RELIABILITY ANALYSIS FOR THE MULTISTATE SYSTEMS

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by

V. M. CHACKO

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA-673635

INDIA

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DECLARATION

I hereby declare that this thesis entitled 'RELIABILITY ANALYSIS FOR THE MULTISTATE SYSTEMS' submitted to the University of Calicut, for the award of Degree of Doctor of Philosophy under the Faculty of Science, is an independent work done by me under the supervision and guidance of Dr. M. Manoharan, Professor, Department of Statistics, University of Calicut .

I also declare that this thesis contains no material which has been accepted for the award of any other degree or diploma of any University or Institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Calicut University Campus **CHACKO, V. M.**

09 September 2009

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT

Dr. M. MANOHARAN, M.Sc., Ph.D.
Professor

CALICUT UNIVERSITY (P. O.)
KERALA, INDIA 673 635.
Phone : 0494 – 2401144 extn. 340 (O)
0494 – 2403403 (H)
Cell : 09447424043
Email : mano30@rediffmail.com

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September,09, 2009

Certificate

This is to certify that the work reported in this Thesis entitled 'RELIABILITY ANALYSIS FOR THE MULTISTATE SYSTEMS' that is being submitted by Sri. CHACKO, V. M. for the award of the Degree of Doctor of Philosophy, to the University of Calicut, is based on the bonafide research work carried out by him under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this Thesis have not been included in any other Thesis submitted previously for the award of any degree or diploma of any other University or Institution.

Dr. M. Manoharan
(Supervising Teacher)

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CHACKO, V. M.

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Chapter 1

INTRODUCTION

1.1 Introduction

Reliability is a critical measure of performance of engineering systems such as power generators, spacecrafts, telecommunication networks, control systems, nuclear reactors, oil and gas pipelines etc.. Conventional reliability theory is built on a framework in which both the system and its components can be in one of only two possible states: "working" or "failed". Consequently, the system structure function is a binary function of binary variables. However, the binary feature of these reliability models places serious limitations on its utility, because most of the engineering systems and their components exhibit many levels of performance between the two extremes of "working" and "failed". Multi-state system reliability models allow both the system and its components to assume more than two levels of performance. While multi-state reliability models provide more precise representations of engineering systems, they are much more complex and present major difficulties in system definition and performance evaluation. The aim of this research work is to advance the state-of-the-art of the highly promising multi-state reliability theory so that it can be applied to design and maintenance of practical engineering systems. Further, the output of this research work will certainly generate major economic benefits to industries through optimal design and maintenance of complex systems.

Reliability is the ability of a system (or an item) to perform a required function, under given environmental and operational condition, and for a stated period of time. For an item to be reliable, it must operate satisfactorily, for a specified period of time in the actual application for which it is intended. The vast majority of the reliability analysis assumes that components and systems are in either of two states: functioning or failed. The reliability of such a system is the probability that the system is functioning for a specified period of time without failure. In such a binary setup, various reliability problems were addressed by different researchers, complementing the work of Barlow and Proschan (1975). The binary state system reliability research is on finding reliability bound for the system state and reliability, obtaining measures of importance for identifying the most important components in the system (series system, parallel system and $k - out - of - n$ system). In order to use statistical distributions such as Exponential, Weibull, Gamma, Lognormal, etc, the life time (from start of functioning to failure) of components are assumed to be continuous random variables with above distributions. So that reliability calculations involves only computation of $P[T > t]$, where T is the lifetime random variable and t is the mission time. The ageing properties (increasing failure rate, increasing failure rate average, decreasing failure rate, decreasing failure rate average, etc) are discussed by several authors, see Brayson and Siddique (1969) and Deshpande et al. (1986). If the component failure rate is independent of time (random failure) Exponential distribution serves as very useful model for reliability computation. The renewal theory and Poisson process theory are used for replacement and maintenance problems. All these research developments are concentrated only on binary systems. Barlow and Proschan (1975) and Barlow and Proschan (1996) are good references for the foundation and developments of binary reliability theory. A comprehensive introduction to system reliability theory is given by Rausand and Hoyland (2004).

However, as mentioned above, in many real life situations the system and their components are actually capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. We are actually able to distinguish among various 'levels of performance' for both system and components. For such systems, the existing dichotomous model is a gross oversimplification and so models assuming degradable (multi-state) systems and components are preferable since they are closer to reality. For example, in a power generation system whose performance is measured in terms of capacity, the performance can be divided into $M + 1$ states, $0, 1, ..., M$ where 'M' is the best state and '0' is the worst state, see

Natvig et al. (1986). In the power generation system, state M corresponds to the performance $100MW$ (perfect functioning), state $M-1$ corresponds to $75MW$,..., state 0 corresponds to $0MW$ (complete failure). We consider the systems with $M+1$ states of output performances where each of the components has also $M+1$ states of performances. Such systems are called multi-state systems (MSSs) with multi-state components. Levitin (2002a) considered the multi-state node acyclic networks (MNAN), each node has different states determined by a set of nodes that receive the signal directly without satisfying the conservation law. A state is a set of performances for each node, where performance is characterized by numbers of nodes where a signal can be received from the given node. Naturally, every state of a node has associated performance level. Another important MSS, the offshore gas pipeline network, can be seen in Natvig and Morch (2003), which constitutes the main parts of the network in the North Sea, as of the end of the eighties, transporting gas to Emden in Germany. Kolowrocki (2004) considered a steal rope of three layer 36 strands, 18 outer strands, 12 inner strands and 6 strands in the next inner level. All strands are composed of 7 steel wires. Considering strands as system's basic components, one may view that the rope is a parallel system composed of components. The state of the system is described as, State 3: a strand is new, without any defects, State 2: a number of brocken wires in the strand is greater than 0% and less than 25% of all of its wires, corrosion is greater than 0% and less than 25%, abrasion is less than 25% and strain is less than 50%, State 1: a number of brocken wires in the strand is greater than or equal to 25% and less than 50% of all its wires or corrosion is greater than or equal to 25% and less than 50%, abrasion is less than 50%, strain is less than

50%, and State 0: otherwise (a strand is failed 0). Kolowrocki (2004) carried out an intensive research in large MSSs using extreme value theory.

In MSSs, the reliability evaluation becomes more complicated than in a binary systems. The methods of MSS reliability assessment are based on four different approaches: the structure function approach, the stochastic process (mainly Markov and semi-Markov) approach, universal generating function (UGF) approach and Monte Carlo simulation approach. The structure function approach is historically the first that was developed and applied for MSS reliability analysis. The main difficulties in the MSS reliability analysis is the dimension damnation since each system components can have many different states. This makes the structure function approach very work-time consuming. However when using the structure function approach one still faces its main disadvantage: inability to investigate dynamic behavior of MSSs.

The stochastic process approach widely used for MSS reliability analysis is more universal. In this approach, one can consider the states of the process as the states of the component and the time between the transition from one state to another is considered to be a random variable. The state of the component at a time depends only on the state from which the last transition to present state occurred. For instance, let the performance process of the component i is a stochastic process $\{X_i(t), t \in \tau\}$, where for each fixed value of $t \in \tau$, $X_i(t)$ is a random variable taking values 0,1,...,M according to the degree of degradation. That is, $X_i(t) = M$ if the component is perfectly functioning, $X_i(t) = j$, if the component i is in jth degraded state and $X_i(t) = 0$, if the component is completely failed. The index τ is contained in $[0, \infty)$. The joint performance process $\{\mathbf{X}(t), t \in \tau\} = \{X_1(t), ..., X_n(t), t \in \tau\}$ for a set of components is a vector of stochastic processes for which ith marginal process $\{X_i(t), t \in \tau\}$ is the performance process for the ith component, $i = 1, ..., n$. We then consider that there is a function ϕ such that the performance process of the system is $\{\phi(\mathbf{X}(t)), t \in \tau\}$. This approach can be applied to relatively small MSSs, because the number of system states increases drastically with the increase in the number of system components. The stochastic process approach is proved to be a very useful tool for time dependent MSSs. Lisnianski and levitin (2003) considered several examples on this topic. For example, they considered an electric generator, for reliability evaluation, that has four possible performance levels 100MW (state 3), 70MW (state 2), 50MW (state 1) and 0MW (state 0). The constant demand is 60MW. The best state with performance rate 100MW is the initial state. Times to transition from one state to another due to failures are distributed exponentially with parameters, $\lambda_{3,2} = 10^{-3}$ (hours(-1)), $\lambda_{2,1} = 5.10^{-3}$ (hours(-1)) and $\lambda_{1,0} = 2.10^{-3}$ (hours(-1)). Hence, times to failures $T_{3,2}$, $T_{2,1}$ and $T_{1,0}$, where $T_{i,j}$ represents the time to transition from state i to state j, are random variables distributed according to the c.d.f., $F_{3,2}(t) = 1 - e^{-\lambda_{3,2}t}$, $F_{2,1}(t) = 1 - e^{-\lambda_{2,1}t}$ and $F_{1,0}(t) = 1 - e^{-\lambda_{1,0}t}$, for $t > 0$. Semi-Markov process is also used when time taken for transition from one state to another is arbitrarily distributed. Markov and semi-Markov modeling and reliability evaluation of MSSs are available in the literature at the beginning of 1980's, see Horjt et al. (1985).

UGF technique is fast enough than the structure function approach and stochastic process approach when the complexity involved in computation of MSS reliability increases. Lisnianski and Levitin (2003) described the structural function approach and stochastic process approach, and proved the advantage of using UGF in reliability evaluation over this methods. This technique allows one to find the entire MSS performance distribution based on the performance distribution of its components by using a fast algebraic procedure. An analyst can use the same recursive procedures for MSSs with different physical nature of performance and different type of component interaction. In practice, however, a variety of products are available in the market. Each product is characterized by its capacity or productivity, reliability and price. The capacity or productivity of a component is the quantitative measure of its performance. It may have different physical nature. Examples of component capacities are: generating capacity of a generator, pipe capacity for a water circulator, carrying capacity for an electric transmission line. If two of such components are connected in parallel, total performance will be sum of individual performances. In traditional reliability theory, the performance of the parallel system is maximum of individual performances. So in systems whose performance is measured in terms of capacity or productivity the traditional reliability analysis is not sufficient, instead one has to use the method of UGF. Also modern large scale systems are distinguished by their structural complexity. The computation of reliability or availability or risk in such systems is complicated and hence the computation of other measures such as importance measures and joint importance measures are also become complicated. In all such cases UGF is found to be a useful tool for evaluation of performance measures and importance measures.

The developments of MSS reliability analysis started at the end of 1970's. The works of Barlow and Wu (1978), El.Neweihi et al. (1978), Ross (1979) and Griffith (1980) gave structural and statistical foundation for the finite state MSSs. In their work they defined the MSS and obtained structural properties similar to binary state systems. This includes, definition of series MSS, parallel MSS, reliability bounds, redundancy at series and parallel level, stochastic performance of MSS and component importance measures in MSSs. An important theoretical development was on component importance measures due to Bueno (1989). Several researchers made fruitful study in importance measures of MSS, see Abouammoh and Al-Khadi (1991) and Vassuer and Llory (1999). The joint importance measures for the two components in a binary system was studied by Hong and Koo (1993) and Hong et al. (1999). Recently joint importance measure for two components in a MSS was proposed by Wu (2005). Later, the bulk of the work has been centered around the reliability analysis of MSSs based on above mentioned literature. Upto date developments in MSS theory can be seen in Hudson and Kapur (1982), El.Neweihi and Proschan (1984), Aven (1985), Aven (1988), Ebrahimi (1991), Abouammoh and Al-Khadi (1991), Aven (1993), Brunelle and Kapur (1999) and Lisnianski and Levitin (2003). The definitions and properties similar to the discrete state MSSs holds for system whose state change in continuous fashion, see Baxter (1984), Block and Savits (1984), Baxter and Kim (1986) and Baxter and Lee (1990). A major application of MSS reliability analysis is in network systems. For example, consider a system with a set of radio relay stations with a single source and a single receiver and n intermediate stations. Each one of consecutively ordered stations can have retransmitters generating signals that reach next k stations $(1 \leq k \leq n)$. Here k is random value depend on availability of retransmitter amplifiers. Therefore each retransmitter is a multi-state component with random performance characterized by k. The aim of the system is to provide propagation of a signal from a source to a receiver. Here the reliability can be defined as the probability that source and receiver are connected by working nodes. It is also interesting to know which component or which group of components are more important to the system performance.

Reliability evaluation in large MSSs (a system with considerably large number of components) has been done by Kolowrocki (2004). They obtained reliability for large MSSs using extreme value theory. The application of this development is in oil/ gas transportation problem in which the system is parallel or series combination of large number of pipe lines of different capacities.

We use the following notations throughout the thesis. The vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ denote the vector of state of components $1, 2, ..., n$. $C = \{1, 2, ..., n\}$ denote the set of component indices. When we discuss about binary system, each variable takes values 0 or 1, and when we discuss about MSSs, each variable takes values $0, 1, ..., M$.

$$(j_i, \mathbf{x}) = (x_1, ..., x_{i-1}, j_i, x_{i+1}, ..., x_n) \text{ and } (\cdot_i, \mathbf{x}) = (x_1, ..., x_{i-1}, \cdot_i, x_{i+1}, ..., x_n).$$

$\mathbf{y} \leq \mathbf{x} \Rightarrow y_i \leq x_i, i = 1, 2, ..., n$, and $\mathbf{y} < \mathbf{x} \Rightarrow y_i \leq x_i, i = 1, 2, ..., n$ and $y_i < x_i$ for some i.

A subset $U \subseteq R^n$ is an upper set if $\mathbf{x} \in U$ and $\mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{y} \in U$.

A subset $L \subseteq R^n$ is an lower set if $\mathbf{x} \in L$ and $\mathbf{y} \leq \mathbf{x} \Rightarrow \mathbf{y} \in L$.

$x \vee y = max(x, y)$ and $x \wedge y = min(x, y)$.

$\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, ..., x_n \vee y_n)$ and $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, ..., x_n \wedge y_n)$.

For $j = 0, 1, ..., M$, $\mathbf{j} = (j, j, ..., j, ..., j)$,

$(j_i, \mathbf{X}) = (X_1, ..., X_{i-1}, j_i, X_{i+1}, ..., X_n)$ and $(\cdot_i, \mathbf{X}) = (X_1, ..., X_{i-1}, \cdot_i, X_{i+1}, ..., X_n)$,

and $\mathbf{Y} \leq \mathbf{X} \Rightarrow Y_i \leq X_i$, for $i = 1, 2, ..., n$ and $Y_i < X_i$ for some i.

We now give some brief mathematical ideas of binary and MSSs.

1.2 Binary State System: An Overview

The theory of binary state systems serves as a unifying foundation for mathematical and statistical theory of reliability, see Barlow and Proschan (1996, 1975). In this theory systems and components are assumed to be in one of two states: functioning or failed. To indicate the state of the ith component, we assign a binary indicator variable x_i to the component i: $x_i = 1$ if the ith component is functioning, 0 if the component i is failed, for $i = 1, 2, ..., n$, where n is the number of components in the system. Similarly the binary variable ϕ indicates the state of the system: $\phi = 1$ if the system is functioning, 0 if the system is failed. We also assume that the state of the system is determined completely by the state of the components, so that we may write $\phi = \phi(\mathbf{x})$ where $\mathbf{x} = (x_1, ..., x_n)$. The function $\phi(\mathbf{x})$ is called the *structure function* of the binary system.

A *series* structure functions if and only if each component functions. The structure function is given by $\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = min(x_1, ..., x_n)$. Similarly a *parallel* structure functions if and only if atleast one of the component functions. The structure function is given by $\phi(\mathbf{x}) = \coprod_{i=1}^{n} x_i = max(x_1, ..., x_n)$.

A component is relevant to the system performance if an improvement in component performance increases the system performance. Now consider a system such that its structure function ϕ is increasing and each component is relevant. Such systems are called *Coherent* systems. Esary and Proschan (1963) provides the properties of coherent structures in reliability study.

A path vector is defined as *a vector* $\mathbf{x}$ such that $\phi(\mathbf{x})= 1$. The corresponding path set is $C_1(\mathbf{x})= \{i|x_i = 1\}$. A *minimal path vector* is a path vector $\mathbf{x}$ such that $\mathbf{y} < \mathbf{x} \Rightarrow \phi(\mathbf{y})= 0$. The corresponding minimal path set is $C_1(\mathbf{x})$.

A *cut vector* is a vector $\mathbf{x}$ such that $\phi(\mathbf{x})= 0$. The corresponding cut set is $C_0(\mathbf{x})$. A *minimal vector* is a cut vector $\mathbf{x}$ such that $\mathbf{y} > \mathbf{x} \Rightarrow \phi(\mathbf{y})= 1$. The corresponding *minimal cut* set is $C_0(\mathbf{x})$.

Suppose now that components are statistically independent and the component reliabilities are available. Suppose that the state X_i of the ith binary component is random with reliability $R_i = P[X_i = 1] = E(X_i)$, $i = 1, 2, ..., n$. The reliability of the binary system is given by

$$R = P[\phi(\mathbf{X}) = 1] = E(\phi(\mathbf{X})). \tag{1.1}$$

In order to order the components based on their contribution to system reliability improvement with respect to component reliability improvement, Birnbaum (1969) proposed the following importance measure to a component i in a binary system.

$$I(i) = \frac{\partial R}{\partial R_i}. \tag{1.2}$$

For example, consider three independent component parallel system with

$$R_1 \leq R_2 \leq R_3,$$

then $R = \coprod_{i=1}^{3} R_i = 1 - \prod_{i=1}^{3}(1 - R_i)$ and $I(1) \leq I(2) \leq I(3)$.

There are other related mathematical and statistical problems (eg. bounds for reliability, maintenance, replacement policies, failure rate distributions etc) addressed by several authors. See Barlow and Proschan (1975, 1996) for further fundamental information.

1.3 Multi-state System: An Overview

MSSs are introduced by El.Neweihi et al. (1978), Barlow and Wu (1978), Ross (1979) and Griffith (1980). In these works, basic concepts of MSS reliability are formulated. Griffith (1980) introduced the properties of MSS in a more elaborated way. A decomposition result is introduced by Block and Savits (1982). A case study on offshore gas pipeline network as an application to MSS reliability is carried out by Natvig et al. (1986). Block et al. (1989) studied L-superadditive structure functions in MSSs. As a generalization to usual MSS, MSS of order k (a system whose component is relevant to atleast k levels of performance) is studied by Abouammoh and Al-kadi (1995).

Stochastic process approach is often used in MSS reliability analysis and it proved to be a rather universal tool. The UGF is widely used in reliability evaluation of many real life MSSs and optimization problems. Lisnianski and Levitin (2003) provides basic concepts of MSSs, defines MSS reliability measures and systematically describes the tools for the reliability assessment and optimization problems.

Structural Definitions

The set theoretic approach is followed by Barlow and Wu (1978), who introduced a class of MSSs based on the concept of minimum path set (minimum cut set) of binary coherent system. Here each component can be in $M + 1$ states, $\{0, 1, ..., M\}$ where 0 is the failed state and M is the maximal or '$perfect$' state. Assume, no minimum path is properly contained in any other minimum path sets.

Let $x_i = j$ if component i is in state j $(0 \leq j \leq M)$, so that $\mathbf{x} = (x_1, x_2, ..., x_n)$ is the component state vector. The specification and determination of component state will in general depend on engineering and system considerations. The component states will be qualitative measures as the concepts 'failed' and 'functioning'.

The performance level of the system, given the component state vector $\mathbf{x}$, will be system dependent and it is unlikely that any one mathematical definition of system performance will be preferred above all others. Hence we concentrate on a fundamental, but necessarily limited measure of system performance.

We recall the following from Barlow and Wu (1978).

Theorem 1.3.1 *For a coherent system with min path sets* $\{P_1, P_2, ..., P_p\}$ *and min cut sets* $\{K_1, K_2, ..., K_k\}$ *and any real valued function* f_i

$$max_{1\leq r\leq p} min_{i\in P_r} f_i = min_{1\leq s\leq k} max_{i\in K_s} f_i.$$

Let ϕ be a function with domain $S^n = \{0, 1, ..., M\}^n$ and range $S = \{0, 1, 2, ..., M\}$, where M and n are positive integers. The following definition shows how a MSS state can be represented using state of components for a coherent system, based on minimum path sets and minimum cut sets.

Definition 1.3.1 *For a coherent system with min path sets* $\{P_1, P_2, ..., P_p\}$ *and min cut sets* $\{K_1, K_2, ..., K_k\}$ *the system state function is*

$$\phi(\mathbf{x}) = max_{1\leq r\leq p} min_{i\in P_r} x_i = min_{1\leq s\leq k} max_{i\in K_s} x_i.$$

With this definition of system state, most of the results for binary coherent systems have a natural generalization. In case of time dependent system, suppose that t_{ij} is the first time that the component i reaches state j starting in state M, then the time until the system first reaches j starting in state M, τ_i, is easily seen to be

$$\tau_i = max_{1\leq r\leq p} min_{i\in P_r} t_{ij} = min_{1\leq s\leq k} max_{i\in K_s} t_{ij}.$$

Apart from the set theoretic approach for defining a MSS, El.Neweihi et al. (1978) introduced the axiomatic definition of multi-state coherent system (MCS) as follows. It was based on extension of binary relevance property to the multi-state case.

Definition 1.3.2 *A system of n components is said to be a multi-state coherent system if its structure function satisfies the following properties.*

1. $\phi(\mathbf{x})$ is increasing for $\mathbf{x} \geq 0$.

2. For level 'j' of component 'i', there exist a vector $(., \mathbf{x})$ such that $\phi(j_i, \mathbf{x}) = j$ while $\phi(l_i, \mathbf{x}) \neq j$ for $l \neq j$, $i = 1, 2, ..., n$ and $j = 0, 1, 2, ..., M$.

3. $\phi(\mathbf{j}) = j$ for $j = 0, 1, 2, ..., M$.

The condition (2) is referred to as the relevance condition.

Using the monotonicity of the structure function, Griffith (1980) introduced the definition of multi-state monotone structure (MMS) function. The definition also gives a bound for the MSS using series and parallel structure functions.

Definition 1.3.3 *The structure function ϕ is a multi-state monotone structure function if it satisfies the following properties.*

1. $\phi(\mathbf{x})$ *is increasing in* $\boldsymbol{x} \geq 0$.

2. $min_{1 \leq i \leq n} x_i \leq \phi(\mathbf{x}) \leq max_{1 \leq i \leq n} x_i$.

Now we consider some important properties of the MSSs defined above. El.Neweihi et al. (1978) obtained the decomposition result of MSS as

$$\phi(\mathbf{x}) = \sum_{j=0}^{M} \phi(j_i, \mathbf{x}) I_{[x_i=j]}$$

where $I_{[x_i=j]} = 1$ if $x_i = j$ and 0 otherwise, $i = 1, 2, ..., n$.

El.Neweihi et al. (1978) obtained bounds for the MCS as,

$$min_{1 \leq i \leq n} x_i \leq \phi(\mathbf{x}) \leq max_{1 \leq i \leq n} x_i$$

and extended the result, 'redundancy at the component level is preferable to the redundancy at the system level', from binary state system theory.

Theorem 1.3.2 *Let ϕ be a structure function of a MCS. Then,*

1. $\phi(\mathbf{x} \vee \mathbf{y}) \geq \phi(\mathbf{x}) \vee \phi(\mathbf{y})$, *and*

2. $\phi(\mathbf{x} \wedge \mathbf{y}) \leq \phi(\mathbf{x}) \wedge \phi(\mathbf{y})$,

equality in (1) and (2) holds for parallel and series system respectively.

Griffith (1980) introduced the concept of strongly coherent and weakly coherent MSSs, based on relevance assumption of components. This relevance assumptions are useful in computation of importance measures. The basic idea is stated below.

Definition 1.3.4 *Let $\phi(\mathbf{x})$ be a MMS.*

1. *For any component i and state j, there exist a $\mathbf{x}$ such that, $\phi(j_i, \mathbf{x}) = j$ while $\phi(l_i, \mathbf{x}) \neq j$ for $l \neq j$, then $\phi(\mathbf{x})$ is said to be strongly coherent.*

2. *For any component i and state $j \geq 1$, there exist a $\mathbf{x}$ such that $\phi((j-1)_i, \mathbf{x})$ $< \phi(j_i, \mathbf{x})$, then $\phi(\mathbf{x})$ is said to be coherent.*

3. *For any component i and state j, there exist a $\mathbf{x}$ such that $\phi(j_i, \mathbf{x}) \neq \phi(l_i, \mathbf{x})$, for $j \neq l$, then $\phi(\mathbf{x})$ is said to be weakly coherent.*

Block and Savits (1982) redefined the existing definition of MSS, so that bounds of $\phi(\mathbf{x})$ is easily obtained as in following theorem.

Theorem 1.3.3 *Let $\phi : S^n \to S$.*

1. *ϕ is non-decreasing if and only if either of the following condition holds*

   (a) *$\phi(\mathbf{x} \vee \mathbf{y}) \geq \phi(\mathbf{x}) \vee \phi(\mathbf{y})$ $\forall$ $\mathbf{x}, \mathbf{y} \in S^n$ or*

   (b) *$\phi(\mathbf{x} \wedge \mathbf{y}) \leq \phi(\mathbf{x}) \wedge \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S^n$.*

2. *If ϕ is non-decreasing, then for all $\mathbf{x} = (x_1, x_2, ..., x_n) \in S^n$,*

   (a) *$min_{1 \leq i \leq n} x_i \leq \phi(\mathbf{x})$ if and only if $\phi(\mathbf{k}) \geq k$ for all $k \in S$,*

   (b) *$\phi(\mathbf{x}) \leq max_{1 \leq i \leq n} x_i$ if and only if $\phi(\mathbf{k}) \leq k$ for all $k \in S$,* *consequently* $min_{1\leq i\leq n}x_i \leq \phi(\mathbf{x}) \leq max_{1\leq i\leq n}x_i$ *if and only if* $\phi(\mathbf{k}) = k$ *for all* $k \in S$.

3. *If* ϕ *is non-decreasing, then*

   (a) $max_{1\leq i\leq n}\phi(x_i, 0) \leq \phi(\mathbf{x}) \leq min_{1\leq i\leq n}\phi(x_i, M)$ *and*

   (b) $\phi(min_{1\leq i\leq n}x_i, ..., min_{1\leq i\leq n}x_i) \leq \phi(\mathbf{x}) \leq \phi(max_{1\leq i\leq n}x_i, ..., max_{1\leq i\leq n}x_i)$.

   Further more, these bounds are not comparable in the sense that there exist a system ϕ *for which (a) is better than (b) and vise versa.*

4. *If* ϕ *is non-decreasing, then,*

   (a) $\phi(\mathbf{x} \vee \mathbf{y}) = \phi(\mathbf{x})\vee\phi(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in S^n$ *if and only if* $\phi(\mathbf{x}) = max_{1\leq i\leq n}h_i(x_i)$ *where* $h_i(j) = \phi(j_i, \mathbf{0})$ *and*

   (b) $\phi(\mathbf{x} \wedge \mathbf{y}) = \phi(\mathbf{x})\wedge\phi(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in S^n$ *if and only if* $\phi(\mathbf{x}) = min_{1\leq i\leq n}H_i(x_i)$ *where* $H_i(j) = \phi(j_i, \mathbf{M})$.

Until now we discussed the structural definition and important properties of a MSS. When the state variables are random, we need to use probabilistic concepts to find reliability and related problems. So we consider the stochastic properties of MSSs in the following section.

Stochastic Properties

Here we discuss the relationship between the stochastic performance of the system and stochastic performance of the components in reliability analysis of MSSs.

Let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be a random vector representing the state of the components $(1, 2, ..., n)$ where the $X_1, X_2, ..., X_n$ are assumed to be stochastically mutually independent. Let $P[X_i = j] = p_{ij}$ and $P[X_i \leq j] = P_i(j)$, for $j = 0, 1, 2, ..., M$, and $i = 1, 2....n$. $P_i(.)$ represent the performance distribution of component i. Clearly, $P_i(j) = \sum_{k=0}^{j} p_{ik}$ and $P_i(M) = \sum_{k=0}^{M} p_{ik} = 1$ for $i = 1, 2, ..., n$. Then $\phi(\mathbf{X})$ is a random variable representing the system state of a MCS having structure function ϕ. Let $P[\phi(\mathbf{X}) = j] = p_j$, $j = 0, 1, 2, ..., M$, and $P[\phi(\mathbf{X}) \leq j] = P(j)$, $j = 0, 1, 2, ..., M$. $P(.)$ represents the performance distribution of the system. Let $h = E\phi(\mathbf{X})$, we may express h as follows, $h \equiv h_p(\overline{\mathbf{p}}_1, ..., \overline{\mathbf{p}}_n)$, since h is a function of $\overline{\mathbf{p}}_1, ..., \overline{\mathbf{p}}_n$, where $\overline{\mathbf{p}}_i = (p_{i0}, ..., p_{iM})$ for $i = 1, 2, ..., n$. We call h, the performance function of the system.

El.Neweihi et al. (1978) investigated the system performance function of n components in terms of the system performance function of the $n-1$ components as,

$$h(\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, ..., \overline{\mathbf{p}}_n) = \sum_{j=0}^{M} p_{ij} h(j_i; \overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, ..., \overline{\mathbf{p}}_n),\ i = 1, 2, ..., n,$$

where $h(j_i; \overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, ..., \overline{\mathbf{p}}_n) = E\phi(j_i, \mathbf{X}) = E\phi(X_1, ..., X_{i-1}, j_i, X_{i+1}, ..., X_n)$.

El.Neweihi et al. (1978) proved that $h(\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, ..., \overline{\mathbf{p}}_n)$ is strictly increasing in p_{ij},

$i = 1, 2, ..., n$ and $j = 0, 1, 2, ..., M$.

Let $P_i(.)$ and $P_i'(.)$ be two possible performance distribution for component i, $i = 1, 2, ..., n$. Assume $P_i(j) \geq P_i'(j)$, $j = 0, 1, 2, ..., M$ and $i = 1, 2, ..., n$. Let $P(.)$ and $P'(.)$ be the corresponding system performance distribution. Then, El.Neweihi et al. (1978) proved that $P(j) \geq P'(j)$ for $j = 0, 1, 2, ..., M$ and $h(\overline{\mathbf{p}}_1, ..., \overline{\mathbf{p}}_n) \leq h(\overline{\mathbf{p}}_1', ..., \overline{\mathbf{p}}_n')$.

Now we relate the properties of P, the system performance distribution, to the properties of h, the system performance function or to the properties of the p_{ij}. El.Neweihi et al. (1978) showed that $h = \sum_{j=0}^{M-1} \bar{P}(j)$, where $\bar{P}(j) = 1 - P(j)$. Let $\bar{P}_i(.) = 1 - P_i(.)$. El.Neweihi et al. (1978) proved that

$$\prod_1^n P_i(j) \leq P(j) \leq 1 - \prod_1^n \bar{P}_i(j),\ and$$

$$\sum_{j=1}^{M} \prod_1^n \bar{P}_i(j-1) \leq h \leq \sum_{j=1}^{M} [1 - \prod_i^n P_i(j-1)].$$

It gives the bounds on both the system performance distribution and system performance function.

A decomposition result useful in computation of importance measures is obtained by El.Neweihi et al. (1978).

Theorem 1.3.4 *Let ϕ be a structure function of a MCS, then*

$$P[\phi(\mathbf{X}) \geq l] = \sum_{j=0}^{M} p_{ij} P[\phi(j_i, \mathbf{X})] \geq l], \quad l = 1, 2, ..., M, \quad i = 1, 2..., n. \tag{1.3}$$

Block and Savits (1982) established the following bounds for reliabilities when component r.v.'s are associated and independent. Let U_k = { all critical path (or upper) vector to the level k } where a vector $\mathbf{x}$ is called an upper (lower) vector for level k of a MMS if $\phi(\mathbf{x}) \geq k(\phi(\mathbf{x}) \leq k)$, it is called critical upper (lower) vector for level k if in addition $\mathbf{y} < \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x} \Rightarrow \phi(\mathbf{y}) < k($ if $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x} \Rightarrow \phi(\mathbf{y}) > k)$. Also denote L_k = {all critical cut (or lower) vectors to the level k }.

Lemma 1.3.1 *Let ϕ be a MMS and k=0,1,...,M-1.*

1. *The following bounds always holds,*

   $max_{\mathbf{Y}\in\cup_{k+1}} P[\cap_{(i,j)\in\cup_{k+1}(\mathbf{Y})}\{X_i > j-1\}] \leq \bar{P}(k) \leq min_{\mathbf{Y}\in L_k} P[\cup_{(i,j)\in L_k(\mathbf{Y})}\{X_i > j\}].$

2. *If the X_i's are associated (see section 1.5 for association of random variables), then*

   $$max_{\mathbf{Y}\in U_{k+1}} \prod_{(i,j)\in U_{k+1}(\mathbf{Y})} \bar{P}_i(j-1) \leq \bar{P}(k) \leq min_{\mathbf{Y}\in L_k} \coprod_{(i,j)\in L_k(\mathbf{Y})} \bar{P}_i(j)$$

   and

   $$\prod_{\mathbf{Y}\in L_k} P[\cup_{(i,j)\in L_k(\mathbf{Y})}\{X_i > j\}] \leq \bar{P}(k) \leq \coprod_{\mathbf{Y}\in U_{k+1}} P[\cap_{(i,j)\in U_{k+1}(\mathbf{Y})}\{X_i > j-1\}]$$

3. *If X_i's are independent, then*

   $$\prod_{\mathbf{Y}\in L_k} \coprod_{(i,j)\in L_k(\mathbf{Y})} \bar{P}_i(j) \leq \bar{P}(k) \leq \coprod_{\mathbf{Y}\in U_{k+1}} \prod_{(i,j)\in U_{k+1}(\mathbf{Y})} \bar{P}_i(j-1).$$

Here the bounds for reliability are obtained when components are independent and associated. The concept of association plays an important role for obtaining bounds for MSS reliability.

As we do have an added concern, in this work, on the importance and joint importance measures for both MSSs and binary systems, we shall briefly present the preliminary ideas and developments of importance measure in the next section.

Importance Measures

When we consider the MSS with multiple components it should be obvious that some components in the system are more important for the system reliability than other components. Measures of importance are quantitative criteria for ordering different components in the coherent system based on their critical roles in the functioning performance or the failure of the system. The definitions of importance for the components in the binary setup are clearly defined in Birnbaum (1969) and Barlow and Proschan (1975). Later El.Neweihi et al. (1978), Griffith (1980), Bueno (1989), and Abouammoh and Al-kadi (1991) extended binary concept of component importance to MSS setup. Barlow and Wu (1978) suggested measures of component importance in a MSS based on the relevance property; a component i is "critical" to a coherent system at state j, if, with component i in state j, the system is in state j and if component i not in state j, the system is not in state j. While studying the importance measures in MSS, Barlow and Wu (1978) proved that the following two statements are equivalent:

1. (i) $X_i = j \Rightarrow \phi(\mathbf{X}) = j$ and (ii) $X_i \neq j \Rightarrow \phi(\mathbf{X}) \neq j$,

2. (i) $X_i \geq j \Rightarrow \phi(\mathbf{X}) \geq j$, (ii) $X_i \leq j \Rightarrow \phi(\mathbf{X}) \leq j$,

   (iii) $X_i > j \Rightarrow \phi(\mathbf{X}) > j$ and (iv) $X_i < j \Rightarrow \phi(\mathbf{X}) < j$.

Needless to say, these statements show how a component is critical to the system.

Now we consider the concepts of component importance of the binary system, see Barlow and Proschan (1975). If $h(\mathbf{p})$, where $\mathbf{p} = (R_1, ..., R_n)$ and $R_i = P[X_i = 1]$, is the reliability function of a binary state system, the importance $I(i)$ of ith component is

$I(i) = \frac{\partial h}{\partial R_i}$ where $h = E[\phi(\mathbf{X})]$ or equivalently

$I(i) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}) = E[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})]$ or equivalently

$I(i) = P[\phi(1_i, \mathbf{X}) = 1$ and $\phi(0_i, \mathbf{X}) = 0]$.

Note that $h(\mathbf{p}) = R_i I(i) + h(0_i, \mathbf{p})$ and it implies $h((R_i + \Delta)_i, \mathbf{p}) = h(\mathbf{p}) + \Delta I(i)$, this demonstrates that a component improvement of $\triangle$ in component i yields a system improvement of $\Delta I(i)$.

In the MSS, as defined in Griffith (1980), consider the expected utility

$$U(\rho) = \sum_{j=1}^{M} a_j p_j(\rho) = \sum_{j=1}^{M} b_j H_j(\rho)$$

where $p_i(\rho) = P[\phi(\mathbf{X}) = i]$, $H_j(\rho) = \sum_{i=j}^{M} p_i(\rho)$, $\rho = (\rho_1, \rho_2, ..., \rho_n)$, $\rho_i = (\rho_{i1}, \rho_{i2}, ..., \rho_{iM})$, $\rho_{ij} = \sum_{l=j}^{M} p_{il}$, $b_1 = a_1$, $b_k = a_k - a_{k-1}$, for $k = 2, 3, ..., M$ and $b_j \geq 0$ for all j. Define

$$I_{lj}(i) = P[\phi(l_i, \mathbf{X}) \geq j] - P[\phi((l-1)_i, \mathbf{X}) \geq j] \;\forall i, \quad \forall\, l \textit{ and } j \geq 1,$$

then, component importance to the level l of component i in the MSS is

$$I_l(i) = \sum_{j=1}^{M} b_j I_{lj}(i),\; l = 1, 2, ..., M.$$

Griffith (1980) obtained the general decomposition

$$U(\rho) = \sum_{j=1}^{M} b_j P[\phi(0_i, \mathbf{X}) \geq j] + \bar{I}(i)\rho_i^T$$

where $\bar{I}(i) = (I_1(i), I_2(i), ..., I_M(i))$ and ρ_i^T is the transpose of ρ_i.

As a particular realization of above arguments, if $a_i = j$ so that $U = E[\phi(\mathbf{X})]$ then, the vector of component importance in the MSS is,

$$\begin{aligned}\bar{I}(i) = \Bigg(&\sum_{j=1}^{M} \{P[\phi(1_i, \mathbf{X}) \geq j] - P[\phi(0_i, \mathbf{X}) \geq j]\}, \dots, \\ &\qquad \sum_{j=1}^{M} \{P[\phi(M_i, \mathbf{X}) \geq j] - P[\phi((M-1)_i, \mathbf{X}) \geq j]\} \Bigg) \\ = &\left(E[\phi(1_i, \mathbf{X})] - E[\phi(0_i, \mathbf{X})], ..., E[\phi(M_i, \mathbf{X})] - E[\phi((M-1)_i, \mathbf{X})]\right).\end{aligned}$$

Bueno (1989) defined the reliability importance of MSS, the reliability importance of level l of the ith component for the level j of the system is

$$I_{lj}(i) = P[\phi(l_i, \mathbf{X}) \geq j] - P[\phi((l-1)_i, \mathbf{X}) \geq j].$$

This definition enables us to obtain the Griffith reliability importance of level l of the ith component for the system,

$$I_l(i) = \sum_{j=1}^{M} I_{lj}(i) = E[\phi(l_i, \mathbf{X}) - \phi((l-1)_i, \mathbf{X})].$$

Note that all the importance measures discussed above are based on component relevancy with respect to a specified level over the system performance. Meng (1993) differentiated some of the existing relevancy assumptions and introduced two new other relevance conditions and importance measures.

Another interesting research area in binary state systems is that of ageing properties of the lifetime random variable. The ageing properties of the binary state systems are discussed in Barlow and Proschan (1975). We recall some important concepts of ageing in the following section.

1.4 Ageing of a Life Time Random Variable

The concept of ageing is very important in reliability theory. 'No ageing' means the age of a component has no effect on the distribution of residual life time. 'Positive ageing' describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of the component. On the otherhand, 'Negative ageing' has an opposite effect on the residual lifetime.

Let X_t be the random variable representing the residual life time of a unit which has attained the age t. Let the survival function be $R_x(t)$. Next it is seen that

$$R_x(t) = \frac{R(t+x)}{R(t)}$$

where $R(x)$ is the survival or reliability function of the life time random variable with distribution function $F(x)$. This is the conditional probability that the unit survived upto time t, will not fail before additional x units of time. Further $R_0(t) = R(t)$.

By positive ageing we mean the phenomenon where by an older system has shorter remaining life time in some statistical sense than a newer or younger one. That is

$$R_x(t) = \frac{R(t+x)}{R(t)} \leq R_x(0) \text{ or } R_x(t) = \frac{R(t+x)}{R(t)}$$

is decreasing in t. Similarly, for negative ageing,

$$R_x(t) = \frac{R(t+x)}{R(t)} \text{ is increasing in } t.$$

Obliviously, any study of the phenomenon of ageing is to be based on $R_x(t)$ and functions related to it.

Another important function used in study of ageing is the conditional failure rate function. The conditional failure rate or failure rate $\lambda(t)$ at time t is defined as

$$\lambda(t) = lim_{x \to 0} \frac{R(t) - R(t+x)}{xR(t)},$$

so that,

$$\lambda(t) = \frac{f(t)}{R(t)},$$

when $F = 1 - R$ is absolutely continuous and $f(t)$ is the probability density function of $F(t)$.

In reliability, we often characterize life distribution through the failure rate function $\lambda(t) = \frac{f(t)}{R(t)}$.

If $\lambda(t)$ is increasing in t (or equivalently $R_x(t) = \frac{R(t+x)}{R(t)}$ is decreasing in t), then F is said to be increasing failure rate (IFR) distribution.

Similarly, if $\lambda(t)$ is decreasing in t (or equivalently $R_x(t) = \frac{R(t+x)}{R(t)}$ is increasing in t), then F is said to be decreasing failure rate (DFR) distribution.

The most commonly considered ageing classes other than IFR and DFR are increasing failure rate average (IFRA), new better than used (NBU), new better than used in expectation (NBUE), and decreasing mean residual life (DMRL) (with their duals).

F is said to be IFRA if $\frac{1}{t}\int_0^t \lambda(x)dx$ is increasing in t.

F is said to be NBU distribution if

$$R(t+x) \leq R(t)R(x),\ t \geq 0, x \geq 0.$$

F is said to be NBUE distribution if

$$\int_0^\infty R(t+x)dx \leq R(t) \int_0^\infty R(x)dx,\ t, x > 0.$$

F is said to be DMRL distribution if

$$\mu(t) = \frac{\int_t^\infty R(x)dx}{R(t)}$$

is decreasing in t.

Details to these ageing properties can be seen in Bryson and Siddiqui (1969) and Barlow and Proschan (1975). The other ageing properties such as Bathtub shaped failure rate, HNBUE (Harmonically new better than used in expectation), etc can be seen in Deshpande et al. (1986), Klefsjo (1982) and Lai et al. (2001).

Suppose, for example, the life time of a component follows two parameter Weibull distribution, then

$$F(t) = 1 - e^{-(t/\theta)^\alpha},\ \theta > 0,\ \alpha > 0,\ t > 0.$$

Then

$$\lambda(t) = \alpha(t/\theta)^{\alpha-1},\ \ \theta > 0,\ \alpha > 0,\ t > 0.$$

Thus if $\alpha = 1$, the model becomes constant failure rate model, $\alpha < 1$ decreasing failure rate model, and $\alpha > 1$ increasing failure rate. In many applications of reliability, maintenance theory, inventory theory and biometry different probabilistic concepts of ageing are of interest. Various authors have studied classes of life distributions based on different concepts of ageing.

If the density function does not exist or if the distribution function is not absolutely continuous, we cannot use failure rate function for the identification of failure rate model. In such cases we use $R_x(t)$ to identify the failure rate model.

When we consider a system governed by stochastic process, it is interesting to study the ageing properties of system lifetime distribution. Ageing properties of first passage time distribution of Markov chain is given by Brown and Chaganty (1983) and aging properties of first passage time distribution of a Markov process is studied by Belzunce et al. (2002). When we model a MSS using semi-Markov process, we need to consider the ageing properties of semi-Markov system in the reliability analysis.

Barlow and Proschan (1975) discussed the bounds of binary system (series and parallel) reliability based on concept of association. Block and Savits (1982) obtained bounds for the reliability when component state random variables are associated. In the following section we discuss the concepts of association among random variables.

1.5 Association of Random Variables

In the classical case of statistical inference, the observed random variables of interest are generally assumed to be independent and identically distributed. However in several real life situations, the random variables need not be independent.

In reliability studies, there are structures in which the components share load, so that failure of one component results in increased load on each of the remaining components. Minimal path structures of a coherent system having components in common behave in a similar manner. Failure of one component will adversely affect the performance of all the minimal path structures containing it. In both the examples given above, the random variables of interest are not independent but are 'associated'.

Hoeffding (1940) [cf. Lehmann (1966)] proved the following result.

Theorem 1.5.1 *Let (X, Y) be a bivariate random vector such that $E(X^2) < \infty$ and $E(Y^2) < \infty$. Then*

$$Cov(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) dx dy \tag{1.4}$$

where

$$\begin{aligned} H(x, y) &= P[X > x, Y > y] - P[X > x]P[Y > y] \\ &= P[X < x, Y < y] - P[X < x]P[Y < y]. \end{aligned}$$

Relation (1.4) is known as the Hoeffding identity. This result is useful in the study of 'association'.

It is customary to consider that two random variables X and Y are associated if

$$Cov(X,Y) = E(XY) - E(X)E(Y) \geq 0.$$

If $Cov(f(X), g(Y)) \geq 0$ for all pair of non-decreasing functions f, g, then X and Y are more strongly associated. Finally, if

$$Cov(f(X,Y), g(X,Y)) \geq 0$$

for all pairs of functions f, g which are non-decreasing in each argument, then X and Y are more strongly associated. See Esary et al. (1967) for more details.

The strongest of these criteria has a natural multivariate extension. We say that random variables X_1,...,X_n are associated if $Cov(f(\mathbf{X}), g(\mathbf{Y})) \geq 0$ for all non-decreasing functions f and g for which $Ef(\mathbf{X})$, $Eg(\mathbf{X})$, $Ef(\mathbf{X})g(\mathbf{X})$ exist, where $\mathbf{X} = (X_1, ..., X_n)$.

The following are some important properties of association.

P1. Any subset of associated random variables are associated.

P2. If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.

P3. The set consisting of single random variable is associated.

P4. Non-decreasing functions of associated random variables are associated.

Barlow and Proschan (1975) used the concept of association for getting reliability bounds for binary system.

When a system performance is a stochastic process, a series of bounds for the availability and unavailability in a fixed time interval, I, for a system of maintained, interdependent components are given in Natvig (1980) in the traditional binary case, and in Funnemark and Natvig (1985) in the multi-state case. For the special case of independent components the only assumption needed to arrive at these bounds is that the marginal performance process of each component is associated in I. When these processes are Markovian and binary, a sufficient condition for this to hold is given by Esary and Proschan (1970). For that, Esary and Proschan (1970) considered devices capable of two states of performance, either functioning or failed. The performance process of a device is a stochastic process $\{X_i(t), t \in \tau\}$, where for each fixed value of $t \in \tau$, $X_i(t)$ is a random variable taking values 0, and 1. $X_i(t) = 1$ if the component is perfectly functioning, and $X_i(t) = 0$, if the component is completely failed. The index τ is contained in $[0, \infty)$. The joint performance process $\{\mathbf{X}(t), t \in \tau\} = \{X_1(t), ..., X_n(t), t \in \tau\}$ for a set of components is a vector of stochastic processes for which ith marginal process $\{X_i(t), t \in \tau\}$ is the performance process for the ith component, $i = 1, ..., n$. The joint performance process $\{\mathbf{X}(t), t \in \tau\}$ of a set of components is associated in time if for each set of times $\{t_1, ..., t_m\} \subset \tau$ the binary random variables in the array

$$\begin{array}{ccc} X_1(t_1) & . & X_1(t_m) \\ . & . & . \\ X_n(t_1) & . & X_n(t_m) \end{array}$$

are associated. Hjort et al. (1985) generalized the condition of association in binary case to the multi-state case, and gave an equivalent and much more convenient condition in terms of the transition intensities.

More generally, let a component consist of k branches in parallel and let its state be the number of functioning branches. Assume that the branches fail and are repaired/replaced independently of each other, all having the same instantaneous failure rate and repair/replacement rate. Then the underlying Markov process is associated. Association concepts play a role in MSS reliability study.

When we compare two pair of random variables based on their degree of association, it is interesting to get a measure of degree of association of each pair. Karlin (1983) proposed a measure for getting degree of association of pair of random variables. But when we compare two MSSs with n multi-state components each of which are governed by Makov process, there is a requirement of a measure based on transition probability function. Some researchers considered dependence notions based on association, i.e., correlation order, see Yi and Weng (2006) and compared dependence of stationary Markov processes, see Hu and Pan (2000). But they used conditional expectation of system structure for the comparison. They did not use transition probability function. Since Kuber and Dharmadhikari (1996) and Dharmadhikari and Dewan (2006) used transition probability function to get the sufficient condition for association in time of Markov and semi-Markov processes, a measure based on transition probability function for comparison of two systems whose performance processes are Markovian will be helpful to system engineers.

Now we consider the most commonly used performance measures of the MSS.

1.6 Performance Measures of MSS

In the thesis, we use four performance measures-reliability, availability, risk (unreliability or unavailability) and expected performance. In this section we give brief details regarding the performance measures.

Let $\mathbf{X} = (X_1, ..., X_n)$, where $X_i \in \{0, 1, ..., M_i\}$ denotes the state of the component i, where M_i is the best state of component i, whose output performance is in $\{x_{i0}, x_{i1}, ..., x_{iM_i}\}$. Here M_i can be equal in some situations, i.e., $M_i = M$ $\forall i = 1, 2, ..., n$. Let $W \in \{w_k, 0 \leq k \leq M\}$ denotes the output performance of the system where $M = max_{1 \le i \le n}\{M_i\}$. Let $\phi(\mathbf{X})$ denotes the state of MSS, i.e., $\phi(\mathbf{X}) = k$ when $W = w_k$. Let, $\forall\, 1 \le i \le n$,

$$p_{ij} = P[X_i = j],\ 0 \le j \le M_i$$

and

$$p_k = P[W = w_k],\ 0 \le k \le M$$

be the probability distribution of components and system.

In a similar fashion we can consider that the states of components and system are time dependent random variables. At time zero, the system begins at its best state and as time passes the system begins to deteriorate. Then, let $\mathbf{X}(t) = (X_1(t), ..., X_n(t))$ represents the vector of component states, $\phi(\mathbf{X}(t))$ represents the system state and $W(t)$ represents the system output performance.

We consider the following performance measures of MSS in this thesis.

1. **System Reliability:** Reliability is the ability of the system to meet the demand. Let w_k be the system demand corresponding to state k of the MSS described above. Then MSS reliability may be defined as

$$R = P[\phi(\mathbf{X}) \ge k] = P[W \ge w_k],\ k \in \{0, 1, ..., M\}. \tag{1.5}$$

For a time dependent MSS, the reliability may be defined as, for a system working without falling below state k upto time t,

$$R(t) = P[\phi(\mathbf{X}(t)) \geq k] = P[W(t) \geq w_k],\ k \in \{0, 1, ..., M\}. \tag{1.6}$$

2. **System Availability:** System availability may be defined as the probability that the system is in working state, above level k, at the time of inspection t. That is,

$$A(t) = P[\phi(\mathbf{X}(t)) \geq k] = P[W(t) \geq w_k],\ k \in \{0, 1, ..., M\}, \tag{1.7}$$

where t is the time of inspection.

3. **System Expected Performance:** System expected performance state is

$$E_s = \sum_{k=0}^{M} P[\phi(\mathbf{X}) \geq k] = \sum_{k=0}^{M} P[W \geq w_k]. \tag{1.8}$$

The system expected output performance is

$$E = \sum_{k=0}^{M} w_k P[W = w_k]. \tag{1.9}$$

In a similar way we can define time dependent system expected performances.

$$E(t) = \sum_{k=0}^{M} w_k P[W(t) = w_k]. \tag{1.10}$$

4. **Risk (System Unreliability or Unavailability)**: The system unreliability may be defined as

$$\bar{R} = 1 - P[\phi(\mathbf{X}) \geq k] = P[W < w_k],\ k \in \{0, 1, ..., M\}. \tag{1.11}$$

For the time dependent system it is

$$\bar{R}(t) = 1 - P[\phi(\mathbf{X}(t)) \geq k] = P[W(t) < w_k],\ k \in \{0, 1, ..., M\}. \tag{1.12}$$

Similarly unavailability is

$$\bar{A}(t) = 1 - P[\phi(\mathbf{X}(t)) \geq k] = P[W(t) < w_k],\ k \in \{0, 1, ..., M\}. \tag{1.13}$$

At steady-state, the probability distribution of the MSS states is:

$$p_i = lim_{t\to\infty} P[W(t) = w_i] = lim_{t\to\infty} P[\phi(\mathbf{X}(t)) = i], 0 \leq i \leq M. \tag{1.14}$$

The two vectors of the system performance realizations $\bar{w} = \{w_i, 0 \leq i \leq M\}$, and system state probabilities, $\bar{p} = \{p_i, 0 \leq i \leq M\}$ define the system output performance distribution.

Suppose that the system (or component) is in operation without break from the start of the operation, then reliability is a suitable measure in reliability analysis. Also in situation of preventive maintenance activities, the definition of reliability is central to the study. But, if we shutdown the system without complete failure for some corrective maintenance or we are interested only in the probability of failure at time t, we use availability as the performance measure. In nuclear science where the risk become an important measure for the system performance, we can use the system unreliability or unavailability. The expected performance or average performance is used as an important performance measure in the system reliability analysis. Lisnianski and Levitin (2003) provides a good account information in reliability analysis of various real life MSSs based on these OPMs.

The MSS OPMs at steady state may be defined as follows.

1. Reliability: $R = \sum_{i=0}^{M} p_i I_{\{w_i \geq w_k\}}$, $I_{\{w_i \geq w_k\}}$ is the indicator function of $\{w_i \geq w_k\}$.

2. Availability: $A = \sum_{i=0}^{M} p_i I_{\{w_i \geq w_k\}}$.

3. Expected State: $E_s = \sum_{i=0}^{M} i p_i$.

4. Expected Output Performance: $E = \sum_{i=0}^{M} w_i p_i$.

5. Unreliability: $\bar{R} = 1 - R$.

6. Unavailability: $\bar{A} = 1 - A$.

Most of the reliability engineers use the above performance measures.

In any statistical problem, after the model specification and derivation of various properties, the inference methods become important such as estimation of unknown parameters using data, testing of some hypothesis regarding unknown parameter, etc. In the following section the concept of Bayesian inference in reliability is discussed.

1.7 Bayesian Inference

An important information regarding unknown parameters involved in any statistical problem is prior information. A useful way of talking about prior information is in terms of probability distribution over the possible values of parameter under consideration. Let p be the unknown parameter with prior density $\pi(p)$. Berger (1985) provided a good account of procedures of prior selection. Bayesian analysis is performed by combining the prior information ($\pi(p)$) and the sample information (x) into what is called posterior distribution of p given x, from which all inferences and decisions are made. The posterior distribution of p given the data may be denoted as $\pi(p|x)$. In general the sample distribution $m(x)$, where $m(x)$ is the sample density function, and $\pi(p|x)$ are not easily calculable. If, for example, x is $N(p, \sigma^2)$ and p is $C(\mu, \beta)$, then $\pi(p|x)$ can only be evaluated numerically. A large part of the Bayesian literature is devoted to finding prior distributions for which $\pi(p|x)$ can be easily calculated. These are called conjugate priors. Beta distribution belongs to conjugate prior family.

In Bayesian reliability literature, if reliability is the unknown parameter, the most of the researchers feels Beta distribution is suitable prior since it is conjugate prior, ie., posterior distribution is again Beta, see Jun et al. (1999) and Hammada et al. (2003). Beta distribution in the interval [0,1] is suitable for prior distribution of reliability.

When we consider MSS made up of n components, each component reliability may be unknown. So we can assign proper prior distributions to them and estimate reliability from posterior distribution.

Now we give the motivation and objectives of the present study in MSS reliability analysis.

1.8 Motivation and Objectives of the Present Study

There are various unsolved problems associated with MSS reliability analysis. In MSS reliability engineering we use four important output performance measures (OPMs) for the MSS. They are reliability, availability, expected performance and risk (unreliability or unavailability). The problem of finding joint importance of two or more components in the sense of Birnbaum, performance achievement worth, performance reduction worth, and Fussel-Vesely with respect to the OPMs in the MSS is an unexplored one. In many engineering applications, these measures have an important role in finding the joint effect of two or more components for getting maximum variation in performance measure, for identifying the performance achievement due to interaction, performance reduction due to lack of interaction and the maximum decrement in the system reliability caused by lack of interaction.

The available MSSs have some specific component relevancy assumptions for their definition and its importance measure. But in some systems these relevance assumption becomes insufficient for system definition and calculation of importance measures. So we need some new relevancy assumptions and its use in computing the importance and joint importance measures. The application of such definitions arises in power generation systems, network systems etc, see Natvig et al. (1986) and Natvig and Morch (2003).

When we are interested in the dynamic behavior of a MSS, Markov/semi-Markov processes turns out to be the good MSS modeling tools. A major research problem related to lifetime of a binary system is the behavior of its failure rates, such as IFR, IFRA, DFR and DFRA. But such behaviors of failure rate in MSS context is to be developed more. So finding a necessary and sufficient conditions for first passage time from acceptable state to unacceptable state in a MSS modeled by semi-Markov process to be IFR, IFRA, DFR and DFRA is a problem of reliability analysis. In order to apply suitable maintenance and replacement policies to the MSS, we can use the ageing properties as in the binary case.

The concepts of association in Markov process and semi-Markov process applicable to MSSs are discussed by Horjt et al. (1985) and Dharmadhikari and Dewan (2006) respectively. But, we need measures to assess the degree of association of system governed by Markov process. If we have some criteria based on transition probability function of the process to find the degree of association, then two processes can be compared based on the degree of association. The application is useful not only in reliability engineering, but also in all other field in which similar models apply.

The evaluation of the reliability, availability, risk, expected performance and importance measures using UGF is now a common practice. But the evaluation of joint importance measures using UGF remains unsolved yet. It would be helpful to system engineers, if the evaluation procedure is available. There are large number of real life MSSs, eg. telecommunication system, oil/gas transportation system, power generation system, signal transmission system, production and manufacturing systems, etc, which can be modeled as multi-state network with multi-state arcs or nodes. It is quite desirable to apply the UGF method of evaluation of joint importance measure evaluation in network systems.

In any statistical decision problem, the Bayesian inference serves as a useful tool for estimation. Hence finding the Bayesian methods in complex multi-state reliability and joint importance measure estimation problems is quite desirable. Bayesian inference will be useful in rare data problem and in presence of prior information of system reliability or component reliability.

Based on the above objectives, we have made an attempt to address the problems and obtained results. These are fundamental to the MSS reliability analysis. The following section briefly discusses the new results under this objectives.

1.9 Main Contribution of the Thesis: An Overview

Importance measure(IM)s of components in MSS reliability is indeed an important topic. In *Chapter* 2, we introduce the joint structural importance measures, and the measures of joint importance with respect to various performance measures such as, system reliability, system availability, system expected output performance, and system risk (unreliability or unavailability). They are joint reliability importance measures for any number of components, joint performance achievement worth, joint performance reduction worth, joint performance Fussel-Vesely measure, for two components and joint performance Birnbaum measure for any number of components. The proposed measure can be used to give priority for safety of operations of group of components (group of two components or group of three components, etc...). Also, we give a characterization result using Schur-convex functions for identifying the sign of the joint reliability importance of a binary imaged MSS.

A key requirement in defining a MCS is the relevance condition of the components. The condition of relevancy in binary coherent system (BCS) is extended in various different ways. Some extensions can be seen in Meng (1993) and Abouammoh and Al-Khadi (1991, 1995). In *Chapter 3*, a new class of MCSs is introduced with a reasonable component relevance condition. Further a more general relevance condition is introduced and compared with some existing component relevance conditions. Based on the new relevance conditions, component importance measures for MCSs are defined. They are most appropriate for comparing components when certain type of system improvement is considered. Further, we introduce new joint importance measures for two or more components with respect to the proposed relevance conditions. Definitions based on structural properties of the new relevance conditions are given. The new MCS classes include several existing MCSs as special case.

In *Chapter 4*, we study the aging properties of the first passage time distribution of the MSS modeled by a semi-Markov process. The states of the system consists of two sets - one of acceptable states and other of down states. We derive a necessary and sufficient condition under which the distribution of the first passage time from acceptable states to down state is IFR and IFRA. The dual results of DFR and DFRA are also discussed.

In reliability engineering, often lifetime random variables are not independent but are associated. A sufficient condition for association when the marginal processes are Markovian is given by Hjort et al. (1985). Lisnianski and Levitin (2003) discussed a large number of real life problems in MSS modeling and reliability assessment, and provided a stochastic process approach (eg. Markov and semi-Markov) for the MSS reliability evaluation. To apply the concept of association to real data one require a measure of the degree of association. Karlin (1983) compared the relative degree (or strength) of association for two sets of random variables. In *Chapter 5*, we address the problem of assessing the degree of association of a Markov process or of comparing the relative strength of association of two Markov processes. We suggest a measure based on transition probability function to obtain and compare the degree of association in time of two processes. The measure is also useful in semi-Markov setup.

The method of UGF (also called the method of generalized generating sequence, see Ushakov (2000)) generalizes the technique that is based on using a well-known ordinary generating function. The basic idea of method is introduced by Ushakov (1987). The approach proved to be very convenient for numerical realization. It ensures relatively small computational resource for evaluating MSS reliability indices and therefore, can be used in complex reliability optimization algorithm. Lisnianski and Levitin (2003) used UGF for the evaluation of importance measures. In *Chapter 6*, we discuss the application of UGF for the evaluation of joint importance measures. The applications of UGF for the evaluation of joint importance measures in network problems are highlighted.

In any statistical problem, inference of probability distributions and parameters is very important. In the case of rare data problem that arise in reliability engineering, we can use the Bayesian results for inference of unknown parameters. In *Chapter 7*, we give the method of Bayesian inference of MSS reliability when the system is modeled as network system. This procedure is useful to get Bayes estimates of joint importance measures.

Finally, we give some concluding remarks and further future research. A detailed reference list is given at the end of the thesis.

Chapter 2

JOINT IMPORTANCE MEASURES

2.1 Introduction

[1]Importance measures (IMs) quantify the criticality of a particular component within a system design. They have been widely used as tools for identifying system weakness, and to prioritize reliability improvement activities. Measures of importance are quantitative criteria for ordering different components in the coherent system whose improvement may result in the greatest improvement for the system based on their critical roles in the functioning or the failure of the system and to provide a checklist

[1]Some contents of this chapter have appeared in Chacko and Manoharan (2008a, 2008c), and Chacko (2008a)

for failure diagnosis. They can also provide valuable information for the safety and efficient operation of the system. From the design point of view, it is crucial to identify the weakness of the system and how failure of each individual component affects proper functioning of the system; so that efforts can be spent properly to improve the system reliability. However, the extend to which a group of component and its states affect the system is a major concern to the system designer and system controller. To solve this problem, methods dependent on the information obtained from joint importance measures can be developed for efficient resource allocation. The knowledge about the joint importance measure can be used as a guide to provide redundancy so that system reliability is increased. It is more informative to the system designers about the interaction effect of two or more components in improving system performance. Information about this type of interaction importance of components constituting a system, with respect to its safety, reliability, availability and risk, is of great practical aid to system designers and managers. Measures of joint importance provide the information on the type and degree of interactions between two or more components by identifying the sign and size of it. A little work has been reported in literature on joint importance measures and the existing measures are extensions of Birnbaum importance measures.

In the binary classical reliability theory Birnbaum (1969) and Barlow and Proschan (1975) proposed some concepts of importance. Although the concept of component importance is very useful one, a few has been systematically generalized it to the multi-state case, see Barlow and Wu (1978), El.Neweihi et al. (1978), Griffith (1980), El.Neweihi and Proschan (1984) and Bueno (1989). Abouammoh and Al-Khadi (1991) reviews the measures on importance for MCSs. Gandini (1990) proposed importance and sensitivity measures for MSSs with binary capacited components. Levitin and Lisnianski (1999) proposed importance and sensitivity measures for MSSs with binary capacited components. These measures account both for MSS performance which is caused by the capacited components and stochastic system demand. Their evaluation method is performed via the UGF method. These approaches have proven to be valuable to the development of multi-state IMs. Wu and Chan (2005) proposed IMs for MSSs with respect to performance utility and related their measure to Griffith's IM. Ramirez-Marquez and Coit (2005b) proposed new importance measures for MSSs from two perspectives 1) how a specific component affects MSS reliability and 2) how a particular state or set of states affects MSS reliability. IMs are widely used in risk informed applications of the nuclear industry to characterize the importance of basic events, i.e., element failures, human errors, common cause failures, etc, with respect to the risk associated to the system. Vassuer and Llory (1999) mentioned reliability achievement worth (RAW), reliability reduction worth (RRW), Fussel-Vesely (FV) measure and Birnbaum measure as the most valuable IMs for binary systems in risk informed applications. Further extensions of these measures to the multi-state case can be seen in Zio and Podofillini (2003) and Ramirez-Marquez and Coit (2005a). Levitin, Podofillini and Zio (2003) proposed similar measures using performance measures, availability and risk. Also Zio and Podofillini (2003) introduced identical measures in terms of system risk-unavailability or unreliability. The use of IMs to analyze probabilistic risk assessment results is discussed in detail by

Cheok, Parry, and Sherry (1998) and Van der Borst and Shoonakker (2001).

However, the joint importance measure provides additional information, which the traditional marginal importance cannot provide, to the system designers, see Hong and Koo (1993). Joint importance measures for binary system can be seen in Armstrong (1995) and Hong et al. (2002). Hong et al. (2000) investigated joint reliability importance (JRI) of two gate events along with its properties in a fault tree. Wu (2005) extended the component IMs to joint importance measures for two multi-state components in a MSS with respect to system structure and expected performance. A limitation of the IMs currently used in reliability and risk analysis is that they rank only individual components or basic events whereas they are not directly applicable to combinations or groups of components or basic events. To partially overcome this limitation, recently, the differential importance measure (DIM), has been introduced for use in risk-informed decision making. The DIM is a first-order sensitivity measure that ranks the parameters of the risk model according to the fraction of total change in the risk that is due to a small change in the parameters' values, taken one at a time. However, it does not account for the effects of interactions among components. Zio and Podofillini (2006) proposed a second-order extension of the DIM, named DIMII, for accounting of the interactions of pairs of components when evaluating the change in system performance due to changes of the reliability parameters of the components.

We recall the existing importance and joint importance measures followed by introducing new joint importance measures-joint structural and reliability importance measures-for two or more components. We also propose joint importance measures as an extension to RAW, RRW, FV and Birnbaum measure of components and generalize it to other performance measures such as availability, and risk-unavailability or unreliability. We find the distribution of the performance of the system, under constraints on the performance of its elements. Once the system performance is determined, one can focus on specific system performance measures. With reference to the predefined threshold of element performance, the element's reachable states are limited to those corresponding to performance either larger or not larger than the threshold level.

The remaining sections of this Chapter are arranged as follows. The joint structural importance measure for more than two components of a MSS is proposed in section 2. Joint reliability importance measure for more than two components of a MSS is proposed in section 3. In section 4, the characterization of joint reliability importance based on Schur-convexity property of binary imaged MSS structure function is given. Joint reliability achievement worth for two components in a MSS is proposed in section 5. Joint reliability reduction worth for two components in a MSS is proposed in section 6. In section 7, the joint Fussel-Vesely measure w.r.t. reliability for two components in a MSS is proposed. The joint reliability Birnbaum measure for two or more than two components is proposed in section 8. The joint risk importance measures based on unreliability or unavailability are proposed in last section.

2.2 Joint Structural Importance Measure

Structural importance measure (SIM) is used to order the components when the component reliabilities are not available while reliability importance measures are used when component reliabilities are available. For a given coherent system, some component are more important than others in determining whether the system functions or not. For example, if a component is in series with the rest of the system, then it would be seen to be at least as important as any other component in the system, see Barlow and Proschan (1975).

Let us consider the structure function of a binary coherent system, $\phi(\mathbf{x})$, where $\mathbf{x} = (x_1, ..., x_n)$ is the state vector of its n components with $x_i = 1$ or 0, $i = 1, 2, ..., n$. Then we would consider component i is more important if

$$\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1 \quad \text{than if} \quad \phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 0,$$

where $(\cdot_i, \mathbf{x}) = (x_1, ..., x_{i-1}, \cdot_i, x_{i+1}, ..., x_n)$. Then Barlow and Proschan (1975) proposed the following measure of the structural importance (SI) of component i:

$$I_\phi(i) = \frac{1}{2^{n-1}} \sum_{\mathbf{x}:x_i=1} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})]. \tag{2.1}$$

It is the proportion of the 2^{n-1} outcomes having $x_i = 1$ which are critical path vectors for ϕ.

In order to get the results in multi-state setup, we shall first assume that a component can degrade one or more state each time, i.e., from state i to $\hat{i}$, $\hat{i} \in \{i-1, i-2, ..., 2, 1, 0\}$ and the system may degrade more than one state. Also assume that both the components and the system have $M+1$ states, $0, 1, ..., M$. Wu (2005) defined SI measure of component i for MSS (SIM) as, for $\hat{m} = m - 1$

$$SIM(i) = \frac{\sum_{\mathbf{x}_i} \chi\left[\phi(m_i, \mathbf{x}_i) = j, \phi(\hat{m}_i, \mathbf{x}_i) < j\right]}{(M+1)^{n-1}},$$

i.e.,

$$SIM(i) = \frac{\sum_{\mathbf{x}_i} \sum_{q=1}^{j} \chi\left[\phi(m_i, \mathbf{x}_i) = j, \phi(\hat{m}_i, \mathbf{x}_i) = j - q\right]}{(M+1)^{n-1}} \tag{2.2}$$

where $\mathbf{x}_i = (\cdot_i, \mathbf{x})$, $(m_i, \mathbf{x}_i) = (x_1, ..., m_i, ..., x_n)$, $[\phi(m_i, \mathbf{x}_i) = j, \phi(\hat{m}_i, \mathbf{x}_i) < j]$ determines the critical path vector to the level j, $(M+1)^{n-1}$ is the total number of state vectors and $\chi[true] = 1$, $\chi[false] = 0$.

Denote JSIM for the joint structural importance measure for MSS. Wu (2005) used the following index (eq. 2.3) to measure the JSIM of component i and l in the MSS.

$$JSIM(i, l) = \sum_{m=1}^{M} \sum_{k=1}^{M} \{SIM(i, l; m, k) - SIM(i, l; m, \hat{k})\} \tag{2.3}$$

where $SIM(i, l; m, k)$ is

$$\frac{\sum_{\mathbf{x}_{il}} \sum_{q=1}^{j} \chi\left[\phi(m_i, k_l, \mathbf{x}_{il}) = j, \phi(\hat{m}_i, k_l, \mathbf{x}_{il}) = j - q\right]}{(M+1)^{n-2}},$$

where $\mathbf{x}_{il} = (\cdot_i, \cdot_l, \mathbf{x})$. Obviously, $SIM(i, l; m, k) - SIM(i, l; m, \hat{k})$ gives information on how two states of two components interact topologically. In order to get the JSIM of more than two components, we shall find the change in the JSIM of two component when the third component changes its state, i.e.,

$$JSIM(i,l,r) = \sum_{m=1}^{M}\sum_{k=1}^{M}\sum_{n=1}^{M}\{JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,k,\hat{n})\} \quad (2.4)$$

where $JSIM(i,l,r;m,k,n) = SIM(i,l;m,k)_{n_r} - SIM(i,l;m,\hat{k})_{n_r}$ while

$$SIM(i,l;m,k)_{n_r} = \frac{\sum_{\mathbf{x}_{ilr}}\sum_{q=1}^{j}\chi\left[\phi(m_i,k_l,n_r,\mathbf{x}_{ilr}) = j, \phi(\hat{m}_i,k_l,n_r,\mathbf{x}_{ilr}) = j-q\right]}{(M+1)^{n-3}}.$$

The joint structural importance of three components i, l and r in a MSS can be measured using (2.4).

Clearly, $JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,k,\hat{n})$ in (2.4) gives information on how states of three components interact topologically. $JSIM(i,l,r)$ indicates how the topological joint importance of two component changes with the third one.

If the JSIM of three components is positive, we can conclude that there is a difference in joint structural importance of two components when the third component change its state from higher level to lower level. It is an indication of joint effect. If this $JSIM(i,l,r) = 0$, we can conclude that the third component has no effect in system performance. Anyway, this type of JSIM gives information regarding joint effect of components, when reliabilities of components are not available, for the system improvement. Similarly by appropriately taking differences we can find the higher order interaction joint structural importance of components.

2.3 Joint Reliability Importance Measure

Joint reliability importance (JRI) of two or more components is a quantitative measure of the interactions of two or more components or states of two or more components. The value of JRI represents the degree of interactions between two or more components with respect to system reliability. JRI indicates how components interact in system reliability, see Armstrong (1999). Consider the vector of component states $\mathbf{X} = (X_1, X_2, ..., X_n)$, where X_i is the random variable representing the state of the ith component.

In the binary setup, the marginal reliability importance of a component is

$$I(i) = \frac{\partial R}{\partial R_i}$$

and the JRI of two components i and j is

$$JRI(i,j) = \frac{\partial^2 R}{\partial R_i . \partial R_j} \tag{2.5}$$

where $R = E(\phi(\mathbf{X}))$ and R_i and R_j are reliabilities of the components i and j respectively. That is, JRI of two binary components is

$$JRI(i,j) = R(1_i, 1_j, \mathbf{p}) - R(1_i, 0_j, \mathbf{p}) - R(0_i, 1_j, \mathbf{p}) + R(0_i, 0_j, \mathbf{p}) \tag{2.6}$$

where $R(\cdot_i, \cdot_j, \mathbf{p}) = E(\phi(X_1, ..., \cdot_i, ..., \cdot_j, ..., X_n))$. In order to generalize this equation for more than two components, i.e., to measure the improvement of reliability importance of the system with respect to the interactive effect of more than two components, at first we shall calculate change in the JRI of two components with respect to the change of reliability of third component. If there is any change in the JRI due to change in state of third component we can say that there is an interactive effect of three components for the system reliability improvement. That is, in the binary setup, the change in the JRI is found to be as follows

$$JRI(i,j,k) = JRI(i,j|1_k,\mathbf{p}) - JRI(i,j|0_k,\mathbf{p}) \tag{2.7}$$

where

$JRI(i,j|q_k,\mathbf{p}) = R(1_i,1_j,q_k,\mathbf{p}) - R(1_i,0_j,q_k,\mathbf{p}) - R(0_i,1_j,q_k,\mathbf{p}) + R(0_i,0_j,q_k,\mathbf{p})$,

$q = 0$ or 1, i.e., change in JRI of two components when third component is improved from its failure state to its functioning state. The value of the $JRI(i,j,k)$ indicates how the JRI of two components changes with the change of the state of third component.

In order to find JRI in MSS, we consider X_i's and system states ϕ take values in the set $\{0,1,2,...,M\}$. Let

$$P[\phi(\mathbf{X}) \geq j] = P[\phi(0_i,\mathbf{X}_i) \geq j] + \sum_{m=1}^{M} \left(P[\phi(m_i,\mathbf{X}_i) \geq j] - P[\phi(\hat{m}_i,\mathbf{X}_i) \geq j]\right) P[X_i \geq m],$$

$$E_s = \sum_{j=1}^{M} P[\phi(\mathbf{X}) \geq j], \text{ and } R_i m = P[X_i \geq m].$$

We shall prove the following lemma. It shows how JRI of three components express in terms of JRI of two components proposed by Wu (2005).

Lemma 2.3.1 *Let ϕ be a structure function of MSS with n components. Then,*

$$\frac{\partial^3 E_s}{\partial R_i m \partial R_l k \partial R_r n} = \frac{\partial^2 E_s}{\partial R_i m \partial R_l k}|_{n_r} - \frac{\partial^2 E_s}{\partial R_i m \partial R_l k}|_{\hat{n}_r}. \tag{2.8}$$

Proof: Since $E_s = \sum_{j=1}^{M} P[\phi(\mathbf{X}) \geq j]$, we get,

$$E_s = \sum_{j=1}^{M} \left(P[\phi(0_i, \mathbf{X}_i) \geq j] + \sum_{m=1}^{M} \left(P[\phi(m_i, \mathbf{X}_i) \geq j] - P[\phi(\hat{m}_i, \mathbf{X}_i) \geq j] \right) R_i m \right).$$

This E_s can be again simplified as

$$\sum_{j=1}^{M} \left[P[\phi(0_i, 0_l, \mathbf{X}_{il}) \geq j] + \sum_{k=1}^{M} \left(P[\phi(0_i, k_l, \mathbf{X}_{il}) \geq j] - P[\phi(0_i, \hat{k}_l, \mathbf{X}_{il}) \geq j] \right) R_l k + \right.$$
$$\sum_{m=1}^{M} \left(P[\phi(m_i, 0_l, \mathbf{X}_{il}) \geq j] + \sum_{k=1}^{M} \left(P[\phi(m_i, k_l, \mathbf{X}_{il}) \geq j] - P[\phi(m_i, \hat{k}_l, \mathbf{X}_{il}) \geq j] \right) R_l k \right) R_i m -$$
$$\left. \sum_{m=1}^{M} \left(P[\phi(\hat{m}_i, 0_l, \mathbf{X}_{il}) \geq j] + \sum_{k=1}^{M} \left(P[\phi(\hat{m}_i, k_l, \mathbf{X}_{il}) \geq j] - P[\phi(\hat{m}_i, \hat{k}_l, \mathbf{X}_{il}) \geq j] \right) R_l k \right) R_i m \right].$$

For convenience and express the ideas we use, $P_{mkn} = P[\phi(m_i, k_l, n_r, \mathbf{X}_{ilr}) \geq j]$ and $P_{im} = P[X_i \geq m]$. Again expanding the expression of E_s above by pivoting at rth component, we get E_s as,

$$\sum_{j=1}^{M} \left[P_{000} + \sum_{n=1}^{M} \left(P_{00n} - P_{00\hat{n}} \right) P_{rn} + \sum_{k=1}^{M} \left(P_{0k0} + \sum_{n=1}^{M} \left(P_{0kn} - P_{0k\hat{n}} \right) P_{rn} \right) P_{lk} - \right.$$
$$\sum_{k=1}^{M} \left(P_{0\hat{k}0} + \sum_{n=1}^{M} \left(P_{0\hat{k}n} - P_{0\hat{k}\hat{n}} \right) P_{rn} \right) P_{lk} + \sum_{m=1}^{M} \left\{ \left(P_{m00} + \sum_{n=1}^{M} \left(P_{m0n} - P_{m0\hat{n}} \right) P_{rn} \right) + \right.$$
$$\left. \sum_{k=1}^{M} \left(P_{mk0} + \sum_{n=1}^{M} \left(P_{mkn} - P_{mk\hat{n}} \right) P_{rn} \right) P_{lk} - \sum_{k=1}^{M} \left(P_{m\hat{k}0} + \sum_{n=1}^{M} \left(P_{m\hat{k}n} - P_{m\hat{k}\hat{n}} \right) P_{rn} \right) P_{lk} \right\} P_{im} -$$
$$\sum_{m=1}^{M} \left\{ \left(P_{\hat{m}00} + \sum_{n=1}^{M} \left(P_{\hat{m}0n} - P_{\hat{m}0\hat{n}} \right) P_{rn} \right) + \sum_{k=1}^{M} \left(P_{\hat{m}k0} + \sum_{n=1}^{M} \left(P_{\hat{m}kn} - P_{\hat{m}k\hat{n}} \right) P_{rn} \right) P_{lk} - \right.$$
$$\left. \left. \sum_{k=1}^{M} \left(P_{\hat{m}k0} + \sum_{n=1}^{M} \left(P_{\hat{m}kn} - P_{\hat{m}k\hat{n}} \right) P_{rn} \right) P_{lk} \right\} P_{im} \right].$$

By differentiating E_s with respect to P_{im}, P_{lk} and P_{rn} we get

$$\begin{aligned}\frac{\partial^3 E_s}{\partial R_i m \partial R_l k \partial R_r n} &= \sum_{j=1}^{M} \{[P_{mkn} - P_{m\hat{k}n} - P_{\hat{m}kn} + P_{\hat{m}\hat{k}n}] - [P_{mk\hat{n}} - P_{m\hat{k}\hat{n}} - P_{\hat{m}k\hat{n}} + P_{\hat{m}\hat{k}\hat{n}}]\} \\ &= \left[\frac{\partial^2 E_s}{\partial R_i m \partial R_l k}\right]_{n_r} - \left[\frac{\partial^2 E_s}{\partial R_i m \partial R_l k}\right]_{\hat{n}_r}. \qquad (2.9)\end{aligned}$$

Thus we proved the lemma. □

Let JRIM represent the JRI for the MSS. Now we define the JRIM for three components. This definition provides a measure for finding JRI of three components in a MSS.

Definition 2.3.1 *The joint reliability importance of three components with respect to state m of component i, state k of component l and state n of component r of a multi-state system is*

$$JRIM(i, l, r; m, k, n) = \frac{\partial^3 E_s}{\partial R_i m \partial R_l k \partial R_r n}. \qquad (2.10)$$

We have expressed the JRI of three components in a MSS using existing JRI measures of Wu (2005). Now we can prove a general theorem of interaction importance of k of components of a system having $n(\geq k)$ components.

Theorem 2.3.1 *Suppose that*

$$JRIM(a_1, ..., a_k; b_1, ..., b_k) = \frac{\partial^k E_s}{\partial R_{a_1} b_1 .... \partial R_{a_k} b_k}, \; k = 2, 3, ..., n \qquad (2.11)$$

represents the interaction importance of k components. Then the joint reliability importance of k of components can be derived as, for k=2,...,n,

$$\frac{\partial^k E_s}{\partial R_{a_1} b_1 ....\partial R_{a_k} b_k} = \frac{\partial^{k-1} E_s}{\partial R_{a_1} b_1 ...\partial R_{a_{k-1}} b_{k-1}}|_{b_{k a_k}} - \frac{\partial^{k-1} E_s}{\partial R_{a_1} b_1 ...\partial R_{a_{k-1}} b_{k-1}}|_{\hat{b_{k a_k}}}. \quad (2.12)$$

Proof. In order to prove this result we shall use the mathematical induction technique. Clearly the result is true for $k = 2$, i.e., joint reliability importance of two components. Assume that the result is true for an integer $k(< n)$, i.e.,

$$JRIM(a_1, ..., a_k; b_1, ..., b_k) = \frac{\partial^{k-1} E_s}{\partial R_{a_1} b_1 ...\partial R_{a_{k-1}} b_{k-1}}|_{b_{k a_k}} - \frac{\partial^{k-1} E_s}{\partial R_{a_1} b_1 ...\partial R_{a_{k-1}} b_{k-1}}|_{\hat{b_{k a_k}}}. \quad (2.13)$$

By observing change in (2.13) w.r.t. change in a state of a_{k+1}th component, i.e., differentiating partially

$$\frac{\partial^k E_s}{\partial R_{a_1} b_1 ...\partial R_{a_k} b_k}$$

with respect to $R_{a_{k+1}} b_{k+1}$ we get

$$\frac{\partial^k E_s}{\partial R_{a_1} b_1 ...\partial R_{a_k} b_k}|_{b_{k+1 a_{k+1}}} - \frac{\partial^k E_s}{\partial R_{a_1} b_1 ...\partial R_{a_k} b_k}|_{\hat{b_{k+1 a_{k+1}}}},$$

since it can be expressed as a relation similar to (2.9).

Thus the result is true for $k + 1$. Hence the result is true for every integer k, $k = 2, 3, ..., n$. $\square$

It motivates to define the JRI of k components in a MSS.

Definition 2.3.2 *The joint reliability importance of k components with respect to state b_1 of the component a_1, state b_2 of the component a_2,...,state b_k of the component a_k of the multi-state system is*

$$JRIM(a_1, ..., a_k; b_1, ..., b_k) = \frac{\partial^k E_s}{\partial R_{a_1} b_1 ... \partial R_{a_k} b_k}, \ k = 2, 3, ..., n. \tag{2.14}$$

As a result we get the joint importance of 2, 3,.., components in the system. We can reach some important conclusions regarding the joint importance such as whether the joint importance is different for different group of components. The size of the joint importance gives information about the degree of interaction. Following the above method we get the module importance.

2.4 Schur-Convexity Property and Joint Importance Measure

In this section we shall prove a characterization result for the joint importance using Schur-convexity property, which will be useful to identify the sign of the joint importance rather than size. Gopal (2002) discussed Schur-convexity of structure function in MSSs.

A vector $\mathbf{a} = (a_1, ..., a_n)$ is said to majorize a vector $\mathbf{b} = (b_1, ..., b_n)$, i.e., $\mathbf{a} \geq \mathbf{b}$ if $\sum_{i=j}^{n} a_{[i]} \geq \sum_{i=j}^{n} b_{[i]}$ for $j = 1, 2, .., n-1$, and $\sum_{i=1}^{j} a_{[i]} \geq \sum_{i=1}^{j} b_{[i]}$, for $j = 2, ..., n$, when $a_{[i]}$ and $b_{[i]}$ are components of $\mathbf{a}$ and $\mathbf{b}$ arranged in decreasing order.

A real valued function f defined over R^n is said to be Schur-convex (Schur-concave) if

$$f(\mathbf{a}) \geq (\leq) f(\mathbf{b}) \quad \text{whenever } \mathbf{a} \geq \mathbf{b}.$$

A characterization of f to be Schur-convex (Schur-concave) is that, for $i \neq j$

$$(a_i - a_j)\left(\frac{\partial f(\mathbf{a})}{\partial a_i} - \frac{\partial f(\mathbf{a})}{\partial a_j}\right) \geq 0 (\leq 0).$$

This characterization is known as Schur-Ostrowaki's condition. If $\mathbf{a}_1, ..., \mathbf{a}_n$ denote n vectors each with p components, then f defined on $(R^p)^n$ is said to be a Schur-convex (Schur-concave) if, for $m \neq k$ and for two components i and l,

$$(a_{im} - a_{lk})\left(\frac{\partial f(\mathbf{a}_1, ..., \mathbf{a}_n)}{\partial a_{im}} - \frac{\partial f(\mathbf{a}_1, ..., \mathbf{a}_n)}{\partial a_{lk}}\right) \geq 0 (\leq 0).$$

For proving Schur-convexity property of performance measure of MSS,
for $\mathbf{X}' = (X'_i, ..., X'_n)$, $X'_i = 1$ if $X_i \geq m$ and zero otherwise and $\phi_j(\mathbf{X}') = 1$ if $\phi(\mathbf{X}) \geq j$ and zero otherwise, we consider

$$(P_{im} - P_{lk})\left(\frac{\partial R}{\partial P_{im}} - \frac{\partial R}{\partial P_{lk}}\right),$$

where $R = E(\phi_j(\mathbf{X}')) = P[\phi(\mathbf{X}) \geq j]$, $P_{im} = P[X_i \geq m]$.

Then

$$
\begin{aligned}
R = & P_{im}P_{lk}P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 1]+ \\
& P_{im}(1 - P_{lk})P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 0]+ \\
& (1 - P_{im})P_{lk}P[\phi_j(\mathbf{X}') = 1, X_i' = 0, X_l' = 1]+ \\
& (1 - P_{im})(1 - P_{lk})P[\phi_j(\mathbf{X}') = 1, X_i' = 0, X_l' = 0].
\end{aligned}
$$

Differentiating R with respect to P_{im}, we get,

$$
\begin{aligned}
\frac{\partial R}{\partial P_{im}} = & P_{lk}P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 1] + (1 - P_{lk})P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 0] \\
& - P_{lk}P[\phi_j(\mathbf{X}') = 1, X_i' = 0, X_l' = 1] - (1 - P_{lk})P[\phi_j(\mathbf{X}') = 1, X_i' = 0, X_l' = 0].
\end{aligned}
$$

Differentiating R with respect to p_{lk}, we get,

$$
\begin{aligned}
\frac{\partial R}{\partial P_{lk}} = & P_{im}P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 1] - P_{im}P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 0]+ \\
& (1 - P_{im})P[\phi_j(\mathbf{X}') = 1, X_i' = 0, X_l' = 1] - (1 - P_{im})P[\phi_j(\mathbf{X}') = 1, X_i' = 0, X_l' = 0].
\end{aligned}
$$

Suppose that $\phi_j(\mathbf{X}')$ is symmetric in its arguments. Then,

$$\phi_j(X_1', ..., p_i, ..., q_l, ..., X_n') = \phi_j(X_1', ..., q_i, ..., p_l, ..., X_n'),\ q, p \in \{0, 1\}.$$

Let $P[\phi_j(1_i, 1_l) = 1] = P[\phi_j(\mathbf{X}') = 1, X_i' = 1, X_l' = 1]$, $k = 0, 1$. We can prove that

$$\left(\frac{\partial R}{\partial P_{im}} - \frac{\partial R}{\partial P_{lk}}\right) = -(P_{im} - P_{lk})\left\{P[\phi_j(1_i, 1_l) = 1] - 2P[\phi_j(1_i, 0_l) = 1] + P[\phi_j(0_i, 0_l) = 1]\right\}$$

and

$$(P_{im} - P_{lk})\left(\frac{\partial R}{\partial P_{im}} - \frac{\partial R}{\partial P_{lk}}\right) = -(P_{im} - P_{lk})^2[JRIBIMS] \tag{2.15}$$

where $JRIBIMS$ represents the joint reliability importance of binary-imaged MSS, i.e., R is Schur-convex (Schur-concave) implies LHS $> (<)0 \Rightarrow JRIBIMS < (>)0$.

Thus we need only to verify the Schur-convexity property of MSS for getting the sign of the joint importance for two components.

We assumed all components and system have the same number of states. If the assumption does not hold, the JSIMs and the JRIMs can still be obtained for varying number of states. If $M = 1$, MSS becomes a binary system. Our idea of the joint importance measures can be extended and explained in many engineering applications. Schur-convexity serves as a characterizing criteria for the joint importance measures to be positive or negative.

In order to define the joint Reliability Achievement Worth (JRAW), joint Reliability Reduction Worth (JRRW), joint Reliability Fussel-Vesely (JRFV) measure for two components and joint Reliability Birnbaum importance (JRBI) measures for any number of components, we first recall the existing importance measures in Ramirez-Marquez and Coit (2005b). The joint importance measures of two components for MSS with the OPMs, reliability and availability, with reference to the existing measures of importance, RAW, RRW, FV, and Birnbaum for individual components are introduced in the following sections. For the sake of better narration, we consider reliability as the output performance measure and introduce joint importance measures. This results are also true for the OPM availability. So we can generalize the results to both reliability and availability. In following sections, for time dependent binary and MSS, we propose JRAW, JRRW, JRFV for two components and JRBI measures for any number of components.

2.5 Joint Reliability Achievement Worth

The RAW measure quantifies the maximum percentage increase in system reliability generated by a particular component. From a binary perspective it is defined as

$$RAW_i = \frac{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1]}{P[\phi(\mathbf{X}(t)) = 1]}.$$

For a constant demand w_k corresponding to state k, multi-state RAW of component i with respect to performance threshold α and corresponding performance state $k_{i\alpha}$ is,

$$MRAW_i = \frac{P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}]}{P[\phi(\mathbf{X}(t) \geq k]}.$$

We propose the joint importance measure, JRAW, of two components i and j of binary state system,

$$JRAW_{ij} = \frac{P_{11} - P_{10} - P_{01}}{P_{1.} + P_{.1}}$$

where

$P_{11} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1, X_j(t) = 1]$, $P_{10} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1, X_j(t) = 0]$, $P_{01} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0, X_j(t) = 1]$, $P_{1.} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1]$ and

$P_{.1} = P[\phi(\mathbf{X}(t)) = 1 | X_j(t) = 1]$, for measuring the joint reliability achievement worth due to interaction. The $JRAW_{ij}$ measure quantifies the maximum percentage increase in system reliability generated by the interaction of two components i and j. Note that $JRAW_{ij} = JRAW_{ji}$.

The multi-state extension of above measures for constant demand w_k corresponding to state k can be defined with respect to performance level α and β for two components i and j, as,

$$MJRAW_{ij} = \frac{P^{\geq\alpha,\geq\beta} - P^{\geq\alpha,<\beta} - P^{<\alpha,\geq\beta}}{P^{\geq\alpha,\cdot} + P^{\cdot,\geq\beta}}.$$

where

$$\begin{aligned}
P^{\geq\alpha,\geq\beta} &= P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) \geq k_{j\beta}] \\
P^{\geq\alpha,<\beta} &= P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) < k_{j\beta}] \\
P^{<\alpha,\geq\beta} &= P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) \geq k_{j\beta}] \\
P^{\geq\alpha,\cdot} &= P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}] \\
\text{and } P^{\cdot,\geq\beta} &= P[\phi(\mathbf{X}(t)) \geq k | X_j(t) \geq k_{j\beta}],
\end{aligned}$$

for measuring the joint reliability achievement worth due to interaction.

2.6 Joint Reliability Reduction Worth

The RRW is an index measuring the potential damage caused to the system by a particular component. The binary expression of the RRW of component i is

$$RRW_i = \frac{P[\phi(\mathbf{X}(t)) = 1]}{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0]}.$$

Then the extension of RRW to the multi-state case for constant demand w_k corresponding to state k can be defined, for the performance level α of component i and corresponding performance state $k_{i\alpha}$, as

$$MRRW_i = \frac{P[\phi(\mathbf{X}(t)) \geq k]}{P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}]}.$$

We propose the joint importance measure, JRRW, of two components i and j of binary state system,

$$JRRW_{ij} = \frac{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0] + P[\phi(\mathbf{X}(t)) = 1 | X_j(t) = 0]}{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0, X_j(t) = 0]}$$

for measuring the joint reliability reduction worth with respect to interaction of the components at below specified levels. The $JRRW_{ij}$ measure quantifies the potential damage caused to the system by interaction of two components i and j at below specified levels. Note that $JRRW_{ij} = JRRW_{ji}$.

The multi-state extension of JRRW for constant demand w_k corresponding to state k, can be defined for performance levels α and β of components i and j, as

$$MJRRW_{ij} = \frac{P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}] + P[\phi(\mathbf{X}(t)) \geq k | X_j(t) < n_{j\beta}]}{P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) < n_{j\beta}]}. \quad (2.16)$$

We next define the joint Fussel-Vesely measure for finding the maximum decrement in system reliability caused by joint effect of two components at below specified levels.

2.7 Joint Reliability Fussel-Vesely Measure

The FV importance measure quantifies the maximum decrement in system reliability caused by a particular component.

The binary expression of FV importance measure is

$$FV_i = \frac{P[\phi(\mathbf{X}(t)) = 1] - P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0]}{P[\phi(\mathbf{X}(t)) = 1]}.$$

It has extended to multi-state case for constant demand w_k corresponding to state k as

$$MFV_i = \frac{P[\phi(\mathbf{X}(t)) \geq k] - P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < x_{ik_{i\alpha}}]}{P[\phi(\mathbf{X}(t)) \geq k]}.$$

We propose the following joint importance measure, JRFV, for two components i and j of the binary state system, $JRFV_{ij} =$

$$\frac{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0] + P[\phi(\mathbf{X}(t)) = 1 | X_j(t) = 0] - P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0, X_j(t) = 0]}{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0] + P[\phi((\mathbf{X}(t)) = 1 | X_j(t) = 0]}$$

for measuring the joint reliability Fussel-Vesely importance with respect to interaction. The $JRFV_{ij}$ measure quantifies the maximum decrement in system reliability caused by joint effect of two components i and j at below specified levels. Note that $JRFV_{ij} = JRFV_{ji}$.

The multi-state extension of JRFV for constant demand w_k corresponding to state k can be defined, with respect to the performance levels α and β of components i and

j, as

$$MJFV_{ij} = \frac{P^{<\alpha,\cdot} + P^{\cdot,<\beta} - P^{<\alpha,<\beta}}{P^{<\alpha,\cdot} + P^{\cdot,<\beta}},$$

where $P^{<\alpha,\cdot} = P[\phi(\mathbf{X}(t)) \geq k|X_i(t) < k_{i\alpha}]$, $P^{\cdot,<\beta} = P[\phi(\mathbf{X}(t)) \geq k|X_j(t) < n_{j\beta}]$, and $P^{<\alpha,<\beta} = P[\phi(\mathbf{X}(t)) \geq k|X_i(t) < k_{i\alpha}, X_j(t) < n_{j\beta}]$.

Now we define the joint Birnbaum importance measure of any number of components with respect to reliability. The component Birnbaum importance measure is the most widely used importance measure by many reliability researchers, engineers and practitioners.

2.8 Joint Reliability Birnbaum Importance Measure

[2]The Birnbaum measure represents the maximum loss in the system reliability when element i switches from the condition of perfect functioning to the condition of certain failure. Let the state X_i of the ith binary component is random with probability $P[X_i = 1] = R_i = EX_i, i = 1, 2, ..., n$. The reliability of the binary system with

[2]Presented at *International Conference on New Trends in Statistics and Optimization*, University of Kashmir, Srinagar, Oct 21-23, 2008 and awarded **M. N. Gopalan Price** for Young Statisticians of *Indian Society for Probability and Statistics 2008*

structure function $\phi(\mathbf{X})$, $\mathbf{X} = (X_1, ..., X_n), \forall i,\ X_i, \phi \in \{0, 1\}$ is

$$P[\phi(\mathbf{X}) = 1] = h(\mathbf{p}) = E\phi(\mathbf{X}), \mathbf{p} = (R_1, ..., R_n).$$

Birnbaum (1969) proposed the following IM for the binary state system.

$$I(i) = \frac{\partial h}{\partial R_i} = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}) = E[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})], i = 1, 2, ..., n.$$

Clearly $I(i)$ describes the rate of improvement of system performance with respect to the improvement in performance of component i.

As an extension of Birnbaum measure to the multi-state case, Griffith (1980) defined the reliability importance of level l of the ith component of the MSS with structure function ϕ as

$$I_l(i) = E[\phi(l_i, \mathbf{X}) - \phi((l-1)_i, \mathbf{X})]$$

where $(l_i, \mathbf{X}) = (X_1, ..., X_{i-1}, l_i, X_{i+1}, ..., X_n)$, and $X_i \in \{0, 1, ..., M\}$, $i = 1, 2, ..., n$.

Multi-state joint reliability Birnbaum importance measure, MJRBI, of two components i and j with respect to performance levels α and β for the MSS can be defined as

$$\begin{aligned} MJRBI_{ij} = {} & P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) \geq n_{j\beta}] \\ & - P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) < n_{j\beta}] \\ & - P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) \geq n_{j\beta}] \\ & + P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) < n_{j\beta}]. \end{aligned} \tag{2.17}$$

It measures the improvement of system reliability due to the interaction effect of two components.

Proceeding like this we can introduce joint reliability Birnbaum importance measures for three components, four components etc. $MJRBI$ of three components i, j and l with respect to performance levels α, β and γ for the MSS can be defined as

$$MJRBI_{ijl} = MJRBI_{ij}(X_l(t) \geq m_{l\gamma}) - MJRBI_{ij}(X_l(t) < m_{l\gamma}) \qquad (2.18)$$

where $MJRBI_{ij}(X_l(t) \geq m_{l\gamma})$ is $MJRBI_{ij}$ when component l is above some predefined threshold γ with corresponding state $m_{l\gamma}$. Similar interpretation holds for $MJRBI_{ij}(X_l(t) < m_{l\gamma})$.

Now we redefine the above joint importance measures with general expression of OPM (reliability or availability)-for the MSS. Let component i be constrained to performance below α, while the rest of components of the MSS are not constrained: we denote by $OM_i^{\leq\alpha}$ the system OPM obtained in this situation. Similarly, we denote by $OM_i^{>\alpha}$ the system OPM resulting from the dual situation in which component i is constrained to performances above α. Also let $OM_{i,j}^{\leq\alpha,\leq\beta}$, $OM_{i,j}^{>\alpha,\leq\beta}$, $OM_{i,j}^{\leq\alpha,>\beta}$ and $OM_{i,j}^{>\alpha,>\beta}$ be the OPMs when both components i and j are restricted in their performance based on performance thresholds α and β respectively. We introduce the following measures for two components in a MSS with respect to performance measure-reliability or availability.

1. Joint Performance Achievement Worth:

$$MJPAW_{ij} = \frac{OM_{i,j}^{>\alpha,>\beta} - OM_{i,j}^{>\alpha,\leq\beta} - OM_{i,j}^{\leq\alpha,>\beta}}{OM_i^{>\alpha} + OM_j^{>\beta}}. \tag{2.19}$$

2. Joint Performance Reduction Worth:

$$MJPRW_{ij} = \frac{OM_i^{\leq\alpha} + OM_j^{\leq\beta}}{OM_{i,j}^{\leq\alpha,\leq\beta}}. \tag{2.20}$$

3. Joint Performance Fussel-Vesely Measure:

$$MJPFV_{ij} = \frac{OM_i^{\leq\alpha} + OM_j^{\leq\beta} - OM_{i,j}^{\leq\alpha,\leq\beta}}{OM_i^{\leq\alpha} + OM_j^{\leq\beta}}. \tag{2.21}$$

4. Joint Performance Birnbaum Importance:

$$MJPBI_{ij} = OM_{i,j}^{\alpha,>\beta} - OM_{i,j}^{\alpha,\leq\beta} \tag{2.22}$$

   where $OM_{i,j}^{\alpha,>\beta}$ represents the Birnbaum importance of the component i when component j is restricted to the performance above level β. Similarly $OM_{i,j}^{\alpha,\leq\beta}$ represents the Birnbaum importance of the component i when component j is restricted to below level β. Similarly we can find third order MJPBI measures by taking differences of MJPBI of two components after restricting the performance of third component below and above some pre-specified performance levels.

Thus we defined four main joint importance measures with respect to reliability and availability. But we can define the joint importance measures of above type with respect to risk also. We define joint risk importance measures in the following section.

2.9 Joint Risk Importance Measures

To compare the joint effect of pair of components with the standardly used risk, one can transform the performance measures into risk measures (unreliability or unavailability). In order to introduce the joint risk importance measures, we define the following indexes in terms of system risk.

$F_i^+(t)$, value of risk metric F when component i has been in state below a specified level throughout the time interval $[0, t]$.

$F_i^-(t)$, value of risk metric F when component i has been in its functioning state (above a specified level) throughout the time interval $[0, t]$.

The definition of the four of the risk importance measures for a system is recalled here with reference to the ith component, see Zio and Podofillini (2003) for details.

1. Birnbaum Risk Importance Measure: $rB_i(t) = F_i^+(t) - F_i^-(t)$, it measures the maximum deviation of risk when ith component shifts from its condition of perfect functioning to condition of certain failure.

2. Risk Achievement Worth (rAW): $rAW_i = \frac{F_i^+(t)}{F(t)}$, it is the ratio of risk when component i is considered always failed in $[0, t]$ to the actual value of risk.

3. Risk Reduction Worth (rRW): $rRW_i = \frac{F(t)}{F_i^-(t)}$, it is the ratio of the nominal value of risk to the risk when component i is always available. It measures the potential of component in reducing the risk, by considering the maximum decrease in risk achievable when optimizing the components to perfection.

4. Risk Fussel-Vesely Measure (rFV): $rFV_i(t) = \frac{F(t)-F_i^-(t)}{F(t)}$, it represents the maximum fractional decrement in risk achievable when component i is always available.

In order to introduce joint risk measures, multi-state joint Risk Birnbaum Importance measure (MJrBI), multi-state joint Risk Achievement Worth (MJrAW), multi-state joint Risk Reduction Worth (MJrRW), and multi-state joint Risk Fussel-Vesely measure (MJrFV), with reference to two components i and j, we define the following indexes in terms of system risk.

$F_{i,j}^{++}(t)$, value of risk metric F when both components i and j have been in state below some specified levels throughout the time interval $[0, t]$.

$F_{i,j}^{+-}(t)$, value of risk metric F when components, i has been in state below some specified level and j has been in state above some specified level, throughout the time interval $[0, t]$.

$F_{i,j}^{-+}(t)$, value of risk metric F when components, i has been in state above some specified level and j has been in state below some specified level, throughout the time interval $[0, t]$.

$F_{i,j}^{--}(t)$, value of risk metric F when both components i and j have been in state above some specified levels throughout the time interval $[0, t]$.

Now we define the multi-state joint risk importance measures to the MSS.

1. Multi-state Joint Risk Birnbaum measure:

$$MJrBI_{ij} = F_{i,j}^{++}(t) - F_{i,j}^{+-}(t) - F_{i,j}^{-+}(t) + F_{i,j}^{--}(t). \tag{2.23}$$

   It is the maximum variation in risk due to the joint effect of components i and j.

2. Multi-state Joint Risk Achievement Worth:

$$MJrAW_{ij} = \frac{F_{i,j}^{++}(t)}{F_i^{+}(t) + F_j^{+}(t)}. \tag{2.24}$$

   It is the ratio of risk when both components i and j is below some specified levels to the risk when either of two components is below some specified levels in $[0, t]$.

3. Multi-state Joint Risk Reduction Worth:

$$MJrRW_{i,j} = \frac{F_i^{-}(t) + F_j^{-}(t)}{F_{i,j}^{--}(t) - F_{ij}^{+-} - F_{ij}^{-+}(t)}. \tag{2.25}$$

It is the ratio of the nominal value of risk when either of two components i and j is available to the risk when both components are always available. It measures the interaction effect of two components in reducing the risk, by considering the maximum decrease in risk achievable with respect to joint effect of two components.

4. Multi-state Joint Risk Fussel-Vesely measure:

$$MJrFV_{ij} = \frac{F_i^-(t) + F_j^-(t) - F_{ij}^{--}(t)}{F_i^-(t) + F_j^-(t)}. \tag{2.26}$$

It represents the maximum fractional decrement in risk achievable when both of two components i and j are always available to the availability of either of two components.

The information about the interaction effect of two or more components in improving system performance can be drawn from the proposed joint importance measures in various different ways. Information about this type of interaction importance of components constituting a system, with respect to its safety, reliability, availability and risk, can be made useful in safety and redundancy operations. The degree of interactions between two or more components provide some guidelines to preference in safety operations to some groups of components. We cannot say one measure is better than the other, each of the measure has specific use, which will depends on the system engineers objective and use.

Chapter 3

A NEW CLASS OF MULTI-STATE COHERENT SYSTEMS

3.1 Introduction

[1]A binary state system (BSS) of n components can be described by a structure function $\phi : \{0,1\}^n \rightarrow \{0,1\}$, which presents the state of system as a function of states of its n components, see Barlow and Proschan (1975). A binary system is statistically coherent if it satisfies the following conditions;

[1]Some contents of this chapter have appeared in Chacko and Manoharan (2008b)

1. $\phi(\mathbf{x})$ is non-decreasing in each argument, where $\mathbf{x} = (x_1, ..., x_n)$ with $x_i \in \{0, 1\}$.

2. For each i, there exist a vector $(._i, \mathbf{x})$, such that $\phi(1_i, \mathbf{x}) > \phi(0_i, \mathbf{x})$.

Note that the condition (1) and (2) gives, $\phi(\mathbf{j}) = j$, $j = 0, 1$, and $\mathbf{j} = (j, .., j)$. There are various approaches which extends the structure function from the BSS case to the MSS case, see El.Neweihi et al. (1978) and Griffith (1980). The effort resulted in, requirement of the extension of non-decreasing binary structure function to MSS structure function. Also note the condition $\phi(\mathbf{j}) = j$, $j = 0, 1$ of the binary coherent system (BCS) is extended to multi-state coherent system (MCS) requiring $\phi(\mathbf{j}) = j, j \in \{0, 1, ..., M\}$. The condition (2) of relevancy in BCS is extended in various different ways. Some extensions can be seen in Natvig (1982), Meng (1993) and Abouammoh and Al-Khadi (1991, 1995).

This chapter is arranged as follows. A new component relevancy and the corresponding class of MCSs are proposed in section 2. The more general component relevancy and its new class of MCSs are also proposed. The importance and joint importance measures for the new classes are introduced. An illustrative example of electrical power generation system is given in section 3.

3.2 Component Relevancy and the New Class of MCSs

In this section we discuss the new relevance condition and its generalization on which the new classes of MCSs are defined. Consider the following existing component relevance conditions.

NAT: For every component i and level $j > 0$, there exist $(._i, \mathbf{x})$ such that $\phi(j_i, \mathbf{x}) \geq j$ and $\phi((j-1)_i, \mathbf{x}) < j$, see Natvig (1993).

GRI.1: For every component i and level $j > 0$, there exist $(._i, \mathbf{x})$ such that $\phi(j_i, \mathbf{x}) > \phi((j-1)_i, \mathbf{x})$, see Griffith (1980).

GRI.2: For every component i, there exist $(._i, \mathbf{x})$ such that $\phi(0_i, \mathbf{x}) < \phi(M_i, \mathbf{x})$, see Griffith (1980).

EP: For every component i and level $j \geq 1$, there exist $(._i, \mathbf{x})$ such that $\phi(j_i, \mathbf{x}) > \phi(0_i, \mathbf{x})$, see El.Neweihi and Proschan (1984).

NAT and GRI.1 indicate degree of relevance of each component to every level of performance; while GRI.2 merely states that $\phi(\mathbf{x})$ is not a constant in any of its arguments.

Now we consider a situation in which some component is not relevant to every level of performances, i.e., system degrades from state j to $j-1$ when component degrade only from state j to $j-2$ or $j-3$ etc. In order to degrade system, component must degrade more than one level of performance. For example, see Natvig et. al. (1986), let $S = \{0, 1, 2, 3, 4\}$, the component takes 0, 2 and 4 when system can take 0, 1, 2, 3 and 4. Consider the structure function ϕ_2 having 5 components, see eq.(3.13). We have, $\phi_2(4_1, 4_2, 2_3, 4_4, 2_5) = 4 > \phi_2(4_1, 4_2, 2_3, 2_4, 2_5) = 3$, from the minimal path vectors of ϕ_2, when the 4th component degrades from state 4 to state 2 the system degrades from state 4 to state 3. Now consider the structure function ϕ_1 with three components, see eq.(3.12). We have, $\phi_1(4_1, 0_2, 4_3) = 4 > \phi_1(4_1, 0_2, 2_3) = 2$, when the third component degrade from state 4 to state 2, the system degrades from state 4 to state 2. Here fourth component must degrade from state 4 to state 2 for the system to degrade from state 4 to state 3 with respect to ϕ_2. The third component must degrade from state 4 to state 2 for the system to degrade with respect to ϕ_1.

We define a new component relevance condition as, degradation of a component from state j to state $j-2$ cause system failure or degradation while degradation of the component from state j to $j-1$ cannot cause system failure or degradation. Now the new class of MCSs, say $CM.1$ class, can be defined using this relevance condition.

Definition 3.2.1 *A MCS of n components with structure function ϕ belonging to class $CM.1$ if ϕ is non-decreasing, $\phi(\mathbf{j}) = j$, and for each component i, there exist*

$(\cdot_i, \boldsymbol{x})$ *such that* $\phi(j_i, \boldsymbol{x}) > \phi((j-2)_i, \boldsymbol{x})$.

Now consider the generalization of the new relevance condition, one or more than one level of degradation of the component cause system degradation, i.e., when the component 'i' degrades from j to $j' \in \{j-1, j-2, ..., 1, 0\}$, the system degrades from state j to any lower state. Thus we define the generalized class of MCSs, say $CM.2$ class, with this relevance condition.

Definition 3.2.2 *A MCS of n components with structure function ϕ belonging to class $CM.2$ if ϕ is non-decreasing, $\phi(\mathbf{j}) = j$, and for each component i, there exist $(\cdot_i, \boldsymbol{x})$ such that $\phi(j_i, \boldsymbol{x}) > \phi(j'_i, \boldsymbol{x}), j' \in \{j-1, j-2, ..., 2, 1, 0\}$.*

Now we introduce component importance and joint importance measures to the new classes of MCSs. We consider the problem of measuring the structural importance and reliability importance of individual components, and joint structural importance and joint reliability importance of two or more components in the new classes of the MCSs. The main advantage of defining a new relevance condition is to obtain the importance measures. At the reliability design phase, the joint importance can improve system designer's understanding of the relationship between components and system and among the components, which is quite desirable.

Now define the structural definition of the component importance (when reliabilities of components are not given) with respect to the new relevance conditions.

Consider $\phi_j(\mathbf{x}) = 1$ if $\phi(\mathbf{x}) \geq j$ and 0 otherwise. We define the structural importance of a component as follows.

Definition 3.2.3 *Let $\phi : S^n \to S$ be the MCS structure function in CM.1 or CM.2 class. Then ϕ is said to have the following measures of structural importance for the level j of component i:*

$$I_{ij}(CM.1) = \frac{1}{(M+1)^{n-1}} \sum_{\mathbf{x}:x_i=j} Max\{0, \phi_j(j_i, \mathbf{x}) - \phi_j((j-2)_i, \mathbf{x})\}$$

$$I_{ij}(CM.2) = \frac{1}{(M+1)^{n-1}} \sum_{\mathbf{x}:x_i=j} Max\{0, \phi_j(j_i, \mathbf{x}) - \phi_j((j')_i, \mathbf{x})\}, j' \in \{j-1, j-2, ..., 0\}.$$

In order to define the joint importance measures for two or more components in the new classes of MCSs, we use the JSIM in previous chapter, see (2.4). Then JSIM for three components i, l, and r, with $\hat{m} = m-2$, $\hat{n} = n-2$ and $\hat{k} = k-2$, in $CM.1$ class is

$$JSIM_{CM.1}(i,l,r) = \sum_{n=1}^{M} \sum_{k=1}^{M} \sum_{m=1}^{M} \{JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,k,n-2). \tag{3.1}$$

Similarly, the JSIM for three components i, l, and r, with $\hat{m} = m' \in \{m-1, ..., 1, 0\}$, $\hat{n} = n' \in \{n-1, ..., 1, 0\}$ and $\hat{k} = k' \in \{k-1, ..., 1, 0\}$, in $CM.2$ class is

$$JSIM_{CM.2}(i,l,r) = \sum_{n=1}^{M} \sum_{k=1}^{M} \sum_{m=1}^{M} \{JSIM(i,l,r;m,k,n) - JSIM(i,l,r;m,k,n')\}. \tag{3.2}$$

Thus we can find JSIM of any number of components w.r.t. both relevance conditions in the MCS classes $CM.1$ and $CM.2$.

Now we consider $\mathbf{X}$=$(X_1, ..., X_n)$ as a random vector of component states. For the BSS with structure function ϕ, the Birnbaum reliability importance of component i is

$$I(i) = P[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1] = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})$$

where $h(\mathbf{p}), \mathbf{p} = (R_1, R_2, ..., R_n)$, is the reliability function of the BCS,

$$h(\mathbf{p}) = R_i h(1_i, \mathbf{p}) + (1 - R_i)h(0_i, \mathbf{p}) = R_i I(i) + h(0_i, \mathbf{p}).$$

Therefore, $I(i) = \frac{\partial h(\mathbf{p})}{\partial R_i}$. It is the major importance measure used in reliability analysis of the binary state system, to order the components according to the their contribution to system reliability. But it has extended to multi-state case, see El.Neweihi et al. (1978) and Griffith (1980). But in that definitions of importance measures, researchers used the existing relevance conditions. When we consider the new classes, the relevance conditions would be CM.1 and CM.2. It overcome the draw backs of the existing importance measures to order the components according to their contribution to system reliability in some systems.

We propose the following component importance measures for the new MCSs.

$$I_i(CM.1) = P[\phi(j_i, \mathbf{X}) > \phi((j-2)_i, \mathbf{X})], \tag{3.3}$$

$$I_i(CM.2) = P[\phi(j_i, \mathbf{X}) > \phi(j'_i, \mathbf{X})],\ j' \in \{j-1, j-2, ..., 1, 0\}. \tag{3.4}$$

Let the distribution of X_i be described by $\overline{\mathbf{p}}_i = (p_{i0}, p_{i1}, ..., p_{iM})$ where $p_{ij} = P[X_i = j]$, $i = 1, 2, ..., n,\ j = 0, 1, ..., M$. The reliability function of the MCS with minimum satisfactory performance level j, is

$$P[\phi(\mathbf{X}) \geq j] = \sum_{j \in S} p_{ij} P[\phi(j_i, \mathbf{X}) \geq j],$$

since $p_{i0} + p_{i1} + ... + p_{iM} = 1$.

Now we prove the following theorems. The theorem says that the rate of improvement of system reliability with respect to p_{ij} will be the importance measures $I_i(CM.1)$ and $I_i(CM.2)$ when the relevance conditions are CM.1 and CM.2.

Theorem 3.2.1 *For the CM.1 class, $I_i(CM.1)$ is the rate of improvement of $P[\phi(\boldsymbol{X}) \geq j]$ with respect to p_{ij}.*

Proof: Clearly

$$P[\phi(\mathbf{X}) \geq j] = \sum_{j \in S/\{j-2\}} p_{ij}\{P[\phi(j_i, \mathbf{X}) \geq j] - P[\phi((j-2)_i, \mathbf{X}) \geq j]\} + P[\phi((j-2)_i, \mathbf{X}) \geq j] \tag{3.5}$$

since $1 - p_{ij-2} = p_{i0} + p_{i1} + ... + p_{ij-3} + p_{ij-1} + ... + p_{iM}$.

Differentiating (3.5) partially with respect to p_{ij}, we get

$$\begin{aligned} \frac{\partial P[\phi(\mathbf{X}) \geq j]}{\partial p_{ij}} &= P[\phi(j_i, \mathbf{X}) \geq j] - P[\phi((j-2)_i, \mathbf{X}) \geq j] \\ &= P[\phi((j-2)_i, \mathbf{X}) < \phi(j_i, \mathbf{X})] \\ &= I_i(CM.1). \square \end{aligned} \tag{3.6}$$

Theorem 3.2.2 *For the CM.2 class, $I_i(CM.2)$ is the rate of improvement of $P[\phi(\boldsymbol{X}) \geq j]$ with respect to p_{ij}.*

Proof: Clearly

$$P[\phi(\mathbf{X}) \geq j] = \sum_{j \in S/\{j'\}} p_{ij}\{P[\phi(j_i, \mathbf{X}) \geq j] - P[\phi((j')_i, \mathbf{X}) \geq j]\} + P[\phi((j')_i, \mathbf{X}) \geq j] \tag{3.7}$$

since $1 - p_{ij'} = p_{i0} + p_{i1} + ... + p_{ij'-1} + p_{ij'+1} + ... + p_{iM}$.

Differentiating (3.7) partially with respect to p_{ij}, we get

$$\begin{aligned} \frac{\partial P[\phi(\mathbf{X}) \geq j]}{\partial p_{ij}} &= P[\phi(j_i, \mathbf{X}) \geq j] - P[\phi((j')_i, \mathbf{X}) \geq j] \\ &= P[\phi((j')_i, \mathbf{X}) < \phi(j_i, \mathbf{X})] \\ &= I_i(CM.2). \square \end{aligned} \tag{3.8}$$

Now we consider the joint reliability importance of k components with respect to the new classes of MCSs. We need only to put $\hat{b}_1 = b_1 - 2$, $\hat{b}_2 = b_2 - 2$,..., $\hat{b}_k = b_k - 2$ in (2.14) in the case of CM.1 class and $\hat{b}_1 = b'_1$, $\hat{b}_2 = b'_2$,..., $\hat{b}_k = b'_k$ in (2.14) where $b'_j \in \{b_j - 1, b_j - 2, ..., 1, 0\}$ in the case of CM.2 class. The usefulness of above JRIM is in power generation system, network systems etc.

Now consider some implications based on the new relevance definitions. In fact one can easily prove the following implications.

Theorem 3.2.3 *$CM.1 \Rightarrow EP$, $CM.2 \Rightarrow NAT \Rightarrow GRI.1 \Rightarrow GRI.2 \Rightarrow EP$, $CM.2 \Rightarrow GRI.1$, $CM.2 \Rightarrow GRI.2$ and $CM.2 \Rightarrow EP$.*

It is clear that all the existing relevance conditions are special cases of $CM.2$ (or $CM.1$) relevance condition. Hence the existing MCSs are special cases of the proposed MCSs.

3.3 Example: Power Generation System

We use the Offshore electrical power generation system considered by Natvig et. al. (1986) as an illustrative example. The purpose of this system is to supply two nearby oilrings with electrical power. Both oilrings have their own main generation, represented by equivalent generators A_1 and A_3 each having capacity of 50MW. In addition the oilrings have a standby generator A_2 that is switched into the network in case of outage of A_1 or A_3, or may be used in extreme load situations in either of the oilrings. The A_2 also has capacity 50MW. The control unit, U, continuously supervises the supply from each of the generators with automatic control of the switches. If for instance the supply from A_3 to oilring 2 is not sufficient, whereas the supply from A_1 to oilring 1 is sufficient, U can activate A_2 to supply oilring 2 with electrical power through the subsea cables L. The components have states 0, 2, 4 and the system has states 0, 1, 2, 3, 4, where 0, 1, 2, 3 and 4 represents the states of system at capacities

0MW, 12.5MW, 25MW, 37.5MW, and 50MW respectively.

The minimal path vectors to the levels are given in table 1 and table 2, of the structure functions,

$$\phi_1(U, A_1, A_2) = I(U > 0)min(A_1 + A_2 I(U = 4), 4),$$

the amount of power that can be supplied to platform 1, and

$$\phi_2(U, A_1, L, A_2, A_3) = I(U > 0)min(A_3 + A_2 I(U = 4)I(A_1 = 4)L/4, 4),$$

the amount of power that can be supplied to platform 2, when $I(.)$ is the indicator function. It is clear from the tables the relevancy of each component in two of the structure functions. The MSS with relevancy $CM.2$ is clearly defined in this example.

Table 1. Minimal path vectors of ϕ_1

Levels	U	A_1	A_2
2	2	2	0
2	4	0	2
4	2	4	0
4	4	0	4
4	4	2	0

Table 2. Minimal path vectors of ϕ_2

Levels	U	A_1	L	A_2	A_3
1	4	4	2	2	0
1,2	2	0	0	0	2
2	4	4	2	4	0
2	4	4	4	2	0
3	4	4	2	2	2
3,4	2	0	0	0	4
4	4	4	2	4	2
4	4	4	4	2	2
4	4	4	4	4	0

We modeled a MSS based on two important relevance conditions, which have application to many engineering systems as it give importance and joint importance measures. The proposed classes of MCSs contains several existing classes. Hence CM.1 class and CM.2 class have special importance. The example of power generation system is a member of CM.2 class, because of relevancy of its components.

Chapter 4

AGEING PROPERTIES OF SEMI-MARKOV SYSTEM

4.1 Introduction

[1] In this chapter, we are concerned with a MSS having $M+1$ states $0, 1, ..., M$ where '0' is the best state and 'M' is the worst state (for convenience). At time zero the system begins at its best state and as time passes system begins to deteriorate. It is assumed that the time spent by the system in each state is random with arbitrary sojourn time distribution. The system stays in some acceptable states for some time

[1]Some contents of this chapter have appeared in Chacko and Manoharan (2009b)

and then it moves to unacceptable (down) state. The first time at which the MSS enters the down state after spending a random amount of time in acceptable states is termed as the first passage time (failure time) to the down state of the MSS.

We study the aging properties of the first passage time distribution of the MSS modeled by the semi-Markov process $\{Y(t), t \geq 0\}$. In the MSS with $M+1$ states $\{0, 1, ..., k-1, k, k+1, ..., M\}$ where $\{0, 1, ..., k-1, k\}$ is the acceptable states, the sojourn time between state 'i' to state 'j' is assumed to be distributed with arbitrary distribution F_{ij}. Markov and semi-Markov modeling of a MSS is given in Lisnianski and Levitin (2003). Our aim is to derive a necessary and sufficient condition for a MSS failure time distribution to be IFR and IFRA and to highlight some potential applications. Deshpande et al. (1986) and Barlow and Proschan (1975) described various aspects of positive aging in terms of conditional probability distributions of residual lifetimes and failure rates. Bryson and Siddiqui (1969) discussed the concept of 'aging' or progressive shortening of an entity's residual lifetime in terms of survival time distribution.

Let F be the c. d. f. of a continuous random variable T representing lifetime of a unit. Then

$$R(t) = 1 - F(t) = \bar{F}(t) = P[T > t]$$

is called its reliability function (or survival function) and

$$R_x(t) = \frac{R(t+x)}{R(t)}$$

is the survival function of a unit of age t, i.e., conditional probability that a unit of age 't' will survive for an additional 'x' unit of time. Obviously, any study of the phenomenon of aging/ no aging (i.e., age has no effect on the residual life time) has to be based on $R_x(t)$ and functions related to it. Following are the definitions of IFR, DFR, IFRA and DFRA distributions, see Barlow and Proschan (1975).

Definition 4.1.1 *Increasing failure rate (IFR) distribution: F is IFR if*

$$R_x(t_1) \geq R_x(t_2), \quad x \geq 0, \quad 0 \leq t_1 \leq t_2 < \infty.$$

Definition 4.1.2 *Increasing failure rate average (IFRA) distributions: F is IFRA if* $-\frac{1}{t} log R(t)$ *is increasing in* t *or equivalently* F *is said to be IFRA if* $(R(t))^{1/t}$ *is decreasing in* t.

Definition 4.1.3 *Decreasing failure rate (DFR) distribution: F is DFR if*

$$R_x(t_1) \leq R_x(t_2), \quad x \geq 0, \quad 0 \leq t_1 \leq t_2 < \infty.$$

Definition 4.1.4 *Decreasing failure rate average (DFRA) distributions: F is DFRA if* $-\frac{1}{t} log R(t)$ *is decreasing in* t *or equivalently* F *is said to be DFRA if* $(R(t))^{1/t}$ *is increasing in* t.

The remaining sections of this Chapter are arranged as follows. The first passage time and its distribution of a semi-Markov system is described in section 2. The necessary and sufficient conditions of ageing properties, IFR, DFR, IFRA and DFRA, of semi-Markov system are proved in section 3. Some applications and examples are given last section.

4.2 First Passage Time of Semi-Markov System

For a continuous time Markov process $\{X(t), t \geq 0\}$ with state space $\mathbf{S}$, a countable set with a partial ordering, and transition matrix $\mathbf{P}$, we say the Markov process is of monotone paths if $P(X(t) > X(s)) = 1$ for $t > s$. Define D a subset of $\mathbf{S}$ to be an increasing set if $i \in D$ and $j \geq i \Rightarrow j \in D$. This Markov process is stochastically monotone if and only if $i \leq j \Rightarrow P(i, D) \leq P(j, D)$ for all increasing sets D. For a state i and a set D define $T_D(i)$ to be first passage time from the state i to D, with $T_D(i) = 0$ if $i \in D$ and $T_D(i) = \infty$ if D is never reached. Brown and Chaganty (1983) proved that, if $\{X_n, n \geq 0\}$ is a stochastically monotone Markov chain with monotone paths on the partially ordered countable set $\mathbf{S}$, and D is an increasing set with the complement of D in $\mathbf{S}$ finite, then $T_D(i)$, the first passage time from state i to set D, is IFRA.

Let $E = \{0, 1, 2, ..., M\}$ be a set representing the state of the MSS and probability space with probability function P, on which we define a bivariate time homogeneous Markov chain $(X, T) = \{X_n, T_n, n \in \{0, 1, 2, ...\}\}$, X_n takes values of E and T_n on the half real line $R^+ = [0, \infty)$, with $0 \leq T_1 \leq T_2 \leq ... \leq T_n \leq ....$ Put $U_n = T_n - T_{n-1}$ for all $n \geq 1$. This Markov process is called a Markov renewal process (MRP) with transition function, the semi-Markov kernel, $\mathbf{Q} = [Q_{ij}]$, where

$$Q_{ij}(t) = P[X_{n+1} = j, U_n \leq t | X_n = i], i, j \in E, t \geq 0$$

and $Q_{ii}(t) = 0, i \in E, t \geq 0.$

Now we consider the semi-Markov process (SMP), as defined in Pyke (1961). It is the generalization of Markov process with countable state space. SMP is a stochastic process which moves from one state to another of a countable number of states with successive states visiting form a Markov chain, and that the process stays in a given state a random length of time, the distribution of which may depend on this state as well as on the one to be visited in the next. Let $N(t) = sup\{n, T_n = U_1 + \cdots + U_n \leq t\}$, define $Z(t) = X_{N(t)}$, it is the semi-Markov process associated with the MRP defined above. In terms of Z, the times T_1, T_2,... are successive times of transitions for Z, and X_0, X_1, X_2,...are successive states visited. If elements of $\mathbf{Q}$ have the form

$$Q_{ij}(t) = P[X_{n+1} = j | X_n = i][1 - e^{-\lambda(i)t}], i, j \in E, t \geq 0,$$

for some function $\lambda(i), i \in E$, then the process $Z(t)$ is a Markov process. That is, in a Markov process, the distributions of the sojourn times are all exponential independent of the next state. The word *semi*-Markov comes from the somewhat limited Markov property which Z enjoys, namely, that the future of Z is independent of its past given the present state provided the 'present' is the time of jump. Limnios (1997) obtained the reliability of a semi-Markov system.

Let I_{ij} =indicator function of $\{i = j\}$. Define the transition probability that system occupied state $j \in E$ at time $t > 0$, given that it is started at state i at time zero, as, $\forall i, j, k \in E$, $t > 0$,

$$p_{ij}(t) = P[Z(t) = j|Z(0) = i] = P[X_{N_t} = j|X_0 = i] = h_i(t)I_{ij} + \mathbf{Q} * \mathbf{P}(t)(i, j),$$

where $h_i(t) = 1-\sum_k Q_{ik}(t), \mathbf{P}(t) = [p_{ij}(t)]$ and $\mathbf{Q} * \mathbf{P}(t)(i, j) = \sum_k \int_0^t Q_{ik}(dx)p_{kj}(t- x)$.

To obtain the reliability function of the semi-Markov system described above, we must define a new process, Y with state space $U \bigcup \{\nabla\}$, where U denotes set of all up states $\{0, 1, ..., k\}$ and ∇ is the absorbing state in which all the states $\{k + 1, ..., M\}$ of the system is united. Let T_D denote the time of first entry to the down states of Z process.

That is, $Y(t) = Z(t)(\omega)$ if $t < T_D(\omega)$ and $Y(t) = \nabla$ if $t \geq T_D(\omega)$.

Let $1 = (1, 1, ..., 1)^T$, a unit row vector with appropriate dimension. The process $Y(t)$ is a semi-Markov process with semi-Markov kernel

$$\begin{pmatrix} \overbrace{Q_{11}(t)}^{Up} & \overbrace{Q_{12}(t)}^{Down} \\ 0 & 0 \end{pmatrix}$$

We denote $\alpha = (\overbrace{\alpha(0), ..., \alpha(k)}^{up,\alpha_1}, \overbrace{\alpha(k + 1), ..., \alpha(M)}^{down,\alpha_2})$ where $\alpha(i) = P[Y(0) = i]$.

The reliability function is

$$
\begin{aligned}
R(t) = P[\forall u \in [0,t], Z(u) \in U] = P[Y(t) \in U] = \sum_{j \in U} P[Y(t) = j] \\
= \sum_{i \in U} \sum_{j \in U} P[Y(t) = j, Y(0) = i] = \sum_{i \in U} \sum_{j \in U} P[Y(t) = j | Y(0) = i] P[Y(0) = i] \\
= \sum_{i \in U} \sum_{j \in U} p_{ij}(t) \alpha(i).
\end{aligned}
$$

4.3 Aging Properties of Semi-Markov System

The sojourn time of the MSS in each state or from one state to another in a semi-Markov setup is a random variable. Consider the random lifetime of the MSS, T_D, the first passage time to the down state from upstate U, with distribution F. In the following we assume that $\forall i, j \in U, p_{ij}(t)$ is either monotone increasing or decreasing in t.

The following theorem give a necessary and sufficient condition for the distribution of semi-Markov system to be IFR.

Theorem 4.3.1 *For a semi-Markov system with monotone decreasing transition probability functions, and first passage time distribution F, the following statements are* *equivalent:*

$$(a) F \text{ is IFR}$$

$$(b) \sum_{i,j\in U} p'_{ij}(t+x)\alpha(i) \leq \sum_{i,j\in U} p'_{ij}(t)\alpha(i), t \geq 0.$$

Proof: In the semi-Markov setup described above, if

$$R_x(t) = \frac{R(t+x)}{R(t)} = \frac{\sum_{i,j\in U} p_{ij}(t+x)\alpha(i)}{R(t)}$$

is decreasing in t then the rate of decrease of

$$\sum_{i,j\in U} p_{ij}(t+x)\alpha(i)$$

is larger than the rate of decrease of $R(t)$. Therefore if $R_x(t)$ is decreasing,

$$\sum_{i,j\in U} p'_{ij}(t+x)\alpha(i) \leq \sum_{i,j\in U} p'_{ij}(t)\alpha(i), t \geq 0.$$

Conversely suppose that (b) holds, then the rate of decrease of

$$\sum_{i,j\in U} p_{ij}(t+x)\alpha(i)$$

is larger than rate of decrease of $R(t)$. Then obviously we have $R_x(t)$ is decreasing in t, which implies that F is IFR. □

However for a DFR distribution F the 'rate of increase' of

$$\sum_{i,j\in U} p_{ij}(t+x)\alpha(i)$$

does not affect that of $R_x(t)$. It is easy to prove $R_x(t)$ is increasing if and only if $\forall i, j \in U, p_{ij}(t+x)$ is increasing in t, because $1/R(t)$ is an increasing function of t

and product of two increasing functions, $\sum_{i,j\in U} p_{ij}(t+x)\alpha(i)$ and $1/R(t)$, is again an increasing function. Hence we have the following theorem.

Theorem 4.3.2 *For a semi-Markov system with monotone increasing transition probability functions, and first passage time distribution F, F is DFR if and only if $\forall i, j \in U, p_{ij}(t+x)$ increasing in t.*

Now we prove a necessary and sufficient condition for the IFRA property of first passage time distribution of the semi-Markov system.

Theorem 4.3.3 *For a semi-Markov system with monotone decreasing transition probability functions, and first passage time distribution F, the following two statements are equivalent:*

$$(a) F \text{ is IFRA}$$

$$(b) t^2 \sum_{i,j\in U} p_{ij}^{'}(t)\alpha(i) \le -1, t \ge 0.$$

Proof: Suppose that, F is IFRA. Then $(R(t))^{1/t}$ is decreasing in t. But

$$(R(t))^{1/t} = (\sum_{i,j\in U} p_{ij}(t)\alpha(i))^{1/t}$$

is decreasing in t only when rate of decrease of $\sum_{i,j\in U} p_{ij}(t)\alpha(i)$ is larger than rate of decrease of $1/t$. That is,

$$\sum_{i,j\in U} p_{ij}^{'}(t)\alpha(i) \le -\frac{1}{t^2}, t \ge 0,$$

equivalently,

$$t^2 \sum_{i,j \in U} p_{ij}^{'}(t)\alpha(i) \leq -1, t \geq 0.$$

Conversely suppose that (b) holds, then $\sum_{i,j \in U} p_{ij}(t)\alpha(i)$ is decreasing at a greater rate than $1/t$, so that $(R(t))^{1/t}$ is a decreasing function of t. Hence, first passage time distribution F of the semi-Markov system is IFRA. □

On a similar lines we prove a necessary and sufficient condition for the DFRA property of first passage time distribution of the semi-Markov system.

Theorem 4.3.4 *For a semi-Markov system with monotone increasing transition probability functions, and first passage time distribution F, the following two statements are equivalent:*

$(a) F$ *is DFRA*

$$(b) \sum_{i,j \in U} p_{ij}^{'}(t)\alpha(i) \geq \sum_{i,j \in U} p_{ij}(t)\alpha(i), t \geq 0.$$

Proof: Suppose that, F is DFRA. Then $(R(t))^{1/t}$ is increasing in t. Now consider the logarithmic transformation of $(R(t))^{1/t}$.

$$log(R(t))^{1/t} = \frac{log(\sum_{i,j \in U} p_{ij}(t)\alpha(i))}{t}$$

is increasing in t only when rate of increase of $log(\sum_{i,j \in U} p_{ij}(t)\alpha(i))$ is larger than rate of increase of t. That is,

$$\frac{\sum_{i,j \in U} p_{ij}^{'}(t)\alpha(i)}{\sum_{i,j \in U} p_{ij}(t)\alpha(i)} \geq 1, t \geq 0,$$

equivalently,

$$\sum_{i,j\in U} p'_{ij}(t)\alpha(i) \geq \sum_{i,j\in U} p_{ij}(t)\alpha(i), t \geq 0.$$

Conversely suppose that (b) holds, then $log(\sum_{i,j\in U} p_{ij}(t)\alpha(i))$ is increasing at a greater rate than increase of t, so that $(R(t))^{1/t}$ is a increasing function of t. Hence, first passage time distribution F of the semi-Markov system is DFRA. $\square$

4.4 Applications

Major application of the above results is in maintenance policies such as age and block replacement policies. A variety of applications of IFR, DFR, IFRA distributions in maintenance policies of a binary system can be seen in Barlow and Proschan (1996). Under the IFR property the expected number of failures will be less under block replacement than under age replacement. When we identify the distribution of semi-Markov system is IFR or DFR, it will be easy to employ suitable maintenance policies according to the above theorems. We consider some examples that arise in practical applications such as power generation system with multi-state performance levels.

Example 4.4.1 *Consider a Markov process in continuous time and discrete state space* $\{1, 2, ..., M\}$, *given in Doob (1953), p.241.* The system start in state '1' at time zero and as it enters 'M', it remains there. Consider the intensity matrix, $\mathbf{Q} = [Q_{ij}]$, with entries $q_{ij} = 0, i \in \{1, 2, ..., M-1\}, j \neq i+1, q_{ii+1} = q$, and $q_M = 0$. The

Kolmogorov's system of differential equation becomes,

for $p_{ij}(t) = P(Y(t) = j|Y(0) = i)$,

$$p'_{ik}(t) = -qp_{ik}(t) + qp_{i+1k}(t), i < M$$

$$p'_{Mk}(t) = 0$$

with initial conditions, $p_{ik}(0) = \delta_{ik}$, the indicator of $\{i = k\}$. Then, $p_{Mk}(t) = 0$, $k \neq M$, $p_{MM}(t) = 1$ and it is easily verified that the solution is

$$\begin{aligned} p_{ik}(t) &= 0 & k < i \\ &= \frac{(qt)^{k-i}e^{-qt}}{(k-1)!}, & i \leq k < M \\ &= e^{-qt}[e^{qt} - 1 - qt - \cdots - \frac{(qt)^{M-i-1}}{(M-i-1)!}], & k = M \end{aligned}$$

which is increasing initially (i. e., system is DFR) for $t < t_0$, where t_0 is the time at which $p'_{ik}(t) = 0$, (i.e., the time at which $p_{ik}(t) = p_{i+1k}(t)$) and then decreasing in $t > t_0$ (i.e., system is not DFR) for $k \in \{1, 2, ..., M-1\}$, the set of acceptable states. Here the process is of monotone paths.

Example 4.4.2 *Consider a continuous time Markov process* $\{X(t), t \geq 0\}$ *with state space* $\{0, 1, ..., M\}$ *and* $Y(0) = 0$, *such that the process stays in state* i *for a random length of time whose distribution is exponential with mean* $1/\lambda_i$ *then moves to state* $(i+1)$, *this continues until down state* M *is reached. Consider the intensity matrix*

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & . & . \\ . & . & . & . \\ 0 & 0 & -\lambda_{M-1} & \lambda_{M-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The last row $(0, 0, ..., 0)$ means that state $'M'$ is absorbing. Bhat (2000), p.197, obtained the forward Kolmogorov's differential equation, with initial conditions $p_{00}(0) = 1$, as

$$p_{0k}^{'}(t) = -\lambda_k p_{0k}(t) + \lambda_{k-1} p_{0k-1}(t), 0 \leq k \leq M - 1.$$

Then,

$$p_{0k}(t) = \lambda_{k-1} e^{-\lambda_k t} \int_0^t e^{\lambda_k x} p_{0k-1}(x) dx, k = 1, 2, ..., M - 1.$$

The nature of the above transition probability functions shows the first passage time is IFRA or IFR or DFR or no aging.

For a numerical realization, Lisnianski and Levitin (2003), p.145, considered an electric generator that has four possible performance levels 100MW (state 0), 70MW (state 1), 50MW (state 2) and 0MW (state 3). The constant demand is 60MW. The best state with performance rate 100MW is the initial state. Times to failures are distributed exponentially with parameters, $\lambda_{0,1} = 10^{-3}$ (hours(-1)), $\lambda_{1,2} = 5.10^{-3}$ (hours(-1)), and $\lambda_{2,3} = 2.10^{-3}$ (hours(-1)). Hence, times to failures $T_{0,1}$, $T_{1,2}$ and $T_{2,3}$ are random variables distributed according to the c.d.f., $F_{0,1}(t) = 1 - e^{-\lambda_{0,1} t}$,

$F_{1,2}(t) = 1 - e^{-\lambda_{1,2}t}$ and $F_{2,3}(t) = 1 - e^{-\lambda_{2,3}t}$, for $t > 0$. The system is having monotone paths. Thus,

$$p_{0,1}(t) = \lambda_{0,1}e^{-\lambda_{1,2}t}\int_0^t e^{\lambda_{1,2}u}p_{0,0}(u)du = \frac{\lambda_{0,1}}{\lambda_{1,2}} = 0.2,$$

$$p_{0,2}(t) = \lambda_{1,2}e^{-\lambda_{2,3}t}\int_0^t e^{\lambda_{2,3}x}\lambda_{0,1}e^{-\lambda_{1,2}x}\int_0^x e^{\lambda_{1,2}u}p_{0,0}(u)dudx = \frac{\lambda_{0,1}}{\lambda_{2,3}} = 0.5,$$

This means that with exponentially distributed sojourn times, the system failure time distribution is IFR as well as DFR, no ageing property of MSS.

On the other hand, when we consider the Weibull distribution for the sojourn times, we can expect specific IFR or DFR property of first passage time distribution. Let $F_{0,1} = (1 - e^{-\lambda_{0,1}t})^{\theta_0}$, $F_{1,2} = (1 - e^{-\lambda_{1,2}t})^{\theta_1}$ and $F_{2,3} = (1 - e^{-\lambda_{2,3}t})^{\theta_2}$, for $t > 0$. Let state 0 and 1 be the up states and 2 and 3 be the down states.

$$p_{01}(t) = \theta_0\lambda_{0,1}\int_0^t (1 - e^{-\lambda_{0,1}x})^{\theta_0-1}(1 - (1 - e^{-\lambda_{1,2}x})^{\theta_1})dx$$

$$p_{02}(t) = \theta_0\lambda_{0,1}\int_0^t (1 - e^{-\lambda_{0,1}x})^{\theta_0-1}\int_0^x \theta_1\lambda_{1,2}(1 - e^{-\lambda_{1,2}u})^{\theta_1-1}(1 - (1 - e^{-\lambda_{2,3}u})^{\theta_2})dudx$$

For $\lambda_{0,1} = 2$, $\lambda_{1,2} = 3$, $\theta_0 = 3$, $\theta_1 = 3$ and $\theta_2 = 3$, the functions $p_{01}(t)$ and $p_{02}(t)$, where

$$\begin{aligned} p_{01}(t) =& k_1((-e^{10t} + \frac{3}{10}e^{3t} + \frac{1}{2}e^{7t} - \frac{1}{13} - \frac{1}{9}e^{4t} + \frac{6}{5}e^{8t} - \frac{3}{4}e^{5t} + \frac{2}{11}e^{2t} \\ &- \frac{3}{7}e^{6t})e^{-13t} + 0.18) \\ p_{02}(t) =& k_2((\frac{1}{225}e^{4t} - \frac{2}{5}e^{14t} + \frac{4}{84}e^{5t} + \frac{1}{3}e^{16t} - \frac{5}{144}e^{7t} + \frac{47}{180}e^{17t} - \\ &\frac{20}{99}e^{8t} + \frac{47}{180}te^{19t} + \frac{10}{81}e^{10t} + \frac{1}{285} + \frac{3}{8}e^{11t} - \frac{2}{225}e^{2t} - \frac{1}{4}e^{13t} \\ &- \frac{5}{192}e^{3t} + \frac{10}{117}e^{6t} - \frac{3}{20}e^{9t} + \frac{1}{7}e^{12t} - \frac{47}{720}e^{15t})e^{-19t} - 0.25) \end{aligned}$$

increases in t for constants k_1 and k_2. This implies that, the first passage time distribution is DFR.

The first passage time random variable has special importance in stochastic process applications. We considered a semi-Markov MSS and obtained a necessary and sufficient condition for IFR/DFR and IFRA/DFRA of the first passage time distribution. The results has theoretical importance, which established in the examples, as well as practical applications.

Chapter 5

DEGREE OF ASSOCIATION OF MARKOV PROCESS

5.1 Introduction

[1] For a complex system it is quite often difficult to calculate system reliability. If the components of the system are maintained, or are interdependent, the calculation of system reliability can become even more difficult, perhaps impossible. Hence, there is a considerable amount of literature in reliability theory which deals with bounds for reliability of MSSs. Associated random variables and time associated

[1] Some contents of this chapter have appeared in Manoharan and Chacko (2009)

stochastic processes are useful in reliability theory for obtaining the reliability bounds for MSSs. The concept of association of random variables is introduced by Esary et al. (1967). A minimal cut lower bound obtained by Esary and Proschan (1970) for a non-maintained system is valid if the joint performance process of the components is associated in time. In particular, if the component lifetime are independent and if the marginal performance process of each component is associated in time, then the joint performance process is also associated in time. A sufficient condition for association when the marginal processes are Markovian is given by Hjort et al. (1985), which has use in MSS reliability study.

Esary and Proschan (1970) proposed sufficient condition for association in time of the Markov performance process of a binary system, in terms of its transition probability functions. Manoharan (1995) discussed about association in time of a Markov process. Kuber and Dharmadhikari (1996) considered a repairable system modeled by semi-Markov process and derived a sufficient condition for the association in time of the process governing the system. Dharmadhikari and Dewan (2006) derived sufficient conditions for the association in time of a vector valued stochastic process. Prakash Rao and Dewan (2001) provided a great deal of information on associated sequences and related inference problems. Lisnianski and Levitin (2003) discussed a large number of real life problems in MSS modeling and reliability assessment, and provided a stochastic process approach (eg. Markov and semi-Markov) for the MSS reliability evaluation.

To apply the concept of association to real data one require a measure of the degree of association. Karlin (1983) compared the relative degree (or strength) of association of two sets of random variables. The problem of assessing the degree of association of a stochastic process (Markov and semi-Markov process) or of comparing the relative strength of association of two stochastic process (Markov and semi-Markov process) is to be studied well. In this chapter, we consider a measure based on transition probability function for the degree of association of the Markov process or to compare the relative degree of association of two Markov process.

The applications of the measure is mainly to compare two MSSs whose performance process is Markov process. It has theoretical and practical interest. Ordering of two processes based on correlation measure in terms of transition probability function is mentioned. The system with high degree of associated components require more careful treatment in its operation, maintenance etc, because failure or imperfect repair of one component will adversely affect the remaining components according to the degree of association among components.

This chapter is arranged as follows. The measure of degree of association of Markov process is proposed in section 2. An illustrative example of a medical data is given in section 3.

5.2 Measure of Degree of Association

Karlin (1983) has provided an approach for assessing the level and form of dependence for multivariate observations that provides a fine tuning in evaluating relationships of pair of random variable by transforming the data in natural manifold ways and then computing the associated correlations whose totality reflects on the nature of dependence between array of transformed variables. The standard approach to measure the degree of dependence between two random variables X and Y involves the computation of a single statistics for the sample, resulting in some estimated measure of "overall" dependence for the distribution of (X, Y).

Karlin (1983) proposed the following definition of ordering bivariate distributions by the strength of their association.

Definition 5.2.1 *For two bivariate distributions corresponding to the random variables* (X, Y) *and* (Z, W) *we say that dependence of* (X, Y) *is stronger than the dependence of* (Z, W) *with respect to classes of non-decreasing functions* $\mathbb{F}$ *and* $\mathbb{G}$ *if*

$$\rho[h(X), g(Y)] \geq \rho[h(Z), g(W)]$$

for all $h \in \mathbb{F}$ *and* $g \in \mathbb{G}$.

Notice that the comparisons are made with respect to the same transformations on the variables (X, Y) and (Z, W).

Now we define a measure which can be used to measure the degree of association of the Markov process. For the sake of better narration, we first discuss the degree of association in discrete time stochastic process, Markov Chain, $\{X_k, k \geq 0\}$ with state space $E = \{1, 2, ..., M\}$. We have,

$$Cov(X_k, X_{k-1}) = E(X_k.X_{k-1}) - E(X_k)E(X_{k-1})$$
$$= \sum_{i,j\in E} P[X_k \geq j, X_{k-1} \geq i] - \sum_{j\in E} P[X_k \geq j] \sum_{i\in E} P[X_{k-1} \geq i].$$

Since for non-negative integer valued random variables, X_k and X_{k-1},

$$E(X_k.X_{k-1}) = \sum_{i=1}^{M}\sum_{j=1}^{M} P[X_k \geq j, X_{k-1} \geq i] \, and \, E(X_k) = \sum_{j=1}^{M} P[X_k \geq j],$$

and each expectation is finite. But, X_k and X_{k-1}, associated if,

$$Cov(X_k, X_{k-1}) \geq 0$$

$$\Rightarrow \left(\sum_{i,j\in E} P[X_k \geq j, X_{k-1} \geq i] - \sum_{i,j\in E} P[X_k \geq j]P[X_{k-1} \geq i] \right) \geq 0$$
$$\Rightarrow \sum_{i,j\in E} \{P[X_k \geq j, X_{k-1} \geq i] - P[X_k \geq j]P[X_{k-1} \geq i]\} \geq 0$$
$$or \sum_{i,j\in E} \{P[X_k \geq j | X_{k-1} \geq i] - P[X_k \geq j]\} P[X_{k-1} \geq i] \geq 0. \tag{5.1}$$

Writing (5.1) in terms of one step transition probability, we get,

$$\sum_{i,j\in E} \sum_{i',j'\in E'} \{P[X_k = j' | X_{k-1} = i'] - P[X_k = j']\} P[X_{k-1} = i'] \geq 0$$

where $E' = \{i', j' : X_{k-1} = i' \geq i, X_k = j' \geq j\}$.

We can use the measure,

$$Cov(X_k, X_{k-1}) = \sum_{i,j\in E} \sum_{i',j'\in E'} \{P[X_k = j' | X_{k-1} = i'] - P[X_k = j']\} P[X_{k-1} = i'],$$

for assessing the association of the discrete time stochastic process.

Standardization of the covariance may be desired to achieve scale invariance and enable meaningful comparisons between different data sets. Karlin (1983) replaced the condition of association, $Cov(h(X), g(Y)) \geq 0$ for all functions $h \in \mathbb{F}$ and $g \in \mathbb{G}$, of two random variables with respect to the classes $\mathbb{F}$ and $\mathbb{G}$ by an equivalent requirement

$$\rho(X, Y) = \frac{Cov(h(X), g(Y))}{\sqrt{Var(h(X)).Var(g(Y))}} \geq 0.$$

For two stochastic processes $\{X_k, k \geq 0\}$ and $\{Y_k, k \geq 0\}$, we can use the following measure for comparing the two processes based on their strength of association.

$$\rho_{(X_k, X_{k-1})} = \frac{Cov(X_k, X_{k-1})}{\sqrt{Var(X_k).Var(X_{k-1})}}$$

where $Var(X_k) = \sum_{i,j \in E} P[X_k \geq max(i,j)] - P[X_k \geq j]P[X_k \geq i]$.

If

$$\rho_{(X_k, X_{k-1})} \geq \rho_{(Y_k, Y_{k-1})} \tag{5.2}$$

the degree of association between X_k and X_{k-1} is larger than association between Y_k and Y_{k-1}. If (5.2) is true for every k, then the stochastic process $\{X_k, k \geq 0\}$ is more associated than $\{Y_k, k \geq 0\}$.

Now consider a continuous time Markov process $\{X(t), t \geq 0\}$. Let us recall the sufficient condition, due to Horjt et al. (1985) which is based on the transition probability function, for the association in time of a Markov process. Denote the transition probabilities as $p_{ij}(s,t) = P[X(t) = j|X(s) = i], s \leq t$, and the transition probability matrix as $P(s,t) = \{p_{ij}(s,t)\}_{i,j \in \{0,1,\ldots,M\}}$. Let $\tau(I) = (0, \infty)$. The transition intensity is defined as $\mu_{ij}(s) = lim_{h \to 0+} \frac{p_{ij}(s,s+h)}{h}$, $i \neq j$.

Let $P_{i,\geq j}(s,t) = P[X(t) \geq j|X(s) = i]$, $\mu_{i,\geq j}(s) = \sum_{k=j}^{M} \mu_{ik}(s)$, $i < j$ and $\mu_{i,<j}(s) = \sum_{k=0}^{j-1} \mu_{ik}(s)$, $i \geq j$.

Horjt et al. (1985) proved the following result.

Theorem 5.2.1 *Let X be a continuous time Markov process with state space $\{0,1,\ldots,M\}$ and transition probability matrix $P(s,t)$. Assume the transition intensities to be continuous. Consider the following statements about X :*

1. *X is associated in time,*
2. *X is conditionally, stochastically, non-decreasing in time, that is*

$$P[X(t) \geq j|X(s_1) = i_1, \ldots, X(s_n) = i_n]$$

   is non-decreasing in $i_1, \ldots, i_n$ for each j and for each choice of $s_1 < \ldots < s_n < t$, $n \geq 1$,
3. *$P_{i,\geq j}(s,t)$ is non-decreasing in i for each j and for each $s < t$,*
4. *for each j and s*

$\mu_{i,\geq j}(s)$ *is non-decreasing in* $i \in \{0, 1, ..., j-1\}$

and

$\mu_{i,<j}(s)$ *is non-decreasing in* $i \in \{j, ..., M\}$.

Then (2), (3) and (4) are equivalent and each of them implies (1).

For the binary case ($M = 1$) it is easily seen that the statement (3) of the above theorem is equivalent to

$$P_{1,1}(s,t) + P_{0,0}(s,t) \geq 1, \; for\, each\, s < t.$$

This was the sufficient condition given by Esray and Proschan (1970) for X to be associated in time.

In order to get the degree of association based on transition probability function, we consider the correlation measure in terms of transition probability function as follows.

Consider the random variables $X(t)$, $X(s)$, $s < t$ in the Markov process. It is clear that if $X(t)$ and $X(s)$, $s < t$ are associated if $E(X(t)^2) < \infty$ and $E(X(s)^2) < \infty$, and

$$Cov(X(t), X(s)) = \sum_{j \in E} \sum_{i \in E} P[X(t) > j, X(s) > i] - P[X(t) > j] P[X(s) > i] \geq 0. \tag{5.3}$$

Using transition probability function, $P[X(t) = j' | X(s) = i']$ of the Markov process, we write (5.3) as,

$Cov(X(t), X(s)) =$

$$\sum_{j\in E}\sum_{i\in E}\sum_{\{i',j':X(s)=i'>i,X(t)=j'>j\}}\{P[X(t)=j'|X(s)=i']-P[X(t)=j']\}P[X(s)=i'] \geq 0. \tag{5.4}$$

Comparison of two Markov processes, $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$, only in terms of transition probabilities is not possible but comparison between covariances in terms of transition probabilities and state probabilities is more reasonable. We can compare the degree of association of two Markov processes using the following correlation function, $\rho_X(t, s)$, since it provide the meaningful comparison with scale invariance as in case discrete time processes.

In the case of the Markov process we have,

$$\rho_X(t, s) = \frac{Cov(X(t), X(s))}{\sqrt{Var(X(t)).Var(X(s))}} \geq 0$$

implies association between $X(s)$ and $X(t)$. In order to measure the degree of association in time of the Markov process we use the correlation $\rho_X(t, s)$ as a function of transition probability function and state probabilities. We can compare the degree of association of two Markov processes using $\rho_X(t, s)$. This gives a stochastic ordering of two Markov processes based on strength of their association. Denote,

$$C_X(t, s) = \sum_{j\in E}\sum_{i\in E}[P[X(t) \geq j|X(s) \geq i] - P[X(t) \geq j]]P[X(s) \geq i],$$

$$C_X(t, t) = \sum_{j\in E}\sum_{i\in E}[P[X(t) \geq max(i, j)] - P[X(t) \geq j]P[X(t) \geq i]]$$

$$\text{and } \rho_X(t, s) = \frac{C_X(t, s)}{\sqrt{C_X(t, t).C_X(s, s)}}.$$

We define, if this correlation function of one process is greater than the other, former process is more associated.

Definition 5.2.2 *For two different Markov processes* $\{X(t), t \geq 0\}$ *and* $\{Y(t), t \geq 0\}$, *we say that association of* $(X(t), X(s))$, $s < t$ *is stronger than the association of* $(Y(t), Y(s))$, $s < t$ *if*

$$\rho_X(t, s) \geq \rho_Y(t, s).$$

Above definition shows the association between random variables of both processes at two time points. But, in the following definition, we compare the degree of association of two Markov processes.

Definition 5.2.3 *For two different Markov processes* $\{X(t), t \geq 0\}$ *and* $\{Y(t), t \geq 0\}$, *we say that association of* X *process is stronger than the association of* Y *process if* $\forall s, t \in R$, $s < t$

$$\rho_X(t, s) \geq \rho_Y(t, s).$$

In the following part we consider some conditions of association in terms of the non-decreasing functions of the classes $\mathbb{F}$ and $\mathbb{G}$ and its distributional properties. It provide a measure for the comparison of the degree of association of two system each consists of n associated components with Markov performance process. For that we use the following definitions, see Prakash Rao and Dewan (2001).

The following definition gives the condition for association of a collection of random variables with respect to non-decreasing functions.

Definition 5.2.4 *A collection of random variables $\{X_n, n \geq 1\}$ is said to be associated if for every n and for every choice of coordinate-wise non-decreasing functions $h(\boldsymbol{x})$ and $g(\boldsymbol{x})$ from R^n to R,*

$$Cov(h(\boldsymbol{X}), g(\boldsymbol{X})) \geq 0$$

whenever it exist, where $\boldsymbol{X} = (X_1, ..., X_n)$.

The use of above definition is to define the association in a continuous time stochastic process which model a performance process of a component. We define the performance process of n components as follows.

Definition 5.2.5 *The performance process of the ith component is a stochastic process $\{X_i(t), t \in \tau\}$ where for each fixed $t \in \tau, X_i(t)$ denotes the state of component i at time t. The joint performance process of the components is given by $\{\boldsymbol{X}(t), t \in \tau\} = \{(X_1(t), ..., X_n(t)), t \in \tau\}$.*

Let $I = [t_A, t_B] \subset [0, \infty), \tau(I) = \tau \cap I$. Using the above definition, we can define the association of joint performance process.

Definition 5.2.6 *The joint performance process $\{\boldsymbol{X}(t),\ t \in \tau\}$ of the components* *is said to be associated in time interval I if and only if, for any integer m and $\{t_1, ..., t_m\} \subset \tau(I)$, the random variables in the array*

$$\begin{array}{ccc} X_1(t_1) & ... & X_1(t_m) \\ ... & ... & ... \\ X_n(t_1) & ... & X_n(t_m) \end{array}$$

are associated.

Here we have a set of mn random variables of the joint performance process of n components. We have to get a single valued measure to compare association of two performance processes of two MSSs with n components.

For the component performance process $\{X_i(t), t \in \tau\}$, $i \in \{1, 2, ..., n\}$ and fixed $t_1 < ... < t_m$, let $h_i(\underline{X}_i) \in \mathbb{F}, g_i(\mathbf{X}_i) \in \mathbb{G}$ are function of random variables from R^m to R, where $\mathbf{X}_i = (X_i(t_1), X_i(t_2), ..., X_i(t_m))$. But we have, $X_i(t_1), X_i(t_2), ..., X_i(t_m)$ are associated if for every $h_i(\mathbf{x}_i)$, and $g_i(\mathbf{x}_i)$, such that $E(h_i(\mathbf{x}_i))^2 < \infty$ and $E(h_i(\mathbf{x}_i))^2 < \infty$, $Cov(h_i(\mathbf{X}_i), g_i(\mathbf{X}_i)) \geq 0$ where

$$Cov(h_i(\mathbf{X}_i), g_i(\mathbf{X}_i)) = \int_R \int_R \{P[h_i(\mathbf{X}_i) > x, g_i(\mathbf{X}_i) > y] - P[h_i(\mathbf{X}_i) > x] P[g_i(\mathbf{X}_i) > y]\} dxdy,$$

$i \in \{1, 2, ..., n\}$.

Now suppose that the component performance stochastic process is Markov process, then we define the association of component Markov performance process.

Definition 5.2.7 *A Markov performance process $\{X_i(t), t \in \tau\}$ of component i is associated if*

$$\int_R \int_R \{P[h_i(\boldsymbol{X}_i) > x | g_i(\boldsymbol{X}_i) > y] - P[h_i(\boldsymbol{X}_i) > x]\} P[g_i(\boldsymbol{X}_i) > y] dx dy \geq 0$$

for every collection of random variables $\boldsymbol{X}_i = (X_i(t_1), ..., X_i(t_m))$ and every choice of coordinate wise non-decreasing function $h_i(\boldsymbol{x}_i)$ and $g_i(\boldsymbol{x}_i)$ from R^m to R such that $E(h_i(\boldsymbol{X}_i))^2 < \infty$ and $E(g_i(\boldsymbol{X}_i))^2 < \infty$.

In a similar way, we can find a condition for association of joint performance process of components, in terms of non-decreasing functions, which is quite desirable. In the following definition, we consider the functions $H \in \mathbb{F}$ and $G \in \mathbb{G}$ from R^{nm} to R. Now we define the association in time of a joint performance process of n components using this non-decreasing functions.

Definition 5.2.8 *The joint performance process of the components $\{\underline{\boldsymbol{X}}(t), t \in \tau\}$ is associated in time if*

$$\int_R \int_R \{P[H(\underline{\boldsymbol{X}}) > x | G(\underline{\boldsymbol{X}}) > y] - P[H(\underline{\boldsymbol{X}}) > x]\} P[G(\underline{\boldsymbol{X}}) > y]) dx dy \geq 0$$

for every collection of random variables,

$\underline{\boldsymbol{X}} = (X_1(t_1), X_2(t_1), ..., X_n(t_1), X_1(t_2), ..., X_n(t_2), ..., X_1(t_m), ..., X_n(t_m))$

and every choice of coordinate wise non-decreasing function $H(\underline{\boldsymbol{x}})$ and $G(\underline{\boldsymbol{x}})$ from R^{nm} to R, such that $E(H(\underline{\boldsymbol{X}}))^2 < \infty$ and $E(G(\underline{\boldsymbol{X}}))^2 < \infty$.

Now we define the measure of degree of association of the MSS consists of n associated components in which each of the components are governed by Markov processes. Denote

$$C_{\underline{\mathbf{X}}}(H,G) = \int_R \int_R \{P[H(\underline{\mathbf{X}}) > x | G(\underline{\mathbf{X}}) > y] - P[H(\underline{\mathbf{X}}) > x]\} P[G(\underline{\mathbf{X}}) > y] dxdy$$

$$C_{\underline{\mathbf{X}}}(H,H) = \int_R \int_R \{P[H(\underline{\mathbf{X}}) > max\{x,y\}] - P[H(\underline{\mathbf{X}}) > x] P[H(\underline{\mathbf{X}}) > y]\} dxdy$$

and $\rho_{\underline{\mathbf{X}}}(H,G) = \dfrac{C_{\underline{\mathbf{X}}}(H,G)}{\sqrt{C_{\underline{\mathbf{X}}}(H,H) C_{\underline{\mathbf{X}}}(G,G)}}$.

We can use the index $\rho_{\underline{\mathbf{X}}}(H,G)$ for measuring the degree of association of the system with n associated components which are governed by Markov processes. A comparison of the degree of association of two performance processes $\{\underline{\mathbf{X}}(t), t \geq 0\}$ and $\{\underline{\mathbf{Y}}(t),\ t \geq 0\}$ of two systems can be made using the measures $\rho_{\underline{\mathbf{X}}}(H,G)$ and $\rho_{\underline{\mathbf{Y}}}(H,G)$.

Definition 5.2.9 *For two performance process $\{\underline{\boldsymbol{X}}(t), t \geq 0\}$ and $\{\underline{\boldsymbol{Y}}(t), t \geq 0\}$, of two systems consists of n associated components each of which are governed by the Markov processes $\{X_i(t), t \geq 0\}$ and $\{Y_i(t), t \geq 0\}$, $i = 1, 2, ..., n$ respectively, we say that association of $\boldsymbol{X}$-system is stronger than the association of $\boldsymbol{Y}$-system if $\forall m\,, n\, and\, H \in \mathbb{F}$, $G \in \mathbb{G}$, from R^{mn} to R such that $E(H^2) < \infty$ and $E(G^2) < \infty$*

$$\rho_{\underline{\mathbf{X}}}(H,G) \geq \rho_{\underline{\mathbf{Y}}}(H,G).$$

The following remark is given by Esray and Proschan (1970) which define the association of system based on n associated components.

Remark 5.2.1 *Let* $\{\underline{\boldsymbol{X}}(t), t \in \tau\}$ *be a time associated joint performance process for a set of* n *components. Let* $\phi_1, ..., \phi_m$ *be the structure functions of a set of coherent systems built from its components, let the joint performance process of a system be* $\{\phi\underline{\boldsymbol{X}}(t), t \in \tau\} = \{\phi_1\underline{\boldsymbol{X}}(t), ..., \phi_m\underline{\boldsymbol{X}}(t), t \in \tau\}$. *For* $\{t_1, ..., t_k\} \subset \tau$, $X_i(t_l), i = 1, 2, ..., n, l = 1, ..., k,$ *are associated. For each* $j = 1, 2, ..., m$ *and fixed* t, $\phi_j\underline{\boldsymbol{X}}(t)$ *is an increasing function of* $X_1(t), ..., X_n(t)$. *By property (*P_4*),* $\phi_j\underline{\boldsymbol{X}}(t_l)$, $j = 1, 2, .., m$, $l = 1, ..., k$ *are associated. Again applying property (*P_4*) to* $\phi_j\underline{\boldsymbol{X}}(t_l)$, $j = 1, ..., m$, $l = 1, ..., k$, *we have the association of* $\phi\underline{\boldsymbol{X}}(t)$, *the structure function of coherent system.*

If we consider two such systems, we can compare the degree of association.

The results described above has theoretical importance in stochastic process theory. So the results are more general. The illustration in this chapter shows its application to medical field also. So the results described in this chapter can be easily applied to compare performance processes of MSSs.

Kuber and Dharmadhikari (1996) discussed association in time for semi-Markov processes and obtained sufficient condition for association in terms of transition probability function. The measure of degree of association in semi-Markov process can be obtained as similar to in Markov process.

The proposed measures may help us (i) to suggest whether a Markov process is associated in time; and (ii) to asses the relative degree (or strength) of association when comparing two different Markov performance processes, and (iii) to asses the relative strength of association of two performance process of two systems consists of n associated components with Markov performance processes. Similar use holds for the semi-Markov case.

5.3 Illustration

We consider an example for the application of proposed measure. In this example, we consider the data set from medical field for the illustration of concept of measure of association. We re-examine the data on an oral hygiene study, discussed in Das and Chattopadhyay (2004)(cf. Dharmadhikari and Dewan (2006)) for the illustration of the association of a vector valued process. Here we calculate the degree of association of each variable process. Dentists recorded the reduction in the amount of plaque on teeth. Each individual in the data was monitored for a couple of days. Two teeth were identified, one on the left lower canine which is in the left lower corner of a jaw, and one on molar at upper right jaw. The reduction in the thickness of plaque for subjects are usually recorded as belonging to four different categories, viz, no reduction, slight reduction, moderate reduction and vast reduction. One of the objects of the study was to evaluate effectiveness of brushing. In such cases natural question can be : Is it possible to reduce the number of records per individual per day? If there is some sort of dependence, it may be possible to reduce the dimension of data. Das and Chattopadhyay (2004) developed a latent mixture regression model to study this categorical multivariate data. Motivated by the data on oral hygiene study Dharmadhikari and Dewan (2006) derived sufficient conditions for association in time of the vector valued process $X = \{\{X_i(m), m \geq 1\}, 1 \leq i \leq k\}$ which takes values on a finite set E^k where $E = \{1, 2, ..., n\}$. Table 1 give a part of dental data analyzed. It gives stain on the same tooth at all four positions before and after brushing, respectively. Numbers under (P_1, P_2, P_3, P_4) indicate the amount of stain at each of the four positions on the selected tooth of an individual. It is easy to verify that data in table 1 are conditionally increasing in its coordinates. Note that first probability is based on only two observations and the departure can be attributed to sampling /measuring errors. With such an understanding, both the data sets can be considered to be associated in time.

The state probabilities in the above example are given in table 2. The conditional probabilities $P[X_k = j | X_{k-1} = i]$ for $i, j \in \{0, 1, 2, 3\}$ for the four sets of data are calculated in table 3.

Table 1. Dental data stain before and after brushing

	Before brushing				After brushing					Before brushing				After brushing			
No.	P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4		P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4
1	1	1	1	2	0	0	0	0	14	2	1	2	2	0	0	1	1
2	1	1	2	2	0	0	0	1	15	2	2	2	2	0	0	1	1
3	1	1	2	2	0	0	0	1	16	2	2	2	2	0	0	1	1
4	1	1	2	2	0	0	0	1	17	2	2	2	2	0	0	1	1
5	1	1	2	2	0	0	0	1	18	2	2	2	2	0	0	1	1
6	1	2	2	2	0	0	0	1	19	2	2	2	2	0	0	1	1
7	1	2	2	2	0	0	0	1	20	2	2	2	2	0	0	1	1
8	1	2	2	2	0	0	0	1	21	2	2	2	2	0	1	1	1
9	1	2	2	2	0	0	0	1	22	2	2	2	2	0	1	1	1
10	1	2	2	2	0	0	0	1	23	2	2	2	2	0	1	1	1
11	1	2	2	2	0	0	0	1	24	2	2	2	3	0	1	1	1
12	1	2	2	2	0	0	0	2	25	2	2	2	3	1	1	1	2
13	1	2	2	3	0	0	0	2									

The information provides useful directions for medical examination and further studies on each position. But in order to get an ordering in terms of association, we have to compute the measure of association (using table 2 and table 3). The values are obtained in table 4. The analysis shows that the data in the third (P_3) position is more associated. This information may be useful to medical practitioners.

Table 2. State probabilities

P_1	P_2	P_3	P_4
$P[X_{k-1} = 1] = \frac{13}{25}$	$P[X_{k-1} = 1] = \frac{6}{25}$	$P[X_{k-1} = 1] = \frac{1}{25}$	$P[X_{k-1} = 1] = 0$
$P[X_{k-1} = 2] = \frac{12}{25}$	$P[X_{k-1} = 2] = \frac{19}{25}$	$P[X_{k-1} = 2] = \frac{24}{25}$	$P[X_{k-1} = 2] = \frac{22}{25}$
$P[X_{k-1} = 3] = 0$	$P[X_{k-1} = 3] = 0$	$P[X_{k-1} = 3] = 0$	$P[X_{k-1} = 3] = \frac{3}{25}$
$P[X_k = 0] = \frac{24}{25}$	$P[X_k = 0] = \frac{20}{25}$	$P[X_k = 0] = \frac{13}{25}$	$P[X_k = 0] = \frac{1}{25}$
$P[X_k = 1] = \frac{1}{25}$	$P[X_k = 1] = \frac{5}{25}$	$P[X_k = 1] = \frac{22}{25}$	$P[X_k = 1] = \frac{21}{25}$
$P[X_k = 3] = 0$	$P[X_k = 3] = 0$	$P[X_k = 3] = 0$	$P[X_k = 2] = \frac{3}{25}$

Table 3. The conditional probabilities $P[X_k = j | X_{k-1} = i]$ for $i, j \in \{0, 1, 2, 3\}$

X_k / X_{k-1}	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	$\frac{11}{12}$	$\frac{1}{12}$	0	0	$\frac{14}{19}$	$\frac{5}{19}$	0	0	$\frac{12}{24}$	$\frac{12}{24}$	0	0	$\frac{1}{22}$	$\frac{20}{22}$	$\frac{1}{22}$	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	0

Here we compute $Cov(X_k, X_{k-1})$, because state space of four positions are assumed to be same.

Table 4. Degree of association

Position	P_1	P_2	P_3	P_4
$Cov(X_k, X_{k-1})$	0	30/625	12/25	2/25

Association in time of a Markov process has been studied by several authors. We introduced a measure to assess the degree of association in time of Markov process based on transition probability function. The result have theoretical importance in the area of ordering of two processes based on degree of association. The practical applications are in reliability study as well in medical field.

Chapter 6

EVALUATION OF JOINT IMPORTANCE MEASURES

6.1 Introduction

[1]In certain MSSs, the performance of different system components can have different physical nature whose performance is measured in terms of productivity or capacity etc. Therefore it is important to measure performance rates of these components by their contribution into the entire MSS output performance. Examples of such

[1]Some contents of this chapter have appeared in Chacko (2008b), Chacko and Manoharan (2009a, 2009c)

MSSs are continuous materials or energy transmission systems, power generation systems. In power generation applications the performance measure is usually defined as productivity or capacity (eg. production capacity of 100MW). The main task of these systems is to provide the desired throughput or transmission capacity for constant energy, material or information flow. The evaluation of system reliability and joint importance measure of such systems are complicated because of multi-state behavior and complexity of configuration of system components. Also we cannot use usual generating function to find out the performance and probability distribution of such systems, since for parallel components the performance will not be usual maximum of individual performances but sum (eg. two parallel power generator with productivity 100MW provide 200MW as total output). UGF is found to be a fine tool for such systems in evaluation of reliability and importance measure, see Lisnianski and Levitin (2003). Some application of UGF can be seen in Levitin (2002a), Levitin (2002b), Levitin (2003), Levitin (2004a) and Yeh (2006). So we will use the UGF for the evaluation of performance measure and joint importance measure of systems whose performance is in terms of productivity or capacity.

This chapter is arranged as follows. The definition of UGF of a component and a complex MSS are given in section 2. The method of evaluation of joint importance measure using UGF is provided in section 3. The application of the evaluation procedure to the network systems is given in section 4. An example is given in last section.

6.2 Universal Generating Function

For a MSS which has a finite number of states, there can be $M+1$ different output performance at each time t,

$$G(t) \in \mathbf{G} = \{G_k, 0 \leq k \leq M\}.$$

The system output performance distribution can be defined by two finite vectors $\mathbf{G}$ and $\mathbf{P} = \{p_k(t) = P[G(t) = G_k], 0 \leq k \leq M\}$.

The UGF, represented by a polynomial $U(z)$ can define MSS OPD, i.e., it represents all the possible states of the system (or component) by relating the probabilities of each state, p_k, to performance G_k of the MSS of that state in the form:

$$U(z) = \sum_{k=0}^{M} p_k(t) z^{G_k},\ z \in R. \tag{6.1}$$

Now we discuss the UGF of complex MSS.

UGF for Complex MSS

Real world MSSs are often complex and consist of large number of components composing different types of structures. UGF is a technique to obtain the entire MSS output performance distribution. This technique uses composition operators for determination of the UGF of a subsystem (component) containing a number of components. These operators determine the subsystem UGF expressed as polynomial $U(z)$

for a group of components using simple algebraic operations over individual UGFs of components. All the composition operators for two different components takes the form

$$\Omega(U_1(z), U_2(z)) = \Omega(\sum_{i=0}^{M} p_{1i} z^{g_{1i}}, \sum_{j=0}^{M} p_{2j} z^{g_{2j}})$$
$$= \sum_{i=0}^{M} \sum_{j=0}^{M} p_{1i} p_{2j} z^{w(g_{1i}, g_{2j})} \tag{6.2}$$

where $U_1(z)$ and $U_2(z)$ are individual UGF of components 1 and 2 with performance distributions $\{g_{1i}, p_{1i}, i \in \{0, 1, ..., M\}\}$ and $\{g_{2j}, p_{2j}, j \in \{0, 1, ..., M\}\}$ respectively. The function $w(.,.)$ in composition operators expresses the entire performance rate of subsystem consisting of different components in terms of the individual performance rates of the components. The definition of the function $w(.,.)$ strictly depends on the type of connection between the components in the reliability logic diagram sense. Here we define composition operators $\Omega\sigma, \Omega\pi$ for subsystems with components connected in series and in parallel respectively. In MSS where the performance measure is defined in terms of capacity or productivity, the total capacity of a pair of components connected in parallel is equal to the sum of the capacities of the components. Therefore the function $w(.,.)$ in composition operator takes the form:

$$w(g_1, g_2) = \pi(g_1, g_2) = g_1 + g_2.$$

For a pair of components connected in series, the component with the least capacity becomes the bottleneck of the system. In this case the function $w(.,.)$ takes the form:

$$w(g_1, g_2) = \sigma(g_1, g_2) = min(g_1, g_2).$$

Note that the composition operators for components connected in parallel and in series satisfies the conditions:

$$\Omega(U_1(z),...,U_k(z),U_{k+1}(z),...,U_n(z)) = \Omega(U_1(z),...,U_{k+1}(z),U_k(z),...,U_n(z))$$

and

$$\Omega(U_1(z),...,U_k(z),U_{k+1}(z),...,U_n(z)) = \Omega(\Omega(U_1(z),...,U_k(z)),U_{k+1}(z),...,U_n(z))$$

for arbitrary k. Consecutively applying the Ω operators with corresponding functions σ or π to the components, one can obtain the UGF for an arbitrary number of components connected in series or in parallel. Combining the two operators one can obtain UGF representing performance distribution of an arbitrary series-parallel system.

6.3 Evaluation of Joint Importance Measure using UGF

[2]Here we propose the component's performance restriction approach (or state space restriction approach) for evaluation of joint importance measures using UGF. Let

[2]This work has been presented at *National Seminar on Recent Advances in Statistics and Analysis of Non-conventional Data,* March 15-17, Farrok College, Kerala and Awarded **Prof. R. N. Pillai Young Statistician Award 2008** of *Kerala Statistical Association*

OM_{ik} be the OPM of the MSS when component i is in a fixed state k while the rest of components evolve stochastically among their corresponding states with steady state performance distributions $\{x_{jl}, p_{jl}\}$, $1 \leq j \leq n$, $0 \leq l \leq M_j$.

The conditional probability of the component i being in a generic state k characterized by a performance x_{ik} not greater than a pre-specified level threshold α (or equivalently $k \leq k_{i\alpha}$) is

$$P[X_i = k | k \leq k_{i\alpha}] = \frac{p_{ik}}{\sum_{r=0}^{k_{i\alpha}} p_{ir}} = \frac{p_{ik}}{p_i^{\leq \alpha}} = p^*_{1ik}\,(say).$$

Similarly, the conditional probability of component i being in a state k when it is known that $k > k_{i\alpha}$ is

$$P[X_i = k | k > k_{i\alpha}] = \frac{p_{ik}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir}} = \frac{p_{ik}}{p_i^{>\alpha}} = p^*_{2ik}\,(say).$$

Consider the following joint probability distribution of two independent components i and j given four additional restrictions, (1)$k > k_{i\alpha}, h > h_{j\beta}$, (2)$k \leq k_{i\alpha}, h > h_{j\beta}$, (3)$k > k_{i\alpha}, h \leq h_{j\beta}$ and (4)$k \leq k_{i\alpha}, h \leq h_{j\beta}$.

$$P[X_i = k, X_j = h | k \leq k_{i\alpha}, h \leq h_{j\beta}] = \frac{p_{ik}p_{jh}}{\sum_{r=0}^{k_{i\alpha}} p_{ir} \sum_{m=0}^{h_{j\beta}} p_{jm}} = p^{**}_{1\ kh}\,(say)$$

$$P[X_i = k, X_j = h | k \leq k_{i\alpha}, h > h_{j\beta}] = \frac{p_{ik}p_{jh}}{\sum_{r=0}^{k_{i\alpha}} p_{ir} \sum_{h_{j\beta}+1}^{M_j} p_{jm}} = p^{**}_{2\ kh}\,(say)$$

$$P[X_i = k, X_j = h | k > k_{i\alpha}, h \leq h_{j\beta}] = \frac{p_{ik}p_{jh}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir} \sum_{m=0}^{m=h_{j\beta}} p_{jm}} = p^{**}_{3\ kh}\,(say)$$

and $$P[X_i = k, X_j = h | k > k_{i\alpha}, h > h_{j\beta}] = \frac{p_{ik}p_{jh}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir} \sum_{m=h_{j\beta}+1}^{M_j} p_{jm}} = p^{**}_{4\ kh}\,(say).$$

Using the above conditional probability distributions, we obtain the following OPMs:

$$OM_i^{\leq\alpha} = \sum_{k=0}^{k_{i\alpha}} \frac{p_{ik}}{p_i^{\leq\alpha}}.OM_{ik}, \tag{6.3}$$

$$OM_i^{>\alpha} = \sum_{k=k_{i\alpha}+1}^{M_i} \frac{p_{ik}}{p_i^{>\alpha}}.OM_{ik}, \tag{6.4}$$

$$OM_{i,j}^{\leq\alpha,\leq\beta} = \sum_{k=0}^{k_{i\alpha}} \sum_{h=0}^{h_{j\beta}} p_1^{**}{}_{hk}.OM_{ik,jh}, \tag{6.5}$$

$$OM_{i,j}^{>\alpha,\leq\beta} = \sum_{k=k_{i\alpha}+1}^{M_i} \sum_{h=0}^{h_{j\beta}} p_3^{**}{}_{hk}.OM_{ik,jh}, \tag{6.6}$$

$$OM_{i,j}^{\leq\alpha,>\beta} = \sum_{k=0}^{k_{i\alpha}} \sum_{h=h_{j\beta}+1}^{M_j} p_2^{**}{}_{hk}.OM_{ik,jh} \text{ and} \tag{6.7}$$

$$OM_{i,j}^{>\alpha,>\beta} = \sum_{k=k_{i\alpha}+1}^{M_i} \sum_{h=h_{j\beta}+1}^{M_j} p_4^{**}{}_{hk}.OM_{ik,jh}, \tag{6.8}$$

where $OM_{ik,jh}$ be the system steady state OPM when component i is in state k and component j is in state h. Substituting (6.3) to (6.8) in (2.19) to (2.22) we get the generalized joint importance measures using steady state probability distribution of components.

In the same way we can express the joint risk importance measures. At steady state, let F_{ik} be the risk associated to the system when component i is in state k. Similarly, let $F_{ik,jh}$ represents the risk associated with the system when component i is in state k and component j is in state h. Then the joint risk importance measures are,

$$
\begin{aligned}
MJrBI_{ij} = &\sum_{r=0}^{k_{i\alpha}}\sum_{m=0}^{k_{j\beta}} p_1^{**}{}_{rm}F_{ir,jm} - \sum_{r=0}^{k_{i\alpha}}\sum_{m=k_{j\beta}+1}^{M_j} p_2^{**}{}_{rm}F_{ir,jm} \\
&- \sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=0}^{k_{j\beta}} p_3^{**}{}_{rm}F_{ir,jm} + \sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=k_{j\beta}+1}^{M_j} p_4^{**}{}_{rm}F_{ir,jm},
\end{aligned} \tag{6.9}
$$

$$
MJrAW_{ij} = \frac{\sum_{r=0}^{k_{i\alpha}}\sum_{m=0}^{k_{j\beta}} p_1^{**}{}_{rm}F_{ir,jm}}{\sum_{r=0}^{k_{i\alpha}} p_{1ir}^{*}F_{ir} + \sum_{m=0}^{k_{j\beta}} p_{1jm}^{*}F_{jm}}, \tag{6.10}
$$

$$
MJrRW_{ij} = \frac{F_1}{F_2 - F_3 - F_4} \tag{6.11}
$$

$$
\begin{aligned}
\text{where } F_1 &= \sum_{r=k_{i\alpha}+1}^{M_i} p_{2ir}^{*}F_{ir} + \sum_{m=k_{j\beta}+1}^{M_j} p_{2jm}^{*}F_{jm} \\
F_2 &= \sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=k_{j\beta}+1}^{M_j} p_4^{**}{}_{rm}F_{ir,jm} \\
F_3 &= \sum_{r=0}^{k_{i\alpha}}\sum_{m=k_{j\beta}+1}^{M_j} p_2^{**}{}_{rm}F_{ir,jm} \\
F_4 &= \sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=0}^{k_{j\beta}} p_3^{**}{}_{rm}F_{ir,jm}, \text{ and}
\end{aligned}
$$

$$
MJrFV_{ij} = \frac{\sum_{r=k_{i\alpha}+1}^{M_i} p_{2ir}^{*}F_{ir} + \sum_{m=k_{j\beta}+1}^{M_j} p_{2jm}^{*}F_{jm} - \sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=k_{j\beta}+1}^{M_j} p_4^{**}{}_{rm}F_{ir,jm}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{2ir}^{*}F_{ir} + \sum_{m=k_{j\beta}+1}^{M_j} p_{2jm}^{*}F_{jm}}. \tag{6.12}
$$

The following recursive algorithm allows to compute the system OPM, see Lisnianski and Levitin (2003).

1. Obtain the UGFs of all of the system components.

2. If the system contains a pair of components connected in parallel or in series, replace this pair with an equivalent macro-component with UGF obtained by

(6.2).

3. If the system contains more than one component, return to step 2.

4. Determine the UGF of the entire series-parallel system as the UGF of the single equivalent macro-component. The system probability and performance distributions are represented by the coefficients and exponents of this UGF, corresponding to the state probabilities and performance, respectively.

5. Compute the system OPM by applying the equations (6.3) to (6.8) with the given vectors of probability distribution and output performances.

In order to obtain the state-space restricted OPMs $OM_i^{\leq\alpha}$, $OM_i^{>\alpha}$, $OM_{ij}^{\leq\alpha,\leq\beta}$, $OM_{ij}^{>\alpha,\leq\beta}$, $OM_{ij}^{\leq\alpha,>\beta}$, and $OM_{ij}^{>\alpha,>\beta}$, one has to modify the UGF of components i and j as follows:

$$U_i^{\leq\alpha}(z) = \sum_{r=0}^{k_{i\alpha}} \frac{p_{ir}}{p_i^{\leq\alpha}} z^{x_{ir}}$$

for $OM_i^{\leq\alpha}$,

$$U_i^{>\alpha}(z) = \sum_{r=k_{i\alpha}+1}^{M_i} \frac{p_{ir}}{p_i^{>\alpha}} z^{x_{ir}}$$

for $OM_i^{>\alpha}$,

$$U_{i,j}^{\leq\alpha,\leq\beta}(z) = \sum_{r=0}^{k_{i\alpha}} \frac{p_{jr}}{p_i^{\leq\alpha}} z^{x_{ir}} * \sum_{m=0}^{k_{j\beta}} \frac{p_{jm}}{p_j^{\leq\beta}} z^{x_{jm}} = \sum_{r=0}^{k_{i\alpha}} \sum_{m=0}^{k_{j\beta}} \frac{p_{ir}p_{jm}}{p_i^{\leq\alpha} p_j^{\leq\beta}} z^{\omega(x_{ir},x_{jm})}$$

for $OM_{ij}^{\leq\alpha,\leq\beta}$,

$$U_{i,j}^{>\alpha,\leq\beta}(z) = \sum_{r=k_{i\alpha}+1}^{M_i} \frac{p_{ir}}{p_i^{>\alpha}} z^{x_{ir}} * \sum_{m=0}^{k_{j\beta}} \frac{p_{jm}}{p_j^{\leq\beta}} z^{x_{jm}} = \sum_{r=k_{i\alpha}+1}^{M_i} \sum_{m=0}^{k_{j\beta}} \frac{p_{ir}p_{jm}}{p_i^{>\alpha} p_j^{\leq\beta}} z^{\omega(x_{ir},x_{jm})}$$

for $OM_{ij}^{>\alpha,\leq\beta}$,

$$U_{i,j}^{\leq\alpha,>\beta}(z)=\sum_{r=0}^{k_{i\alpha}}\frac{p_{ir}}{p_i^{\leq\alpha}}z^{x_{ir}}*\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{jm}}{p_j^{>\beta}}z^{x_{jm}}=\sum_{r=0}^{k_{i\alpha}}\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{ir}p_{jm}}{p_i^{\leq\alpha}p_j^{>\beta}}z^{\omega(x_{ir},x_{jm})}$$

for $OM_{ij}^{\leq\alpha,>\beta}$, and

$$U_{i,j}^{>\alpha,>\beta}(z)=\sum_{r=k_{i\alpha}+1}^{M_i}\frac{p_{ir}}{p_i^{>\alpha}}z^{x_{ir}}*\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{jm}}{p_j^{>\beta}}z^{x_{jm}}=\sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{ir}p_{jm}}{p_i^{\leq\alpha}p_j^{>\beta}}z^{\omega(x_{ir},x_{jm})}$$

for $OM_{ij}^{>\alpha,>\beta}$, then apply the algorithm given above. We use the coefficients of above UGFs for the evaluation of joint importance measures in (2.19) to (2.22) and (6.9) to (6.12).

6.4 Application in Network Systems

Since the very early times of reliability engineering, the network reliability is one of the main subjects of research. The network reliability theory has been applied extensively in many real-world systems such as computer and communication systems, electric power generation system, power transmission and distribution systems, telecommunication network system, transportation systems, oil/gas production systems etc. Network reliability evaluation approaches exploit a variety of tools for system modeling and reliability index calculation. Network reliability problems are generally classified based on the method used to transfer the flow (or signal) and how the flow conservation law is satisfied. Typically there are two categories; the multi-state arc network (MAN) and the multi-state node network (MNN). In MAN, each arc has a non-negative integer valued discrete random variable capacity (multi-state arc) and all flows in the network obey the conservation law. Apparently in MNN, each node is a multi-state node with discrete states determined by a set of nodes receiving the signal directly from it without satisfying conservation law. Both have their own applications; for example electrical power distribution system can be modeled by MAN, and computer networks or cellular phone networks can be modeled as MNN. Thripathy et al (1996) and Patra and Misra (1996) provide information of MSS modeling of networks and reliability evaluation.

Let $G = (N, A)$ represents a stochastic capacited network made up of n arcs with known demand w_k from a specified source node s to a specific sink node t where N represents the set of all nodes and $A = \{a_i | 1 \leq i \leq n\}$ represents the set of all arcs. Each arc i may be in one of M_i+1 states $\{0, 1, ..., M_i\}$, $i \in \{1, 2, ..., n\}$. Let $W(t)$ is the random output performance of the multi-state network G at time t which takes the values $w_i, i \in \{0, 1, ..., M\}$, where $M = max_i\{M_i\}$, depending on the network state i at time t. Let $p_i(t)$ is the probability of the network system being in state i at time t. The two vectors of the network performance realizations, $\bar{w} = \{w_i, 0 \leq i \leq M\}$, and of the network state probabilities, $\{p_i(t), 0 \leq i \leq M\}$ define the network output performance distribution at time t. The current capacity of arc a_i at steady state is represented by x_{ij}, when the arc is in state $j, 0 \leq j \leq M_i$. The vector $(p_{i0}, p_{i1}, ..., p_{iM_i})$ represents the steady state probability associated to each of the values of capacity of arc a_i. The network state vector $\mathbf{x} = (x_1, ..., x_n)$ denotes the state of all the arcs of the network system. Function $\phi(\mathbf{x}) : Z^n \rightarrow Z$, where $Z = \{0, 1, ..., M\}$, $M = max_i\{M_i\}$

maps the network state vector into network state. That is, $\phi(\mathbf{x})$, is the state of the network from source to sink under system state vector $\mathbf{x}$, which represents a MSS structure function.

We shall make the following assumptions for the network reliability system.

1. Arc states are stochastically independent.
2. The structure function is statistically coherent. That is, improving an arc performance cannot cause to degrade the performance of the network system and all arcs are relevant.

JRI of the two edges in an undirected network in binary nature is an extension of the marginal reliability importance (MRI) of edges, Hong and Lie (1993). In an undirected binary network, reliability is the probability that source and terminal are connected by working edges. For an undirected stochastic network $G(N, E)$, where $E = \{e_i | 1 \leq i \leq n\}$ is set of all edges and N is the set of nodes, let $R(G)$ represents the probability that the source and terminal are connected by working edges and $\mathbf{q} = (q_1, ..., q_n)$ where $q_i = P\{e_i \in E \text{ is working}\}$. Marginal reliability importance of edge e_i in an undirected network is defined as $I_G(i) = \frac{\partial R(G)}{\partial q_i}$. Again JRI of two edges is defined as follows.

Definition 6.4.1 *The JRI of two edges e_i and e_j is the second order partial derivative* *of reliability $R(G)$ of an undirected network with respect to reliabilities q_i and q_j of both edges:* $I_G(i,j) = \frac{\partial^2 R(G)}{\partial q_i \partial q_j}$.

When we consider the multi-state network with multi-state arc (or nodes), we can use (2.19)-(2.22) for finding the various joint importance measures in network systems. In all the measures arcs (or nodes) are referred to components of the MSS. In order to illustrate the use of UGF in finding joint importance measures in network systems we consider the following example.

6.5 Example: Sliding Window System

Consider a multi-state multiple sliding window system (MSWS) with $n = 5$, number of multi-state components, see Levitin (2004b). It generalizes the linear consecutive $k-out-of-r-from-n : F$ system consists of n linearly ordered multi-state components. Each multi-state component can have several different states: from complete failure up to perfect functioning. A performance rate is associated with each state. A set of integer numbers is defined such that any $r = 3$, or $r = 4$ corresponds to the number of consecutive multi-state components (length of sliding window). For each r the function, $f_3(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i - 5$ and $f_4(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} x_i - 6$, where x_i is the performance rate of ith component, named the acceptability function, is defined in such a manner that $f_r < 0$ constitutes the system failure. The MSWS fails if at least one of the functions f_r over the performance rates of any r consecutive components is negative. Each multi-state component has a total failure (corresponding to performance rate 0) and functioning with nominal performance rates 2, 2, 3, 1, and 2, respectively. The UGF of the individual multi-state components are

$$U_1(Z) = p_{1,1}Z^0 + p_{1,2}Z^2,$$

$$U_2(Z) = p_{2,1}Z^0 + p_{2,2}Z^2,$$

$$U_3(Z) = p_{3,1}Z^0 + p_{3,2}Z^3,$$

$$U_4(Z) = p_{4,1}Z^0 + p_{4,2}Z^1$$

$$\text{and } U_5(Z) = p_{5,1}Z^0 + p_{5,2}Z^2.$$

The UGF technique is used for evaluating the MSWS OPM, reliability R, without restriction to components. The system reliability is

$$R = p_{1,2}p_{2,2}p_{3,2}p_{4,1}p_{5,2} + (p_{1,1} + p_{1,2})p_{2,2}p_{3,2}p_{4,2}p_{5,2}. \tag{6.13}$$

We consider the following probability distribution of 5 multi-state components in Table 1. The non-zero OPM with component restriction are computed using (6.13) in Table.2. The proposed joint importance measures for pairs $(i, i+1)$, $i = 1, ..., 4$ are evaluated in table 3 and table 4. A numerical comparison can be made for pair of components using the size of the value of relevant measure with regard to their impact on system reliability and unreliability. It is also clear that the greatest joint importance is assigned to the pair (2,3) based on MRJBI and MJRAW.

Table 1. Probability distributions of components

Multi-state component i	$p_{i,0}$	$p_{i,1}$	$p_{i,2}$	$p_{i,3}$
1	0.2	0	0.8	0
2	0.3	0	0.7	0
3	0.24	0	0	0.76
4	0.39	0.61	0	0
5	0.01	0	0.99	0

Table 2. MSWS OPM-reliability with restriction to components performance,

$\alpha = 0.8, \beta = 0.9$

Multi-state components	OPM	-reliability
i=1	$OM_1^{\geq\alpha} =$	$2.p_{2,2}p_{3,2}p_{4,1}p_{5,2} = 0.4108104$
i=2	$OM_2^{\geq\alpha} =$	$p_{1,2}p_{3,2}p_{4,1}p_{5,2} + p_{3,2}p_{4,2}p_{5,2} = 0.6937128$
i=3	$OM_3^{\geq\alpha} =$	$p_{1,2}p_{2,2}p_{4,1}p_{5,2} + p_{2,2}p_{4,2}p_{5,2} = 0.760914$
i=4	$OM_4^{\geq\alpha} =$	$p_{2,2}p_{3,2}p_{5,2} = 0.52668$
i=5	$OM_5^{\geq\alpha} =$	$p_{1,2}p_{2,2}p_{3,2}p_{4,1} + p_{2,2}p_{3,2}p_{4,2} = 0.490504$
i=1	$OM_1^{<\alpha} =$	$p_{1,2}p_{2,2}p_{3,2}p_{5,2} = 0.421344$
i=1, j=2	$OM_{1,2}^{\geq\alpha,\geq\beta} =$	$p_{3,2}p_{4,1}p_{5,2} + p_{3,2}p_{4,2}p_{5,2} = 0.897336$
i=2, j=3	$OM_{2,3}^{\geq\alpha,\geq\beta} =$	$p_{1,2}p_{4,1}p_{5,2} + (p_{1,1} + p_{1,2})p_{4,2}p_{5,2} = 0.91278$
i=3, j=4	$OM_{3,4}^{\geq\alpha,\geq\beta} =$	$(p_{1,1} + p_{1,2})p_{2,2}p_{3,2}p_{4,2}p_{5,2} = 0.3212748$
i=1, j=2	$OM_{1,2}^{<\alpha,\geq\beta} =$	$p_{2,2}p_{3,2}p_{4,2}p_{5,2} = 0.3212748$
i=4, j=5	$OM_{4,5}^{<\alpha,\geq\beta} =$	$p_{1,2}p_{2,2}p_{3,2} = 0.4256$
i=3, j=4	$OM_{3,4}^{\geq\alpha,<\beta} =$	$p_{1,2}p_{2,2}p_{5,2} = 0.5544$

Table 3. MJRBI, MJRAW, MJRRW, MJRFV.

Multi-state components	MJRBI	MJRAW	MJRRW	MJRFV
i=1, j=2	0.5760612	0.5216	0	0
i=2, j=3	0.91278	0.8264	0	0
i=3, j=4	-0.2331252	0.2909	0	0
i=4, j=5	-0.1043252	0.2909	0	0

Table 4. MJrBI, MJrAW, MJrRW, MJrFV.

multi-state Components	MJrBI	MJrAW	MJrRW	MJrFV
i=1, j=2	-0.5760612	0	8.7224	0.8854
i=2, j=3	0.08722	0	6.2585	0.8401
i=3, j=4	0.2331252	0	1.04962	0.0473
i=4, j=5	0.1043252	0	1.4480	0.3094

Again pair (1,2) have greatest MJrAW and MJrFV. This pairs needs more safety and redundancy operations.

In any statistical problem where the complexity involved, one has to get some simple evaluation method. We gave a method for evaluation of joint importance measures, proposed in the previous chapters, based on UGF. The method is illustrated in a network system (signal transmission system).

Chapter 7

BAYESIAN INFERENCE FOR MULTI-STATE SYSTEMS

7.1 Introduction

In a non-Bayesian statistical reliability analysis, the analyst avoids the historical data that is available on similar systems. But in Bayesian analysis, it permits to include the information from past data in the overall assessment of system reliability. Let p denotes the reliability of a certain component at time t. Then p is assumed to belong to $[0, 1]$, but no values of p in this interval is given preference, even if one is quite certain that p is close to 1, say. This prior knowledge easily get lost in the model.

In Bayesian inference, one can introduce this kind of knowledge into the model by interpreting p as a random variable with some distribution $\pi(p)$, expressing what one thinks (believes) about the actual value of p. In MSS reliability evaluation, we have several advantages by adopting Bayesian approach; firstly, the expert opinion can be effectively reflected into the model even in the situation of rare data, see Aven and Hjorteland (2003). The basics of Bayesian theory can be seen in Berger (1985). This theory is used in MSS reliability analysis. In this chapter we consider the Bayesian inference of MSS reliability and joint importance measures. For that we shall consider the whole system reliability in terms of component's probability distributions. Then we find the Bayesian estimate of component's probability distributions to get Bayesian estimate of whole system reliability. This estimation procedure is mainly applicable to the multi-state signal transmission systems (network systems).

This chapter is arranged as follows. The role of Bayesian inference in MSS reliability is described in section 2. The Bayes estimation of MSS reliability and joint importance measure is described in section 3. An illustration in network system is given in section 4.

7.2 Bayesian Inference in MSS Reliability

We consider Bayesian estimation of MSS reliability and joint importance measures. In network systems, if we get the Bayesian estimate of component reliability, we can easily find the estimate of system reliability. It is possible only when system reliability is expressed in terms of component reliability. Otherwise the Bayesian inference is no longer simple. In the case of network systems, the system reliability can be expressed as combination of component reliabilities, so our approach is useful in network MSSs. We can use some well known statistical distributions, eg. Beta distribution, for obtaining the Bayesian estimate of reliability of components. Beta distribution has many advantage in Bayesian analysis over other distributions in interval [0,1], because it is in conjugate prior family, see Huang et al. (2006) for Beta prior selection in reliability analysis.

Weber and Jouffe (2006) presented a methodology that will help developing Bayesian Networks in complex dynamic models. Robinson (2001) discussed a hierarchical Bayes approach to system reliability analysis. Hamada et al. (2003) summarized the information about the probability of occurrence of each basic event in a multi-state fault tree using a probability distribution. The prior information for a basic event probability is in the form of a beta distribution denoted by Beta(a,b). If there are also basic event data available in the form of x failures, say, in n trials, then the posterior information for the basic event can be expressed as Beta(a+x,b+(n-x)).

7.3 Reliability and Joint Importance Measure Estimation

Suppose that the system reliability can be expressed as function of component's probability distribution, i.e.,

$$R = f(p_{1,0}, ..., p_{1,M}, p_{2,0}, ..., p_{2,M}, ..., p_{n,0}, ..., p_{n,M}),$$

where $p_{i,j}$ is the probability that the component i is in state j. Let p_{ij} is interpreted as the proportion of times component i would be in state j when considering an infinite or very large number of similar situations to the one analyzed.

The parameter p_{ij} is unknown, considered to be a random variable over $[0, 1]$, and our uncertainty related to its value is specified through a prior distribution $H_{ij}(p_{ij})$.

We assume that the the state of the component is either in state j or failed. Let the density function of prior distribution be

$$h_{ij}(p_{ij}) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p_{ij}^{\alpha-1}(1-p_{ij})^{\beta-1},\ \alpha, \beta \geq 0,\ 0 \leq p_{ij} \leq 1,$$

and p_{ij}'s are such that $p_{i0} + \cdots + p_{ij} + \cdots + p_{iM} = 1$, $j \in \{0, 1, 2, ..., M\}$. Suppose given n observations for each component, x_{ij} is the number times component type i is in state j. Then the posterior distribution of p_{ij} would be

$$h_{ij}(p_{ij}|x_{ij}) = \frac{\Gamma(n+\alpha+\beta)}{\Gamma(x_{ij}+\alpha)\Gamma(n-x_{ij}+\beta)} p_{ij}^{x_{ij}+\alpha-1}(1-p_{ij})^{n+\beta-x_{ij}-1},\ \alpha, \beta \geq 0,\ 0 \leq p_{ij} \leq 1.$$

Bayes point estimate of p_{ij} is $\hat{p}_{ij}$, which is the mean of the posterior distribution

$$\hat{p}_{ij} = \frac{x_{ij} + \alpha}{n + \alpha + \beta}, \tag{7.1}$$

if we consider squared error loss function.

Hence we get the Bayes estimate of MSS reliability

$$\hat{R} = f(\hat{p}_{1,1}, ..., \hat{p}_{1,M}, \hat{p}_{2,1}, ..., \hat{p}_{2,M}, ..., \hat{p}_{n,1}, ..., \hat{p}_{n,M}). \tag{7.2}$$

Substituting the Bayes estimates (7.1) of p_{ij}'s in various joint importance measures, we can easily obtain the Bayes estimate of multi-state joint importance measures.

In the next section, we give an illustration of the Bayes estimation for MSS reliability and joint importance measures.

7.4 Example: Network System

Once the estimate of the probability distribution of components is obtained we get the estimate of importance and joint importance measures also. We describe the evaluation of the output performance measure with reference to the five component MSWS discussed in Chapter 6. Each multi-state component can have several different states: from complete failure up to perfect functioning. A performance rate is associated with each state. A set of integer numbers is defined such that any $r = 3$, or $r = 4$ corresponds to the number of consecutive multi-state components (length of sliding window). For each r the function, $f_3(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i - 5$ and $f_4(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} x_i - 6$, where x_i is the performance rate of ith component, is defined in such a manner that $f_r < 0$ constitutes the system failure. The MSWS fails if at least one of the functions f_r over the performance rates of any r consecutive components is negative. Each multi-state component has a total failure (corresponding to performance rate 0) and functioning with nominal performance rates 2, 2, 3, 1, and 2, respectively. The UGF of the individual multi-state components are $U_1(Z) = p_{1,1}Z^0 + p_{1,2}Z^2$, $U_2(Z) = p_{2,1}Z^0 + p_{2,2}Z^2$, $U_3(Z) = p_{3,1}Z^0 + p_{3,2}Z^3$, $U_4(Z) = p_{4,1}Z^0 + p_{4,2}Z^1$ and $U_5(Z) = p_{5,1}Z^0 + p_{5,2}Z^2$.

The Bayes estimate of reliability is

$$\hat{O}M = \hat{R} = \hat{p}_{1,2}\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,1}\hat{p}_{5,2} + (\hat{p}_{1,1} + \hat{p}_{1,2})\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,2}\hat{p}_{5,2}.$$

Table 1. provides the Bayes estimates of performance measures with restriction to two components. For a numerical illustration, consider $p_{1,2} \sim Beta(2,3)$, $p_{2,2} \sim Beta(2,2)$, $p_{3,2} \sim Beta(2,1)$, $p_{4,1} \sim Beta(2,3)$, and $p_{5,2} \sim Beta(2,2)$. Suppose there are 5 observations on each component and $x_{1,2} = 3$, $x_{2,2} = 4$, $x_{3,3} = 3$, $x_{4,1} = 2$, and $x_{5,2} = 2$. Then the Bayes estimates would be $\hat{p}_{1,2} = 0.5$, $\hat{p}_{2,2} = 0.67$, $\hat{p}_{3,2} = 0.625$, $\hat{p}_{4,1} = 0.4$, and $\hat{p}_{5,2} = 0.444$. The Bayes estimate of joint importance measures is calculated in table 2.

Table 1. Bayes estimate of MSWS-reliability with restriction to components performance, $\alpha = .8, \beta = .9$

multi-state Components	OPM	reliability estimate
i=1	$\widehat{OM}_1^{\geq\alpha} =$	$2.\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,1}\hat{p}_{5,2}$
i=2	$\widehat{OM}_2^{\geq\alpha} =$	$\hat{p}_{1,2}\hat{p}_{3,2}\hat{p}_{4,1}\hat{p}_{5,2} + \hat{p}_{3,2}\hat{p}_{4,2}\hat{p}_{5,2}$
i=3	$\widehat{OM}_3^{\geq\alpha} =$	$\hat{p}_{1,2}\hat{p}_{2,2}\hat{p}_{4,1}\hat{p}_{5,2} + \hat{p}_{2,2}\hat{p}_{4,2}\hat{p}_{5,2}$
i=4	$\widehat{OM}_4^{\geq\alpha} =$	$\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{5,2}$
i=5	$\widehat{OM}_5^{\geq\alpha} =$	$\hat{p}_{1,2}\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,1} + \hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,2}$
i=1	$\widehat{OM}_1^{<\alpha} =$	$\hat{p}_{1,2}\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{5,2}$
i=1,j=2	$\widehat{OM}_{1,2}^{\geq\alpha,\geq\beta} =$	$\hat{p}_{3,2}\hat{p}_{4,1}\hat{p}_{5,2} + \hat{p}_{3,2}\hat{p}_{4,2}\hat{p}_{5,2}$
i=2,j=3	$\widehat{OM}_{2,3}^{\geq\alpha,\geq\beta} =$	$\hat{p}_{1,2}\hat{p}_{4,1}\hat{p}_{5,2} + (\hat{p}_{1,1} + \hat{p}_{1,2})\hat{p}_{4,2}\hat{p}_{5,2}$
i=3,j=4	$\widehat{OM}_{3,4}^{\geq\alpha,\geq\beta} =$	$(\hat{p}_{1,1} + \hat{p}_{1,2})\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,2}\hat{p}_{5,2}$
i=1,j=2	$\widehat{OM}_{1,2}^{<\alpha,\geq\beta} =$	$\hat{p}_{2,2}\hat{p}_{3,2}\hat{p}_{4,2}\hat{p}_{5,2}$
i=4,j=5	$\widehat{OM}_{4,5}^{<\alpha,\geq\beta} =$	$\hat{p}_{1,2}\hat{p}_{2,2}\hat{p}_{3,2}$
i=3,j=4	$\widehat{OM}_{3,4}^{\geq\alpha,<\beta} =$	$\hat{p}_{1,2}\hat{p}_{2,2}\hat{p}_{5,2}$

The Bayes estimates of joint importance measures are given below.

1. Joint Performance Achievement Worth:

$$MJPAW_{ij} = \frac{\widehat{OM}_{i,j}^{>\alpha,>\beta} - \widehat{OM}_{i,j}^{>\alpha,\leq\beta} - \widehat{OM}_{i,j}^{\leq\alpha,>\beta}}{\widehat{OM}_i^{>\alpha} + \widehat{OM}_j^{>\beta}}$$

2. Joint Performance Reduction Worth:

$$MJPRW_{ij} = \frac{\widehat{OM}_i^{\leq\alpha} + \widehat{OM}_j^{\leq\beta}}{\widehat{OM}_{i,j}^{\leq\alpha,\leq\beta}}$$

3. Joint Performance Fussel-Vesely Measure:

$$MJPFV_{ij} = \frac{\widehat{OM}_i^{\leq\alpha} + \widehat{OM}_j^{\leq\beta} - \widehat{OM}_{i,j}^{\leq\alpha,\leq\beta}}{\widehat{OM}_i^{\leq\alpha} + \widehat{OM}_j^{\leq\beta}}$$

4. Joint Performance Birnbaum Importance Measure:

$$MJPBI_{ij} = \widehat{OM}_{i,j}^{\alpha,>\beta} - \widehat{OM}_{i,j}^{\alpha,\leq\beta}$$

Table 2. Bayes estimate of joint importance measures

Pair of components	MJPAW	MJPBI
1,2	-0.2405	-0.07437
2,3	2.7517	0.9328
3,4	-0.1974	-.0.07437
4,5	-1.5487	-0.2394

The Bayes estimate of joint importance measures indicates that the pair $(2,3)$ need more safety and redundancy operations.

The above procedure provides a method of Bayes estimation of MSS reliability and joint importance measures by computing the posterior mean of components reliability. This procedure is applicable to MSSs like signal transmission system since the state of the signal transmission node will be in j or failed, the example illustrate the procedure. It also emphasis the use of Beta prior distribution when the system reliability is expressed in terms of components probability distributions.

Concluding Remarks and Future Research Directions

In many real life systems the states of the system are not binary, instead multi-state possessing some degradation process over time. When we consider the system state as random, the probabilistic concepts are incorporated into the system states for finding MSS performance measures. The most useful and frequently used measures are reliability, availability, risk(unreliability or unavailability) and expected performance. For instance, when the system is in continuous operation from the start of the installation, possessing some degradation from best state to worst state, reliability is used as the performance measure. Apart from this notion, suppose that we give some corrective maintenance actions after MSS is shutdown for a specified period of time or the system is not required in continuous operation, availability is used as the performance measure. In nuclear systems, the unreliability or unavailability is the suitable performance measure. There are large probabilistic safety systems in which the performance measure is usually risk. Similarly expected performance or average performance is also important as reliability or availability in system engineering. Under this requirement, we used the OPMs to obtain new results.

In the first chapter, we gave some preliminary details on BSS, MSS, association of random variables, ageing properties of lifetime random variables, and Bayesian inference. We gave details of the development of MSSs from that of BSSs. The definitions and some main properties of the existing MSSs are recalled there. The notion of *most commonly used importance measures is also drawn up. The motivation and the objectives for the current study are briefly sketched there. The main contribution of the present study is given at the end of the chapter.*

Recently, importance and joint importance measure got the attention of many researchers. Their query was, how can we measure the importance or joint importance of two components with respect to MSS output performance measures mentioned above. But we considered the problem of obtaining joint importance of more than two components with respect to system expected performance. When component reliabilities are unknown, we defined joint structural importance measures. The main advantage of such measures is the valuable information provided by them to provide safety and redundancy operations for a group of components based on their critical roles in interaction for the better performance of the whole system. Also we proposed a characterization result to identify the sign of multi-state JRI measure. Sometimes the system engineer may want only to know whether the group of components is more (or less) important, but no need of the exact importance. In such a situation this characterization becomes useful. The importance measures other than Birnbaum measure used in engineering and nuclear system applications is performance achievement worth, performance reduction worth and performance Fussel-Vesely measure. We defined joint performance achievement worth, joint performance reduction worth, joint performance Fussel-Vesely measure and joint performance Birnbaum measure for MSSs with respect to reliability, availability and risk. These measures give information regarding which pair provide more (or less) interaction effect for achievement *of performance measure, reduction of performance measure etc. Indeed, this chapter gave some basic foundation to the joint importance of two or more components in MSS reliability theory. The joint performance achievement worth, joint performance reduction worth, and joint Fussel-Vesely measure for more than two components are quite worthwhile objectives for future research. We can use these importance and joint importance measures in maintenance and replacement policies, which will be our next consideration.*

As stated in introductory chapter, there are various definitions of MSSs each of which are differentiated based on the component relevancy. In all such definitions, either all components are relevant to all states or all components have same relevancy structure. But many MSSs do not come under the existing classes. So reliability analysis using the existing MSS definitions caught problems when we compute importance and joint importance measures, since the main advantage of defining a new class of MSSs is to get importance and joint importance measures. In chapter 3, we proposed a new relevance condition and its generalization to overcome this difficulty. Also we defined the new classes in such a way that the existing classes are special cases of the new classes. The structural properties of the new classes of MSSs, and the problem of redundancy and bounds for the reliability are objectives for further research.

Ageing and its ramifications are well developed in binary reliability theory when the lifetime random variables are assumed to follow some statistical distributions.

Ageing properties of first passage time distribution in Markov chain/ Markov process have been appeared in reliability literature. But when we consider a MSS, it is interesting to get ageing properties of first passage time distribution of semi-Markov system. In chapter 4, we derived a necessary and sufficient condition for IFR/DFR and IFRA/DFRA properties of a semi-Markov system. The study of first passage time distribution or other relevant distributions with respect to other ageing properties may be a quite worthwhile future work.

Reliability bounds for the MSS can be obtained when components are associated. But this problem has been solved when MSS is modeled using Markov and semi-Markov process. In Chapter 5, we proposed the correlation measure in terms of transition probability function for measure the degree of association of Markov process, which is quite interesting in engineering and medical field. This measure is illustrated using a data set from medical field. Still a quite few problems of association in components of the MSSs are unexplored. The dependence ordering based on correlation in terms of transition probability function may open up new research directions. We need to explore stochastic ordering of two semi-Markov systems based on correlation order in the future period.

The new method of evaluation of reliability indices using UGF is found to be a fast procedure which avoid complexities arising in structure function approach, and stochastic process approach. In Chapter 6, we proposed a method for evaluating the joint importance measure for two components in the MSSs. It is illustrated in a radio *relay signal transmission system. This method can be further extended to three or more components, but the complications will increase. However we find good scope in UGF for the reliability analysis of MSSs.*

In any statistical study based on stochastic models, there is an associated inferential problem one needs to be studied. Since expert opinion can influence the engineering system reliability analysis, we can incorporate prior information into component reliability, so that Bayesian analysis for the inference is easily applied. In Chapter 7, we gave Beta prior to component reliability and obtained Bayes estimate for the MSS reliability and hence of various joint importance measures. The application of hyper priors in MSS reliability may also be of more practical utility.

The theory of MSSs are now developing, because many of the practitioners and engineers found it useful to model many systems. The combination of various types of multi-state systems with different criteria and constraints can produce many different interesting optimization problems. For instance, one may incorporate economical indices associated with different levels of system performance. This provides a wide range of models in which design, maintenance activity, warranty policy etc are optimized.

REFERENCES

1. Abouammoh, A. M. and Al-kadi (1991). On measures of importance for components in multi-state coherent system, *Microel. Reliab.*, **31**, 109-122.

2. Abouammoh, A. M. and Al-kadi (1995). Multi-state coherent system of order k, *Microel. Reliab.*, **35**, 1415-1421.

3. Armstrong, M. J. (1995). Joint reliability importance of components, *IEEE Trans. Reliab.*, **44**, 408-412.

4. Aven, T. (1985). Reliability evaluation of multi-state systems with multi-state components, *IEEE Trans. Reliab.*, **34**, 473-479.

5. Aven, T. (1988). Some considerations on reliability theory and its applications, *Reliab. Eng. Sys. Saft.*, **21**, 215-223.

6. Aven, T. (1993). On performance measures for multi-state monotone systems, *Reliab. Eng. Sys. Saft.*, **41**, 259-266.

7. Aven, T. and Hjorteland, A. (2003). A predictive Bayesian approach to multi-state reliability analysis, *In. J. Reliab. Qual. Saft. Eng.*, **10**, 221-234.

8. Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, New York.

9. Barlow, R. E. and Proschan, F. (1996). *Mathematical Theory of Reliability*, SIAM, New York.

10. Barlow, R. E. and Wu, A. S. (1978). Coherent system with multi-state components, *Math. Oper. Res.*, **3**, 275-281.

11. Baxter, L. A. (1984). Continuum structures. I, *J. Appl. Prob.*, **21**, 802-815.

12. Baxter, L. A. and Kim, C. (1986). Bounding the stochastic performance of continuum structure functions I, *J. Appl. Prob.*, **23**, 660-669.

13. Baxter, L. A. and Lee, S. M. (1990). A weak convergence theorem for continuum structure function, *J. Math. Anal. Appl.*, **148**, 403-408.

14. Belzunce, F., Ortega, E. M. and Ruizon, J. M. (2002). Ageing properties of first-passage times of increasing Markov processes, *Adv. Appl. Prob.*, **34**, 241-259.

15. Berger, J. O. (1985). *Statistical Decision Theory and Bayesian Analysis*, Springer, Berlin.

16. Bhat, B. R. (2000). *Stochastic Models-analysis and applications*, New Age Int., New Delhi.

17. Birnbaum, Z. W. (1969). On the importance of different components in a multicomponent system, in *Multivariate analysis II*, Ed. P. R. Krishnaiah, Academic Press, 591-592, NewYork.

18. Block, H. W., Griffith, W. and Savits, T. H. (1989). L-supperadditive structure functions, *Ann. Appl. Prob.*, **21**, 919-929.

19. Block, H. W. and Savits, T. H. (1982). A decomposition for multi-state monotone systems, *J. Appl. Prob.*, **19**, 391-402.

20. Block, H. W. and Savits, T. H. (1984). Continuous multi-state structure functions, *Oper. Res.*, **32**, 703-714.

21. Brown, M. and Chaganty, N. R. (1983). On the first passage time distribution for a class of Markov chains, *Ann. Prob.*, **11**, 1000-1008.

22. Brunelle, R. D. and Kapur, K. C. (1999). Review and classification of reliability measures for multi-state and continuum models, *IIE Trans.,* **31**, 1171-1180.

23. Bryson, M. C. and Siddiqui, M. M. (1969). Some criteria for aging, *J. Amer. Stat. Asso.*, **64**, 1472-1483.

24. Bueno, V. C. (1989). On the importance of components for multi-state monotone systems, *Stat. Prob. Lett.*, **7**, 51-59.

25. Chacko, V. M. (2008a). Joint Birnbaum importance measure to the multi-state system, Presented at *International Conference on New Trends in Statistics and Optimization*, University of Kashmir, Srinagar, Octo. 20-23.

26. Chacko, V. M. (2008b). Joint importance measures of the multi-state *k-out-of-r-from-n: F* system, *J. Kerala Stat. Asso.*, **19**, 19-30.

27. Chacko, V. M. and Manoharan, M. (2008a). Joint importance measures for the multi-state system, *Advances in Performance and Safety of Complex systems*, Eds. A. K. Verma, P. K. Kapur and S. G. Ghadge, Macmillan, 308-314,

New Delhi.

28. Chacko, V. M. and Manoharan, M. (2008b). A new class of multi-state coherent systems, Presented at *International Conference on Statistics and its Applications to Management*, Indian Institute of Management, Kozhikode, May 1-3.

29. Chacko, V. M. and Manoharan, M. (2008c). On importance measures of multi-state systems, Presented at *National Seminar on Recent Developments in Probability and Mathematical Statistics*, Department of Statistics, University of Mysore, Mysore, Oct 29-31.

30. Chacko, V. M. and Manoharan, M. (2009a). Joint criticality importance measures for components in MEMS devices, *Proc. ICMEMS 2009* (ISBN-978-81-908468-0-6), Indian Institute of Technology Madras, Chennai, PO.3.

31. Chacko, V. M. and Manoharan, M. (2009b). Ageing properties of first passage time distribution of multi-state reliability system, *ProbStat Forum: e-journal*, **2**, 22-30.

32. Chacko, V. M. and Manoharan, M. (2009c). Joint importance measures in network system, Presented at *National Seminar on Recent Advances in Statistics*, NAS college, Kanjhangad, Kerala, Feb. 14-15.

33. Cheok, M. C., Parry, G. W. and Sherry, R. R. (1998). Use of importance measures in risk informed applications, *Reliab. Eng. Sys. Saft.*, **60**, 213-226.

34. Deshpande, J. V., Kochar, S. C. and Singh, H. (1986). Aspects of positive aging, *J. Appl. Prob.*, **23**, 748-758.

35. Dharmadhikari, A. D. and Dewan, I. (2006). Association in time of a vector valued process, *Stat. Prob. Lett.* **76**, 1147-1155.

36. Doob, J. L. (1953). *Stochastic Process*, John Wiley, New York.

37. Ebrahimi, E. (1991). How to compare two systems, *J. Appl. Prob.*, **28**, 940-946.

38. El.Neweihi, E. and Proschan, F. (1984). Degradable systems: a survey of multi-state reliability theory, *Comm. Stat. Theory and Methods*, **13**, 403-432.

39. El-Neweihi, E., Proschan, F., and Sethuraman, J. (1978). Multi-state coherent systems, *J. Appl. Prob.*, **15**, 675-688.

40. Esary, J. D. and Proschan, F. (1963). Coherent structures of nonidentical components, *Technometrics*, **5**, 191-209.

41. Esary, J. D. and Proschan, F. (1970). A reliability bound for systems of maintained independent components, *J. Amer. Stat. Asso.,* **65**, 329-398.

42. Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables with applications, *Ann. Math. Stat.*, **38**, 1466-1474.

43. Funnemark, E. and Natvig, B. (1985). Bounds for the availabilities in a fixed time interval for multi-state monotone systems, *Adv. Appl. Prob.*, **17**, 638-655.

44. Gandini, A. (1990). Importance and Sensitivity analysis in assessing system reliability, *IEEE Trans. Reliab.*, **39**, 61-69.

45. Gopal, G. (2002). Schur property of the performance function for the multi-state coherent system, *Proc. 3 In. Conf. Math. Methods. Reliab.*, 255-258, Trondheim, Norvey.

46. Griffith, W. S. (1980). Multi-state reliability models, *J. Appl. Prob.*, **17**, 735-744.

47. Hamada, M., Martz, H. F., Reese, C. S., Graves, T., Johnson, V. and Wilson, A. G. (2003). A fully Bayesian approach for combining multilevel failure information in fault tree quantification and corresponding optimal resource allocation, *Paper*, **19**, School of Public health, University of Michigam.

48. Hjort, N. B., Natvig, B. and Funnemark, E. (1985). The association in time of a Markov process with applications to multi-state reliability theory, *J. Appl. Prob.*, **22**, 473-479.

49. Hong, J. S. and Lie, C. H. (1993). Joint reliability importance of two edges in an undirected network, *IEEE Trans. Reliab.*, **47**, 97-101.

50. Hong, J. S., Koo, H. Y. and Lie, C. H. (2002). Joint reliability importance of *k-out-n* systems, *Europ. J. Oper. Res.,* **42**, 539-547.

51. Hong, J. S., Koo, H. Y. and Lie, C. H. (2000). Computation of joint reliability importance measures of two gate events in a fault tree, *Reliab. Eng. Sys. Saft.*, **68**, 1-5.

52. Huang, H. Z., Zio, M. J. and Sun, Z. Q. (2006). Bayesian reliability analysis for fuzzy lifetime data, *Fuzzy sets. Sys.*, **157**, 1674-1686.

53. Hu, T. and Pan, X. (2000). Comparisons of dependence for stationary Markov processes, *Prob. Eng. Inform. Sci.*, **14**, 299-315.

54. Hudson, J. C. and Kapur, K. C. (1982). Reliability theory for multi-state systems with multi-state components, *Microel. Reliab.*, **22**, No.1, 1-7.

55. Jun, C. H., Chang, S. Y., Hong, Y. and Yang, H. (1999). A Bayesian approach to prediction of system failure rates by criticalities under event trees, *In. J. Prod. Econ.*, **61**, 623-628.

56. Karlin, S. (1983). Association arrays in assessing forms of dependence between bivariate random variables, *Proc. Natl. Acad. Sci*, **80**, 647-651.

57. Klefjo, B. (1982) The HNBUE and HNWUE classes of life distributions, *Nav. Res. Log. Quart.*, **29**, 331-344.

58. Kuber, M. and Dharmadhikari, A. D. (1996). Association in time of a finite semi-Markov process, *Stat. Prob. Lett.*, **26**, 125-133.

59. Kolowrocki, K. (2004). *Reliability of Large Systems*, Elsevier, NewYork.

60. Lai, C. D., Xie, M. and Murthy, D. N. P. (2001). Bathtub-shaped failure rate life distributions, *Handbook of Statistics*, Eds. N. Balakrishnan and C. R. Rao, **20**, 69-103.

61. Lehmann, E. (1966). Some concepts of dependence, *Ann. Math. Stat.*, **37**, 1137-1153.

62. Levitin, G. (2002a). Optimal reliability enhancement for multi-state transmission networks with fixed transmission times, *Reliab. Eng. Syst. Saft.*, **76**, 287-299.

63. Levitin, G. (2002b). Optimal allocation of multi-state elements in Linear consecutively connected system with delays, *In. J. Reliab. Qual. Saft. Eng.*, **9**, 89-108.

64. Levitin, G. (2003). Common supplay failures in linear multi-state sliding window system, *Reliab. Eng. Sys. Saft.*, **82**, 55-62.

65. Levitin, G. (2004a). A universal generating function approach for analysis of multi-state systems with dependent elements, *Reliab. Eng. Sys. Saft.*, **84**, 285-292.

66. Levitin, G. (2004b). Reliability of linear multi-state multiple sliding window system, *Reliab. Eng. Sys. Saft*, **85**, 385-395.

67. Levitin, G. and Lisnianski, A. (1999). Importance and sensitivity analysis of multi-state systems using universal generating functions method, *Reliab. Eng. Sys. Saft.*, **65**, 271-282 .

68. Levitin, G., Podofillini, L. and Zio, E. (2003). Generalized importance measures for multi-state elements based on performance level restrictions, *Reliab. Eng. Sys. Saft.*, **82**, 287-298.

69. Limnios, N. (1997). Dependability analysis of semi-Markov systems, *Reliab. Eng. Sys. Saft.*, **55**, 203-207.

70. Lisnianski, A. and Levitin, G. (2003). *Multi-state System Reliability-assessment, optimization and applications*, World Scientific, Singapore.

71. Manoharan, M. (1995). On a sufficient condition for association in time of a process, *Proc. XV Indian Society of Probability and Statistics Conference* (M. S. University, Tirunelveli, India), 65-66.

72. Manoharan, M. and Chacko, V. M. (2009). On association in time of a Markov process, Communicated.

73. Meng, F. C. (1993). Component relevancy and characterization results in multi-state systems, *IEEE Trans. Reliab.*, **42**, 478-483.

74. Natvig, B. (1982). Two suggestions of how to define a multi-state coherent system, *Adv. Appl. Prob.*, **14**, 434-455.

75. Natvig, B. (1993). Strict and exact bounds for the availabilities in a fixed time interval for multi-state monotone systems, *Scand. J. Stat.*, **20**, 171-175.

76. Natvig, B. and Morch, H. W. H. (2003). An application of multi-state reliability theory to an offshore gas pipeline network, *In. J. Reliab. Qual. Saft. Eng.*, **10**, 361-381.

77. Natvig, B., Sormo, S., Holen, A. T. and Hogansen, G. (1986). Multi-state reliability theory-a case study, *Adv. Appl. Prob.,* **18**, 921-932.

78. Patra, S. and Misra, R. B. (1996). Evaluation of probability mass function of flow in a communication network considering a multi-state model of network

links, *Microel. Reliab.*, **36**, 415-621.

79. Prakash Rao, B. L. S. and Dewan, I. (2001). Associated sequences and related inference problems, *Hand book of Statistics*, Eds. D. N. Shanbhag and C. R. Rao, **19**, 693-729.

80. Pyke, R. (1961). Markov renewal processes: definitions and preliminary properties, *Ann. Math. Stat.*, **32**, 1231-1242.

81. Ramirez-Marquez, J. E. and Coit, D. W. (2005a). Multi-state component criticality analysis in multi-state systems, *Advances in Safety and Reliability*, Ed. K. Kolowrocki, Taylor and Francis group, 671-1677, London.

82. Ramirez-Marquez, J. E. and Coit, D. W. (2005b). Composite importance measures for multi-state systems with multi-state components, *IEEE Trans. Reliab.*, **54**, 517-528.

83. Rausand, M. and Hoyland, A. (2004). *System Reliability Theory-models, statistical methods and application*, Wiley Int., Canada.

84. Ross, S. M. (1979). Multivariate state component systems, *Ann. Prob.*, **7**, 379-383.

85. Robinson, D. G. (2001). A hierarchical Bayes approach to system reliability analysis, *Sandia Report*, Sandia National Laborataries, NewYork.

86. Thripathy, C. R., Patra, S., Misra, R. B. and Mahapatra, N. (1996). Reliability evaluation of multi-state interconnection networks with multi-state elements,

Microel. Reliab., **36**, 423-428.

87. Ushakov, I. (1987). A universal generating function, *Soviet J. Comp. Syst. Sci.*, **24**, 37-49.

88. Ushakov, I. (2000). The method of generalized generating sequences, *Euro. J. Oper. Res.*, **125**, 316-323.

89. Van der Borst, M. and Shoonakker, H. (2001). An overview of PSA importance measures, *Reliab. Eng. Sys. Saft.*, **72**, 241-245.

90. Vasseur, D. and Llory, M. (1999). International survey on PSA figures of merit, *Reliab. Eng. Sys. Saft.*, **66**, 261-274.

91. Weber, R. and Jouffe, L. (2006). Complex system reliability modelling with dynamic object oriented Bayesian networks, *Rel. Eng. Sys. Saft.*, **91**, 149-162.

92. Wu, S. (2005). Joint importance of multi-state systems, *Comp. Indust. Eng.*, **49**, 63-75.

93. Wu, S. and Chan, L. Y. (2005) Performance utility-analysis of multi-state systems, *IEEE Trans. Reliab.*, **50**, 14-20.

94. Yeh, W. C. (2006). The *k-out-of-n* acyclic multi-state node networks reliability evaluation using universal generating function method, *Reliab. Eng. Syst. Saft.*, **91**, 800-808.

95. Yi, Z. and Weng, C. (2006). On the correlation order, *Stat. Prob. Lett.*, **76**, 1410-1416.

96. Zio, E. and Podofillini, L. (2003). Importance measures of multi-state components in multi-state systems, *In. J. Reliab. Qual. Saft. Eng.*, **10**, 289-310.

97. Zio, E. and Podofillini, L. (2006). Accounting for components interactions in the differential importance measure, *Reliab. Eng. Sys. Saft.*, **91**, 1163-1174.

CONFERENCE PRESENTATIONS

1. International Conference on Reliability, Quality and Safety Engineering, *Nuclear Power Corporation India (Ltd) (jointly with IIT Bombay)*, Jan 5-8, 2008, (with Manoharan, M) Joint importance measures of the multi-state system

2. National Seminar on Recent Advances in Statistics and Analysis of Non-conventional Data, *Farook College, Calicut*, March 15-17, 2008, Joint importance measure of a multi-state k-out-of-r-from-n: F system

3. International Conference on Statistics and its Application to Management, *Indian Institute of Management, Kozhikode*, May 1-3, 2008, (with Manoharan, M) A new class of multi-state coherent systems

4. International Conference on new trends in Statistics and Optimization, *University of Kashmir, Srinagar*, Octo 20-23, 2008, Joint Birnbaum importance measure of the multi-state system

5. National Seminar on Recent Developments in Probability and Mathematical Statistics, *University of Mysore, Mysore,* Octo 29-31, 2008, (with Manoharan, M) On importance measure of the multi-state system

6. International Conference on MEMS 2009, *Indian Institute of Technology Madras, Chennai,* Jan 3-5, 2009, (with Manoharan, M) Joint criticality importance measures of components in MEMS devices

7. National Seminar on Recent trends in Statistics, *Nehru College, Kanjhangad,* Feb 11, 2009, (with Manoharan, M) Joint importance measure multi-state network system