by Pareto (1897) as a model for the distribution of incomes. Several different forms of Pareto distribution have been studied by many authors including Lomax (1954), Davis and Feldstein (1979), Grimshaw (1993) and Nadarajah and Gupta (2008). One of the popular hierarchy of Pareto distribution is Pareto Type II which has been named as Lomax distribution. Lomax distribution has been applied in a variety of fields such as engineering, reliability and life testing. In statistical literature, there are several methods to propose new distribution by the use of some baseline distribution. Dinesh et al. (2015) proposed a method, DUS transformation, to get new distribution by the use of Exponential baseline distribution and studied its properties with application to survival data analysis. If $f(x)$ and $F(x)$ be the pdf and cdf of some baseline distribution, then the pdf $g(x)$ of the corresponding DUS Transformation distribution is given by

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)}. \tag{4.1.1}$$

The cdf and failure rate function corresponding to the pdf $g(x)$ is given by

$$G(x) = \frac{1}{e-1} [e^{F(x)} - 1] \tag{4.1.2}$$

and

$$h(x) = \frac{1}{e - e^{F(x)}} f(x) e^{F(x)} \tag{4.1.3}$$

respectively. It is a transformation, not a generalization, hence it produces a parsimonious distribution in terms of computation and interpretation as it never contains any new parameter other than the parameter(s) involved in the baseline distribution.

The aim of this chapter to derive DUS Transformation of Lomax distribution which possesses the upside-down bathtub-shaped failure rate function. The proposed distribution is thus capable of modeling the real problems.

The following sections are organized as follows. The pdf, distribution function, failure rate function and its characteristics are given in section 4.2. In section 4.3, shapes of the pdf and failure rate function are given. Moments, moment generating function, characteristic function, quantile function, entropy, skewness and kurtosis are discussed in section 4.4. In section 4.5, distribution of maximum and minimum order statistics are discussed. The maximum likelihood estimation is discussed in section 4.6. In section 4.7, stress-strength reliability and its MLE are derived. In section 4.8, a simulation study is given. Three real data sets are analyzed in section 4.9. Conclusions are given in section 4.10.

4.2 DUS Transformation of Lomax Distribution

In this section, we consider DUS transformation of Lomax distribution with two parameters.

Consider Lomax distribution with pdf,

$$f(x) = \alpha\beta(1+\beta x)^{-(\alpha+1)}, \quad x > 0,\ \alpha > 0,\ \beta > 0 \tag{4.2.1}$$

and the corresponding cdf is given by,

$$F(x) = 1-(1+\beta x)^{-\alpha}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \tag{4.2.2}$$

Using (4.2.1) in (4.1.1), the pdf of DUS transformation of Lomax distribution is obtained by,

$$g(x) = \frac{1}{e-1}\alpha\beta(1+\beta x)^{-(\alpha+1)}e^{1-(1+\beta x)^{-\alpha}}, \quad x > 0, \alpha > 0,\ \beta > 0. \tag{4.2.3}$$

The distribution having pdf (4.2.3) is named as DUS-Lomax distribution and is denoted by DUS-Lomax(α, β). Here α and β are the shape and scale parameters respectively. The cdf and failure rate function of DUS-Lomax(α, β) are, respectively, given by

$$G(x) = \frac{1}{e-1}\left[e^{1-(1+\beta x)^{-\alpha}} - 1\right], \quad x > 0, \alpha > 0,\ \beta > 0 \tag{4.2.4}$$

and

$$r(x) = \alpha\beta(1+\beta x)^{-(\alpha+1)}\left[e^{(1+\beta x)^{-\alpha}} - 1\right]^{-1}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \tag{4.2.5}$$

4.3 Shapes

Here, we discuss the shapes of the pdf and failure rate function of DUS-Lomax(α,β) distribution.

4.3.1 Shape of Probability Density Function

We can see from (4.2.3) that

$$\lim_{x\to 0} g(x) = \begin{cases} \frac{\alpha\beta}{(e-1)}, & \alpha < 1 \\ \frac{\beta}{(e-1)}, & \alpha = 1 \\ \frac{\alpha\beta}{(e-1)}, & \alpha > 1 \end{cases}$$

and

$$\frac{1}{(1+\beta x)} \text{tends to zero, as } x \to \infty.$$

So $\lim_{x\to\infty} g(x) = 0$. The first derivatives of $g(x)$ is

$$g'(x) = \frac{e}{e-1}\alpha\beta e^{-(1+\beta x)^{-\alpha}} \left[-(\alpha+1)\alpha(1+\beta x)^{-\alpha-2} + \alpha\beta(1+\beta x)^{-2\alpha-2}\right].$$

So the mode of DUS-Lomax(α,β) is $\frac{1}{\beta}\left[\left(1+\frac{1}{\alpha}\right)^{-\frac{1}{\alpha}} - 1\right]$. Clearly, $g(x)$ is unimodal. Figure 4.1 shows the pdf of DUS-Lomax(α,β) for various choices of the parameters.

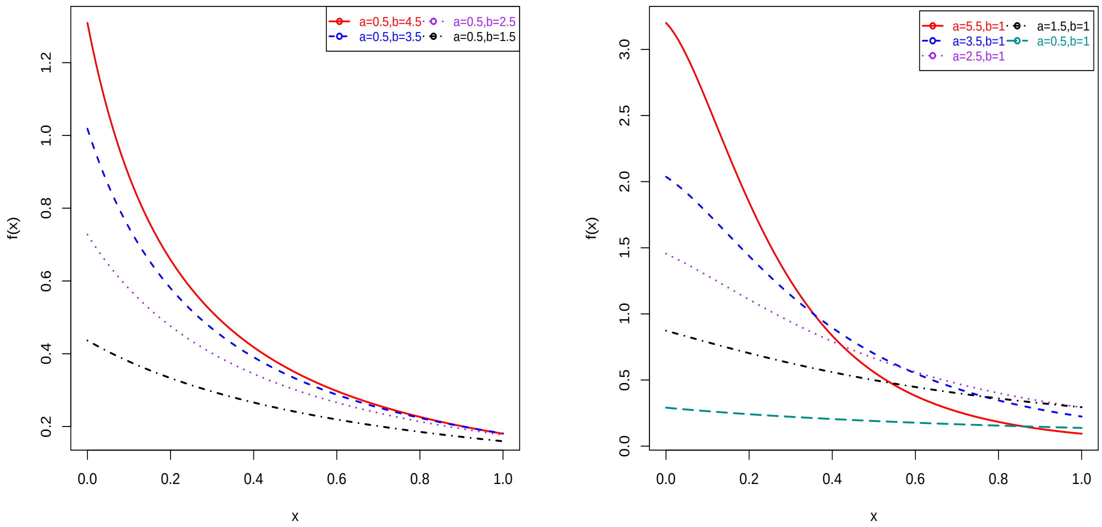


Figure 4.1: PDF of DUS-Lomax(α, β) for values of parameters $\alpha = 0.5$ and $\beta = 4.5, 3.5, 2.5, 1.5$ with color shapes red, blue, purple, black (left) and $\alpha = 5.5, 3.5, 2.5, 1.5, 0.5$ and $\beta = 1$ with color shapes red, blue, purple, black, dark cyan, respectively (right).

4.3.2 Shape of Failure Rate Function

For discussing the shape property of failure rate function, we apply Glaser's technique, see Glaser (1980). Let $\eta(x) = \frac{-g'(x)}{g(x)}$ where $g(x)$ is the density function and $g'(x)$ is the first derivative of $g(x)$ with respect to x. Then

$$\eta(x) = \frac{(\alpha+1)\beta(1+\beta x)^{-\alpha-2} - \alpha\beta(1+\beta x)^{2\alpha-2}}{(1+\beta x)^{\alpha+1}}$$

and its first derivative is

$$\eta'(x) = -(\alpha+1)\beta^2(1+\beta x)^{-\alpha-2}\Bigg[(1+\beta x)^{\alpha} - \alpha\Bigg].$$

If $\alpha > 1$, $\beta > 0$, $\eta'(x) > 0$ for $x \in (0, x_0)$, $\eta'(x_0) = 0$, $\eta'(x) < 0$ for $x \in (x_0, \infty)$ where $x_0 = \frac{\alpha^{1/\alpha} - 1}{\beta}$, the shape of failure rate function, $r(x)$, appears UBFR shapes if $\alpha > 1$. If $\alpha \leq 1$, $\beta > 0$, $\eta'(x) < 0$, the shape of failure rate function appears monotonically decreasing. The failure rate function of the DUS-Lomax(α, β) distribution exhibit monotonically decreasing and UBFR shapes, see Figure 4.2.

From (4.2.5),

$$\lim_{x \to 0} r(x) = \begin{cases} \alpha\beta(e-1), & \alpha < 1 \\ \beta(e-1), & \alpha = 1 \\ \alpha\beta(e-1), & \alpha > 1 \end{cases}$$

and

$$\frac{1}{(1+\beta x)} \text{tends to zero,} \quad e^{\frac{1}{(1+\beta x)^\alpha}} \text{ tends to 1, as } x \to \infty.$$

So $\lim_{x \to \infty} r(x) = 0$.

4.4 Statistical Properties

In this section, we study the statistical properties for the two parameter DUS-Lomax(α, β) distribution, moments, moment generating function, characteristic function, quantile function, skewness, kurtosis etc.

4.4.1 Moments

If X be a random variable having the pdf in (4.2.3), then the r^{th} raw moment is

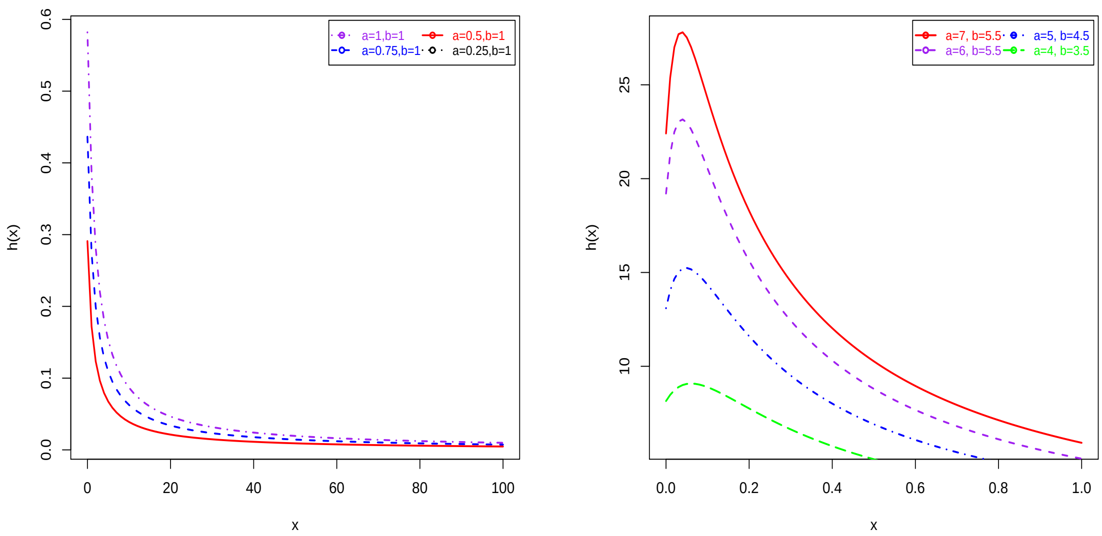


Figure 4.2: Failure rate function of the DUS-Lomax(α, β) for different parameter values $\alpha = 1, 0.75, 0.5, 0.25$ and $\beta = 1$ with color shapes purple, blue, red, black (left) and $\alpha = 7, 6, 5, 4$ and $\beta = 5.5, 5.5, 4.5, 3.5$ with color shapes red, purple, blue, green respectively (right).

$$
\begin{aligned}
\mu'_r &= \frac{e}{e-1}\alpha\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j}{j}\beta^{j+1}\int_0^{\infty}x^{j+r}e^{-(1+\beta x)^{-\alpha}}dx \\
&= \frac{e}{e-1}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j}{j}\beta^{j+1}\frac{1}{\beta^{j+r+1}}\sum_{k=0}^{j+r}(-1)^k\binom{j+r}{r}\int_0^1 u^{-\frac{j+r-k+1}{\alpha}-1}e^{-u}du \\
&= \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+r}(-1)^{j+k+n}\binom{\alpha+j}{j}\binom{j+r}{r}\frac{1}{n!\beta^r}\frac{\alpha}{\alpha n-r+k-j-1}.
\end{aligned}
$$

The mean μ and variance σ^2 of DUS-Lomax(α, β) distribution are, respectively,

$$
\mu = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{j+k+n}\binom{\alpha+j}{j}\frac{1}{n!\beta}\frac{(j+1)\alpha}{\alpha n+k-j-2}
$$

and

$$\sigma^2 = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+2}(-1)^{j+k+n}\binom{\alpha+j}{j}\binom{j+2}{2}\frac{1}{n!\beta^2}\frac{\alpha}{\alpha n+k-j-3}$$
$$-\left(\frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{j+k+n}\binom{\alpha+j}{j}\frac{1}{n!\beta}\frac{(j+1)\alpha}{\alpha n+k-j-2}\right)^2.$$

The skewness and kurtosis can be obtained using

$$\text{Skewness} = \frac{(\mu_3' - 3\mu\mu_2' + 2\mu^3)^2}{(\mu_2' - \mu^2)^3}\text{and Kurtosis} = \frac{(\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4)}{(\mu_2' - \mu^2)^2}.$$

4.4.2 Moment Generating Function

The mgf of DUS-Lomax(α, β) distribution is

$$M_X(t) = \frac{e}{e-1}\alpha\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\beta^{k+1}\int_0^{\infty}x^k e^{tx}e^{-(1+\beta x)^{-\alpha}}dx$$
$$= \frac{e}{e-1}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\frac{1}{\beta^m}\sum_{j=0}^{k+m}(-1)^j\binom{k+m}{j}\sum_{m=0}^{\infty}\frac{t^m}{m!}$$
$$\int_0^1 u^{-\frac{k+m-j+1}{\alpha}-1}e^{-u}du$$
$$= \frac{e}{e-1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{k+m}\sum_{m=0}^{\infty}\frac{t^m}{m!}(-1)^{k+j+n}\binom{\alpha+k}{k}\binom{k+m}{j}$$
$$\frac{1}{n!\beta^m}\frac{\alpha}{\alpha n+j-k-m-1}.$$

4.4.3 Characetristic Function

The characteristic function of DUS-Lomax(α, β) distribution is

$$\begin{aligned}
\phi_X(t) &= \frac{e}{e-1}\alpha\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\beta^{k+1}\int_0^{\infty} x^k e^{itx} e^{-(1+\beta x)^{-\alpha}}dx,\ i=\sqrt{-1} \\
&= \frac{e}{e-1}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\frac{1}{\beta^m}\sum_{j=0}^{k+m}(-1)^j\binom{k+m}{j}\sum_{m=0}^{\infty}\frac{(it)^m}{m!} \\
&\int_0^1 u^{-\frac{k+m-j+1}{\alpha}-1}e^{-u}du \\
&= \frac{e}{e-1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{k+m}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}(-1)^{k+j+n}\binom{\alpha+k}{k}\binom{k+m}{j} \\
&\frac{1}{n!\beta^m}\frac{\alpha}{\alpha n+j-k-m-1}.
\end{aligned}$$

4.4.4 Quantile Function

For any $p \in (0,1)$, the p^{th} quantile $Q(p)$ of DUS-Lomax(α, β) is

$$Q(p) = \frac{1}{\beta}\Big[\Big(1-\log(1+p(e-1))\Big)^{-\frac{1}{\alpha}}-1\Big]. \tag{4.4.1}$$

Setting $p = 0.5$ in (4.4.1), we get the median of DUS-Lomax(α, β) as follows

$$\text{Median} = \frac{1}{\beta}\Big[\Big(1-\log(1+0.5(e-1))\Big)^{-\frac{1}{\alpha}}-1\Big]. \tag{4.4.2}$$

Setting $p = \frac{1}{4}$ in (4.4.1), we get the 1$^{\text{st}}$ quartile of DUS-Lomax(α, β) as follows

$$Q_1 = \frac{1}{\beta}\Big[\Big(1-\log(1+\frac{1}{4}(e-1))\Big)^{-\frac{1}{\alpha}}-1\Big].$$

Setting $p = \frac{3}{4}$ in (4.4.1), we get the 3rd quartile of DUS-Lomax(α, β) as follows

$$Q_3 = \frac{1}{\beta}\Big[\Big(1 - \log(1 + \frac{3}{4}(e-1))\Big)^{-\frac{1}{\alpha}} - 1\Big].$$

A random sample X with DUS-Lomax(α, β) distribution can be simulated using

$$X = \frac{1}{\beta}\Big[\Big(1 - \log(1 + u(e-1))\Big)^{-\frac{1}{\alpha}} - 1\Big], \text{where } u \sim U(0,1). \tag{4.4.3}$$

4.4.5 Entropy

Suppose X is the DUS-Lomax(α, β), first we consider

$$\begin{aligned}\int f^{\gamma}(x)dx &= \left(\frac{e}{e-1}\right)^{\gamma} \alpha^{\gamma}\beta^{\gamma+i} \sum_{i=0}^{\infty}(-1)^i \binom{\gamma\alpha + \gamma + i - 1}{i} \int_0^{\infty} x^i e^{-\gamma(1+\beta x)^{-\alpha}} dx \\ &= \left(\frac{e}{e-1}\right)^{\gamma} \alpha^{\gamma-1}\beta^{\gamma-1} \sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{m=0}^{\infty}(-1)^{i+k+m} \\ &\quad \binom{\gamma\alpha + \gamma + i - 1}{i}\binom{i}{k}\frac{\gamma^m}{m!}\frac{\alpha}{\alpha m + k + i - 1}\end{aligned}$$

where $\gamma > 0$ and $\gamma \neq 1$. Then the Renyi entropy is

$$\begin{aligned}\tau_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\} \\ &= \frac{1}{1-\gamma}\log\Bigg\{\left(\frac{e}{e-1}\right)^{\gamma} \alpha^{\gamma-1}\beta^{\gamma-1} \sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{m=0}^{\infty}(-1)^{i+k+m} \\ &\quad \binom{\gamma\alpha + \gamma + i - 1}{i}\binom{i}{k}\frac{\gamma^m}{m!}\frac{\alpha}{\alpha m + k + i - 1}\Bigg\}.\end{aligned}$$

4.5 Distribution of Maximum and Minimum

In order to conduct reliability analysis in series structure, parallel structure, series-parallel structure, parallel-series structure and complex structures, the theory of order statistics is used as tool for analyzing life time data. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from DUS-Lomax(α, β) with cdf and pdf as in (4.2.4) and (4.2.3), respectively and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the corresponding order statistics. The pdf of the r^{th} order statistic is

$$
\begin{aligned}
f_{X_{(r)}}(x;\alpha,\beta) &= \frac{n!}{(r-1)!(n-r)!} f(x;\alpha,\beta) F^{r-1}(x;\alpha,\beta) \bar{F}^{n-r}(x;\alpha,\beta) \\
&= \frac{n!}{(r-1)!(n-r)!} \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e-1} e^{1-(1+\beta x)^{-\alpha}} \\
&\left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{r-1}\left[1-\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{n-r}, \\
&\qquad x>0,\ \alpha>0,\ \beta>0.
\end{aligned}
$$

The cdf of r^{th} order statistic is

$$
\begin{aligned}
F_{(r)}(x;\alpha,\beta) &= \sum_{j=r}^{n} \binom{n}{j} F^{j}(x;\alpha,\beta)[1-F(x;\alpha,\beta)]^{n-j} \\
&= \sum_{j=r}^{n} \binom{n}{j} \left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{j} \\
&\left[1-\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{n-j}, x>0,\ \alpha>0,\ \beta>0.
\end{aligned}
$$

The pdf of the 1st order statistics $X_{(1)}$, is

$$f_{X_{(1)}}(x;\alpha,\beta) = \frac{n\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e-1} e^{1-(1+\beta x)^{-\alpha}} \left[1 - \frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n-1},\ x>0,\ \alpha>0,\ \beta>0.$$

The pdf of the n^{th} order statistics $X_{(n)}$, is

$$f_{X_{(n)}}(x;\alpha,\beta) = \frac{n\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e-1} e^{1-(1+\beta x)^{-\alpha}} \left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n-1},\ x>0,\ \alpha>0,\ \beta>0.$$

The cdf of $X_{(1)}$ is

$F_{X_{(1)}}(x;\alpha,\beta) = P(X_{(1)} \leq x) = 1 - \left[1 - \frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n}, x>0,\ \alpha > 0,\ \beta>0.$

The cdf of $X_{(n)}$ is

$F_{X_{(n)}}(x;\alpha,\beta) = P(X_{(n)} \leq x) = \left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n},\ x>0,\ \alpha>0,\ \beta > 0.$

Reliability, $R(x;\alpha,\beta)$, of series and parallel system having n components with DUS-Lomax (α,β), respectively, are

$$\left[1 - \frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n} \text{and } 1 - \left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n}.$$

4.6 Parameter Estimation

In this section, we discuss method of moments and method of maximum likelihood for the estimation of parameters. Asymptotic bounds of the unknown parameter are also discussed.

Let $X_1, X_2, \ldots, X_n$ be an observed random sample from DUS-Lomax(α, β) with unknown parameters α and β. Let $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ be the first two sample moments. Equating sample moments with population moments, we get the moment estimators of the parameters,

$$m_1 = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{j+k+n}\binom{\alpha+j}{j}\frac{1}{n!\beta}\frac{(j+1)\alpha}{\alpha n - r + k - j - 2}$$

and

$$m_2 = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+2}(-1)^{j+k+n}\binom{\alpha+j}{j}\binom{j+2}{2}\frac{1}{n!\beta^2}\frac{\alpha}{\alpha n - r + k - j - 3}.$$

We derive MLE of the parameters of the DUS-Lomax(α, β) distribution as below.

The likelihood function is

$$l(x; \alpha, \beta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta) = \frac{e^n \alpha^n \beta^n}{(e-1)^n} e^{-\sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}} \prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)},$$

so that the log-likelihood function becomes

$$\log l = K + n\log\alpha + n\log\beta - (\alpha+1)\sum_{i=1}^{n}\log(1+\beta x_i) - \sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}, \quad (4.6.1)$$

where $K = n\log(\frac{e}{e-1})$. Then the partial derivatives of $\log L$ with respect to unknown parameters α and β are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log(1+\beta x_i) + \sum_{i=1}^{n} \log(1+\beta x_i)(1+\beta x_i)^{-\alpha} \tag{4.6.2}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{(1+\beta x_i)} + \alpha \sum_{i=1}^{n} x_i(1+\beta x_i)^{-(\alpha+1)}. \tag{4.6.3}$$

Setting the left side of the above two equations to zero, we get the likelihood equations as a system of two non-linear equations in α and β.

$$\frac{n}{\alpha} - \sum_{i=1}^{n} \log(1+\beta x_i) + \sum_{i=1}^{n} \log(1+\beta x_i)(1+\beta x_i)^{-\alpha} = 0 \tag{4.6.4}$$

$$\frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{(1+\beta x_i)} + \alpha \sum_{i=1}^{n} x_i(1+\beta x_i)^{-(\alpha+1)} = 0. \tag{4.6.5}$$

Solving these systems, (4.6.4) and (4.6.5), in α and β gives the MLE of α and β. These equations cannot be solved analytically and statistical software can be used to solve them numerically, by taking initial value arbitrarily.

4.6.1 Asymptotic distribution and Confidence bounds

In this section, we derived the asymptotic distribution and confidence intervals of the parameters $\alpha > 0$ and $\beta > 0$, when the MLEs of the unknown parameters α and β cannot be obtained in closed forms, using variance covariance matrix I^{-1}, where I^{-1} is the inverse of the observed information matrix which is defined as follows

$$I^{-1} = \begin{bmatrix} E(-\frac{\partial^2 \log L}{\partial \alpha^2}) & E(-\frac{\partial^2 \log L}{\partial \alpha \partial \beta}) \\ E(-\frac{\partial^2 \log L}{\partial \beta \partial \alpha}) & E(-\frac{\partial^2 \log L}{\partial \beta^2}) \end{bmatrix}^{-1}.$$

For a large sample, the asymptotic distribution of $\hat{h}$, $\hat{h} = (\alpha, \beta)$ is defined by $\sqrt{n}(\hat{h} - h) \rightarrow N(0, I^{-1})$. The second partial derivatives are as follows

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} \left[\log(1+\beta x_i)\right]^2 (1+\beta x_i)^{-\alpha} \tag{4.6.6}$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} x_i (1+\beta x_i)^{-(\alpha+1)} - \alpha \sum_{i=1}^{n} x_i \log(1+\beta x_i)(1+\beta x_i)^{-(\alpha+1)} - \sum_{i=1}^{n} \frac{x_i}{1+\beta x_i} \tag{4.6.7}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n}{\beta^2} + (\alpha+1) \sum_{i=1}^{n} \frac{x_i^2}{(1+\beta x_i)^2} - \alpha(\alpha+1) \sum_{i=1}^{n} x_i^2 (1+\beta x_i)^{-(\alpha+2)}. \tag{4.6.8}$$

The approximate $100\%(1-\eta)$ confidence intervals of the parameters α and β, by using variance-covariance matrix, are $\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{var(\hat{\alpha})}$ and $\hat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{var(\hat{\beta})}$ where $Z_{\frac{\eta}{2}}$ is the upper $100(\frac{\eta}{2})^{\text{th}}$ percentile of the standard Normal distribution.

4.7 Stress-Strength Reliability Estimation

Consider two independent random variables X and Y, where Y represents the 'stress' and X represents the 'strength'. The reliability of the stress-strength model is $R = P(Y < X)$, which is used in engineering statistics, quality control and other fields.

Suppose X and Y have DUS-Lomax(α, β) distribution with parameters (α, β_1) and (α, β_2) respectively. The reliability of the system is

$$\begin{aligned}
R = P[Y < X] &= \int_0^\infty f(x)F_Y(x)dx \\
&= \left(\frac{e}{e-1}\right)^2 \alpha\beta_1 \int_0^\infty (1+\beta_1 x)^{-(\alpha+1)} e^{-(1+\beta_1 x)^{-\alpha}} \left(e^{-(1+\beta_2 x)^{-\alpha}} - 1\right) dx \\
&= \left(\frac{e}{e-1}\right)^2 \alpha\beta_1^{n+1} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+n}{n} \int_0^\infty x^n e^{-(1+\beta_1 x)^{-\alpha}} \left(e^{-(1+\beta_2 x)^{-\alpha}} - 1\right) dx \\
&= \left(\frac{e}{e-1}\right)^2 \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+n}{n} \left\{ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{n+i} \sum_{r=0}^{\infty} \frac{\beta_2^i}{\beta_1^i} \frac{(-1)^{m+i+j+r}}{m!r!} \binom{\alpha m + i - 1}{i} \right. \\
&\left. \binom{n+i}{j} \frac{\alpha}{\alpha r - n + j - i - 1} - \sum_{k=0}^{n} \sum_{l=0}^{\infty} \frac{(-1)^{k+l}}{l!} \binom{n}{k} \frac{\alpha}{\alpha l - n + k - 1} \right\}. \qquad (4.7.1)
\end{aligned}$$

The Maximum Likelihood Estimation of R

Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$ be two independent random samples from DUS-Lomax(α, β_1), and DUS-Lomax(α, β_2) respectively. The log-likelihood function of α, β_1 and β_2 for the observed samples is

$$\begin{aligned}
\log l(x, y, \alpha, \beta_1, \beta_2) &= n \log\left(\frac{e}{e-1}\right) + (n+m)\log\alpha + n\log\beta_1 \\
&- (\alpha+1)\sum_{i=1}^{n} \log(1+\beta_1 x_i) - \sum_{i=1}^{n} (1+\beta_1 x_i)^{-\alpha} + m\log\left(\frac{e}{e-1}\right) + m\log\beta_2 \\
&- (\alpha+1)\sum_{j=1}^{m} \log(1+\beta_2 y_j) - \sum_{j=1}^{m} (1+\beta_2 y_j)^{-\alpha}. \qquad (4.7.2)
\end{aligned}$$

The estimators $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$ of the parameters of α, β_1 and β_2 respectively can then be obtained as the solution of the following non-linear equations.

$$\frac{\partial \log l}{\partial \alpha} = \frac{n+m}{\alpha} - \sum_{i=1}^{n} \log(1+\beta_1 x_i) - \sum_{j=1}^{m} \log(1+\beta_2 y_j) + \sum_{i=1}^{n} \frac{\log(1+\beta_1 x_i)}{(1+\beta_1 x_i}\Big)^{\alpha} + \sum_{j=1}^{m} \frac{\log(1+\beta_2 y_j)^{\alpha}}{(1+\beta_2 y_j)}$$

$$\frac{\partial \log l}{\partial \beta_1} = \frac{n}{\beta_1} - (\alpha_1+1) \sum_{i=1}^{n} \frac{x_i}{1+\beta_1 x_i} + \alpha_1 \sum_{i=1}^{n} x_i (1+\beta_1 x_i)^{-(\alpha_1+1)}$$

$$\frac{\partial \log l}{\partial \beta_2} = \frac{m}{\beta_2} - (\alpha_2+1) \sum_{j=1}^{m} \frac{y_j}{1+\beta_2 y_j} + \alpha_2 \sum_{j=1}^{m} y_j (1+\beta_2 y_j)^{-(\alpha_2+1)}.$$

MLE of R, denoted by $\hat{R}^{ML}$, can obtained by replacing α, β_1 and β_2 by their MLEs.

Then $\hat{R}^{ML}$ is given by

$$\hat{R}^{ML} = \left(\frac{e}{e-1}\right)^2 \sum_{n=0}^{\infty} (-1)^n \binom{\hat{\alpha}+n}{n} \Bigg\{ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{n+i} \sum_{r=0}^{\infty} \frac{\hat{\beta}_2^{\,i}}{\hat{\beta}_1^{\,i}} \frac{(-1)^{m+i+j+r}}{m!r!} \binom{\hat{\alpha}m+i-1}{i} \binom{n+i}{j} \frac{\hat{\alpha}}{\hat{\alpha}r-n+j-i-1} - \sum_{k=0}^{n} \sum_{l=0}^{\infty} \frac{(-1)^{k+l}}{l!} \binom{n}{k} \frac{\hat{\alpha}}{\hat{\alpha}l-n+k-1} \Bigg\}. \quad (4.7.3)$$

The asymptotic variance of $\hat{R}^{ML}$ is given by

$$AV(\hat{R}^{ML}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial R}{\partial \theta_i} \frac{\partial R}{\partial \theta_j} I^{-1}(\underline{\theta}), \quad (4.7.4)$$

where $\underline{\theta} = (\alpha, \beta_1, \beta_2)$ and $I^{-1}(\underline{\theta})$ is the inverse of Fisher Information Matrix. Therefore, an asymptotic $100(1-\nu)\%$ confidence interval for R can obtain as $\hat{R}^{ML} \pm Z_{\frac{\nu}{2}}\sqrt{AV(\hat{R}^{ML})}$ where $Z_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}-$ quantile of standard Normal distribution.

4.8 Simulation Study

A simulation study is performed to verify the MLEs work for different sample sizes and different parameter values for the proposed DUS-Lomax(α, β) distribution using inversion method. Eq. (4.4.3) is used to generate a random sample from the DUS-Lomax with parameter α and β. The different sample sizes considered in the simulation are $n = 10,\ 25,\ 50,\ 100,\ 250,\ 500,\ 750$ and 1000. We have used 'optim' package in R language to find the estimate. We replicated the process 5000 times and reported the average estimates and the associated mean squared errors in Table 4.1, 4.2, 4.3 and 4.4.

The simulation is conducted for four different cases using varying true parameter values. The selected true parameter values are $\alpha = 0.5$ and $\beta = 0.01$; $\alpha = 1$ and $\beta = 0.5$; $\alpha = 2$ and $\beta = 1.5$; and $\alpha = 0.5$ and $\beta = 2$ for the first, second, third and fourth cases, respectively.

As the sample size increases, the mean square error decreases for all selected parameter values as in Tables 4.1, 4.2, 4.3 and 4.4. The bias caused by the estimates are nearer to zero. Also, when the sample size increases, absolute bias decreases. Thus the estimates tends to the true parameter values with the increase in sample size.

Table 4.1: Simulation study at $\alpha = 0.5$ and $\beta = 0.01$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 0.4785$	-4.299×10^{-6}	9.240×10^{-8}
	$\hat{\beta} = 0.0213$	2.252×10^{-6}	2.536×10^{-8}
25	$\hat{\alpha} = 0.5349$	6.975×10^{-6}	2.432×10^{-7}
	$\hat{\beta} = 0.0126$	5.181×10^{-7}	1.342×10^{-9}
50	$\hat{\alpha} = 0.5222$	4.440×10^{-6}	9.857×10^{-8}
	$\hat{\beta} = 0.0105$	9.865×10^{-8}	4.866×10^{-11}
100	$\hat{\alpha} = 0.5147$	2.939×10^{-6}	4.319×10^{-8}
	$\hat{\beta} = 0.0103$	6.017×10^{-8}	1.811×10^{-11}
250	$\hat{\alpha} = 0.5051$	1.010×10^{-6}	5.102×10^{-9}
	$\hat{\beta} = 0.0101$	2.743×10^{-8}	3.762×10^{-12}
500	$\hat{\alpha} = 0.5032$	6.418×10^{-7}	2.06×10^{-9}
	$\hat{\beta} = 0.0100$	6.533×10^{-9}	2.134×10^{-13}
750	$\hat{\alpha} = 0.5015$	2.986×10^{-7}	4.458×10^{-10}
	$\hat{\beta} = 0.0101$	1.305×10^{-8}	8.510×10^{-13}
1000	$\hat{\alpha} = 0.5013$	2.529×10^{-7}	3.198×10^{-10}
	$\hat{\beta} = 0.0101$	1.067×10^{-8}	5.699×10^{-13}

4.9 Data Analysis

In this section, we illustrate the use of DUS-Lomax(α, β) distribution using three real data sets. We fit DUS-Lomax(α, β) distribution to these data sets and compare with Lomax distribution, Gompertz Lomax (GoL) distribution, Kumaraswamy Lomax (KL) distribution, DUS-Exponential distribution and Inverse Lindley (IL) distribution. The first data-sets, considered here, represent the survival times of two groups of patients suffering from head and neck cancer disease. The patients in one group were treated using radiotherapy ((RT), see Table 4.5), whereas the patients belonging to other group were treated using a combined RT

Table 4.2: Simulation study at $\alpha = 1$ and $\beta = 0.5$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 1.3044$	6.088×10^{-5}	1.853×10^{-5}
	$\hat{\beta} = 0.3013$	-3.974×10^{-5}	7.897×10^{-6}
25	$\hat{\alpha} = 1.4531$	9.061×10^{-5}	4.105×10^{-5}
	$\hat{\beta} = 0.5667$	1.334×10^{-5}	8.90×10^{-7}
50	$\hat{\alpha} = 1.1232$	2.464×10^{-5}	3.035×10^{-6}
	$\hat{\beta} = 0.5086$	1.724×10^{-6}	1.487×10^{-8}
100	$\hat{\alpha} = 1.0526$	1.051×10^{-5}	5.528×10^{-7}
	$\hat{\beta} = 0.5074$	1.487×10^{-6}	1.106×10^{-8}
250	$\hat{\alpha} = 1.0199$	3.975×10^{-6}	7.899×10^{-8}
	$\hat{\beta} = 0.502$	3.997×10^{-7}	7.987×10^{-10}
500	$\hat{\alpha} = 1.009$	1.798×10^{-6}	1.616×10^{-8}
	$\hat{\beta} = 0.501$	2.806×10^{-7}	3.938×10^{-10}
750	$\hat{\alpha} = 1.0060$	1.203×10^{-6}	7.237×10^{-9}
	$\hat{\beta} = 0.5012$	2.328×10^{-7}	2.711×10^{-10}
1000	$\hat{\alpha} = 1.0033$	6.501×10^{-7}	2.113×10^{-9}
	$\hat{\beta} = 0.502$	3.128×10^{-7}	4.891×10^{-10}

and chemotherapy ((CT + RT), see Table 4.7) (Efron (1988)). Another one concerns 46 observations reported on active repair times ((hours), see Table 4.9) for an airborne communication transceiver (Chhikara and Folks (1977)).

The required numerical evaluations are carried out using the R software. Table 4.6, Table 4.8 and Table 4.10 provide the MLEs of the model parameters. The model selection is carried out using the AIC and the BIC:

$$\text{AIC} = -2l + 2k,$$

$$\text{BIC} = -2l + k \log n,$$

Table 4.3: Simulation study at $\alpha = 2$ and $\beta = 1.5$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 19.2540$	0.00345	0.05954
	$\hat{\beta} = 2.0681$	0.000114	6.455×10^{-5}
25	$\hat{\alpha} = 5.3742$	0.000675	0.00228
	$\hat{\beta} = 1.6073$	2.147×10^{-5}	2.305×10^{-5}
50	$\hat{\alpha} = 2.9633$	0.000193	0.000186
	$\hat{\beta} = 1.5316$	6.317×10^{-6}	1.995×10^{-7}
100	$\hat{\alpha} = 2.3108$	6.217×10^{-5}	1.933×10^{-5}
	$\hat{\beta} = 1.4932$	-1.358×10^{-6}	9.215×10^{-9}
250	$\hat{\alpha} = 2.0899$	1.798×10^{-5}	1.616×10^{-6}
	$\hat{\beta} = 1.4981$	-3.833×10^{-7}	7.345×10^{-10}
500	$\hat{\alpha} = 2.0414$	8.286×10^{-6}	3.433×10^{-7}
	$\hat{\beta} = 1.4984$	-3.283×10^{-7}	5.389×10^{-10}
750	$\hat{\alpha} = 2.0269$	5.388×10^{-6}	1.452×10^{-7}
	$\hat{\beta} = 1.5023$	4.682×10^{-7}	1.096×10^{-9}
1000	$\hat{\alpha} = 2.0191$	3.810×10^{-6}	7.260×10^{-8}
	$\hat{\beta} = 1.5004$	8.41×10^{-8}	3.536×10^{-11}

where l denotes the log-likelihood function, k is the number of parameters and n is the sample size. Moreover, perfection of competing models is also tested using the K-S test. K-S test statistic is

$$KS = \max\left\{\frac{i}{m} - z_i, z_i - \frac{i-1}{m}\right\}, i = 1, ..., n,$$

where m denotes the number of classes and $z_i = \text{cdf}(x_i)$, the $x_i^{'}$s being the ordered observations.

Table 4.4: Simulation study at $\alpha = 0.5$ and $\beta = 2$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 1.0819$	0.000116	6.772×10^{-5}
	$\hat{\beta} = 2.9260$	0.000185	0.000172
25	$\hat{\alpha} = 0.5657$	1.314×10^{-5}	8.636×10^{-7}
	$\hat{\beta} = 2.2879$	5.757×10^{-5}	1.657×10^{-5}
50	$\hat{\alpha} = 0.5314$	6.276×10^{-6}	1.970×10^{-7}
	$\hat{\beta} = 2.1124$	2.248×10^{-5}	2.528×10^{-6}
100	$\hat{\alpha} = 0.5149$	2.987×10^{-6}	4.462×10^{-8}
	$\hat{\beta} = 2.0359$	7.178×10^{-6}	2.576×10^{-7}
250	$\hat{\alpha} = 0.5045$	9.096×10^{-7}	4.137×10^{-9}
	$\hat{\beta} = 2.0277$	5.548×10^{-6}	1.539×10^{-7}
500	$\hat{\alpha} = 0.5015$	3.093×10^{-7}	4.784×10^{-10}
	$\hat{\beta} = 2.0223$	4.457×10^{-6}	9.931×10^{-8}
750	$\hat{\alpha} = 0.5015$	3.091×10^{-7}	4.778×10^{-10}
	$\hat{\beta} = 2.0065$	1.296×10^{-6}	8.396×10^{-9}
1000	$\hat{\alpha} = 0.5008$	1.677×10^{-7}	1.407×10^{-10}
	$\hat{\beta} = 2.0091$	1.830×10^{-6}	1.674×10^{-8}

4.9.1 Complete Data radiotherapy (RT)

The data set is given below: The values of the AIC, BIC and K-S Statistic are

Table 4.5: Survival times of patients treated using RT:

6.53	7	10.42	14.48	16.1	22.7	34	41.55	42	45.28
49.4	53.62	63	64	83	84	91	108	112	129
133	133	139	140	140	146	149	154	157	160
160	165	146	149	154	157	160	160	165	173
176	218	225	241	248	273	277	297	405	417
420	440	523	583	594	1101	1146	1417		

listed in Table 4.6. The variance covariance matrix of the MLEs under the DUS-

Table 4.6: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics of the fitted models in Data set 1.

Model	MLEs	$\log \mathbf{L}$	AIC	BIC	KS-Statistic	p-value
DUS-Lomax	$\hat{\alpha} = 4.195$ $\hat{\beta} = 0.0018$	-370.85	745.7	749.82	0.139	0.210
Lomax	$\hat{\alpha} = 6.668$ $\hat{\beta} = 0.00078$	-371.61	747.219	751.34	0.145	0.175
KL	$\hat{a} = 27.93$ $\hat{b} = 112.22$ $\hat{\alpha} = 0.1851$ $\hat{\lambda} = 0.0085$	-371.01	750.03	758.27	0.154	0.126
GoL	$\hat{\theta} = 0.0042$ $\hat{\alpha} = 0.689$ $\hat{\beta} = 1.812$ $\hat{\gamma} = 1.405$	-372.381	752.76	761.004	0.158	0.111
DUS-E(θ)	$\hat{\theta} = 0.0056$	-373.82	749.647	751.71	0.201	0.0188
ILD	$\hat{\theta} = 60.094$	-385.70	773.41	775.47	22.629	2.2×10^{-16}

Lomax distribution for the Data set 1 is computed as

$$= \begin{pmatrix} 3.2382 & -0.001953 \\ -0.001953 & 1.0991 \times 10^{-6} \end{pmatrix}.$$

Thus, the variances of the MLE of α and β is $\text{Var}(\hat{\alpha}) = 3.2382$ and $\text{Var}(\hat{\beta}) = 1.0991 \times 10^{-6}$. Therefore, 95% confidence intervals for α and β are [1.235, 7.155] and [0.000107, 0.00356] respectively. Histogram and Empirical cdf of DUS-Lomax (α, β) are given in Figure 4.3.

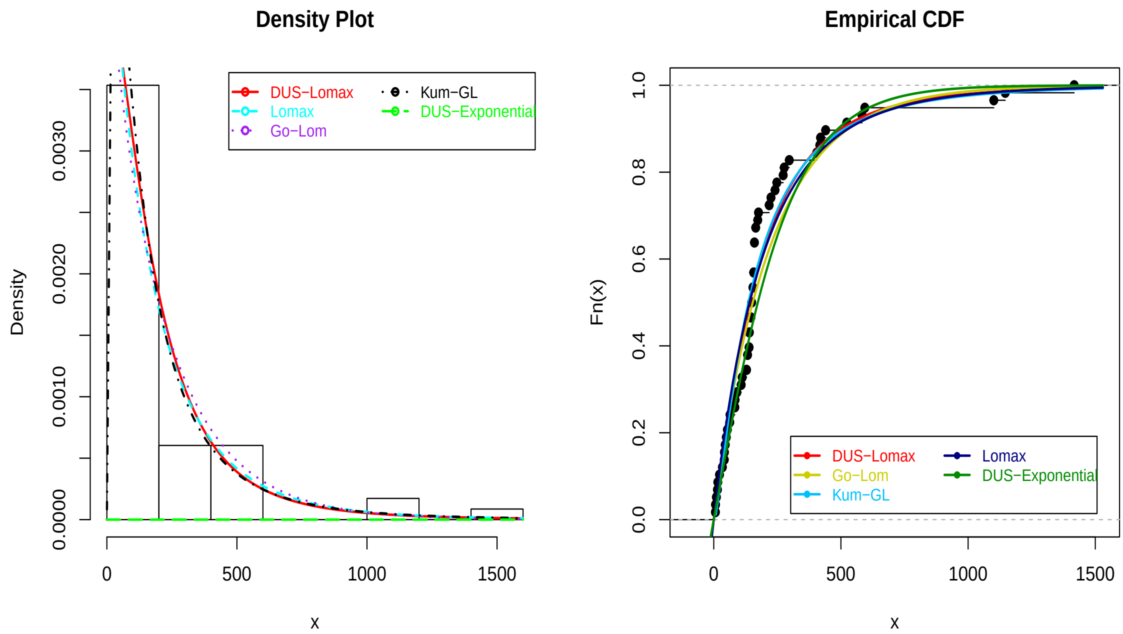


Figure 4.3: Histogram with fitted pdfs (left) and Empirical cdf with fitted cdfs (right) for the Data set 1.

4.9.2 Complete Data RT and chemotherapy (RT+CT)

The data set is given below: The values of the AIC, BIC and K-S Statistic are

Table 4.7: Survival times of patients treated using RT+CT:

12.2	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46
58.36	63.47	68.46	78.26	74.47	81.43	84	92	94
110	112	119	127	130	133	140	146	155
159	173	179	194	195	209	249	281	319
339	432	469	519	633	725	817	1776	

listed in Table 4.8. The variance covariance matrix of the MLEs under the DUS-

Table 4.8: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics of the fitted models in Data set 2.

Model	**MLEs**	$\log \mathbf{L}$	**AIC**	**BIC**	**KS-Statistic**	**p-value**
DUS-Lomax	$\hat{\alpha} = 3.165$ $\hat{\beta} = 0.0028$	-279.91	563.81	567.38	0.093	0.806
Lomax	$\hat{\alpha} = 4.40$ $\hat{\beta} = 0.0013$	-280.45	564.91	568.48	0.104	0.695
KL	$\hat{a} = 23.902$ $\hat{b} = 0.125$ $\hat{\alpha} = 8.675$ $\hat{\lambda} = 44.97$	-281.91	571.82	578.95	0.211	0.034
GoL	$\hat{\theta} = 0.0185$ $\hat{\alpha} = 0.467$ $\hat{\beta} = 0.719$ $\hat{\gamma} = 1.99$	-281.77	571.54	578.68	0.1297	0.414
DUS-E(θ)	$\hat{\theta} = 0.0056$	-283.91	569.82	571.60	0.198	0.0208
IL	$\hat{\theta} = 77.68$	-279.58	561.16	562.94	29.01	5.551×10^{-16}

Lomax distribution for the Data set 2 is computed as

$$= \begin{pmatrix} 41.1184 & -0.05016 \\ -0.05016 & 6.0923 \times 10^{-5} \end{pmatrix}.$$

Thus, the variances of the MLE of α and β is $\text{Var}(\hat{\alpha}) = 41.118$ and $\text{Var}(\hat{\beta}) = 6.0923 \times 10^{-5}$. Therefore, 95% confidence intervals for α and β are [-7.382, 13.713] and [-0.0101, 0.0156] respectively. Histogram and Empirical cdf of DUS-Lomax (α, β) are given in Figure 4.4.

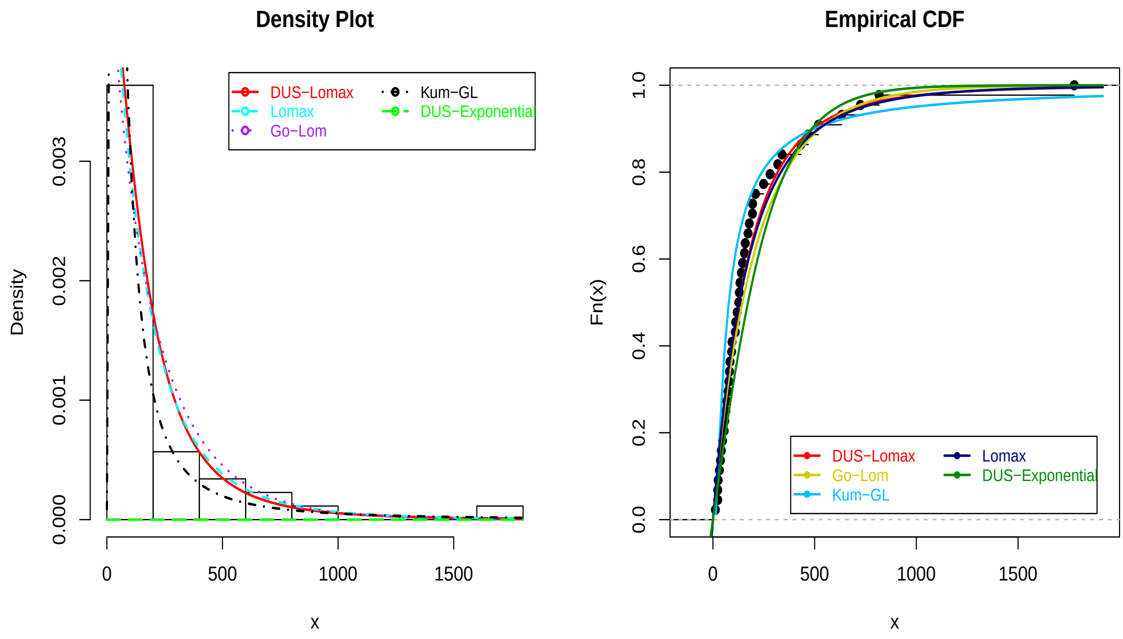


Figure 4.4: Histogram with fitted pdf (left) and Empirical cdf with fitted cdf (right) for the Data set 2.

4.9.3 Complete Data Repair Time

The data set is given below:

Table 4.9: Repair Time:

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6
0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0
1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0
2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3
4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5		

The values of the AIC, BIC and K-S Statistic are listed in Table 4.10. The variance

Table 4.10: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics of the fitted models in Data set 3.

Model	MLEs	$\log \mathbf{L}$	AIC	BIC	KS-Statistic	p-value
DUS-Lomax	$\hat{\alpha} = 2.610$ $\hat{\beta} = 0.227$	-102.70	209.40	213.06	0.118	0.548
Lomax	$\hat{\alpha} = 3.549$ $\hat{\beta} = 0.108$	-102.95	209.91	213.57	0.127	0.446
GoL	$\hat{\theta} = 1.776$ $\hat{\alpha} = 1.165$ $\hat{\beta} = 0.189$ $\hat{\gamma} = 0.245$	-102.95	213.96	221.27	0.129	0.432
DUS-E(θ)	$\hat{\theta} = 0.344$	-107.66	217.31	219.14	0.211	0.033
ILD	$\hat{\theta} = 1.577$	-101.17	204.34	206.17	0.883	2.2×10^{-16}

covariance matrix of the MLEs under the DUS-Lomax distribution for the Data set 3 is computed as

$$= \begin{pmatrix} 1.3693 & -0.15959 \\ -0.15959 & 0.0203 \end{pmatrix}.$$

Thus, the variances of the MLE of α and β is $\text{Var}(\hat{\alpha}) = 1.369$ and $\text{Var}(\hat{\beta}) = 0.0203$. Therefore, 95% confidence intervals for α and β are [0.6855, 4.535] and [-0.00767, 0.4610] respectively. Histogram and Empirical cdf of DUS-Lomax (α, β) are given in Figure 4.5.

Table 4.6, Table 4.8 and Table 4.10 show that, DUS-Lomax (α, β) has lowest AIC, BIC, KS-Statistic, and largest Log-likelihood value and p-value based on K-S Statistic. The second lowest AIC, BIC, K-S Statistic and second largest log-likelihood value and p value are obtained by the Lomax distribution. The

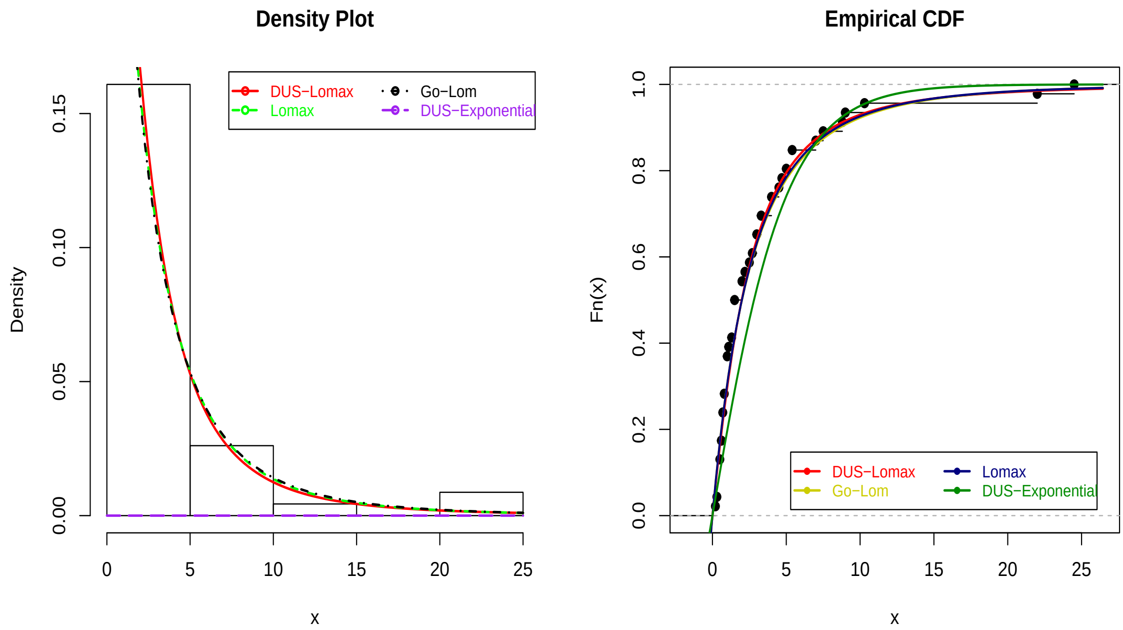


Figure 4.5: Histogram with fitted pdfs (left) and Empirical cdf with fitted cdfs (right) for the Data set 3.

proposed distribution, DUS-Lomax (α, β) can be used when failure rate pattern of lifetime distribution is upside-down bathtub shaped. In Data set 1, 2 and 3 seems that DUS-Lomax (α, β) is more appropriate than Lomax distribution, GoL distribution, KL distribution, DUS-Exponential distribution and IL distribution. So DUS-Lomax (α, β) is better alternative in the situations in which upside-down bathtub distributions arises.

4.10 Summary

DUS-transformation is a kind of parsimonious distribution. That is, we can do computation and interpretation very easily even without changing the parameters. Then if we apply this transformation into Lomax, its failure rate behaviour is changing into an upside down bathtub one. A new distribution, DUS-Lomax (α, β) distribution, is proposed and its properties are studied. The DUS-Lomax (α, β) has UBFR function. We derived the moments, moment generating function, characteristic function, quantiles, entropy etc., of the proposed distribution. Distributions of minimum and maximum are obtained. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of stress-strength models is derived. A simulation study is performed for validate the MLE. DUS-Lomax (α, β) distribution is applied to three real data sets and shows that DUS-Lomax (α, β) distribution is a better fit than other well-known distributions.

CHAPTER 5

STRESS-STRENGTH RELIABILITY

5.1 Introduction

[1] The estimation of stress-strength reliability is a very common problem in statistical literature. This model is used in many applications of physics and engineering such as strength failure and the system collapse. In many practical situations, the components of a system are of different structure so that the assumption of identical strength distributions may not be quite realistic.

The term stress is defined as a failure inducing variable. It is defined as stress (load) which tends to produce a failure of a component or of a device of a material. The term load may be defined as mechanical load, environment, temperature and electric current etc.

[1] Some contents of this chapter are based on Deepthi and Chacko (2020).

The term strength is defined as it is failure resisting variable. The ability of component, device or a material to accomplish its required function (mission) satisfactorily without failure when subjected to the external loading and environment.

In reliability and survival analysis, the stress-strength model describes the probabilistic behavior of life of a component that has a random strength X and is subjected to random stress Y. The system fails if and only if the stress is greater than strength at any time. The reliability parameter, for a single component stress-strength (SSS) model, is

$$R = P(Y < X) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x, y)\ dy\ dx,$$

where $f(x, y)$ is the joint pdf of X and Y. If the r.v's X and Y are independent, then $f(x, y) = f(x)\ g(y)$, where $f(x)$ and $g(y)$ are the marginal pdfs of X and Y, so that

$$R = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x)\ g(y)\ dy\ dx.$$

Let $G_y(x) = \int_{-\infty}^{x} g(y)\ dy$, then R becomes

$$R = \int_{-\infty}^{\infty} G_y(x) f(x)\ dx.$$

The survival probability of a SSS model has been considered by several authors for different distributions. Birnbaum (1956) introduced the stress-strength model and proposed a non-parametric estimator of R. Guttman et al. (1988) and Weerahandi and Johnson (1992) considered the estimation of R, and also obtained the associated confidence interval of R, when both stress and strength depend on some known covariates. Sun et al. (1998) obtained a Bayesian approach for estimating stress-strength reliability.

Raqab and Kundu (2005) studied the estimation of stress-strength reliability, when Y and X two independent scaled Burr type X distribution. Kundu and Gupta (2005) studied stress-strength reliability based on independent generalized exponential distributions with different shape parameters but having the same scale parameters. Kundu and Gupta (2006) studied the estimation of R based od Weibull distribution. Baklizi and Eidous (2006) proposed an estimator of stress-strength reliability based on kernel estimators. Raqab et al. (2008) discussed estimation of R based on three-parameter generalized Exponential distribution. Zhou (2008) illustrated estimation of stress-strength reliability using bootstrap method. Jing et al. (2009) estimated stress-strength reliability using empirical likelihood method. Kundu and Raqab (2009) proposed estimation of R based on three-parameter Weibull distribution. Rezaei et al. (2010) studied the estimation of stress-strength reliability based on two independent generalized Pareto random variables. Baklizi (2012) studied inference on stress-strength reliability in the two-parameter Weibull model.

Recently Jose et al. (2019) and Xavier and Jose (2020) studied the stress-strength reliability estimation of single and multi-component systems using various generalizations of half logistic distribution. Joby et al. (2020) studied estimation of stress-strength reliability of single and multi-component systems based on discrete phase type distribution. Domma et al. (2019) proposed the stress-strength reliability based on the m-generalized order statistics and the corresponding concomitant. Krishna et al. (2019) studied estimation of R using inverse Weibull distribution based on progressive first failure censoring. Kohansal and Nadarajah (2019) considered estimation of R using Kumaraswamy distribution based on Type-II hybrid progressive censored samples. Musleh et al. (2019) studied inference on R in bivariate Lomax model.

Bai et al. (2018) considered reliability inference of stress-strength model under progressively Type-II censored samples when stress and strength have truncated proportional hazard rate distributions. Asgharzadeh et al. (2017) considered estimation of stress-strength reliability based on the generalized exponential distribution. Bi and Gui (2017) studied Bayesian estimation of R using inverse Weibull distribution. Estimation of stress-strength parameter using record values from proportional hazard model was considered by Basirat et al. (2016). Two-parameter bathtub shaped life time distribution based on upper record values was presented by Tarvirdizade and Ahmadpour (2016). Ghitany et al. (2014) discussed inference on stress-strength reliability based on Power Lindley distributions. Sharma (2014) proposed an upside-down bathtub shape distribution and estimate of stress-strength reliability of inverse Lindley distribution.

But, in reality, many of the system consist of two or more components. The reliability analysis of multi-component system having various lifetime distributions for the strength of its components is important for the researchers and engineers. The multi-component stress-strength (MSS) reliability modeling is quite desirable in various real life situations. Bhattacharyya and Johnson (1974) observed the performance of a system depends on more than one component and these components have their own strength. For example, an aircraft generally contains more than one engines (k) and assume that for take off at least $s(1 \leq s \leq k)$ engines are needed. So, the aircraft will take off smoothly, if s out of k engines work. In engineering, a power system powering a manufacturing unit has k fuse cut-outs arranged in a parallel way. The power system will keep powering the manufacturing unit as long as at least $s(1 \leq s \leq k)$ fuse cut-outs are working. In suspension bridges, the deck is supported by a series of vertical cables hung from the towers. Suppose a suspension bridge consists of k number of vertical cable pairs. The bridge will only survive if a minimum s number of vertical cables through the deck are not damaged when subjected to stresses due to wind loading, heavy traffic, corrosion etc.

To find the reliability of a k component system, let the random samples $Y, X_1, X_2, \ldots, X_k$ be independent, $G(y)$ be the continuous distribution function of stress Y and $F(x)$ be the common continuous distribution function of strength $X_1, X_2, \ldots, X_k$ of components $1, 2, \ldots, k$ respectively. The reliability in a MSS model developed by Bhattacharyya and Johnson (1974) is given by

$$\begin{aligned} R_{s,k} &= P[\text{at least } s \text{ of the } (X_1, X_2, \ldots, X_k) \text{ exceed } Y] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F(y)]^i [F(y)]^{k-i} dG(y), \quad s = 1,\, 2, \ldots, k \end{aligned}$$

where $X_1, X_2, \ldots, X_k$ independent and identically distributed (iid) with common distribution function $F(x)$ and subjected to the common random stress Y. Several researchers developed inferential procedures for the reliability of MSS model. Mokhlis and Khames (2011) studied the reliability of some parallel and series MSS model using multivariate Marshall-Olkin Exponential distribution. Rao (2012) developed the MSS reliability based on Generalized Exponential distribution.

Recently, Rao et al. (2015) discussed the MSS reliability with the Burr-XII distribution. Dey et al. (2016) studied the estimation of reliability of a MSS model based on Kumaraswamy distribution. The MSS model using Lindley distribution, when the system consists of k components experiencing a random stress is considered by Khalil (2017). Kohansal (2017) investigated the estimation of MSS reliability by assuming the Kumaraswamy distribution based on progressively Type-II censored samples. Abouelmagd et al. (2018) studied the estimation of reliability of a MSS model based on both classical and Bayesian approaches assuming that the components follow power Lindley model. Hassan and Alohal (2018) studied estimation of MSS reliability based on generalized linear failure rate distribution.

Pandit and Joshi (2018) studied estimation of MSS reliability based on generalized Pareto distribution. Fatma (2019) studied estimation of MSS reliability using Topp-Leone distribution. Jamal et al. (2019) studied estimation of MSS reliability using Pareto distribution based on upper record values. Pak et al. (2019) investigated Bayesian estimation of the reliability of an MSS system for the bathtub-shaped distribution when the available data are reported in terms of record values. Jha et al. (2020) investigated the Bayesian estimation of MSS reliability under progressive Type II censoring when stress and strength variables follow unit Gompertz distributions. Hassan et al. (2020) studied Bayesian estimation of the reliability of a MSS system with Weibull distribution based on upper record values.

In this chapter, we consider the two different cases for stress-strength reliability

- Case 1: If $X \sim \text{TPGL}(\alpha, \beta_1, \lambda_1)$ and $Y \sim \text{TPGL}(\alpha, \beta_2, \lambda_2)$.
- Case 2: If $X \sim \text{TPGL}(\alpha, \beta, \lambda_1)$ and $Y \sim \text{PL}(\alpha, \lambda_2)$.

The procedure of estimating reliability of SSS model is considered in section 5.2. In section 5.3, estimating reliability of MSS model is considered. In section 5.4, a simulation study to investigate the merits of the proposed methods is given. Real data sets are analyzed in section 5.5. Conclusions are given in section in 5.6.

The aim of this chapter is to develop the inferential procedure for estimating the stress-strength reliability $R = P[X > Y]$, where X represents the strength and Y denotes the stress. It is further assumed that X and Y are independent Three Parameter Generalized Lindley (TPGL) and Power Lindley (PL) random variables, having bathtub shaped failure rate function. Stress-strength reliability plays a very important role in the reliability analysis, and has nice probabilistic interpretation. The reliability $R = P[X > Y]$ is the probability that failure will occur a high stress. Many authors developed the estimation procedures for estimating the stress-strength reliability from various lifetime models. In this chapter we discuss stress-strength reliability analysis of bathtub shaped failure rate models.

5.2 Estimation of SSS Reliability

In this section, the procedure of estimating reliability of SSS model using two different cases. That is, when $X \sim \text{TPGL}(\alpha, \beta_1, \lambda_1)$, $Y \sim \text{TPGL}(\alpha, \beta_2, \lambda_2)$ and $X \sim \text{TPGL}(\alpha, \beta, \lambda_1)$, $Y \sim \text{PL}(\alpha, \lambda_2)$. The system fails if and only if the applied stress is greater than its strength. In section 5.2.1 and 5.2.2 obtain the MLE of R in both cases and obtain its asymptotic distribution in both cases in section 5.2.3. The asymptotic distribution has been used to construct an asymptotic confidence interval.

Case 1: Suppose X and Y are random variables independently distributed as $X \sim \text{TPGL}(\alpha, \beta_1, \lambda_1)$ and $Y \sim \text{TPGL}(\alpha, \beta_2, \lambda_2)$.

Nosakhare and Opone (2018) introduced a TPGL distribution, which exhibits bathtub shape for its failure rate function. These distributions are generated using the exponentiation and power transformations to the Lindley distribution. Reliability estimation of SSS and MSS model using TPGL distribution is an unexplored problem.

The pdf of TPGL distribution is

$$f(x; \alpha, \beta, \lambda) = \frac{\alpha\lambda^2}{1+\lambda\beta}(\beta + x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}, \quad x > 0,\ \alpha > 0, \beta > 0, \lambda > 0.$$

Here β and λ are scale parameters, and α is the shape parameter. The cdf is given by

$$F(x; \alpha, \beta, \lambda) = 1 - \left(\frac{1+\beta\lambda+\lambda x^\alpha}{1+\beta\lambda}\right)e^{-\lambda x^\alpha}, \quad x > 0,\ \alpha > 0, \beta > 0, \lambda > 0$$

and the reliability function is given by

$$R(x; \alpha, \beta, \lambda) = \left(\frac{1+\beta\lambda+\lambda x^\alpha}{1+\beta\lambda}\right)e^{-\lambda x^\alpha}, \quad x > 0,\ \alpha > 0, \beta > 0, \lambda > 0.$$

The failure rate function of TPGL distribution is

$$r(x;\alpha,\beta,\lambda) = \frac{\alpha\lambda^2(\beta + x^\alpha)x^{\alpha-1}}{1+\beta\lambda+\lambda x^\alpha}, \quad x>0,\ \alpha>0, \beta>0, \lambda>0.$$

Then, SSS reliability is

$$\begin{aligned}
R = P(Y<X) &= \int_0^\infty f(x)F_y(x)\ dx \\
&= \frac{\alpha\lambda_1^2}{1+\beta_1\lambda_1}\int_0^\infty \left(\beta_1 x^{\alpha-1}e^{-\lambda_1 x^\alpha} + x^{2\alpha-1}e^{-\lambda_1 x^\alpha}\right)\left[1-\left(1+\frac{\lambda_2 x^\alpha}{1+\beta_2\lambda_2}\right)e^{\lambda_2 x^\alpha}\right] dx \\
&= \frac{\lambda_1^2\lambda_2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)}\left[(1+\beta_1)\left(1+\frac{1}{\lambda_1+\lambda_2}\right)+\frac{2}{(\lambda_1+\lambda_2)^2}\right].
\end{aligned} \tag{5.2.1}$$

Case 2: Suppose X follows TPGL distribution with parameters $(\alpha, \beta, \lambda_1)$ and Y follows PL distribution with parameters (α, λ_2) and they are independent random variables. The PL distribution proposed by Ghitany et al. (2013) an extension of the Lindley distribution. The pdf of PL distribution is

$$f(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}, \quad x>0,\ \alpha>0,\ \lambda>0. \tag{5.2.2}$$

Here λ and α are scale and shape parameters. The corresponding cdf is given by

$$F(x;\alpha,\lambda) = 1-\left(1+\frac{\lambda x^\alpha}{1+\lambda}\right)e^{-\lambda x^\alpha}, \quad x>0,\ \alpha>0,\ \lambda>0. \tag{5.2.3}$$

Suppose that X represent the strength of a component exposed to Y stress, then the single component stress-strength reliability is obtained as follows,

$$
\begin{aligned}
R &= P(X > Y) = \int_0^\infty P[X > Y|Y = y] f_y(y)\, dy \\
&= \int_0^\infty \frac{\alpha\lambda_2^2}{1+\lambda_2} \left\{ \int_0^x (1 + y^\alpha) y^{\alpha-1} e^{\lambda_2 y^\alpha}\, dy \right\} \frac{\alpha\lambda_1^2}{1+\beta\lambda_1} (\beta + x^\alpha) x^{\alpha-1} e^{-\lambda_1 x^\alpha}\, dx \\
&= \frac{\alpha\lambda_2^2}{1+\lambda_2} \frac{\alpha\lambda_1^2}{1+\beta\lambda_1} \int_0^\infty \left\{ \frac{e^{-\lambda_2 x^\alpha}}{\alpha\lambda_2} + \frac{x^\alpha e^{-\lambda_2 x^\alpha}}{\alpha\lambda_2} - \frac{e^{-\lambda_2 x^\alpha}}{\alpha\lambda_2^2} \right\} \left((\beta + x^\alpha) x^{\alpha-1} e^{-\lambda_1 x^\alpha} \right) dx \\
&= \frac{\alpha^2\lambda_1^2\lambda_2^2}{(1+\beta\lambda_1)(1+\lambda_2)} \left\{ \frac{\beta}{\alpha\lambda_2} \int_0^\infty x^{\alpha-1} e^{-(\lambda_1+\lambda_2)x^\alpha}\, dx + \frac{1}{\alpha\lambda_2} \int_0^\infty x^{2\alpha-1} e^{-(\lambda_1+\lambda_2)x^\alpha}\, dx \right. \\
&+ \frac{\beta}{\alpha\lambda_2} \int_0^\infty x^{2\alpha-1} e^{-(\lambda_1+\lambda_2)x^\alpha}\, dx + \frac{1}{\alpha\lambda_2} \int_0^\infty x^{3\alpha-1} e^{-(\lambda_1+\lambda_2)x^\alpha}\, dx \\
&- \left. \frac{\beta}{\alpha\lambda_2^2} \int_0^\infty x^{\alpha-1} e^{-(\lambda_1+\lambda_2)x^\alpha}\, dx - \frac{1}{\alpha\lambda_2^2} \int_0^\infty x^{2\alpha-1} e^{-(\lambda_1+\lambda_2)x^\alpha}\, dx \right\} \\
&= \frac{\lambda_1^2\lambda_2}{(1+\beta\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)} \left\{ \left(2 + \beta - \frac{1}{\lambda_2}\right) \left(1 + \frac{1}{\lambda_1+\lambda_2}\right) + \frac{2}{(\lambda_1+\lambda_2)^2} \right\}.
\end{aligned}
\tag{5.2.4}
$$

Remark 5.2.1.

- The stress-strength reliability parameter R in (5.2) and (5.2.4) does not depend on the common shape parameter α.
- If $R = 0.5$ which means there is an equal chance that strength X is greater than stress Y.
- If $R > 0.5$ which means there is a small chance that X is greater than Y.
- If $R < 0.5$ which means there is a high chance that X is greater than Y.

5.2.1 Maximum Likelihood Estimation of R (Case 1:)

Let X be the strength r.v following TPGL $(\alpha, \beta_1, \lambda_1)$ distribution and Y be the stress r.v. following TPGL$(\alpha, \beta_2, \lambda_2)$ distribution. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be two ordered random samples of size n, m respectively, taken from TPGL distribution. Then the likelihood function based on the combined random sample is given by

$$L = \prod_{i=1}^{n} \frac{\alpha\lambda_1^2}{1+\beta_1\lambda_1}(\beta_1 + x_i^\alpha)x_i^{\alpha-1}e^{-\lambda_1 x_i^\alpha} \prod_{j=1}^{m} \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}(\beta_2 + y_j^\alpha)y_j^{\alpha-1}e^{-\lambda_2 y_j^\alpha}.$$

The log-likelihood function is

$$\begin{aligned} l = \log L = (n+m)\log\alpha + 2n\log\lambda_1 - n\log(1+\beta_1\lambda_1) + \sum_{i=1}^{n}\log(\beta_1 + x_i^\alpha) \\ + (\alpha-1)\sum_{i=1}^{n}\log x_i - \lambda_1\sum_{i=1}^{n} x_i^\alpha + 2m\log\lambda_2 - m\log(1+\beta_2\lambda_2) \\ + \sum_{j=1}^{m}\log(\beta_2 + y_j^\alpha) + (\alpha-1)\sum_{j=1}^{m}\log y_j - \lambda_2\sum_{j=1}^{m} y_j^\alpha. \end{aligned}$$

The MLE of the parameters is the solution of following non-linear equations

$$\frac{\partial l}{\partial \beta_1} = -\frac{n\lambda_1}{1+\beta_1\lambda_1} + \sum_{i=1}^{n}\frac{1}{\beta_1 + x_i^\alpha} \tag{5.2.5}$$

$$\frac{\partial l}{\partial \beta_2} = -\frac{m\lambda_2}{1+\beta_2\lambda_2} + \sum_{j=1}^{m}\frac{1}{\beta_2 + y_j^\alpha} \tag{5.2.6}$$

$$\frac{\partial l}{\partial \lambda_1} = \frac{2n}{\lambda_1} - \frac{n\beta_1}{1+\beta_1\lambda_1} - \sum_{i=1}^{n} x_i^{\alpha} \tag{5.2.7}$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{2m}{\lambda_2} - \frac{m\beta_2}{1+\beta_2\lambda_2} - \sum_{j=1}^{m} y_j^{\alpha} \tag{5.2.8}$$

and

$$\frac{\partial l}{\partial \alpha} = \frac{n+m}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta_1+x_i^{\alpha})} + \sum_{i=1}^{n} \log x_i - \lambda_1 \sum_{i=1}^{n} x_i^{\alpha}\log x_i + \sum_{j=1}^{m} \frac{y_j^{\alpha}\log y_j}{(\beta_2+y_j^{\alpha})} + \sum_{j=1}^{m} \log y_j - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha}\log y_j. \tag{5.2.9}$$

The second partial derivatives are

$$\frac{\partial^2 l}{\partial \beta_1^2} = \frac{n\lambda_1^2}{(1+\beta_1\lambda_1^2)^2} - \sum_{i=1}^{n} \frac{1}{(\beta_1+x_i^{\alpha})^2}, \quad \frac{\partial^2 l}{\partial \beta_1 \partial \alpha} = -\sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta_1+x_i^{\alpha})^2},$$

$$\frac{\partial^2 l}{\partial \beta_2^2} = \frac{m\lambda_2^2}{(1+\beta_2\lambda_2^2)^2} - \sum_{j=1}^{m} \frac{1}{(\beta_2+y_j^{\alpha})^2}, \quad \frac{\partial^2 l}{\partial \beta_2 \partial \alpha} = -\sum_{j=1}^{m} \frac{y_j^{\alpha}\log y_j}{(\beta_2+y_j^{\alpha})^2},$$

$$\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 l}{\partial \beta_1 \partial \lambda_2} = \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_1} = 0,$$

$$\frac{\partial^2 l}{\partial \lambda_1^2} = \frac{n\beta_1^2}{(1+\beta_1\lambda_1)^2} - \frac{2n}{\lambda_1^2}, \quad \frac{\partial^2 l}{\partial \beta_1 \partial \lambda_1} = -\frac{n}{(1+\beta_1\lambda_1^2)^2}, \quad \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} = -\sum_{i=1}^{n} x_i^{\alpha}\log x_i,$$

$$\frac{\partial^2 l}{\partial \lambda_2^2} = \frac{m\beta_2^2}{(1+\beta_2\lambda_2)^2} - \frac{2m}{\lambda_2^2}, \quad \frac{\partial^2 l}{\partial \beta_2 \partial \lambda_2} = -\frac{m}{(1+\beta_2\lambda_2^2)^2}, \quad \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} = -\sum_{j=1}^{m} y_j^{\alpha}\log y_j$$

and

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n+m}{\alpha^2} + \sum_{i=1}^{n} \left\{ \frac{x_i^{\alpha}\log^2(x_i)}{\beta_1+x_i^{\alpha}} - \frac{x_i^{2\alpha}\log^2 x_i}{(\beta_1+x_i^{\alpha})^2} \right\} - \lambda_1 \sum_{i=1}^{n} x_i^{\alpha}\log^2(x_i) - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha}\log^2(y_j) + \sum_{j=1}^{m} \left\{ \frac{y_j^{\alpha}\log^2(y_j)}{\beta_2+y_j^{\alpha}} - \frac{y_j^{2\alpha}\log^2 y_j}{(\beta_2+y_j^{\alpha})^2} \right\}.$$

The MLE of SSS reliability R is obtained by

$$\hat{R}^{ML} = \frac{\hat{\lambda_1}^2\hat{\lambda_2}}{(1+\hat{\beta}_1\hat{\lambda_1})(1+\hat{\beta}_2\hat{\lambda_2})(\hat{\lambda_1}+\hat{\lambda_2})}\left[(1+\hat{\beta}_1)\left(1+\frac{1}{\hat{\lambda_1}+\hat{\lambda_2}}\right)+\frac{2}{(\hat{\lambda_1}+\hat{\lambda_2})^2}\right]. \quad (5.2.10)$$

5.2.2 Maximum Likelihood Estimation of R (Case 2:)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from TPGL$(\alpha, \beta, \lambda_1)$ and $Y_1, Y_2, \ldots, Y_m$ is a random sample of size m from PL(α, λ_2). Then the likelihood function is given by

$$L = \prod_{i=1}^{n} \frac{\alpha\lambda_1^2}{1+\beta\lambda_1}(\beta + x_i^\alpha)x_i^{\alpha-1}e^{-\lambda_1 x_i^\alpha} \prod_{j=1}^{m} \frac{\alpha\lambda_2^2}{1+\lambda_2}(1+y_j^\alpha)y_j^{\alpha-1}e^{-\lambda_2 y_j^\alpha}.$$

Then the log-likelihood function is

$$\begin{aligned} l = \log L = (n+m)\log\alpha + 2n\log\lambda_1 - n\log(1+\beta\lambda_1) + \sum_{i=1}^{n}\log(\beta + x_i^\alpha) \\ + (\alpha-1)\sum_{i=1}^{n}\log x_i - \lambda_1\sum_{i=1}^{n} x_i^\alpha + 2m\log\lambda_2 - m\log(1+\lambda_2) \\ + \sum_{j=1}^{m}\log(1+y_j^\alpha) + (\alpha-1)\sum_{j=1}^{m}\log y_j - \lambda_2\sum_{j=1}^{m} y_j^\alpha. \quad (5.2.11) \end{aligned}$$

The MLE of the parameters is the solution of non-linear equations as follows

$$\frac{\partial l}{\partial \beta} = -\frac{n\lambda_1}{1+\beta\lambda_1} + \sum_{i=1}^{n}\frac{1}{\beta + x_i^\alpha} \quad (5.2.12)$$

$$\frac{\partial l}{\partial \lambda_1} = \frac{2n}{\lambda_1} - \frac{n\beta}{1+\beta\lambda_1} - \sum_{i=1}^{n} x_i^{\alpha} \tag{5.2.13}$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{2m}{\lambda_2} - \frac{m}{1+\lambda_2} - \sum_{j=1}^{m} y_j^{\alpha} \tag{5.2.14}$$

and

$$\begin{aligned}\frac{\partial l}{\partial \alpha} &= \frac{n+m}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta + x_i^{\alpha})} + \sum_{i=1}^{n} \log x_i - \lambda_1 \sum_{i=1}^{n} x_i^{\alpha} \log x_i \\ &+ \sum_{j=1}^{m} \frac{y_j^{\alpha}\log y_j}{(1+y_j^{\alpha})} + \sum_{j=1}^{m} \log y_j - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha} \log y_j. \end{aligned} \tag{5.2.15}$$

The second partial derivatives are

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{n\lambda_1^2}{(1+\beta\lambda_1^2)^2} - \sum_{i=1}^{n} \frac{1}{(\beta + x_i^{\alpha})^2}, \quad \frac{\partial^2 l}{\partial \beta \partial \alpha} = -\sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta + x_i^{\alpha})^2},$$

$$\frac{\partial^2 l}{\partial \lambda_1^2} = \frac{n\beta^2}{(1+\beta\lambda_1)^2} - \frac{2n}{\lambda_1^2}, \quad \frac{\partial^2 l}{\partial \beta \partial \lambda_1} = -\frac{n}{(1+\beta\lambda_1^2)^2}, \quad \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} = -\sum_{i=1}^{n} x_i^{\alpha} \log x_i,$$

$$\frac{\partial^2 l}{\partial \lambda_2^2} = \frac{m}{(1+\lambda_2)^2} - \frac{2m}{\lambda_2^2}, \quad \frac{\partial^2 l}{\partial \beta \partial \lambda_2} = 0, \quad \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} = -\sum_{j=1}^{m} y_j^{\alpha} \log y_j$$

and

$$\begin{aligned}\frac{\partial^2 l}{\partial \alpha^2} &= -\frac{n+m}{\alpha^2} + \sum_{i=1}^{n} \left\{ \frac{x_i^{\alpha}\log^2(x_i)}{\beta + x_i^{\alpha}} - \frac{x_i^{2\alpha}\log^2 x_i}{(\beta + x_i^{\alpha})^2} \right\} \\ &- \lambda_1 \sum_{i=1}^{n} x_i^{\alpha} \log^2(x_i) - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha} \log^2(y_j) + \sum_{j=1}^{m} \left\{ \frac{y_j^{\alpha}\log^2(y_j)}{1+y_j^{\alpha}} - \frac{y_j^{2\alpha}\log^2 y_j}{(1+y_j^{\alpha})^2} \right\}. \end{aligned}$$

The MLE of single component stress-strength reliability R is obtained by

$$\hat{R}^{ML} = \frac{\hat{\lambda_1}^2\hat{\lambda_2}}{(1+\hat{\beta}\hat{\lambda_1})(1+\hat{\lambda_2})(\hat{\lambda_1}+\hat{\lambda_2})}\left\{\left(2+\hat{\beta}-\frac{1}{\hat{\lambda_2}}\right)\left(1+\frac{1}{\hat{\lambda_1}+\hat{\lambda_2}}\right)+\frac{2}{(\hat{\lambda_1}+\hat{\lambda_2})^2}\right\}. \tag{5.2.16}$$

5.2.3 Asymptotic distribution and Confidence Intervals

In this section, the asymptotic distribution and confidence interval of the MLE of R is obtained from case 1. To find an asymptotic variance of the $\hat{R}^{ML}$ in (5.2.10), let us denote the Fisher information matrix of $\theta = (\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ as $I(\theta) = [I_{ij}(\theta);\ i, j = 1, 2, \ldots, 5]$, i.e.,

$$I(\theta) = E\begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\beta_1} & -\frac{\partial^2 l}{\partial\alpha\partial\beta_2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\beta_1\partial\alpha} & -\frac{\partial^2 l}{\partial\beta_1^2} & -\frac{\partial^2 l}{\partial\beta_1\partial\beta_2} & -\frac{\partial^2 l}{\partial\beta_1\partial\lambda_1} & -\frac{\partial^2 l}{\partial\beta_1\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\beta_2\partial\alpha} & -\frac{\partial^2 l}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2 l}{\partial\beta_2^2} & -\frac{\partial^2 l}{\partial\beta_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\beta_2\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_1\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_1\partial\beta_1} & -\frac{\partial^2 l}{\partial\lambda_1\partial\beta_2} & -\frac{\partial^2 l}{\partial\lambda_1^2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_2\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_2\partial\beta_1} & -\frac{\partial^2 l}{\partial\lambda_2\partial\beta_2} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_2^2} \end{bmatrix}.$$

In order to establish the asymptotic Normality of R, we further define

$$d(\theta) = \left(\frac{\partial R}{\partial\alpha}, \frac{\partial R}{\partial\beta_1}, \frac{\partial R}{\partial\beta_2}, \frac{\partial R}{\partial\lambda_1}, \frac{\partial R}{\partial\lambda_2}\right)' = (d_1, d_2, d_3, d_4, d_5)',$$

where

$$\frac{\partial R}{\partial\alpha} = 0, \quad \frac{\partial R}{\partial\beta_1} = -\frac{\lambda_1^2\lambda_2\left((\lambda_1-1)\lambda_2^2+(2\lambda_1^2-\lambda_1-1)\lambda_2+\lambda_1^3+\lambda_1\right)}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^3},$$

$$\frac{\partial R}{\partial \beta_2} = -\frac{\lambda_1^2\lambda_2^2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)}\left[(1+\beta_1)\left(1+\frac{1}{\lambda_1+\lambda_2}\right)+\frac{2}{(\lambda_1+\lambda_2)^2}\right],$$

$$\frac{\partial R}{\partial \lambda_1} = \lambda_1\lambda_2\left[\frac{\left((\beta_1^2+\beta_1)\lambda_2-\beta_1^2+1\right)\lambda_1^3+\left((2\beta_1^2+2\beta_1)\lambda_2^2+(4\beta_1+4)\lambda_2-4\beta_1\right)\lambda_1^2}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^4}\right.$$
$$+\frac{\left((\beta_1^2+\beta_1)\lambda_2^3+(\beta_1^2+6\beta_1+5)\lambda_2^2+(4\beta_1+2)\lambda_2-2\right)\lambda_1}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^4}$$
$$\left.+\frac{(2\beta_1+2)\lambda_2^3+(2\beta_1+2)\lambda_2^2+4\lambda_2}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^4}\right]$$

and

$$\frac{\partial R}{\partial \lambda_2} = -\lambda_1^2\left[\frac{(1+\beta_1)\beta_2\lambda_2^4+((2\beta_1+2)\beta_2\lambda_1+(2\beta_1+2)\beta_2)\lambda_2^3}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4}\right.$$
$$+\frac{((-2\beta_1-2)\lambda_1^2+4)\lambda_2-2\lambda_1}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4}$$
$$+\frac{((1+\beta_1)\beta_2\lambda_1^2+((2\beta_1+2)\beta_2-\beta_1-1)\lambda_1+6\beta_2+\beta_1+1)\lambda_2^2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4}$$
$$\left.+\frac{(-\beta_1-1)\lambda_1^3+(-\beta_1-1)\lambda_1^2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4}\right].$$

We obtain the asymptotic distribution of $\hat{R}^{ML}$ as

$$\sqrt{n+m}(\hat{R}^{ML}-R) \xrightarrow{d} N(0, d'(\theta)I^{-1}(\theta)d(\theta)).$$

Asymptotic variance of $\hat{R}^{ML}$ is obtained as

$$AV(\hat{R}^{ML}) = \frac{1}{n+m}d'(\theta)I^{-1}(\theta)d(\theta)$$

$$AV(\hat{R}^{ML}) = V(\hat{\alpha})d_1^2 + V(\hat{\beta_1})d_2^2 + V(\hat{\beta_2})d_3^2 + V(\hat{\lambda_1})d_4^2 + V(\hat{\lambda_2})d_5^2$$
$$+ 2d_1d_2Cov(\hat{\alpha}, \hat{\beta_1}) + 2d_1d_3Cov(\hat{\alpha}, \hat{\beta_2}) + \ldots + 2d_4d_5Cov(\hat{\lambda_1}, \lambda_2). \tag{5.2.17}$$

Asymptotic $100(1-\gamma)\%$ confidence interval for R can be obtained as

$$\hat{R}^{ML} \pm Z_{\frac{\gamma}{2}} \sqrt{AV(\hat{R}^{ML})},$$

where $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ quantile of the standard Normal distribution.

Case 2: To find an asymptotic variance of $\hat{R}^{ML}$ in (5.2.16). Let us denote the Fisher information matrix of $\theta = (\alpha, \beta, \lambda_1, \lambda_2)$ as $I(\theta) = I_{ij}(\theta);\ i, j = 1, 2, 3, 4.$

$$I(\theta) = E\begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\beta} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\beta\partial\alpha} & -\frac{\partial^2 l}{\partial\beta^2} & -\frac{\partial^2 l}{\partial\beta\partial\lambda_1} & -\frac{\partial^2 l}{\partial\beta\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_1\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_1\partial\beta} & -\frac{\partial^2 l}{\partial\lambda_1^2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_2\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_2\partial\beta} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_2^2} \end{bmatrix}.$$

In order to establish the asymptotic normality of R, we further define

$$d(\theta) = \left(\frac{\partial R}{\partial\alpha}, \frac{\partial R}{\partial\beta}, \frac{\partial R}{\partial\lambda_1}, \frac{\partial R}{\partial\lambda_2}\right)' = (d_1, d_2, d_3, d_4)',$$

where $\frac{\partial R}{\partial\alpha} = 0,$

$$\frac{\partial R}{\partial\beta} = \frac{-\lambda_1^2\left((2\lambda_1 - 1)\lambda_2^3 + (4\lambda_1 - \lambda_1 - 1)\lambda_2^2 + (2\lambda_1^3 - \lambda_1^2)\lambda_2 - \lambda_1^3 - \lambda_1^2\right)}{(1+\lambda_2)(\lambda_1+\lambda_2)^3(1+\beta\lambda_1)^2},$$

$$\frac{\partial R}{\partial \lambda_1} = \lambda_1 \Bigg[\frac{\left((\beta^2+2\beta)\lambda_2^2 + (-\beta^2-2\beta+2)\lambda_2 + \beta - 1\right)\lambda_1^3}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2}$$
$$+ \frac{\left((2\beta^2+4\beta)\lambda_2^3 + (2\beta+8)\lambda_2^2 + (-4\beta-4)\lambda_2\right)\lambda_1^2}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2}$$
$$+ \frac{\left((\beta^2+2\beta)\lambda_2^4 + (\beta^2+6\beta+10)\lambda_2^3 + (3\beta-1)\lambda_2^2 - 4\lambda_2\right)\lambda_1}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2}$$
$$+ \frac{(2\beta+4)\lambda_2^4 + (2\beta+2)\lambda_2^3 + 2\lambda_2^2}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2} \Bigg],$$

$$\frac{\partial R}{\partial \lambda_2} = -\lambda_1^2 \Bigg[\frac{(\beta+2)\lambda_2^4 + ((2\beta+4)\lambda_1 + 2\beta + 2)\lambda_2^3}{(1+\beta\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^4}$$
$$+ \frac{((\beta+2)\lambda_1^2 + (\beta-3)\lambda_1 + \beta + 4)\lambda_2^2}{(1+\beta\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^4}$$
$$+ \frac{((-2\beta-8)\lambda_1^2 - 6\lambda_1 + 2)\lambda_2 + (-\beta-3)\lambda_1^3 + (-\beta-4)\lambda_1^4 - 4\lambda_1}{(1+\beta\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^4} \Bigg].$$

The asymptotic variance of $\hat{R}^{ML}$ is obtained as

$$\begin{aligned} AV(\hat{R}^{ML}) &= \frac{1}{n+m} d'(\theta) I^{-1}(\theta) d(\theta) \\ &= V(\hat{\alpha})d_1^2 + V(\hat{\beta})d_2^2 + V(\hat{\lambda_1})d_3^2 + V(\hat{\lambda_2})d_4^2 \\ &+ 2d_1 d_2 Cov(\hat{\alpha}, \hat{\beta}) + \ldots + 2d_3 d_4 Cov(\hat{\lambda_1}, \lambda_2). \end{aligned}$$

Hence, an asymptotic $100(1-\eta)\%$ confidence interval for R can be obtained as

$$\hat{R}^{ML} \pm Z_{\frac{\eta}{2}} \sqrt{AV(\hat{R}^{ML})},$$

where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}$ quantile of the standard Normal distribution.

5.3 Estimation of Reliability in MSS Model

Suppose that $Y, X_1, X_2, \ldots, X_k$ are independent, $G(y)$ is the cdf of Y and $F(x)$ is the common cdf of $X_1, X_2, \ldots, X_k$. The reliability of MSS model with TPGL distribution is

$$R_{s,k} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}\sum_{i=s}^{k}\binom{k}{i}\int_0^\infty\left[\left(1+\frac{\lambda_1 x^\alpha}{1+\beta_1\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^i \left[1-\left(1+\frac{\lambda_1 x^\alpha}{1+\beta_1\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^{k-i}(\beta_2+x^\alpha)x^{\alpha-1}e^{-\lambda_2 x^\alpha}\ dx.$$

Expanding the terms inside the integral, we get

$$= \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3} \int_0^\infty \beta_2 x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\ dx + \int_0^\infty x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\ dx.$$

After the simplification, we get

$$R_{s,k} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3} \left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}. \quad (5.3.1)$$

By using the invariance property, MLE of MSS reliability $R_{s,k}$, $s = 1, 2, \ldots, k$, is obtained by

$$\hat{R}_{s,k}^{ML} = \frac{\hat{\alpha}\hat{\lambda_2}^2}{1+\hat{\beta}_2\hat{\lambda_2}} \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\hat{\lambda_1}}{1+\hat{\beta}_1\hat{\lambda_1}}\right)^{l_1+l_3}$$
$$\left\{\frac{\hat{\beta}_2(l_1+l_3)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+2}}\right\}. \quad (5.3.2)$$

In order to establish the asymptotic Normality of $R_{s,k}$, $1 \le s \le k$, we further define

$$d(\theta) = \left(\frac{\partial R_{s,k}}{\partial \alpha}, \frac{\partial R_{s,k}}{\partial \beta_1}, \frac{\partial R_{s,k}}{\partial \beta_2}, \frac{\partial R_{s,k}}{\partial \lambda_1}, \frac{\partial R_{s,k}}{\partial \lambda_2}\right)' = (d_1, d_2, d_3, d_4, d_5)',$$

where, $\frac{\partial R_{s,k}}{\partial \alpha} = 0$,

$$\frac{\partial R_{s,k}}{\partial \beta_1} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2} \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3+1}$$
$$(-(l_1+l_3))\left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\},$$

$$\frac{\partial R_{s,k}}{\partial \beta_2} = \alpha \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3}$$
$$\left[\frac{-\lambda_2}{(1+\beta_2\lambda_2)}\left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}\right.$$
$$\left. + \frac{\lambda_2^2}{(1+\beta_2\lambda_2)} \frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}\right],$$

$$\frac{\partial R_{s,k}}{\partial \lambda_1} = \frac{\alpha \lambda_2^2}{1+\beta_2\lambda_2} \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3}$$
$$\left[\frac{(l_1+l_3)(\beta_1-1)}{\lambda_1(1+\beta_1\lambda_1)}\left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}\right.$$
$$\left. + (i+l_2)\left\{\frac{\beta_2(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}} + \frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}}\right\}\right],$$

$$\frac{\partial R_{s,k}}{\partial \lambda_2} = \alpha\frac{\lambda_2^2+2\lambda_2}{1+\beta_2\lambda_2} \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3}$$
$$\left[(\beta_2-1)\left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}\right.$$
$$\left. + \left\{\frac{\beta_2(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}} + \frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}}\right\}\right].$$

Asymptotic variance of $\hat{R}_{s,k}^{ML}$ is

$$AV(\hat{R}_{s,k}^{ML}) = \sum_{i=1}^{5}\sum_{j=1}^{5} \frac{\partial R_{s,k}}{\partial \theta_i}\frac{\partial R_{s,k}}{\partial \theta_j} I^{-1}(\theta) \tag{5.3.3}$$

where $\theta = (\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ and $I^{-1}(\theta)$ is the Fisher Information Matrix. Therefore, asymptotic $100(1-\nu)\%$ confidence interval for $R_{s,k}$ can be obtained as $\hat{R}_{s,k}^{ML} \pm Z_{\frac{\nu}{2}}\sqrt{AV(\hat{R}_{s,k}^{ML})}$ where $Z_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}-$ quantile of standard Normal distribution.

Case 2: Suppose that $Y, X_1, X_2, \ldots, X_k$ are independent, $G(y)$ is the cumulative function of Y and $F(x)$ is the common cumulative function of $X_1, X_2, \ldots, X_k$. Here, $X \sim$ TPGL$(\alpha, \beta, \lambda_1)$ and $Y \sim$ PL(α, λ_2). The reliability in multi-component stress strength is

$$R_{s,k} = \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\binom{k}{i}\int_0^\infty \left[\left(1+\frac{\lambda_1 x^\alpha}{1+\beta\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^i \left[1-\left(1+\frac{\lambda_1 x^\alpha}{1+\beta\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^{k-i}(1+x^\alpha)x^{\alpha-1}e^{-\lambda_2 x^\alpha}\,dx.$$

Expanding the terms inside the integral, we get

$$\begin{aligned} R_{s,k} &= \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3} \\ &\quad \int_0^\infty x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\,dx + \int_0^\infty x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\,dx \\ &= \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3} \\ &\quad \left\{\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}. \end{aligned} \quad (5.3.4)$$

By using the invariance property, MLE of $R_{s,k}$ is obtained by

$$\begin{aligned} \hat{R}_{s,k}^{ML} &= \frac{\hat{\alpha}\hat{\lambda_2}^2}{1+\hat{\lambda_2}}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\hat{\lambda_1}}{1+\hat{\beta}\hat{\lambda_1}}\right)^{l_1+l_3} \\ &\quad \left\{\frac{(l_1+l_3)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+2}}\right\}. \end{aligned} \quad (5.3.5)$$

In order to establish the asymptotic Normality of $R_{s,k}$, we further define

$$d(\theta) = \left(\frac{\partial R_{s,k}}{\partial\alpha}, \frac{\partial R_{s,k}}{\partial\beta}, \frac{\partial R_{s,k}}{\partial\lambda_1}, \frac{\partial R_{s,k}}{\partial\lambda_2}\right)' = (d_1, d_2, d_3, d_4)',$$

where, $\frac{\partial R_{s,k}}{\partial \alpha}=0$

$$\frac{\partial R_{s,k}}{\partial \beta}=\frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3+1}$$
$$(-(l_1+l_3))\left\{\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}$$

$$\frac{\partial R_{s,k}}{\partial \lambda_1}=\frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3}$$
$$\left[\frac{(l_1+l_3)(\beta-1)}{\lambda_1(1+\beta\lambda_1)}\left\{\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}\right.$$
$$\left.+(i+l_2)\left\{\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}+\frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}}\right\}\right]$$

$$\frac{\partial R_{s,k}}{\partial \lambda_2}=\alpha\frac{\lambda_2^2+2\lambda_2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3}$$
$$\left\{\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}+\frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}}\right\}.$$

The asymptotic variance of $\hat{R}_{s,k}^{ML}$ is

$$AV(\hat{R}_{s,k}^{ML})=\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial R_{s,k}}{\partial\theta_i}\frac{\partial R_{s,k}}{\partial\theta_j}I^{-1}(\theta) \tag{5.3.6}$$

where $\theta=(\alpha,\beta,\lambda_1,\lambda_2)$ and $I^{-1}(\theta)$ is the Fisher Information Matrix. Therefore, an asymptotic $100(1-\zeta)\%$ confidence interval for $R_{s,k}$ can be obtained as $\hat{R}_{s,k}^{ML}\pm Z_{\frac{\zeta}{2}}\sqrt{AV(\hat{R}_{s,k}^{ML})}$ where $Z_{\frac{\zeta}{2}}$ is the upper $\frac{\zeta}{2}-$ quantile of standard Normal distribution.

5.4 Simulation Study

This section consists a simulation study to compare the performances of the estimators proposed in the previous sections. Here, we studied the behavior of the estimators of parameters, R and $R_{s,k}$ on the basis of simulated sample with varying sample size and various combinations of the parameters. All the results are based on 1000 replications.

For this purpose, we need a simulation algorithm for generating a random sample from TPGL and PL distributions. The simplest method used for this purpose is inverse cdf method that utilizes probability integral transformation. Since the probability integral transformation under TPGLD and PLD cannot be applied explicitly, one can apply either Newton's method to solve the the Lambert W function as suggested by (Jorda (2010)). First, we perform the simulation study when $X \sim TPGL(\alpha, \beta_1, \lambda_1)$ and $Y \sim TPGL(\alpha, \beta_2, \lambda_2)$ distributions (case 1) in section 5.4.1. Second, we perform the simulation study when $X \sim TPGL(\alpha, \beta, \lambda_1)$ and $Y \sim PL(\alpha, \lambda_2)$ distributions (case 2) in section 5.4.2.

5.4.1 MLE of R and $R_{s,k}$ (Case 1)

In this section, we perform simulation study R and $R_{s,k}$ when (s, k) = (1,3) and (2,4) respectively, when $X \sim TPGL(\alpha, \beta_1, \lambda_1)$ and $Y \sim TPGL(\alpha, \beta_2, \lambda_2)$. Now to study the behavior of $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$ we use the following algorithm.

Algorithm

1. For given values of $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ compute R, $R_{1,3}$ and $R_{2,4}$ from (5.2.1) and (5.3.1).

2. Using Newton-Raphson formula to generate 1000 random sample.

3. Compute $\hat{\beta}_1, \hat{\beta}_2, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\alpha}$ from (5.2.5) to (5.2.9).

4. Compute $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$.

5. Compute the average bias, average MSE and asymptotic 95% confidence interval of R, $R_{1,3}$ and $R_{2,4}$. $\text{Bias}_1 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)$, $\text{Bias}_2 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})$, $\text{MSE}_1 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)^2$, $\text{MSE}_2 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})^2$ when (s, k) = (1,3) and (2,4). Compute asymptotic 95% confidence interval of R and $R_{s,k}$.

We considered two sets of parameter values $(\alpha, \beta_1, \lambda_1)$ and $(\alpha, \beta_2, \lambda_2)$: (3,5.5,1.25) and (3,0.5,2.5), (2.5,4.5,1.75) and (2.5,3,1), (2,2.5,1.5) and (2,0.5,1.5), (1.5,6,1) and (1.5,0.05,2), and different choice of sample sizes (n, m)= (10,10), (15,15), (15,25), (25,25), (25,30), (30,50), (50,50). From each samples, we compute the estimates of $(\alpha, \beta_1, \lambda_1, \beta_2, \lambda_2)$ using ML estimation. Once we estimate $(\alpha, \beta_1, \lambda_1, \beta_2, \lambda_2)$, we obtain the estimates of R by substituting in (5.2.1). Also obtain the estimates of $R_{s,k}$ by substituting in (5.3.1) for (s, k)=(1,3) and (2,4) respectively. These parameter values correspond to the R values 0.492 (moderate), 0.244 (small), 0.662 (high) and 0.827 (high), respectively. When (s, k) = (1,3), corresponding $R_{s,k}$ values are 0.858 (high), 0.540 (moderate), 0.662 (high) and 0.799 (high) respectively. When $(s,k) = (2{,}4)$, corresponding $R_{s,k}$ values are 0.73 (high), 0.386 (small), 0.489 (small) and 0.631 (high) respectively.

5.4.2 MLE of R and $R_{s,k}$ (Case 2)

In this section, we perform our simulation study of R and $R_{s,k}$ when $(s,k) = (1{,}3)$ and (2,4) respectively, when $X \sim TPGL(\alpha, \beta, \lambda_1)$ and $Y \sim PL(\alpha, \lambda_2)$. To study the behavior of $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$ we use the following algorithm.

Algorithm

1. For given values of $(\alpha, \beta, \lambda_1, \lambda_2)$ compute R, $R_{1,3}$ and $R_{2,4}$ from (5.2.4) and (5.3.4).

2. Using Newton-Raphson formula to generate 1000 random sample.

3. Compute $\hat{\beta}, \hat{\lambda_1}, \hat{\lambda_2}$ and $\hat{\alpha}$ from (5.2.12) to (5.2.15)

4. Compute $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$.

5. Compute the average bias, average MSE and asymptotic 95% confidence interval of R, $R_{1,3}$ and $R_{2,4}$. Where, $\text{Bias}_{1,3} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)$, $\text{Bias}_{2,4} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})$, and $\text{MSE}_{1,3} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)^2$, $\text{MSE}_{2,4} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})^2$ when $(s,k) = (1{,}3)$ and (2,4), and asymptotic 95% confidence interval of R and $R_{s,k}$.

We considered two sets of parameter values $(\alpha, \beta, \lambda_1)$ and (α, λ_2): (2.5,5,1.5) and (2.5,0.5), (3,0.5,2) and (3,1.5), (3.5,2,1.25) and (3.5,1.75), (2.75,1,2) and (2.75,2.25), and different choice of sample sizes (n,m)= (10,10), (15,15), (15,25), (25,25), (25,30), (30,50), (50,50). From each samples, we compute the estimates of $(\alpha, \beta, \lambda_1, \lambda_2)$ using ML estimation. Using the estimate of $(\alpha, \beta, \lambda_1, \lambda_2)$, we obtain the estimates of R by substituting in (5.2.4). Also obtain the estimates of $R_{s,k}$ by substituting in (5.3.4) for (s, k) = (1,3) and (2,4) respectively. These parameter values correspond to the R values 0.353 (small), 0.864 (high), 0.454 (moderate) and 0.710 (high), respectively. When (s, k) = (1,3), corresponding $R_{s,k}$ values are 0.257 (small), 0.682 (high), 0.823 (high) and 0.788 (high) respectively. When (s, k) = (2,4), corresponding $R_{s,k}$ values are 0.153 (small), 0.530 (moderate), 0.687 (high) and 0.646 (high) respectively.

In Tables 5.1-5.16, the average biases, MSE and confidence intervals of the estimates of R and $R_{s,k}$ based on MLE method are given.

Table 5.1: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(3, 5.5, 1.25)$ and $Y \sim TPGL(3, 0.5, 2.5)$

(n, m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.6893	0.000197	0.0000388	(0.6459, 0.7327)
(15,15)	0.4669	-0.0000255	0.0000065	(0.4464, 0.4874)
(15,25)	0.6679	0.000176	0.0000308	(0.6371, 0.6987)
(25,25)	0.6455	0.000153	0.0000234	(0.6248, 0.6662)
(25,30)	0.9421	0.000450	0.000202	(0.9259, 0.9583)
(30,50)	0.3931	-0.0000993	0.0000099	(0.3720, 0.4142)
(50,50)	0.7441	0.0000252	0.0000634	(0.7136, 0.7746)

From the simulation results, it is observed that as the sample size (n, m) increases, the biases and the MSEs decreases. That means when the sample size increases, then the estimated reliability reaches nearest to true value. Thus the consistency properties of all the methods are verified.

Table 5.2: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(3, 5.5, 1.25)$ and $Y \sim TPGL(3, 0.5, 2.5)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.7101	-0.000148	0.0000219	(0.6875, 0.7327)
	(15,15)	0.9426	0.0000845	0.00000714	(0.9135, 0.9717)
	(15,25)	0.9447	0.0000866	0.00000749	(0.9287, 0.9607)
	(25,25)	0.8788	0.0000207	0.00000043	(0.8511, 0.9065)
	(25,30)	0.9563	0.0000982	0.00000964	(0.9315, 0.9811)
	(30,50)	0.8861	0.000028	0.00000078	(0.8499, 0.9223)
	(50,50)	0.8824	0.0000243	0.00000059	(0.8463, 0.9185)
(2,4)	(10,10)	0.5592	-0.000171	0.0000292	(0.5366, 0.5818)
	(15,15)	0.8707	0.000141	0.0000198	(0.8548, 0.8866)
	(15,25)	0.8780	0.000148	0.0000219	(0.8609, 0.8951)
	(25,25)	0.7664	0.0000364	0.00000133	(0.7315, 0.8013)
	(25,30)	0.8914	0.000162	0.0000261	(0.8731, 0.9097)
	(30,50)	0.7730	0.000043	0.00000185	(0.7371, 0.8089)
	(50,50)	0.7707	0.0000407	0.00000165	(0.7586, 0.7828)

Table 5.3: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2.5, 4.5, 1.75)$ and $Y \sim TPGL(2.5, 3, 1)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.6649	0.000421	0.000178	(0.6431, 0.6867)
(15,15)	0.4039	0.000160	0.0000257	(0.3824, 0.4254)
(15,25)	0.2758	0.0000322	0.00000104	(0.2613, 0.2903)
(25,25)	0.4273	0.000184	0.0000338	(0.4079, 0.4467)
(25,30)	0.4380	0.000915	0.0000378	(0.4188, 0.4572)
(30,50)	0.7933	0.000550	0.000302	(0.7844, 0.8022)
(50,50)	0.2079	-0.0000357	0.00000127	(0.1919, 0.2239)

Table 5.4: MLE of $\hat{R}_{s,k}^{ML}$, Bias and MSE using $X \sim TPGL(2.5, 4.5, 1.75)$ and $Y \sim TPGL(2.5, 3, 1)$

(s,k)	(n,m)	$\hat{R}_{s,k}^{ML}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.6012	0.0000608	0.0000037	(0.5855, 0.6169)
	(15,15)	0.5200	-0.0000204	0.00000042	(0.5054, 0.5346)
	(15,25)	0.6060	0.0000657	0.00000431	(0.5912, 0.6208)
	(25,25)	0.4636	-0.0000768	0.00000589	(0.4477, 0.4795)
	(25,30)	0.6754	0.000135	0.0000182	(0.6432, 0.7076)
	(30,50)	0.5294	-0.0000110	0.000000121	(0.5066, 0.5522)
	(50,50)	0.5534	0.0000131	0.000000171	(0.5433, 0.5635)
(2,4)	(10,10)	0.4405	0.0000543	0.00000295	(0.4200, 0.4610)
	(15,15)	0.3851	-0.0000011	0.0000000011	(0.3532, 0.4170)
	(15,25)	0.4491	0.0000629	0.00000396	(0.4282, 0.4700)
	(25,25)	0.3212	-0.0000650	0.00000422	(0.3076, 0.3348)
	(25,30)	0.5194	0.000133	0.0000178	(0.4947, 0.5441)
	(30,50)	0.3566	-0.0000296	0.000000879	(0.3373, 0.3759)
	(50,50)	0.4000	0.0000138	0.000000191	(0.3839, 0.4161)

Table 5.5: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2, 2.5, 1.5)$ and $Y \sim TPGL(2, 0.5, 1.5)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.6598	-0.0000019	0.0000000036	(0.6347, 0.6849)
(15,15)	0.6370	-0.0000247	0.00000061	(0.6199, 0.6541)
(15,25)	0.4395	-0.000222	0.0000494	(0.4109, 0.4681)
(25,25)	0.6120	-0.0000497	0.00000247	(0.6004, 0.6236)
(25,30)	0.7054	0.0000437	0.00000191	(0.6893, 0.7215)
(30,50)	0.5911	-0.0000706	0.00000498	(0.5799, 0.6023)
(50,50)	0.7032	0.0000415	0.00000172	(0.6871, 0.7193)

5.5 Data Analysis

In this section, we consider two real data sets of the breaking strengths of jute fiber at two different gauge lengths (see Xia et al. (2009)). Two sets of real data

Table 5.6: MLE of $\hat{R}_{s,k}^{ML}$, Bias and MSE using $X \sim TPGL(2, 2.5, 1.5)$ and $Y \sim TPGL(2, 0.5, 1.5)$

(s,k)	(n,m)	$\hat{R}_{s,k}^{ML}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.4161	-0.000246	0.0000603	(0.3896, 0.4426)
	(15,15)	0.5462	-0.000115	0.0000133	(0.5202, 0.5722)
	(15,25)	0.6214	-0.0000402	0.00000162	(0.6098, 0.6330)
	(25,25)	0.4636	-0.0000768	0.00000589	(0.4358, 0.4914)
	(25,30)	0.5353	-0.000126	0.0000160	(0.5180, 0.5526)
	(30,50)	0.6482	-0.0000135	0.000000182	(0.6205, 0.6759)
	(50,50)	0.7806	0.000119	0.0000142	(0.7754, 0.7858)
(2,4)	(10,10)	0.3018	-0.000188	0.0000352	(0.2928, 0.3108)
	(15,15)	0.3706	-0.000119	0.0000141	(0.3662, 0.3750)
	(15,25)	0.4569	-0.0000324	0.00000105	(0.4129, 0.5009)
	(25,25)	0.4839	-0.0000054	0.000000029	(0.4665, 0.5013)
	(25,30)	0.3693	-0.000120	0.0000144	(0.3544, 0.3842)
	(30,50)	0.5052	0.000016	0.00000025	(0.4907, 0.5197)
	(50,50)	0.6299	0.000141	0.0000198	(0.6069, 0.6529)

Table 5.7: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(1.5, 6, 1)$ and $Y \sim TPGL(1.5, 0.05, 2)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.3659	-0.000462	0.000213	(0.3422, 0.3896)
(15,15)	0.4616	-0.000366	0.000134	(0.4404, 0.4828)
(15,25)	0.5579	-0.000269	0.0000726	(0.5484, 0.5674)
(25,25)	0.8876	0.0000603	0.00000363	(0.8610, 0.9146)
(25,30)	0.7866	-0.0000407	0.00000166	(0.7780, 0.7952)
(30,50)	0.6834	-0.000144	0.0000206	(0.6726, 0.6942)
(50,50)	0.9782	0.0001508	0.0000228	(0.9632, 0.9932)

are shown as follows:

Data set I: Breaking strength of jute fiber length 10 mm (variable X). 693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16,

Table 5.8: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(1.5, 6, 1)$ and $Y \sim TPGL(1.5, 0.05, 2)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.6560	-0.000143	0.0000206	(0.6405, 0.6715)
	(15,15)	0.8787	0.0000793	0.0000063	(0.8544, 0.9030)
	(15,25)	0.8064	0.0000070	0.00000079	(0.7975, 0.8153)
	(25,25)	0.7580	-0.0000414	0.00000171	(0.7479, 0.7681)
	(25,30)	0.9255	0.000126	0.0000159	(0.9190, 0.9320)
	(30,50)	0.8522	0.0000528	0.00000278	(0.8387, 0.8657)
	(50,50)	0.8405	0.0000411	0.00000169	(0.8293, 0.8517)
(2,4)	(10,10)	0.4984	-0.000133	0.0000176	(0.4844, 0.5124)
	(15,15)	0.7533	0.000122	0.0000149	(0.7324, 0.7742)
	(15,25)	0.6674	0.0000363	0.00000132	(0.6331, 0.7017)
	(25,25)	0.5816	-0.0000496	0.00000256	(0.5618, 0.6014)
	(25,30)	0.8290	0.000198	0.0000391	(0.8131, 0.8449)
	(30,50)	0.7182	0.0000870	0.00000757	(0.7005, 0.7359)
	(50,50)	0.7021	0.0000709	0.00000503	(0.6902, 0.7140)

Table 5.9: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2.5, 5, 1.5)$ and $Y \sim PL(2.5, 0.5)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.1381	-0.000215	0.0000462	(0.0119, 0.2643)
(15,15)	0.2229	-0.00013	0.0000169	(0.1768, 0.2690)
(15,25)	0.3503	-0.0000026	0.0000000068	(0.1614, 0.5392)
(25,25)	0.3240	-0.0000289	0.00000084	(0.1139, 0.5342)
(25,30)	0.4003	0.0000474	0.00000225	(0.0956, 0.7050)
(30,50)	0.3607	0.00000772	0.000000060	(0.2117, 0.5097)
(50,50)	0.3663	0.0000134	0.000000179	(0.0601, 0.6726)

671.49, 183.16, 257.44, 727.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, 177.25.

Table 5.10: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(2.5, 5, 1.5)$ and $Y \sim PL(2.5, 0.5)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.1339	-0.000123	0.0000151	(0.1187, 0.1491)
	(15,15)	0.1509	-0.000106	0.0000112	(0.1268, 0.1750)
	(15,25)	0.2363	-0.0000206	0.000000422	(0.2014, 0.2712)
	(25,25)	0.2274	-0.0000294	0.000000863	(0.1860, 0.2688)
	(25,30)	0.3359	0.0000791	0.00000626	(0.2917, 0.3801)
	(30,50)	0.3061	0.0000493	0.00000243	(0.2445, 0.3677)
	(50,50)	0.3110	0.0000632	0.00000399	(0.2780, 0.3440)
(2,4)	(10,10)	0.0747	-0.0000785	0.00000617	(0.0015, 0.1479)
	(15,15)	0.0884	-0.0000648	0.0000042	(0.0361, 0.1407)
	(15,25)	0.1418	-0.0000114	0.00000013	(0.0463, 0.2327)
	(25,25)	0.1367	-0.0000165	0.000000272	(0.1110, 0.1624)
	(25,30)	0.2238	0.0000706	0.00000498	(0.2037, 0.2439)
	(30,50)	0.1949	0.0000417	0.00000174	(0.1589, 0.2309)
	(50,50)	0.2152	0.0000620	0.00000384	(0.1717, 0.2587)

Table 5.11: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(3, 0.5, 2)$ and $Y \sim PL(3, 1.5)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.7027	-0.000162	0.0000261	(0.6816, 0.7238)
(15,15)	0.6747	-0.000190	0.0000359	(0.6259, 0.7235)
(15,25)	0.5274	-0.000337	0.000113	(0.5078, 0.5470)
(25,25)	0.5548	-0.000309	0.0000957	(0.5271, 0.5825)
(25,30)	0.6049	-0.000259	0.0000672	(0.5886, 0.6212)
(30,50)	0.7467	-0.000117	0.0000138	(0.7061, 0.7873)
(50,50)	0.9789	0.000115	0.0000132	(0.9624, 0.9954)

Data set II: Breaking strength of jute fiber length 20 mm (variable Y). 71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70,

Table 5.12: MLE of $\hat{R}_{s,k}^{ML}$, Bias and MSE using $X \sim TPGL(3, 0.5, 2)$ and $Y \sim PL(3, 1.5)$

(s,k)	(n,m)	$\hat{R}_{s,k}^{ML}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.6085	-0.0000733	0.00000538	(0.5736, 0.6434)
	(15,15)	0.7038	0.0000221	0.000000486	(0.6903, 0.7173)
	(15,25)	0.6388	-0.000043	0.00000185	(0.6128, 0.6648)
	(25,25)	0.7109	0.0000292	0.00000085	(0.6849, 0.7369)
	(25,30)	0.6632	-0.0000185	0.000000344	(0.6313, 0.6951)
	(30,50)	0.6898	0.00000801	0.000000064	(0.6505, 0.7291)
	(50,50)	0.6575	-0.0000242	0.000000587	(0.6372, 0.6778)
(2,4)	(10,10)	0.4486	-0.0000815	0.00000664	(0.4006, 0.4966)
	(15,15)	0.5475	0.0000174	0.000000304	(0.5134, 0.5816)
	(15,25)	0.4765	-0.0000536	0.00000287	(0.4370, 0.5160)
	(25,25)	0.5522	0.0000221	0.000000486	(0.5340, 0.5704)
	(25,30)	0.5011	-0.0000290	0.000000843	(0.4810, 0.5212)
	(30,50)	0.5375	0.00000744	0.000000055	(0.5077, 0.5673)
	(50,50)	0.5085	-0.0000216	0.000000466	(0.4774, 0.5396)

Table 5.13: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(3.5, 2, 1.25)$ and $Y \sim PL(3.5, 1.75)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.4557	0.0000018	0.0000000031	(0.4194, 0.4920)
(15, 15)	0.3193	-0.000135	0.0000181	(0.0512, 0.5875)
(15,25)	0.4487	-0.00000527	0.000000028	(0.4160, 0.4814)
(25,25)	0.3450	-0.000109	0.0000119	(0.3287, 0.3613)
(25,30)	0.4061	-0.0000478	0.00000229	(0.3851, 0.4271)
(30,50)	0.5237	0.0000698	0.00000487	(0.5186, 0.5288)
(50,50)	0.4074	-0.0000466	0.00000217	(0.0337, 0.7810)

547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55.

These data were first used by Xia et al. (2009) and later by Saracoglu et al.

Table 5.14: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(3.5, 2, 1.25)$ and $Y \sim PL(3.5, 1.75)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.9414	0.000118	0.0000140	(0.9183, 0.9645)
	(15, 15)	0.8443	0.0000212	0.000000448	(0.8050, 0.8836)
	(15, 25)	0.8390	0.0000159	0.000000252	(0.7921, 0.8859)
	(25, 25)	0.8435	0.0000204	0.00000042	(0.8019, 0.8851)
	(25,30)	0.9196	0.0000965	0.00000932	(0.8867, 0.9525)
	(30,50)	0.7758	-0.0000473	0.00000224	(0.7120, 0.8396)
	(50,50)	0.7610	-0.0000621	0.00000385	(0.7289, 0.7931)
(2,4)	(10, 10)	0.8723	0.000185	0.0000343	(0.8376, 0.9070)
	(15,15)	0.7118	0.0000248	0.00000061	(0.6949, 0.7287)
	(15,25)	0.7091	0.0000221	0.000000488	(0.6899, 0.7283)
	(25,25)	0.7118	0.0000247	0.00000061	(0.7003, 0.7233)
	(25,30)	0.8327	0.000146	0.0000212	(0.8131, 0.8523)
	(30,50)	0.6278	-0.0000592	0.00000351	(0.5919, 0.6637)
	(50,50)	0.6059	-0.0000811	0.00000659	(0.5855, 0.6263)

Table 5.15: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2.75, 1, 2)$ and $Y \sim PL(2.75, 2.25)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.4675	-0.000242	0.0000587	(0.2107, 0.7243)
(15, 15)	0.6207	-0.0000890	0.00000793	(0.4591, 0.7823)
(15, 25)	0.7498	0.0000401	0.00000161	(0.7164, 0.7832)
(25, 25)	0.7738	0.0000641	0.00000411	(0.5773, 0.9703)
(25, 30)	0.5643	-0.000145	0.0000212	(0.4451, 0.6835)
(30, 50)	0.8633	0.000154	0.0000236	(0.8399, 0.8867)
(50, 50)	0.7907	0.0000810	0.00000655	(0.7292, 0.8522)

(2012). Shahsanaei and Daneshkhah (2013) used the data to study the estimation of stress-strength parameter for generalized linear failure rate (GLFR) distribution under progressive type-II censoring and studied the validity of GLFR for both data

Table 5.16: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(2.75, 1, 2)$ and $Y \sim PL(2.75, 2.25)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10, 10)	0.9482	0.000160	0.0000255	(0.9136, 0.9828)
	(15,15)	0.8412	0.0000528	0.00000279	(0.7914, 0.8910)
	(15,25)	0.8527	0.0000643	0.00000413	(0.8170, 0.8884)
	(25,25)	0.7268	-0.0000616	0.00000379	(0.7095, 0.7441)
	(25,30)	0.7535	-0.0000349	0.00000122	(0.7165, 0.7905)
	(30,50)	0.7748	-0.0000136	0.000000185	(0.7291, 0.8205)
	(50,50)	0.7402	-0.0000482	0.00000233	(0.7008, 0.7796)
(2,4)	(10,10)	0.8830	0.000237	0.0000560	(0.8079, 0.9581)
	(15,15)	0.7142	0.0000679	0.00000461	(0.6931, 0.7353)
	(15,25)	0.7349	0.0000886	0.00000785	(0.6990, 0.7708)
	(25,25)	0.5745	-0.0000718	0.00000516	(0.5246, 0.6244)
	(25,30)	0.6007	-0.0000456	0.00000208	(0.5895,0.6119)
	(30,50)	0.6312	-0.0000151	0.000000229	(0.6018, 0.6606)
	(50,50)	0.5897	-0.0000567	0.00000321	(0.5425, 0.6369)

sets.

In Table 5.17 we provided the MLEs of the parameters of TPGL and PL, i.e., α, β, λ as well as the results of K-S and A-D goodness of fit tests.

The unknown parameters of case 1 are $\hat{\alpha}$=0.928, $\hat{\beta}_1$=1.492, $\hat{\beta}_2$=3.495, $\hat{\lambda}_1$=0.0085 and $\hat{\lambda}_2$=0.00895. The MLE of R becomes $\hat{R}$=0.2388 and the 95% interval of R is (0.2110, 0.2576). The MLEs and 95% confidence interval of $R_{s,k}$ are provided in Table 5.18.

The unknown parameters of case 2 are $\hat{\alpha}$=0.9232, $\hat{\beta}$=2.112, $\hat{\lambda}_1$=0.00878 and $\hat{\lambda}_2$=0.00933. The MLE of R becomes $\hat{R}$=0.0117 and the 95% interval of R is (0.0078, 0.0156). The MLEs and 95% confidence interval of $R_{s,k}$ are provided in

Table 5.18.

Table 5.17: MLEs and K-S and A-D tests

Plane	Model	MLEs	K-S	p-value	A-D	p-value
length 10 mm (X)	**TPGL**	$\hat{\alpha} = 0.954$ $\hat{\beta} = 0.9845$ $\hat{\lambda} = 0.0073$	0.1005	0.8928	0.4573	0.7893
	PL	$\hat{\alpha} = 0.954$ $\hat{\lambda} = 0.0073$	0.1005	0.8931	0.4573	0.7893
length 20 mm (Y)	**TPGL**	$\hat{\alpha} = 0.889$ $\hat{\beta} = 0.985$ $\hat{\lambda} = 0.0115$	0.1523	0.4456	0.7577	0.5114
	PL	$\hat{\alpha} = 0.889$ $\hat{\lambda} = 0.011$	0.1523	0.4457	0.7578	0.5112

Table 5.18: Estimates of $R_{s,k}$

	Case 1		Case 2	
(s, k)	$\hat{R}^{ML}_{s,k}$	95% ACI	$\hat{R}^{ML}_{s,k}$	95% ACI
(1, 3)	0.7705	(0.7103, 0.8307)	0.7712	(0.7042, 0.8382)
(2, 4)	0.6257	(0.6067, 0.6497)	0.6253	(0.5957, 0.6549)
(3, 5)	0.5268	(0.4758, 0.5778)	0.5253	(0.4188, 0.6318)

5.6 Summary

We estimated $R = P(Y < X)$ in two cases. First, when Y and X both follow TPGL distribution. Second, when Y and X follows PL distribution and TPGL distribution, respectively. We provided MLE to estimate the unknown parameters and used this to estimate of $R$and $R_{s,k}$. Also obtained asymptotic $100(1-\nu)\%$ CI for the reliability parameter. Also obtain asymptotic CI for the reliability parameter. The simulation results indicate that, when increasing the sample sizes, MSE caused by the estimates are nearer to zero. The MLE of R is 0.2388 in case 1, which means there is a small chance that strength is greater than stress. Then the SSS reliability in case 2 is comparatively low than in case 1.

The MLE of MSS reliability is 0.771 in both cases for (1,3) component system, which means there is a high chance that strength is greater than stress. The MLE of MSS reliability is 0.527 in case 1 and 0.525 in case 2 for (3,5) component system, which means there is an equal chance that strength is greater than stress.

CHAPTER 6

TOTAL TIME ON TEST TRANSFORM AND ORDERING OF LIFE DISTRIBUTIONS

6.1 Introduction

[1] The concept of total time on test (TTT) transform was studied in the early 1970s (see Barlow and Campo (1975)). When several units are tested for studying their life lengths, some of the units would fail while others may survive the test duration. The sum of all observed and incomplete life lengths is generally visualized as the TTT statistic. When the number of items placed on test tends to infinity, the limit of this statistic is called the TTT transform. The plots provided information about the identification of failure rate model of the lifetime r.v. Incomplete censored data can be analyzed using TTT transform.

[1] Some contents of this chapter are based on Deepthi and Chacko (2021).

Many research papers on TTT concentrate on its engineering applications. Aarset (1985) derived the exact distribution of TTT transform under the null hypothesis of exponentiality. Gupta and Michalek (1985) developed an explicit method to determine the reliability function by using the TTT transform. Abouammoh and Khalique (1997) investigated the properties of scaled TTT for some test statistics for testing exponentially against all these mean residual life criteria. Bergman (1977) studied the exact and asymptotic distributions of the number of crossings are given under the hypothesis of exponentiality.

Recently, Vera and Lynch (2005) introduced higher-order TTT transforms by applying definition of TTT recursively to the transformed distributions. Nair et al. (2008) studied the properties of TTT transform of order n and examined their applications in reliability analysis. Nair and Sankaran (2013) listed some known characterizations of common aging notions in terms of the TTT transform function. Franco-Pereira and Shaked (2013) derived two characterizations of the decreasing percentile residual life ($DPRL(\alpha)$) of order and aging notion in terms of the TTT function.

TTT transform provide the central value of censored data. In order to get the dispersion values in the censored situation, we need the distributions of the increasing convex (concave) functions of lifetime random variables. The problem of fitting an appropriate distribution for the function of r.v can be addressed through the identification of failure rate model. The problem of identification of failure rate behavior of increasing convex (concave) function of r.v based on distributional properties of the lifetime variable is also an unexplored one. So, we consider TTT transform of increasing convex (concave) function of r.v and study its properties. The behavior of TTT transform of increasing convex (concave) function with the behavior of failure rate function of the lifetime r.v need to be undergo more investigation.

In this chapter, we considered increasing convex (concave) total time on test (ICXTTT (ICVTTT)) transform of a lifetime r.v and its properties. In section 6.2, ordering of life distribution is discussed. In section 6.3, the concept of the TTT processes is discussed. In section 6.4, we defined increasing convex (concave) TTT (ICXTTT (ICVTTT)) transform of the random variable. Some results about the ageing patterns are given in section 6.5. In section 6.6, we defined ICXTTT (ICVTTT) transform order and obtained its relationship with stochastic ordering. Illustrative examples are given in section 6.7.

6.2 Ordering of life distributions

By the ageing of a mathematical unit, component or some other physical or biological system, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one. Many criteria of ageing have been developed in the literature. The stochastic comparison of distributions has been an important area of research in many diverse areas of statistics and probability. We are comparing two lifetime variables X and Y in terms of their failure rates $r_F(t)$ and $r_G(t)$, density functions $f(t)$ and $g(t)$, survival functions $\bar{F}(t)$ and $\bar{G}(t)$, mean residual lives $\mu_F(t)$ and $\mu_G(t)$ or other ageing characteristics. Ageing classes can often be characterized by some partial ordering. For example, in Barlow and Proschan (1975), IFR and IFRA classes are characterized by convex ordering and star-shaped ordering respectively. Many different types of stochastic orders have been studied in the literature; for example Deshpande et al. (1986) and a comprehensive discussion of ordering is available in Shaked and Shanthikumar (2007). It is often easy to make value judgements when such ordering exist. Stochastic ordering between two probability distributions, if it holds, is more informative than simply comparing their means or medians only. Similarly, if one wishes to compare the dispersion or spread between two distributions, the simplest way would to be to compare their standard deviations or some such other measures of dispersion.

6.2.1 Stochastic order

The r.v X is stochastically larger than the random variable Y, written $X \geq_{\text{st}} Y$, if

$$P(X > a) \geq P(Y > a), \qquad \forall\ a. \tag{6.2.1}$$

If X and Y have distributions F and G respectively, then (6.2.1) is equivalent to

$$\bar{F}(a) \geq \bar{F}(a), \qquad \forall\ a$$

denoted by $X \leq_{\text{st}} Y$.

6.2.2 Hazard rate order

Let X and Y be two nonnegative r.v's with absolutely continuous distribution functions and with failure rate functions r and q, respectively, such that

$$r(t) \geq q(t), \quad t \in \mathbb{R}. \tag{6.2.2}$$

Then X is said to be smaller than Y in the hazard rate order (denoted as $X \leq_{\text{hr}} Y$).

The hazard rate order can be trivially (but beneficially) used to characterize IFR random variables.

6.2.3 Convex (Concave) order

If two variables have the same mean, they can still be compared by how spread out the distributions are. This is captured to limit extend by the variance, but more fully by a range of stochastic orders. Convex order is a special kind of variability order.

DEFINITION 6.2.1. *Let X and Y be two random variables such that*

$$E[\phi(X)] \leq E[\phi(Y)] \textit{ for all increasing convex [concave] functions } \phi : \mathbb{R} \to \mathbb{R}, \tag{6.2.3}$$

provided the expectations exist. Then X is said to be smaller than Y in the increasing convex [concave] order (denoted by $X \leq_{icx} Y[X \leq_{icv} Y]$).

Roughly speaking, if $X \leq_{\text{icx}} Y$ then X is both smaller and less variable than Y in some stochastic sense. Similarly, $X \leq_{\text{icv}} Y$ then X is both smaller and more variable than Y in some stochastic sense. Decreasing convex (concave) order by requiring (6.2.1) to hold for all decreasing convex (concave) functions ϕ (denoted as $X \leq_{\text{dcx}} Y[X \leq_{\text{dcv}} Y]$). The term decreasing convex and decreasing concave are counter intuitive in the sense that if X is smaller than Y in the sense of either of these two orders then X is larger than Y in some stochastic sense.

6.3 Total Time on Test Transform

The concept of the TTT transform processes was first defined by Barlow and Campo (1975). Given a sample of size n from the non-negative r.v X having distribution F, let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(k)} \leq \ldots \leq X_{(n)}$ be the order statistics corresponding to the sample. Total time test to the r^{th} failure is,

$$\begin{aligned} T(X_{(r)}) &= nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + \ldots + (n-r+1)(X_{(r)} - X_{(r-1)}) \\ &= \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}. \end{aligned}$$

Let $H_n^{-1}(\frac{r}{n}) = \frac{1}{n}T(X_{(r)})$

$$\text{i.e., } H_n^{-1}(\frac{r}{n}) = \int_0^{F_n^{-1}(\frac{r}{n})} (1 - F_n(u))du.$$

The empirical distribution function defined in terms of the order statistics is

$$F_n(u) = \begin{cases} 0, & u < X_{(i)} \\ \frac{i}{n}, & X_{(i)} \leq u < X_{(i+1)} \\ 1, & X_{(n)} > u. \end{cases}$$

If there exist an inverse function $F_n^{-1}(x) = \inf\{x : F_n(x) \geq u\}$, the fact that $F_n(u) \xrightarrow{a.s.} F(u)$ implies, by Glivenko Candelli theorem,

$$\lim_{\substack{\frac{r}{n}\to t \\ n\to\infty}} \int_0^{F_n^{-1}(\frac{r}{n})} (1 - F_n(u))\, du = \int_0^{F^{-1}(t)} (1 - F(u))\, du \tag{6.3.1}$$

uniformly in $t \in [0,1]$. Barlow and Campo (1975) defined TTT transform of F as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(u))\, du \quad t \in [0,1]. \tag{6.3.2}$$

There is a one to one correspondence between distribution F and their transform H_F^{-1}. Suppose F has density f, then

$$\begin{aligned} \frac{d}{dt} H_F^{-1}(t) &= \frac{d}{dt} \int_0^{F^{-1}(t)} (1 - F(u))\, du \\ &= (1-t) \frac{d}{dt} F^{-1}(t). \end{aligned}$$

So that

$$\frac{d}{dt} F^{-1}(t) = \frac{\frac{d}{dt} H_F^{-1}(t)}{(1-t)}.$$

Note that H_F is a distribution with support on $[0, \mu]$, where μ is the mean of F, since

$$\begin{aligned} H_F^{-1}(1) &= \int_0^{F^{-1}(1)} (1 - F(u))\, du \\ &= \mu, \qquad \text{when } \bar{F}(0) = 0. \end{aligned}$$

It is easy to verify that the scaled TTT transform is

$$\phi(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)} = \frac{H_F^{-1}(t)}{\mu}$$

is continuous increasing function on $[0, 1]$ which is 0 at $t = 0$ and 1 at $t = 1$.

The curve $\phi(t)$ versus $0 \leq t \leq 1$ is called the scaled TTT transform curve. Using the scaled TTT transform curve, the shape of the failure rate function of the distribution can be classified as one of the following.

- If the scaled TTT transform curve is concave above the 45^o line, the failure rate is increasing.
- If the scaled TTT transform curve is convex below the 45^o line, then the failure rate is decreasing.
- If the scaled TTT transform curve is first convex below the 45^o line then concave above the line the shape of the failure rate is a bathtub shaped.
- The shape of the failure rate will be unimodal shaped if the scaled TTT transform curve is first concave above the 45^o line followed by convex below the 45^o line.

Figure 6.1 summarizes the different shapes of the scaled TTT transform curve for distributions with increasing, decreasing, bathtub and unimodal failure rate functions. For an ordered sample $x_{0:n}, x_{1:n}, x_{2:n}, \ldots, x_{n:n}$, the total time one test

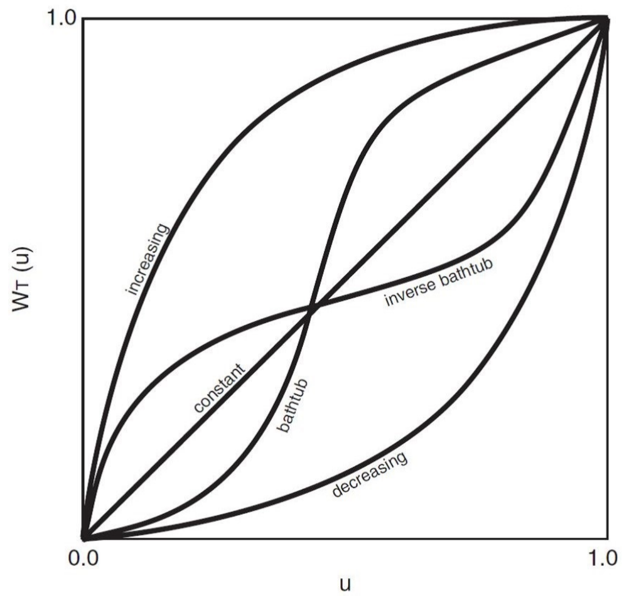


Figure 6.1: Theoretical aspects of TTT plots.

statistics is given by

$$TTT_r = \sum_{i=1}^{r}(n-i+1)(x_{i:n} - x_{i-1:n}), \quad r = 1, 2, \ldots, n.$$

The empirical scaled TTT transform is

$$TTT_r^* = \frac{TTT_r}{TTT_n},$$

where $0 \leq TTT_n \leq 1$. The TTT-plot can be drawn by plotting $(\frac{r}{n})$ against TTT_r^*.

6.4 Increasing Convex (Concave) TTT transform

Let $g(x)$ be an increasing convex (concave) function of X. Let $G(x)$ be the distribution function of $g(X)$. Total observed values of transformed variables $g(X)$ under type 2 censored scheme is

$$\begin{aligned} Tg(X_{(r)}) &= n\ g(X_{(1)}) + \ldots + (n-r+1)\big(g(X_{(r)}) - g(X_{(r-1)})\big) \\ &= \sum_{i=1}^{r} g(X_{(i)}) + (n-r)g(X_{(r)}). \end{aligned}$$

For $g(x)$, define

$$(H_n^{-1})g\left(\frac{r}{n}\right) = \int_0^{(H_n^{-1}(\frac{r}{n}))} \big(1 - H_n(w)\big)dw = \int_0^{g(F_n^{-1}(\frac{r}{n}))} \big(1 - F_n(u)\big)du$$

where

$$H_n(u) = \begin{cases} 0, & g(u) < g(X_{(i)}) \\ \frac{i}{n}, & g(X_{(i)}) \leq g(u) < g(X_{(i+1)}) \\ 1, & g(X_{(n)}) > g(u). \end{cases}$$

$H_n^{-1}(x) = \inf\{x : H_n(x) \geq g(u)\}$ and the fact that $F_n(u) \xrightarrow{a.s.} F(u)$ implies, $g(F_n^{-1}(u)) \xrightarrow{a.s.} g(F^{-1}(u))$, then by Glivenko Candelli theorem,

$$\lim_{\substack{\frac{r}{n}\to t \\ n\to\infty}} \int_0^{g(F_n^{-1}(\frac{r}{n}))} (1 - F_n(u))\ du = \int_0^{g(F^{-1}(t))} (1 - F(u))\ du.$$

We define TTT transform of increasing convex (concave) function $g(X)$ as

$$(H_F^{-1})g(t) = \int_0^{g(F^{-1}(t))} (1 - F(u))\ du \quad t \in [0, 1]. \tag{6.4.1}$$

But, $\frac{\frac{d}{dt}H_F^{-1}(t)}{1-t} = \frac{d}{dt}F^{-1}(t)$ and $\frac{d}{dt}H_F^{-1}(t)|_{t=F(x)} = \frac{1}{r(x)}$. Then

$$\begin{aligned}
\frac{d}{dt}(H_F^{-1})g(t) &= \frac{d}{dt}\int_0^{g(F^{-1}(t))} (1 - F(u))\ du \\
&= \left[1 - \int_0^{g(F^{-1}(t))} f(u)\ du\right] g'(F^{-1}(t))\frac{d}{dt}F^{-1}(t) \\
&= \left[1 - \int_0^{g(F^{-1}(t))} f(u)\ du\right] g'(F^{-1}(t))\frac{\frac{d}{dt}H_F^{-1}(t)}{1-t} \\
\frac{d}{dt}(H_F^{-1})g(t)|_{t=F(x)} &= \left[1 - \int_0^{g(x)} f(u)\ du\right] \frac{g'(x)}{\bar{F}(x)r(x)}.
\end{aligned}$$

That is,

$$\frac{d}{dt}(H_F^{-1})g(t)|_{t=F(x)} = \frac{\bar{F}(g(x))}{\bar{F}(x)}.\frac{g'(x)}{r(x)}. \tag{6.4.2}$$

Note that $H_F g(.)$ (the inverse of $H_F^{-1}g(.)$) is a distribution with support on $[0, \mu]$,

$$(H_F^{-1})g(1) = \int_0^{g(F^{-1}(1))} (1 - F(u))\ du = \mu.$$

It is easy to verify that the scaled transform $\frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is continuous increasing function on $[0, 1]$.

Example 6.4.1. Let $g(x) = x^2$, then

$$(H_F^{-1})^2(t) = \int_0^{(F^{-1}(t))^2} (1 - F(u))\ du \quad t \in [0, 1]. \tag{6.4.3}$$

$$\text{Then,} \quad \frac{d}{dt}(H_F^{-1})^2(t) = \frac{d}{dt}\int_0^{(F^{-1}(t))^2} (1 - F(u))\ du$$

$$= \left[1 - \int_0^{(F^{-1}(t))^2} f(u)\ du\right] 2\ F^{-1}(t)\ \frac{\frac{d}{dt}H_F^{-1}(t)}{1-t}$$

$$\frac{d}{dt}(H_F^{-1})^2(t)|_{t=F(x)} = \left[1 - \int_0^{x^2} f(u)\ du\right] \frac{2x}{\bar{F}(x)r(x)}.$$

That is,

$$\frac{d}{dt}(H_F^{-1})^2(t)|_{t=F(x)} = \frac{\bar{F}(x^2)}{\bar{F}(x)}.\frac{2x}{r(x)}.$$

Note that (H_F^2) (the inverse of $(H_F^{-1})^2$) is a distribution with support on $[0, \mu]$.

$$(H_F^{-1})^2(1) = \int_0^{(F^{-1}(1))^2} (1 - F(u))\ du = \mu.$$

It is easy to verify that the scaled TTT transform is $\frac{(H_F^{-1})^2(t)}{(H_F^{-1})^2(1)}$ is continuous increasing function on $[0, 1]$.

Example 6.4.2. Let $F(x) = 1 - e^{-x/\theta},\ x > 0,\ \theta > 0$ be the distribution function of Exponential distribution with mean θ. Then

$$(H_F^{-1})^2(t) = \int_0^{(F^{-1}(t))^2} (1 - F(x))\ dx$$

$$
\begin{aligned}
&= \int_0^{(F^{-1}(t))^2} e^{-x/\theta}\ dx \\
&= \int_0^{(F^{-1}(t))^2} \theta\ dF(x) \\
(H_F^{-1})^2(t)|_{t=F(x)} &= \theta F((F^{-1}(t))^2) \\
\therefore \frac{(H_F^{-1})^2(t)}{(H_F^{-1})^2(1)} &= F((F^{-1}(t))^2).
\end{aligned}
$$

Now, we consider simulated data and plot TTT transform of Exponential distribution and its convex transform $g(x) = x^2$. From Figure 6.2, the TTT transform plot of Exponential data set indicates constant failure rate, but ICXTTT transform plot indicates that the transformed data follows the decreasing failure rate pattern.

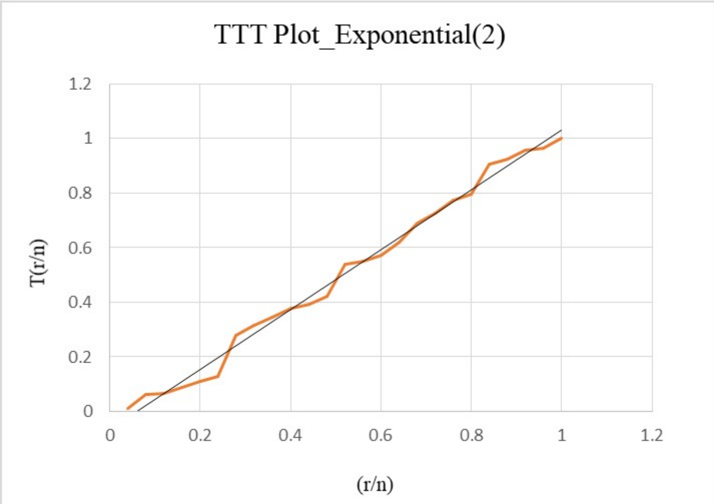
So that, square of Exponential r.v follows some decreasing failure rate model. Thus we can choose any DFR model to square of Exponential data.

6.5 Ageing Properties

We prove some general results about the ageing patterns of function $g(X)$ using $(H_F^{-1})g(t)/(H_F^{-1})g(1)$, which is based on the failure rate function $r(x)$ of X having distribution F.

Proposition 6.5.1. G is IFR if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$ is smaller than the rate of increase of $r(x)$. G is DFR if $r(x)$ is decreasing in $x \geq 0$.

Proof. Clearly $\frac{d}{dt}(H_F^{-1})g(t)$ is decreasing in $t \in [0, 1]$, if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$



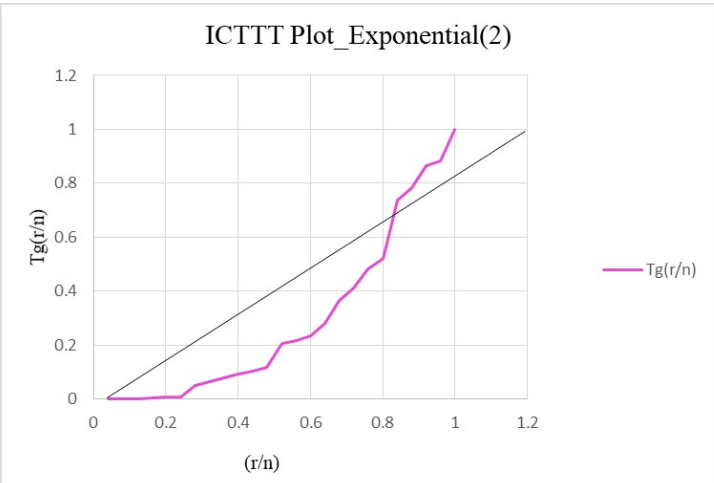


Figure 6.2: TTT plot (top) and ICXTTT plot (bottom) for the Exponential Simulated data with parameter θ=2.

is smaller than the rate of increase of $r(x) \Rightarrow \frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is concave in $t \in [0,1]$, if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$ is smaller than the rate of increase of $r(x)$. Hence G is IFR, if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$ is smaller than the rate of increase of $r(x)$.

Similarly,

$\frac{d}{dt}(H_F^{-1})g(t)$ is increasing in $t \in [0,1]$, if $r(x)$ is decreasing in $x \geq 0$.

That is, $\frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is convex in $t \in [0, 1]$, if $r(x)$ is decreasing in $x \geq 0$. G is DFR, if $r(x)$ is decreasing in $x \geq 0$. □

Proposition 6.5.2. Let X has distribution F and $Y = g(X)$ has distribution $G(y)$. G is IFRA (DFRA) $\Rightarrow$ $\frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))\ (H_F^{-1})g(1)}$ is decreasing (increasing) in $t \in [0, 1]$.

Proof. Let $Y = g(X)$ and $G(y)$ be the distribution function of Y. G has IFRA $\Rightarrow \frac{1}{y}\int_0^y r(u)\ du$ is increasing in $y \geq 0$.

Let $T(y) = \int_0^y \bar{G}(u)\ du$. $\frac{T(y)}{y}$ is decreasing in $y \geq 0$, since it is an average of the decreasing function $\bar{G}(y)$.
Then, $\frac{\int_0^y r(u)\ dT(u)}{T(y)}$ is increasing in $y \geq 0$. Hence

$$\frac{G(y)}{\int_0^y \bar{G}(u)\ du} \text{ is increasing in } y \geq 0$$

and

$$\frac{\int_0^y \bar{G}(u)\ du}{G(y)} \text{ is decreasing in } y \geq 0.$$

Then,

$$\frac{\int_0^y \bar{G}(u)\ du}{G(y)} = \frac{\int_0^{g(x)} \bar{F}(w)\ dw}{F(g(x))} \text{ is decreasing in } x \geq 0$$

since $\bar{G}(u) = P(g(X) > u) = P(X > w) = \bar{F}(w)$ for $w = g^{-1}(u)$ corresponding to u.

Now make the change of variables $t = F(x)$ and $x = F^{-1}(t)$ and finally we have

$$\frac{\int_0^{g(F^{-1}(t))} \bar{F}(w)\ dw}{F(g(F^{-1}(t)))} = \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))} \text{ is decreasing in } t \in [0, 1].$$

$$\Rightarrow \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))\ (H_F^{-1})g(1)} \text{ is decreasing in } t \in [0,1].$$

Similarly, for G is DFRA,

$$\frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))} \text{ is increasing in } t \in [0,1].$$

$$\Rightarrow \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))\ (H_F^{-1})g(1)} \text{ is increasing in } t \in [0,1].$$

□

6.6 Increasing Convex (Concave) TTT transform order

In this section we defined the increasing convex (concave) TTT transform order. Let X and Y be two nonnegative random variables with distributions F and H respectively. If

$$\int_0^{F^{-1}(t)} (1 - F(u))\ du \le \int_0^{H^{-1}(t)} (1 - G(u))\ du \quad t \in [0,1] \tag{6.6.1}$$

then X is said to be smaller than Y in the TTT order (denoted by $X \le_{\text{ttt}} Y$). A sufficient condition for the order $\le_{\text{ttt}}$ is the usual stochastic order:

$$X \le_{\text{st}} Y \implies X \le_{\text{ttt}} Y. \tag{6.6.2}$$

In order to verify (6.6.2) one may just notice that if $X \leq_{\mathrm{st}} Y$, then $F^{-1}(u) \leq G^{-1}(u)$ for all $u \in (0,1)$ (see, Shaked and Shanthikumar (2007)). By letting $u \to 1$ in (6.6.2) it is seen that

$$X \leq_{\mathrm{ttt}} Y \implies E(X) \leq E(Y). \tag{6.6.3}$$

Let X and Y be two random variables such that $Tg(X_{(n)}) \leq Tg(Y_{(n)})$ for all increasing convex (concave) functions $g : \mathbb{R} \to \mathbb{R}$ and all samples of size n. Then X is smaller than Y in some stochastic sense, since $\frac{1}{n}Tg(X_{(n)})$ is average of total observed increasing convex (concave) transformed time of a test.

DEFINITION 6.6.1. *Let X and Y be two non-negative random variables with absolutely continuous distribution functions F and H respectively. If*

$$(H_F^{-1})g(t) \leq (H_H^{-1})g(t) \quad \forall\ t \in [0,1]$$

where g is an increasing convex (concave) function, then X is smaller than Y in increasing convex (concave) TTT transform order (denoted as $X \leq_{icxttt} Y$ ($X \leq_{icvttt} Y$)).

Now we prove the relationship of ICXTTT (ICVTTT) transform orders to stochastic orders.

Theorem 6.6.1. *Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Then*

$$X \leq_{st} Y \implies X \leq_{icxttt} Y.$$

Proof. Let g be the increasing convex (concave) function $g : \mathbb{R} \to \mathbb{R}$. Since, $X \leq_{\text{st}} Y$, $g(F^{-1}(t)) \leq g(G^{-1}(t))$ for all $t \in [0,1]$.

Hence,

$$\int_0^{g(F^{-1}(t))} (1 - F(u))\, du \leq \int_0^{g(G^{-1}(t))} (1 - G(u))\, du, \ \forall t \in [0,1]$$

then $X \leq_{\text{icxttt}} Y$. □

6.7 Examples

Usually the TTT transform plot is drawn by plotting $T(\frac{r}{n}) = \frac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{\sum_{i=1}^{r} X_{(i)}}$ against $(\frac{r}{n})$, where $i = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, n$. A TTT transform curve may be concave (convex) if corresponding distribution is IFR (DFR) distribution. A TTT transform curve is straight line if the distribution is exponential. If the shape of TTT transform is concave (convex) and then convex (concave), then the distribution has a bathtub (upside down bathtub) shaped failure rate function.

Then the ICXTTT (ICVTTT) transform plot is drawn by plotting $Tg(\frac{r}{n}) = \frac{\sum_{i=1}^{r} g(X_{(i)}) + (n-r)g(X_{(r)})}{\sum_{i=1}^{r} g(X_{(i)})}$ against $(\frac{r}{n})$, where $i = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, n$,

Figure 6.3 and 6.4 shows scaled TTT transforms and scaled ICXTTT transforms of bathtub shaped failure rate data (Aarset data (Aarset (1987))) and Weibull simulated data respectively. From Figure 6.3, the failure rate pattern of transformed data still shows bathtub shape. It means that, even after transformation, we may be able to identify the failure rate pattern. It is useful because, selection of statistical distribution to the transformed variables is a problem for

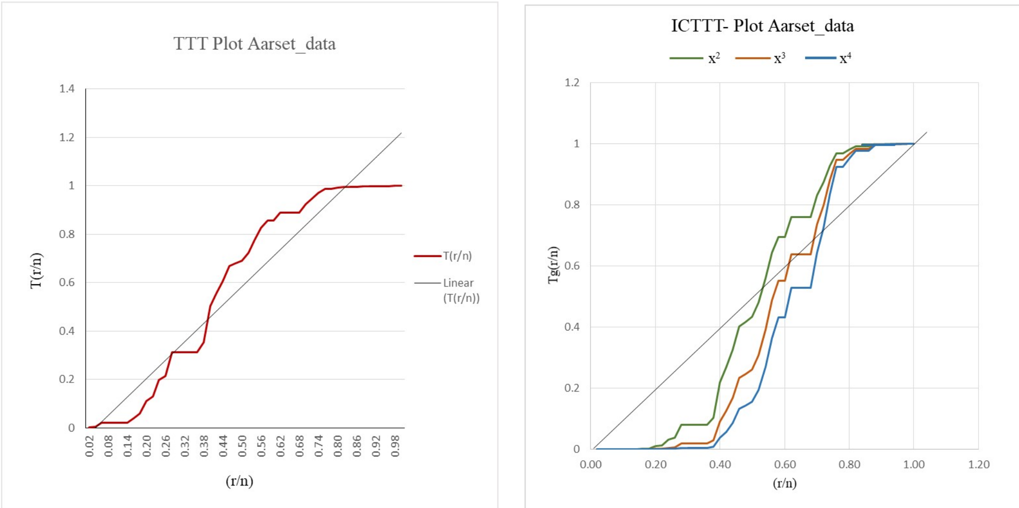


Figure 6.3: TTT plot (left) and ICXTTT plot (right) for the Aarset data.

many researchers. The researchers need only to search for a particular class of distribution, if they could identify the failure rate pattern using ICXTTT (ICVTTT) transform.

Another advantage of defining ICXTTT (ICVTTT) transform is that, the transform statistic can be used for estimating the dispersion parameters, variance etc of censored data.

$$(H_n^{-1})g\left(\frac{r}{n}\right) = \int_0^{(H_n^{-1}(\frac{r}{n}))} \big(1 - H_n(w)\big)dw = \int_0^{g(F_n^{-1}(\frac{r}{n}))} \big(1 - F_n(u)\big)du$$

is actually mean of censored-transformed data from F. This can be used for the purpose of estimation and testing the parameters of distribution of transformed data. Figure 6.4, shows that the TTT transform plot of Weibull($\alpha = 1.5, \lambda = 1$) data set indicates IFR behavior, but ICXTTT transform plot for Weibull($\alpha = 1.5, \lambda = 1$) indicates an upside down BFR pattern for the failure rate. The TTT

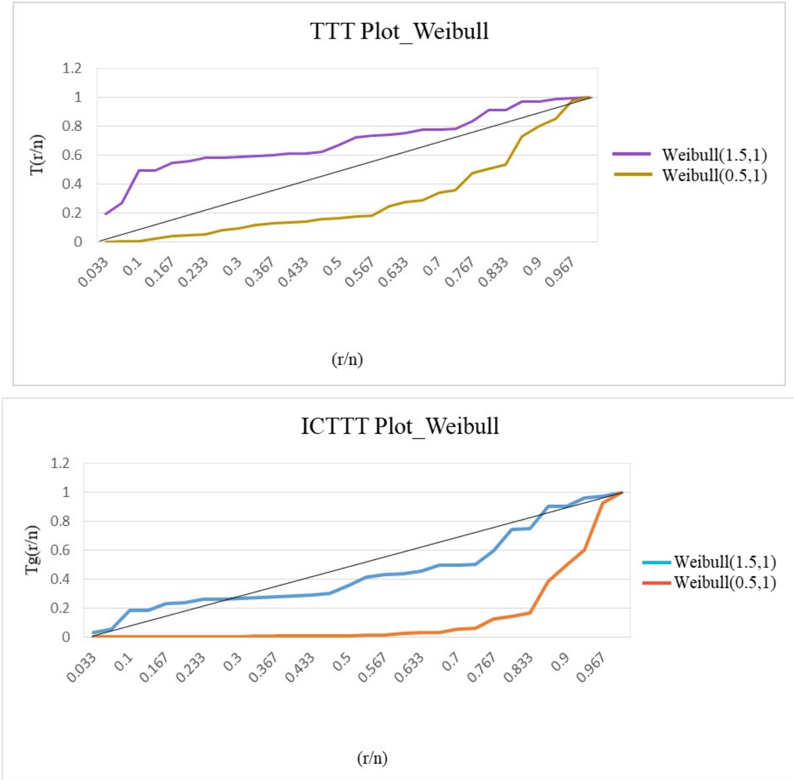


Figure 6.4: TTT plot (top) and ICXTTT plot (bottom) for the Weibull Simulated data with parameter $\alpha = 1.5, 0.5$ and λ=1 respectively.

transform plot of Weibull($\alpha = 0.5, \lambda = 1$) shows $F(t)$ has decreasing failure rate (DFR) while ICXTTT transform plot based on a Weibull($\alpha = 0.5, \lambda = 1$) shows decreasing behavior for failure rate.

6.8 Summary

We defined increasing convex (concave) TTT transform. The procedure of identification of the failure rate model of functions of random variables, using failure rate function of random variable is discussed. IFR (DFR) and IFRA (DFRA) properties of distribution of increasing convex (concave) transformations of the variable are explained. Illustrative examples are provided.

CHAPTER 7

BURN-IN PROCESS USING BFR DISTRIBUTIONS

7.1 Introduction

Burn-in testing (BIT) is a widely accepted method for many years to detect and eliminate early failures. BIT is mandatory in high-reliability storage contracts such as military and aerospace applications, and is also essential for automotive, medical, long-distance telecommunications, and other electronic materials, packages and systems. BIT is usually performed at the component level because the cost involved in inspecting and replacing parts is small.

Testing plays an important role in controlling and ensuring the required quality and reliability of built-in integrated circuits (IC). The IC fabrication process involves several tests at different stages: pre-burn-in, burn-in and final test (see Kececioglu and Sun (1997)). The four main types of burn-in tests used in the industry are static, dynamic, monitored and test-in burn-in (TIBI) (Ooi et al. (2007)). In static burn-in (also known as traditional burn-in) equipment under test (EUT) is subjected to high temperatures. Dynamic burn-in is similar to static one which involves exercising EUTs by applying test vectors or stimulus sets to toggle the device's internal nodes. Static or dynamic burn-in types provide no monitoring of EUT responses. Consequently faulty ICs are not detected until a subsequent final test stage.

The burn-in procedure stops when we get pre-determined reliability. In Mi (1994a) it was shown that the optimal burn-in time, say b^*, for maximizing the mean residual life function $\mu(b) = E(X-b|X>b)$ satisfies $b^* \leq t_1$, where t_1 is the first change point, if F is BFR. Since burn-in is usually expensive, an important issue is deciding how long the procedure should continue. The time to stop the burn-in process to optimize a given criterion is known as optimal burn-in time (see, Jensen and Petersen (1982)).

For burn-in to be effective, it must have a high failure rate early in life. Items that survive the burn-in have burn-in effect that eliminates the part of the lifetime with a high initial risk of failure. A class of life time distributions with bathtub-shaped failure rates has this property. Some other mechanical and electronic lifetimes can also be analyzed by BFR distributions.

An engineer thinking about burn-in use needs to answer a number of questions related to the purpose of the burn-in test, the type of lifetime supply, the availability of data, and the logistics of running the procedure. Lawrence (1966) and Chandrasekaran (1977) investigated the burn-in problems. Park (1985) learned about the mean residual life of the product. Plaser and Fied (1977), and Nguyen and Murthy (1982) studied the Economic design of burn-in procedures. Li and Cheng (2010) studied the best designs of accelerated life tests for lives that are distributed as exponential under advanced censoring.

In this chapter we discussed optimal burn-in process and expression of long run average cost function per unit time for obtaining optimal burn-in time and optimal age using WL and GXE distribution.

7.2 Optimal Burn-in

Traditionally, burn-in has been used to increase the mean residual life of items that survive the burn period. There are situations where increasing the mean residual life expectancy is not an appropriate criterion. For example, when considering an item going on a space mission, the goal is to minimize the chances of failing on a mission over a period of time. The optimal burn-in time may be different from the optimal burn-in time with the maximum mean residual life. Another goal of burn-in is to achieve a certain degree of reliability. Costs often need to be considered in the goal of a burn-in procedure. The cost of a component failure is higher for a satellite than for a vehicle battery (see, Myung and Young (2002)).

In this section some basic criteria for determining the optimal burn-in time for a lifetime is discussed. In Section 7.3.1 performance based criteria is discussed.

The maintenance policy in burn-in is considered in Section 7.3.2.

7.2.1 Performance-Based Criteria

Consider the performance-based criteria by maximizing the average remaining life, in which there is no more general understanding of the cost structure. We now list several criteria for determining burn-in (see, Block and Savits (1997)).

C1: Let T be a fixed mission time and let $\bar{F}$ be the survival function of the lifetime random variable. Find b which maximizes $\bar{F}(b+T)/\bar{F}(b)$, that is, find b such that, given survival to time b, the probability of completing the mission is as large as possible.

C2: Let X be a lifetime random variable. Find the burn-in time b which maximizes $E(X-b|X>b)$, that is, find the burn-in time which gives the largest mean residual life.

C3: Let $\{N_b(t),\ t \geq 0\}$ be a renewal process of lifetimes which are burned in for b units of time (i.e., if F is the original lifetime distribution and the inter-arrival distribution has survival function $\bar{F}_b(t) = \bar{F}(b+Tt)/\bar{F}(b)$. For fixed mission time T, find b which minimizes $E\left[N_b(t)\right]$, which is the mean number of burn-in components which fail during the mission time T.

C4: For a fixed α, $0 < \alpha < 1$, find the burn-in time b which maximizes $T = q_\alpha(b)$, where $q_\alpha(b) = F_b^{-1}(\alpha) = \inf\{x \geq 0 : \bar{F}_b(x) \leq 1-\alpha\}$ is α-percentile residual life (see Joe and Proschan(1984)), i.e., find the burn-in time which gives the maximal warranty period T for which at most $\alpha\%$ of items will fail.

Criteria C1, C2, and C4 deal with only one component. Criterion C3 deal with replacement components with other similar components when they fail. Mi (1994b) achieved similar results to the C1 and C3 criteria. Launer (1993) showed that optimal burn-in time occurs before t_1.

7.2.2 Burn-in and Maintenance Policy

The most common replacement policy would be age replacement policy, in which component is replaced at time T or at the time of failure which occurs first. Once a cost structure has been established, to model the total cost related to the maintenance policy adopted, an optimal T is determined (denoted by T^* and is called optimal maintenance policy) such that costs will be minimized. Assuming that the failure rate increases, Barlow and Proschan (1975) have shown that an optimal age replacement policy exists, but it may be infinite. The optimal maintenance policy, however, depends on the distribution of the component used in the operation.

Mi (1994a) consider the following procedure. Consider burn-in a new component. If the component fails before burn-in time b, repair it, and then re-burn the component. If this element survives the time b, it can be used for operation. Cha (2000) adopted a block replacement policy with fewer failures.

For new component having burn-in time b, if failure occurs before the b, then minimal repair will carry out with cost $c_s > 0$, and the burn-in procedure will continue for the repaired component. Let c_o be the cost to burn which is calculated in proportion to the total burn-in time, the total expected cost incurred by burn-in is the sum of the cost for burn-in $c_o b$, and the expected cost of minimal repairs $c_s \int_0^b r(t)\ dt$.

$$i.e., C_1(b) = c_o b + c_s \int_0^b r(t)\ dt \tag{7.2.1}$$

where $\int_0^b r(t)\ dt$ is the expected no.of minimal repairs during the burn-in period.

Let c_f indicates the cost incurred by the replacement at age T and c_r be the cost incurred by failure replacement before T^*, $0 < c_r < c_f$. Then the total expected replacement cost is the sum of the expected cost incurred by replacement at age T and the expected cost incurred by failure replacement before T^*,

$$C_2(T) = c_f F_b(T) + c_r \bar{F}_b(T) \tag{7.2.2}$$

where $\bar{F}_b(T)$ is the conditional survival function $\frac{\bar{F}(b+T)}{\bar{F}(b)}$, then $F_b(T) = 1 - \bar{F}_b(T)$.

The mean residual life function for a general repairable product is $\mu(b) = \frac{\int_b^\infty \bar{F}(t)\ dt}{\bar{F}(b)}$. The total expected cycle length is the sum of the expected length of a replacement for non-failed item and the expected length of failure cycle;

$$T\bar{F}_b(t) + \int_0^T t f_b(t)\ dt = \int_0^T \bar{F}_b(t)\ dt. \tag{7.2.3}$$

Hence, from (7.2.1), (7.2.2) and (7.2.3) the long-run average cost per unit time $C(b, T)$ is

$$\begin{aligned} C(b,T) &= \frac{c_o + c_s \int_0^b r(t)\ dt + c_f F_b(T) + c_r \bar{F}_b(T)}{\int_0^T \bar{F}_b(t)\ dt} \\ &= \frac{\left(c_o + c_s \int_0^b r(t)\ dt\right) \bar{F}(b) + c_f \left(\bar{F}(b) - \bar{F}(b+T)\right) + c_r \bar{F}(b+T)}{\int_0^T \bar{F}(b+t)\ dt}. \end{aligned}$$

The optimal burn-in time b^* and the optimal age T^* which satisfy $C(b^*, T^*) = \min_{b\geq 0, T>0} C(b, T)$.

7.3 Optimal Burn-in Procedure for WL and GXE distributions

WL Distribution:- Let X be a lifetime r.v. following WL distribution with failure rate function

$$r(x) = \alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta} + e^{x^\beta}\right), \quad x > 0,\ \alpha > 0,\ \beta > 0$$

and cdf

$$F(x; \alpha, \beta) = 1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

The total expected cost incurred by burn-in is

$$\begin{aligned} C_1(b) &= c_o b + c_s \int_0^b \alpha\left(\beta t^{\beta-1}(1+t)e^{t^\beta} + e^{t^\beta}\right) dt \\ &= c_o b + c_s\{\alpha(1+b)e^{b^\beta}\} \end{aligned} \tag{7.3.1}$$

$$\bar{F}_b(T) = \frac{e^{-\alpha\left((1+(b+T))e^{(b+T)^\beta}-1\right)}}{e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}} \tag{7.3.2}$$

$$F_b(T) = 1 - \left\{\frac{e^{-\alpha\left((1+(b+T))e^{(b+T)^\beta}-1\right)}}{e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}}\right\}. \tag{7.3.3}$$

Substituting (7.3.2) and (7.3.3) in (7.2.2), we get total expected replacement cost as

$$C_2(T) = c_f \left(1 - \left\{ \frac{e^{-\alpha\left((1+(b+T))e^{(b+T)^\beta}-1\right)}}{e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}} \right\}\right) + c_r \left\{ \frac{e^{-\alpha\left((1+(b+T))e^{(b+T)^\beta}-1\right)}}{e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}} \right\}. \tag{7.3.4}$$

The total expected cycle length is

$$T\frac{e^{-\alpha\left((1+(b+t))e^{(b+t)^\beta}-1\right)}}{e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}} + \int_0^T t\alpha \left[\beta(b+t)^{\beta-1}(1+(b+t))e^{(b+t)^\beta}\right]$$
$$\times \frac{e^{-\alpha\left((1+(b+t))e^{(b+t)^\beta}-1\right)}}{e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}} \, dt = \int_0^T \bar{F}_b(t)\, dt. \tag{7.3.5}$$

Hence, from (7.3.1), (7.3.4) and (7.3.5), the long-run average cost per unit time is

$$C(b,T) = \frac{\left(c_o + c_s\{\alpha(1+b)e^{b^\beta}\}\right) e^{-\alpha\left((1+b)e^{b^\beta}-1\right)}}{\int_0^T e^{-\alpha\left((1+(b+t))e^{(b+t)^\beta}-1\right)}\, dt}$$

$$+ \frac{c_f\left(e^{-\alpha\left((1+b)e^{b^\beta}-1\right)} - e^{-\alpha\left((1+(b+T))e^{(b+T)^\beta}-1\right)}\right)}{\int_0^T e^{-\alpha\left((1+(b+t))e^{(b+t)^\beta}-1\right)}\, dt}$$
$$+ \frac{c_r e^{-\alpha\left((1+(b+T))e^{(b+T)^\beta}-1\right)}}{\int_0^T e^{-\alpha\left((1+(b+t))e^{(b+t)^\beta}-1\right)}\, dt}.$$

GXE Distribution:- Let X be a lifetime r.v. following the failure rate function GXE distribution with $x > 0,\ \alpha, \lambda > 0$,

$$r(x) = \frac{\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1) - 2\lambda x)\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}}{1-(1-(1+\lambda x^2)e^{-\lambda(x^2+x)})^{\alpha}}$$

and cdf

$$F(x) = \left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}, \quad x > 0,\ \alpha > 0,\ \lambda > 0.$$

The total expected cost incurred by burn-in is

$$C_1(b) = c_o b + c_s \int_0^b \frac{\alpha e^{-\lambda(t^2+t)}(\lambda(1+\lambda t^2)(2t+1) - 2\lambda t)\left(1 - (1+\lambda t^2)e^{-\lambda(t^2+t)}\right)^{\alpha-1}}{1 - (1 - (1+\lambda t^2)e^{-\lambda(t^2+t)})^{\alpha}}\, dt \tag{7.3.6}$$

$$\bar{F}_b(T) = \frac{1 - \left(1 - (1+\lambda(b+T)^2)e^{-\lambda((b+T)^2+(b+T))}\right)^{\alpha}}{1 - (1 - (1+\lambda b^2)e^{-\lambda(b^2+b)})^{\alpha}} \tag{7.3.7}$$

$$F_b(T) = 1 - \left\{ \frac{1 - \left(1 - (1+\lambda(b+T)^2)e^{-\lambda((b+T)^2+(b+T))}\right)^{\alpha}}{1 - (1 - (1+\lambda b^2)e^{-\lambda(b^2+b)})^{\alpha}} \right\}. \tag{7.3.8}$$

Substitute these two results in Eq.(7.2.2), we get total expected replacement cost as

$$C_2(T) = c_f \left(1 - \left\{ \frac{1 - \left(1 - (1+\lambda(b+T)^2)e^{-\lambda((b+T)^2+(b+T))}\right)^{\alpha}}{1 - (1 - (1+\lambda b^2)e^{-\lambda(b^2+b)})^{\alpha}} \right\} \right) + c_r \left\{ \frac{1 - \left(1 - (1+\lambda(b+T)^2)e^{-\lambda((b+T)^2+(b+T))}\right)^{\alpha}}{1 - (1 - (1+\lambda b^2)e^{-\lambda(b^2+b)})^{\alpha}} \right\}. \tag{7.3.9}$$

The total expected cycle length is

$$
\begin{aligned}
& T\frac{1-\left(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))}\right)^{\alpha}}{1-(1-(1+\lambda b^2)e^{-\lambda(b^2+b)})^{\alpha}} \\
& + \int_0^T t\alpha e^{-\lambda((b+t)^2+(b+t))}\left(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))}\right)^{\alpha-1} \\
& \left[\frac{(\lambda(1+\lambda(b+t)^2)(2(b+t)+1)-2\lambda(b+t))}{1-(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))})^{\alpha}}\right] \\
& \times\frac{1-\left(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))}\right)^{\alpha}}{1-(1-(1+\lambda b^2)e^{-\lambda(b^2+b)})^{\alpha}}\ dt \\
& = \int_0^T \bar{F}_b(t)\ dt. \qquad (7.3.10)
\end{aligned}
$$

Hence, from (7.3.6), (7.3.9) and (7.3.10) the long-run average cost per unit time is

$$
\begin{aligned}
C(b,T) = & \frac{\left(c_o + c_s\int_0^b \frac{\alpha e^{-\lambda(t^2+t)}(\lambda(1+\lambda t^2)(2t+1)-2\lambda t)\left(1-(1+\lambda t^2)e^{-\lambda(t^2+t)}\right)^{\alpha-1}}{1-\left(1-(1+\lambda t^2)e^{-\lambda(t^2+t)}\right)^{\alpha}}\ dt\right)}{\int_0^T\left(1-(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))})^{\alpha}\right)\ dt} \\
& \times\left(1-\left(1-(1+\lambda b^2)e^{-\lambda(b^2+b)}\right)^{\alpha}\right) \\
& + \frac{c_f\left(1-\left(1-(1+\lambda b^2)e^{-\lambda(b^2+b)}\right)^{\alpha}\right)}{\int_0^T\left(1-(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))})^{\alpha}\right)\ dt} \\
& - \frac{c_f\left(1-\left(1-(1+\lambda(b+T)^2)e^{-\lambda((b+T)^2+(b+T))}\right)^{\alpha}\right)}{\int_0^T\left(1-(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))})^{\alpha}\right)\ dt}
\end{aligned}
$$

$$+\frac{c_r\left(1-\left(1-(1+\lambda(b+T)^2)e^{-\lambda((b+T)^2+(b+T))}\right)^{\alpha}\right)}{\int_0^T\left(1-(1-(1+\lambda(b+t)^2)e^{-\lambda((b+t)^2+(b+t))})^{\alpha}\right)dt}$$

7.4 Summary

We discussed about burn-in process. Expressions for obtain optimal Burn-in time and optimal age under age replacement policy are derived for WL and GXE distributions.

CHAPTER 8

CONCLUSION AND FUTURE WORK

8.1 Conclusion

The theory of reliability is well established scientific discipline with its own principles and methods of problem solving. Probability theory and mathematical statistics play an important role in most problems in reliability theory. Many authors discussed the benefits of ageing properties like increasing failure rate, decreasing failure rate, etc. The bathtub failure rate distributions are widely used to address the lifetime inference and testing problems. The bathtub shape is 'characteristic of the failure rate curve of many well designed products and components including the human body'. Many bathtub shaped failure rate distributions are available in literature, but some of them are appropriate for given data. So identification of distribution or failure rate model shall lead to more accurate probability computations. In the present work we tried to enhanced the application of bathtub shaped failure rate distribution, proposed new BFR and UBFR distributions, compared existing bathtub shaped failure rate models, studied on the stress-strength reliability models, developed the theory and application of TTT transform in identification of BFR model of increasing convex (concave) function of lifetime random variable, and derived expression for optimal burn-in time and optimal age using proposed distributions.

In *Chapter 1*, the relevance and scope of the study and basic information about concepts are given. In *Chapter 2* presented the review of various bathtub shaped distributions.

In *Chapter 3*, two new distributions *viz.*, GXE Distribution and WL distribution have been proposed and its properties studied. We shows that these distributions exhibits bathtub shaped failure rates. For avoiding the scale problem, a three parameter WL Distribution is introduced and shown that this distribution had BFR function. The flexibility of these two proposed distributions are illustrated using two real data sets. For each data set, the proposed distributions was shown to give better fit than several other competitors exhibiting BFR function.

In *Chapter 4*, a new distribution, based on DUS transformation using Lomax distribution as baseline has been proposed and its properties are studied. This distribution exhibited UBFR function. This situation was analyzed by Efron (1988) in the context of head and neck cancer data, in which the failure rate initially increased, attended a maximum and then decreased before it finally stabilized because of a therapy. Three data sets are considered and shown that DUS-Lomax is more appropriate than some other well-known distributions (Lomax distribution, Gompertz Lomax distribution, Kumaraswamy Lomax distribution, DUS-Exponential distribution and Inverse Lindly distribution). We have shown that, DUS-Lomax has the lowest AIC, BIC, KS-Statistic, and highest Log-likelihood value and p-value. Therefore DUS-Lomax is a better alternative in situations where upside-down bathtub shaped distributions occur.

In *Chapter 5*, we compare two methods of estimating $R = P(Y < X)$ in two cases. First, when Y and X both follow three parameter generalized Lindley distribution. Second, when Y and X follows Power Lindley distribution and three parameter generalized Lindley distribution. We provide MLE procedure to estimate the unknown parameters and use this to estimate of R. Also obtain asymptotic $100(1 - \nu)\%$ CI for the reliability parameter. The simulation results indicate that MLE in the average bias and average MSE for different choices of the parameters.

Whereas to estimate the multi-component stress-strength reliability in two cases. First, when Y and X both follow three parameter generalized Lindley distribution. Second, when Y and X follows Power Lindley distribution and three parameter generalized Lindley distribution. We provide MLE procedure to estimate the unknown parameters and use this to estimate of $R_{s,k}$. Also obtain asymptotic CI for the reliability parameter. The simulation results indicate that MLE in the average bias and average MSE for different choices of the parameters. When increasing the sample sizes, MSE caused by the estimates are nearer to zero by extensive simulation.

Two sets of data are considered and shown that TPGL and PL distributions are fit with this data. For analysis, the stress-strength reliability result for the case 1 is a value of 0.2388, and for case 2, the stress-strength reliability result is 0.012, which means there is a small chance that X is greater than Y. For case 1, when $(s, k) = (1, 3)$ and $(2, 4)$ corresponding multi-component stress-strength reliability value are 0.7705 and 0.6257 respectively, and for case 2 also when $(s, k) = (1, 3)$ and $(2, 4)$ corresponding multi-component stress-strength reliability value are 0.7712 and 0.6253 respectively, which means there is a high chance that X is greater than Y. If we consider $(s, k) = (3, 5)$ corresponding MSS are for case 1 is 0.5268 and for case 2 is 0.5253, which means there is a equal chance that X is greater than Y.

In *Chapter 6*, we defined the increasing convex (concave) TTT transformation. The procedure for identifying the failure rate model is illustrated with example. If we take lifetime random variables which follows bathtub shaped failure rate distribution and we need to check whether the result is same when we apply concave and convex transformations. If yes, can the same distribution be used to fit $g(X)$. Those reasons are explained hereby using failure rate patterns. ICXTTT (ICVTTT) transform can be used for estimating the dispersion parameters, of censored data.

Finally, we discussed the burn-in process and maintenance policy often used in field work. Optimal Burn-in time and optimal age under age replacement policy for WL and GXE distributions can be obtained using the derived expression.

8.2 Future Work

On the basis of the present study some important questions are: If the distribution of system lifetime is Bathtub shaped failure rate model, identification of change points (from decreasing to constant and constant to increasing), is crucial to the system engineering. In industry burn in process, the change point estimation is very important to separate week and strong components, before send into market. We can answer questions like, how long the burn-in process needs to be continued? What is the period of useful life? etc. In inventory theory, the number of inventory to be kept for repair or replacement can be decided according to the failure behavior.

Possible future works are to (i) provide a review of known upside-down bathtub shaped distributions; (ii) develop the problem of classical and Bayesian estimation of stress-strength reliability life distribution based on upper record values; (iii) develop time-dependent stress-strength reliability models subject to random stresses at random time cycles. Each run of the system changes the power of the system over time; (iv) examine ICXTTT transformation ordering and express the hazard ordering, likelihood ratio ordering and mean residual ordering, their mutual relationships and expressions for the ICXTTT transform in terms of these ordering; (v) derive the upper and lower bounds for the optimal burn-in time. It is also desirable to study on MCMC methods for censored data, regression issues with covariates.

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LIST OF PUBLISHED WORKS

(a) **Published:**

(1) Deepthi, K.S. and Chacko, V.M. (2021). Identification of Failure rate behavior of Increasing Convex (Concave) Transformations. *Reliability: Theory and Applications*, Vol. 16, **1(61)**, pp. 109-116. $https://doi.org/10.24412/1932-2321-2021-161-109-116$. (Scopus and UGC Care)

(2) Deepthi, K.S. and Chacko, V.M. (2020). Reliability Estimation of Stress-Strength Model using three parameter Generalized Lindley distribution, *Advances and Applications in Statistics*, **65(1)**, pp. 69-89. $http://dx.doi.org/10.17654/AS065010069$. (Web of Science and UGC Care).

(3) Deepthi, K.S. and Chacko, V.M. (2020). An Upside-down Bathtub Shaped failure rate model using DUS Transformation of Lomax Distri-

bution. Chapter 6, Cui, L. (Ed.), Frenkel, I. (Ed.) and Lisnianski, A. (Ed.). *Stochastic Models in Reliability Engineering*, Boca Raton: CRC press, Taylor & Francis Group, London.
https : //*doi.org*/10.1201/9780429331527.

(4) Deepthi, K.S. and Chacko, V.M. (2019). Estimation of Stress-Strength model using Three parameter Generalized Lindley Distribution. *Proceedings of National Seminar on Statistical Approaches in Data Science*, pp. 55-63, ISBN: 978-81-935819-2-6.

(5) Chacko, V.M. and Deepthi, K.S. (2019). Generalized X-Exponential Bathtub Shaped Failure Rate Distribution. *Journal of the Indian Society for Probability and Statistics*, **20(2)**, pp. 157-171, e-ISSN 2364-9569.
https : //*doi.org*/10.1007/*s*41096 − 019 − 00066 − 7. (University List and UGC Care at the time of submission).

(6) Chacko, V.M. and Deepthi, K.S. and Beenu, T. (2018). Weibull-Lindly Distribution: A bathtub shaped failure rate model. *Reliability: Theory and Applications*, Vol.13, **4(51)** , pp. 9-20.
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(7) Chacko, V.M., Beenu, T. and Deepthi K.S. (2017) A One parameter Bathtub shaped failure rate distribution, *Reliability: Theory and Applications*, Vol.12, **3(46))** , pp. 38-43.
http : //*www.gnedenko.net*/*Journal*/2017/032017/*RTA*_3_2017−04.*pdf*

(b) **Presentations in Conferences/Seminars:**

(1) A Generalization of Weibull-Lindley distribution: Two parameter Bathtub Shaped Model, *International Conference on Changing Paradigms and Emerging Challenges in Statistical Sciences (ICPECS-2018)* in conjunction with Bi-Decennial Convention of Society of Statistics, Computer and Applications organized by the Department of Statistics, Pondicherry University, Puducherry, India, January 29-30, 2018.

(2) A Generalization of Exponential distribution with Bathtub Shaped Failure rate Model, *National Seminar on Innovative Approaches in Statistics* in conjunction with the *Annual Conference of the Kerala Statistical Association* organized by the Department of Statistics, St. Thomas' College (Autonomous), Thrissur, Kerala, India, February 15-17, 2018.

(3) Generalized X-Exponential Bathtub Shaped Failure rate Distribution, *International Conference on Mathematics in collaboration with International Multidisciplinary Research Foundation (IMRF)* organized by the Department of Mathematics, St. Thomas' College (Autonomous), Thrissur, Kerala, June 29-30, 2018.

(4) Estimation of Stress-Strength Reliability using Three parameter Generalized Lindley Distribution, *National Seminar on Statistical Approaches in Data Science*, organized by the Department of Statistics, St. Thomas' College (Autonomous), Thrissur, February 6-7, 2019.

(5) An Upside-down Bathtub Shaped Failure rate model using DUS Trans-

formation of Lomax Distribution, *National Seminar on Recent Trends in Statistical Sciences* in conjunction with 40^{th} *Annual Conference of Kerala Statistical Association* organized by the Department of Statistics, University of Kerala, Trivandrum, Kerala, March 7-9, 2019.

(6) Identification of Failure rate behavior of Increasing Convex (Concave) TTT Transformations, *National Web based Seminar on Recent Trends in Statistical Theory and Applications-2020* organized by Indian Society for Probability and Statistics, Kerala Statistical Association (KSA) and Department of Statistics, University of Kerala, Trivandrum, June 29-July 1, 2020.

(c) **Achievements:**

(1) Dr. R.N. Pillai Young Statistician Award for Research Paper presentation in *National Seminar on Innovative Approaches in Statistics* in conjunction with the Annual Conference of the Kerala Statistical Association conducted by the Kerala Statistical Association at St. Thomas' College (Autonomous), Thrissur on 16^{th} February 2018.