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CERTIFICATE

This is to certify that the thesis titled Integral Transforms and Generalized Functions submitted by Ms. Geetha K. V., part-time research scholar, Department of Mathematics, St. Joseph's College, Irinjalakuda, in partial fulfillment of the requirement for the degree of Doctor of Philosophy in Mathematics, to the University of Calicut, is a bonafide record of research work undertaken by her in this department under my supervision during the period 2006–2009 and that no part thereof has been presented before for any other degree.

Irinjalakuda **Mangalambal N. R.**

18-02-2009 (Supervising Teacher)

DECLARATION

I hereby declare that this thesis titled *Integral Transforms and Generalized Functions* is the record of bona-fide research I carried out in this Centre under the supervision of Dr. N. R. Mangalambal, Reader, Department of Mathematics, St. Joseph's College,
Irinjalakuda. I further declare that this thesis, or any part thereof, has not previously formed the basis for the award of any other degree, diploma or any other similar title of recognition.

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Introduction

Ever since Paul Dirac introduced the so-called delta function and made use of the notion in the study of quantum mechanics in the late 1920's attempts were made by several mathematicians to give a mathematical basis for the idea. This led to the formulation and study of the concept called 'generalized functions'. Early attempts in this direction were made by Bochner S. in 1932 and Sobolev S. L. in 1936. With the publication of the monograph 'Theorie des Distributions' by L. Schwartz in 1950–51, the study of generalized functions attained wide popularity. The first major contribution in this area which can be qualified as a milestone was a book of the same name in two volumes by L. Schwartz [**37**]. From then on, the theory of generalized functions was developed intensively by many mathematicians. The application of integral transforms to generalized functions has been effectively used to solve problems which can be expressed in terms of differential equations or boundary value problems involving generalized functions.

Several attempts are being made to develop the theory of generalized functions and application of integral transforms to generalized functions. Zemanian [**50**] has extended several integral transforms to generalized functions using the method of adjoints. The idea of Boehmians motivated by T. K. Boehme [**3**] and studied and developed by P. Mikusinski [**21**] and others has been used for the application of integral transforms to generalized functions. J. F.

Colombeau [**7**] has studied the generalized functions by admitting an operation of multiplication. Recently, the theory and application of more general objects than generalized functions called hyperfunctions have been developed by Sato [**34**], Zahrinov [**48**], Komatsu et al [**16**].

The extensions of various integral transforms like the Laplace, the Hankel, the Mellin, the Weierstrass and the convolution transforms have been done by Zemanian [**50**]. V. S. Vladimirov [**44**] has studied the properties of generalized functions, the application of integral transforms and Tauberian theorems for the Laplace transform. H. H. Schaefer [**36**], Anthony L. Peressini [**29**] have studied ordered topological vector spaces, the order relation being introduced through the notion of positive cones. Based on the works of Zemanian and Vladimirov we have applied the notion of ordered topological vector spaces to the test function spaces and their duals.

The space $\mathcal{D}$ of smooth functions with compact support in $\mathbb{R}^p$ with the inductive limit topology and its dual $\mathcal{D}'$, the space of distributions are the basic background in the study of distribution theory [**33**]. It has been observed that when $\mathcal{D}'$ is ordered by the dual cone C' of the cone of non-negative functions in $\mathcal{D}$, $\mathcal{D}^+$, the order dual of $\mathcal{D}$ is a proper subset of $\mathcal{D}'$ with the topology of pointwise convergence assigned to $\mathcal{D}'$. However, when the topology of bounded convergence is assigned to $\mathcal{D}'$, since the dual cone C' is a normal cone it follows that $\mathcal{D}'$ is a reflexive space ordered by a closed normal cone, the order dual and the topological dual of $\mathcal{D}$ coincide and that $\mathcal{D}'$ is both order complete and topologically complete. Thus $\mathcal{D}'$ is a vector lattice.

We have applied the above idea to the spaces $\mathscr{E}$, ζ, $\mathcal{L}_{a,b}$, $\mathcal{L}(w,z)$, $M^m_{a,b}$, $M^m_{a,b,c}$ and to the corresponding dual spaces $\mathscr{E}'$, ζ', $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w,z)$, $(M^m_{a,b})'$, $(M^m_{a,b,c})'$ and extended the Fourier, the Laplace, the Stieltjes transforms to the corresponding dual spaces. The order properties and continuity properties of the above integral transforms and the inverses defined on these dual spaces with respect to the topology of bounded convergence are subjected to study. Comparison of the solutions of differential equations which can be solved by the application of the above integral transforms to generalized functions is done using illustrations.

The preliminary chapter contains the definitions and basic properties of ordered topological vector spaces, order topology, multinormed spaces, countable union spaces, Schwartz topology on $\mathcal{D}$.

In the first chapter, the notions of ordered multinormed space, ordered countable union space, normal cone, b-cone, topology of bounded convergence in the dual of ordered multinormed space, ordered countable union space are defined. The convolution and direct product of elements in $\mathcal{D}'$ are defined and their order properties and continuity properties are studied. The compatibility of the above operations with the lattice properties in $\mathcal{D}'$ is also proved.

In the second chapter, the spaces $\mathscr{E}$, ζ and their duals $\mathscr{E}'$, the space of distributions with compact support and ζ', the space of tempered distributions are studied as ordered topological vector spaces.

$\mathscr{E}'$, ζ' are equipped with the topology of bounded convergence. Order property and continuity of convolution and direct product on $\mathscr{E}'$, ζ' are proved. Fourier transform is applied to the elements of ζ'. Comparison of fundamental solutions of two differential equations which can be solved by the application of the Fourier transform is also done.

In the third chapter, the ordered multinormed spaces $\mathcal{L}_{a,b}$, the ordered countable union space $\mathscr{L}(w,z)$ and their respective duals $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w,z)$ with the topology of bounded convergence assigned to them are studied. The Laplace transform is applied to the ordered topological space of Laplace transformable functions and the order properties of the transform and its inverse are studied. Comparison of the solutions of differential equations solved by the application of Laplace transform is also done.

In the fourth chapter the Stieltjes transform is applied to the ordered linear space $(M^m_{a,b})'$ of generalized functions to which the topology of bounded convergence is assigned. The Abelian and Tauberian theorems for the Stieltjes transform in the new context are proved. Corollaries extending the result to monotone nets are also proved.

In the fifth chapter, we have applied a combination of the Laplace and the Stieltjes transforms to an ordered vector space of generalized functions to which the topology bounded convergence is assigned. Some of the order properties of the transform and its inverse are studied. Also we establish the operational transform formula.

In the sixth chapter, the notions of the asymptotic of a function of order α in the wedge W, that of the strongasymptotic of a generalized function in $\zeta'(W)$ of order α at ∞ are defined and the compatibility of these notions with the lattice properties in $\mathcal{D}'(W)$, $\zeta'(W)$ is proved. The holomorphic functions defined on T^V, the tube region, form a convolution algebra $H(W)$ which is isomorphic to $\zeta'(W)$ via the Laplace transformation. We define an order relation on $H(W)$ by identifying a cone in $H(W)$ and assign a topology to $H(W)$ with respect to which the above cone is normal. The notion of elements in $H(W)$ having strongasymptotic is defined and is observed to be compatible with the lattice properties in $H(W)$. The Tauberian and Abelian theorems in this new background for the Laplace transform are proved. Corollaries extending the result of the theorems to monotone nets are also proved. A special case of the Tauberian theorem applied to the one dimensional case is also proved.

Relevance of the study. In all the ordered linear spaces we have studied the respective positive cones are generating so that it was sufficient if the results were studied on the positive cone. In the present background the duals of all test function spaces form topological vector lattices. So, further lattice theoretic properties of these spaces, order preserving transforms on them and concerned characterization theorems may be taken up for study. We have illustrated that solutions of differential equations under various boundary value conditions are comparable. These comparable solutions form a chain. Further lattice properties of such chains can also be subjected to study.

CHAPTER 0

Preliminary

The basic background used in our study are ordered topological vector spaces, countable union spaces, order topology, the topological vector space $\mathcal{D}$ and the space of distributions $\mathcal{D}'$. The preliminary chapter contains the definitions and some properties of the above ideas which have been taken from [**29, 33, 36**] and [**50**].

DEFINITION 0.1.1. [**29**] An ordered vector space is a real vector space E equipped with a transitive, reflexive, antisymmetric relation $\leq$ satisfying the following conditions

(1) If $x, y, z \in E$ and $x \leq y$ then $x + z \leq y + z$

(2) If $x, y \in E$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$ then $x \leq y \Rightarrow \alpha x \leq \alpha y$.

Note. Schaefer [**36**] defines a real underlying space of a vector space as follows: If L is a vector space (or a topological vector space) over a field $K = H(i)$ containing i then the restriction of scalar multiplication to $H \times L$ to L turns L into a vector space (or a tvs) L_0 over H. L_0 is called the real underlying space of L. A vector space E over $\mathbb{C}$, the complex numbers is said to be ordered if its underlying real space E_0 is an ordered vector space over $\mathbb{R}$.

DEFINITION 0.1.2. [**29**] The positive cone C in an ordered real vector space E is defined by $C = \{x \in E : x \geq 0\}$.

Note. The cone C has the following properties.
(1) $C + C \subseteq C$
(2) $\alpha C \subseteq C$, for $\alpha > 0$, $\alpha \in \mathbb{R}$
(3) $C \cap (-C) = \{0\}$.

DEFINITION 0.1.3. Let E be a vector space over $\mathbb{C}$, the complex numbers. If C is the positive cone of E when the field of scalars is restricted to $\mathbb{R}$ then the cone in E is $C + iC$ which is also denoted as C.

Note. [**29**] A subset C of E as defined in 0.1.2 defines an order relation $\leq$ on E as follows: for $x, y \in E$, $x \leq y$ if $y - x \in C$. With respect to this order relation E is an ordered vector space whose positive cone is C.

DEFINITION 0.1.4. [**29**] A subset W of E containing 0 and satisfying
(1) $W + W \subseteq W$
(2) $\alpha W \subseteq W$, $\alpha > 0$, $\alpha \in \mathbb{R}$
is called a wedge.

DEFINITION 0.1.5. [**29**] If E is an ordered vector space and $x, y \in E$, $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is the order interval between x and y.

DEFINITION 0.1.6. [**29**] If E is an ordered vector space with positive cone C and if A is subset of E, the full hull of A denoted as

$[A]$ is defined as

$$[A] = \{z \in E : x \leq z \leq y, x, y \in A\}$$

i.e., $[A] = (A + C) \cap (A - C)$. If $A = [A]$, A is said to be full.

DEFINITION 0.1.7. [**29**] A subset B of E is said to be order-bounded if there exists $x, y \in E$ such that $B \subseteq [x, y]$.

DEFINITION 0.1.8. [**29**] A subset D of E is majorized (resp. minorized) in D if there exists an element $z \in E$ such that $z \geq d$ (resp. $z \leq d$) for all $d \in D$.

DEFINITION 0.1.9. [**29**] If every pair of elements of a subset D of E is majorized (minorized) in D then D is directed ($\leq$) (resp. directed $\geq$).

DEFINITION 0.1.10. [**50**] If $\mathscr{V}$ is a linear space, a seminorm on $\mathscr{V}$ is a rule γ that assigns a real number $\gamma(\phi)$ to each $\phi \in \mathscr{V}$ and satisfies
(1) $\gamma(\alpha\phi) = |\alpha|\gamma(\phi)$, $\phi \in \mathscr{V}$, $\alpha \in \mathbb{C}$
(2) $\gamma(\phi + \psi) \leq \gamma(\phi) + \gamma(\psi)$, $\phi, \psi \in \mathscr{V}$.

Note. [**50**] (1) Clearly $\gamma(0) = 0$, $\gamma(\phi) \geq 0$, $\forall \phi \in \mathscr{V}$.
(2) A seminorm is a norm if it satisfies the additional condition $\gamma(\phi) = 0 \Rightarrow \phi = 0$.

DEFINITION 0.1.11. [**50**] Let $S = \{\gamma_\alpha\}_{\alpha\in J}$ be a collection of seminorms on a linear space $\mathscr{V}$. The collection S is said to be separating if for every $\phi \in \mathscr{V}$, $\phi \neq 0$, there is at least one $\gamma_\alpha \in S$ such that $\gamma_\alpha(\phi) \neq 0$. If S is separating, S is called a countable multinorm.

DEFINITION 0.1.12. [**50**] Let $\mathscr{V}$ be a linear space and $S = \{\gamma_\alpha\}_{\alpha\in J}$ be a collection of seminorms, not necessarily separating. Given a finite collection of seminorms $\{\gamma_{\alpha_i}\}_{i=1}^n$, a balloon centered at $\psi \in \mathscr{V}$ is the set of all $\phi \in \mathscr{V}$ such that $\gamma_{\alpha_i}(\phi - \psi) < \epsilon_i$, $i = 1, 2, \ldots, n$ where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are arbitrary positive real numbers.

Note. [**50**] Clearly the intersection of two ballons centered at the same point ψ is also a ballon at ψ.

DEFINITION 0.1.13. [**50**] A neighbourhood of $\psi \in \mathscr{V}$ is any set in $\mathscr{V}$ that contains a balloon centered at ψ. The collection of all neighbourhoods of all points of $\mathscr{V}$ is the topology of $\mathscr{V}$ generated by the multinorm S.

DEFINITION 0.1.14. [**50**] A multinormed space $\mathscr{V}$ is a linear space having a topology generated by a multinorm S. If S is countable, $\mathscr{V}$ is called a countably multinormed space.

DEFINITION 0.1.15. [**50**] Let $\{\mathscr{V}_m\}_{m=1}^\infty$ be a sequence of countably multinormed spaces such that $\mathscr{V}_1 \subseteq \mathscr{V}_2 \subseteq \ldots$. Also assume that the topology of each $\mathscr{V}_m$ is stronger than the topology induced on it by $\mathscr{V}_{m+1}$.

Let $\mathscr{V} = \cup_{m=1}^{\infty}\mathscr{V}_m$. $\mathscr{V}$ is a linear space. A sequence $(\phi_i)_{i=1}^{\infty}$ is said to converge to $\phi \in \mathscr{V}$ if all ϕ_i and ϕ belong to the same $\mathscr{V}_m$ and $\phi_i \to \phi$ in $\mathscr{V}_m$ (and hence in $\mathscr{V}_{m+1}, \mathscr{V}_{m+2}, \ldots$). In this case $\mathscr{V}$ is called a countable union space.

DEFINITION 0.1.16. [**50**]A countable union space $\mathscr{V} = \cup_{m=1}^{\infty}\mathscr{V}_m$ is called a strict countable union space if for each m the topology of $\mathscr{V}_m$ is identical to the topology induced on it by $\mathscr{V}_{m+1}$.

DEFINITION 0.1.17. [**33**] A multi-index is an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers and $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$ whose order is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\alpha = 0$, $D^\alpha f = f$.

DEFINITION 0.1.18. [**33**] A complex-valued function f defined on some non-empty open set $I \subseteq \mathbb{R}^n$ belongs to $\mathscr{C}^\infty(I)$ if $D^\alpha f \in \mathscr{C}(I)$ for every multi-index α, where $\mathscr{C}(I)$ denotes the set of all continuous functions defined on I.

DEFINITION 0.1.19. [**33**] If K is any compact set in $\mathbb{R}^n$, $\mathcal{D}_K$ denotes the linear space of all $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ whose support lies in K.

DEFINITION 0.1.20. [**33**] Let (K_i) be a sequence of compact sets in $\mathbb{R}^n$ such that $K_i \subseteq K_{i+1}^0$ for $i \in \mathbb{N}$ and let $I = \cup_{i=1}^{\infty} K_i$. Define $p_n(f) = \max\{|D^\alpha f(x)| : x \in K_n, |\alpha| \leq n\}$. The seminorms $\{p_n\}_{n=1}^{\infty}$ define a metrizable, locally convex topology on $\mathscr{C}^\infty(I)$

with respect to which $\mathscr{C}^\infty(I)$ is a Frechet space. For each $K \subseteq I$ $\mathcal{D}_K$ is a closed subspace of $\mathscr{C}^\infty(I)$ and hence is a Frechet space.

DEFINITION 0.1.21. [**33**] If I is any non-empty open set in $\mathbb{R}^n$, $\mathcal{D}(I) = \cup \mathcal{D}_K$ as K varies over compact subsets of I is the test function space $\mathcal{D}(I)$. When $I = \mathbb{R}^n$, $\mathcal{D}(I)$ is denoted as $\mathcal{D}$.

Note. [**33**] $\mathcal{D}(I)$ is a linear space with respect to the addition and scalar multiplication of complex functions. $\phi \in \mathcal{D}(I)$ if and only if $\phi \in \mathscr{C}^\infty(I)$ and the support of ϕ is a compact subset of I.

DEFINITION 0.1.22. [**33**] For $\phi \in \mathcal{D}(I)$ define

$$\|\phi\|_n = \max\{|D^\alpha \phi(x)| : x \in I,\ |\alpha| \leq n\},\ \phi \in \mathcal{D}(I),$$

$n = 0, 1, 2, \ldots$. The collection $\{\| \ \|_n\}$ is a collection of seminorms on $\mathcal{D}(I)$ and induce a topology on $\mathcal{D}^n(I)$ such that the restriction of these norms to any fixed $\mathcal{D}_K$ induce the same topology on $\mathcal{D}_K$ as defined in 0.1.8. This topology on $\mathcal{D}(I)$ is called the Schwartz topology on $\mathcal{D}(I)$.

DEFINITION 0.1.23. [**29**] If A, B are subsets of a vector space E then A absorbs B if there is a constant $\lambda_0 > 0$ such that $\lambda B \subset A$ for all scalars λ such that $|\lambda| \leq \lambda_0$.

DEFINITION 0.1.24. [**29**] A subset B of a topological vector space $E(\tau)$ is τ-bounded if B is absorbed by each neighbourhood of 0 in $E(\tau)$.

DEFINITION 0.1.25. [**29**] Let $\mathcal{L}(E, F)$ denote the linear space of all continuous linear mappings of a topological vector space $E(\tau_1)$ into $F(\tau_2)$ and let $N(G, V) = \{T \in \mathcal{L}(E, F) : T(G) \subset V\}$, where G is a τ_1-bounded subset of $E(\tau_1)$ and V is a neighbourhood of 0 in F for τ_2. If V is convex, $N(G, V)$ is convex for each τ_1-bounded subset G of E. The topology on $\mathcal{L}(E, F)$ having the collection of all sets $N(G, V)$ as mentioned above as a neighbourhood basis for 0 is a locally convex topology if τ_2 on F is locally convex. This topology on $\mathcal{L}(E, F)$ is called the topology of bounded convergence on $\mathcal{L}(E, F)$.

DEFINITION 0.1.26. [**29**] A linear map $T : E_1 \to E_2$ where E_1, E_2 are orderd topological vector spaces is said to be orderbounded if T maps every orderbounded set in E_1 to an orderbounded set in E_2.

DEFINITION 0.1.27. [**29**] In $\mathcal{L}(E, \mathbb{R})$, the linear space of all continuous linear functionals defined on an ordered vector space E, let E^b denote the linear subspace of $\mathcal{L}(E, \mathbb{R})$ of all orderbounded linear functionals. If $C^* = C(E, \mathbb{R})$ is the wedge of all non-negative linear functionals of E, the order dual of E, denoted as E^+, is defined to be the linear hull of C^* in E^b. *i.e.*, $E^+ = C^* - C^*$ in E^b. In $\mathcal{L}(E, \mathbb{C})$, the order dual of E is $E^+ + iE^+$ which is also denoted as E^+.

DEFINITION 0.1.28. [**29**] If E is an ordered vector space the order topology τ_0 on E is the finest locally convex topology for which every orderbounded set is τ-bounded.

CHAPTER 1

Duals of ordered multinormed spaces, ordered countable union spaces

In this chapter, we define the notions of ordered multinormed spaces, ordered countable union spaces, thus assigning an 'order' to multinormed spaces, countable union spaces. The notions of normal cone, b-cone and strict b-cone on ordered multinormed spaces are also specified. We assign the topology of bounded convergence to the dual $\mathscr{V}'$ of the ordered multinormed space $\mathscr{V}$. As illustration, the space of test functions $\mathcal{D}$ and its dual D', the space of distributions are studied. The convolution and direct product of elements in D' are proved to be continuous and order preserving with respect to the topology of bounded convergence. We also prove that these operations are compatible with the lattice properties in $\mathcal{D}'$.

1.1. Duals of ordered multinormed spaces, ordered countable union spaces

Zemanian [**50**] has studied in detail the notions of multinormed spaces, countable union spaces. We assign an order relation to these spaces by identifying positive cones. An order relation is assigned to their dual spaces also via the dual cone. It is observed by Peressini [**29**] that in the case of the test function space $\mathcal{D}$ and its dual $\mathcal{D}'$, when an order relation is defined on $\mathcal{D}'$ via the dual cone of non-negative functions in $\mathcal{D}$, the order dual $\mathcal{D}^+$ is only a subspace of the topological dual $\mathcal{D}'$, when $\mathcal{D}'$ is assigned the topology of pointwise convergence. But when the topology of bounded convergence is assigned to $\mathcal{D}'$, the order dual and the topological dual of $\mathcal{D}$ become identical [**29**]. In what follows we apply the above ideas to duals of ordered multinormed spaces, ordered countable union spaces. We define first some basic notions on multinormed spaces, countable union spaces thus extending notions like positive cone, normal cone, b-cone defined by Peressini [**29**] on ordered vector spaces to ordered multinormed spaces.

DEFINITION 1.1.1. A multinormed space $\mathscr{V}$ on which a positive cone C is specified is an ordered multinormed space.

DEFINITION 1.1.2. The positive cone C generates the multinormed space $\mathscr{V}$ if $\mathscr{V}$ is spanned by C *i.e.*, if $\mathscr{V} = C - C$.

DEFINITION 1.1.3. Let $\mathscr{V}(\tau)$ be an ordered multinormed space with positive cone C. C is said to be normal for the topology τ generated by the multinorm S if there is a neighborhood basis of 0 for τ consisting of full sets. (Refer Definition 0.1.6)

DEFINITION 1.1.4. Let $\mathcal{S}$ be a saturated class of τ-bounded subsets of an ordered multinormal space $\mathscr{V}(\tau)$ such that $\mathscr{V} = \cup\{S : S \in \mathcal{S}\}$. The positive cone C in $\mathscr{V}(\tau)$ is a strict $\mathcal{S}$-cone (an $\mathcal{S}$-cone) if the class

$\mathcal{S}_C = \{(S \cap C) - (S \cap C) : S \in \mathcal{S}\}$ ($\bar{\mathcal{S}}_C = \{\overline{(S \cap C) - (S \cap C)} : S \in \mathcal{S}\}$) is a fundamental system for $\mathcal{S}$. A strict $\mathcal{S}$-cone (an $\mathcal{S}$-cone) for the class of all τ-bounded sets in $\mathscr{V}(\tau)$ is called a strict b-cone (b-cone).

Note 1. [**36**] A family $\mathcal{S} \neq \{\phi\}$ of bounded sets of a locally convex space E is said to be saturated if

(1) it contains arbitrary subsets of each of its members
(2) it contains scalar multiples of each of its members
(3) it contains the closed, convex, circled hull of the union of each finite sub family.

Note 2. [**36**] A fundamental system of bounded sets of a topological vector space E is a family $\mathcal{B}$ of bounded sets such that every bounded subset of E is contained in a suitable member of $\mathcal{B}$.

DEFINITION 1.1.5. If the supremum, $\sup\{\phi, \psi\} = \phi \vee \psi$ and the infimum $\inf\{\phi, \psi\} = \phi \wedge \psi$ of every pair $\phi, \psi \in \mathscr{V}$ an ordered multinormed space $\mathscr{V}$, exists in $\mathscr{V}$, then $\mathscr{V}$ is a multinormed vector lattice.

DEFINITION 1.1.6. A subset B of $\mathscr{V}$, an ordered multinormed space is order complete if every directed subset D of B that is majorized in B has a supremum in B.

DEFINITION 1.1.7. If in an ordered multinormed space $\mathscr{V}$ the property $[0, \phi] + [0, \psi] = [0, \phi + \psi]$, for all $\phi, \psi \in \mathscr{V}$ is satisfied then $\mathscr{V}$ is said to have the decomposition property.

DEFINITION 1.1.8. A net $(\phi_\alpha)_{\alpha \in J}$ in a multinormed vector lattice $\mathscr{V}$ order converges to $\phi_0 \in \mathscr{V}$ if
(i) $\{\phi_\alpha\}_{\alpha \in J}$ is an order bounded subset of $\mathscr{V}$
(ii) there is a net $(\psi_\alpha)_{\alpha \in J}$ in $\mathscr{V}$ that decreases to 0 such that

$$|\phi_\alpha - \phi_0| \leq |\psi_\alpha|, \quad \forall \alpha \in J.$$

Order and topology on $\mathscr{V}'$. Let $\mathscr{V}'$ denote the linear space of all continuous linear functionals defined on an ordered multinormed space $\mathscr{V}$ and ordered by the dual cone $C'(\mathscr{V}') = \{f \in \mathscr{V}', f(\phi) \geq 0, \forall \phi \in C\}$. Let $\sigma(\mathscr{V}, \mathscr{V}')$ denote the topology of pointwise convergence on $\mathscr{V}'$. The collection of all polars B^0 of B as B varies over all $\sigma(\mathscr{V}, \mathscr{V}')$-bounded subsets of $\mathscr{V}$ form a neighborhood basis of 0 in $\mathscr{V}'$. This topology on $\mathscr{V}$ denoted as $\beta(\mathscr{V}', \mathscr{V})$ is the topology of bounded convergence on $\mathscr{V}'$.

The topology $\beta(\mathscr{V}', \mathscr{V})$ on $\mathscr{V}'$ may also be described as follows: using Definitions 0.1.23 and 0.1.24, a $\sigma(\mathscr{V}, \mathscr{V}')$-bounded subset B of $\mathscr{V}$ is of the form $B = \{\psi : |f(\psi)| < t\epsilon$ for some $f \in \mathscr{V}'\}$, $\epsilon > 0$ for all $t > s$, $t, s \in \mathbb{R}$. Then $B^0 = \{f \in \mathscr{V}' : |f(\psi)| < 1, \forall \psi \in B\}$ is the polar of B. The class of all B^0 as B varies over $\sigma(\mathscr{V}, \mathscr{V}')$-bounded subsets of $\mathscr{V}$ is a neighborhood basis of 0 in $\mathscr{V}'$ for a locally convex topology $\beta(\mathscr{V}', \mathscr{V})$ on $\mathscr{V}'$.

By Proposition 3.7, Chapter 2, [**29**] it follows that if $\mathscr{V}'$ is equipped with the topology of bounded convergence and is ordered by the dual cone C' of the cone C of non negative functions in $\mathscr{V}$, then $\mathscr{V}'$

is order complete if the dual cone C' is a closed normal cone with respect to $\beta(\mathscr{V}', \mathscr{V})$.

THEOREM 1.1.1. *The topology of pointwise convergence is weaker than the topology of bounded convergence on $\mathscr{V}'$ where $\mathscr{V}'$ is the dual of the ordered multinormed space $\mathscr{V}$.*

PROOF. It is enough if we prove that every sequence (f_n) of elements in $\mathscr{V}'$ which converges to 0 with respect to the topology of pointwise convergence converges to 0 with respect to the topology of bounded convergence in $\mathscr{V}'$. We say that (f_n) converges to 0 with respect to the topology of pointwise convergence if, for every $\epsilon > 0$ there exists $k \in N$ such that

$$|f_n(\psi)| < \epsilon \quad \text{for all } \psi \in \mathscr{V},\ n \geq k. \tag{1}$$

We say that a sequence (f_n) in $\mathscr{V}'$ converges to 0 with respect to the topology of bounded convergence on $\mathscr{V}'$ if given any neighborhood basis of 0, B^0, if there exists $k \in N$ such that $f_n \in B^0$ for $n \geq k$.

$$\text{i.e., } |f_n(\psi)| < 1, \quad \forall \psi \in B,\ n \geq k.$$

Taking $\epsilon = 1$, $\psi \in B$ in (1) it follows that (fn) converges to 0 with respect to the topology of bounded convergence on $\mathscr{V}'$ if (f_n) converges to 0 with respect to the topology of pointwise convergence on $\mathscr{V}'$. □

We assign an order to the (strict) countable union space as follows:

DEFINITION 1.1.9. Let $\mathscr{V} = \cup_{m=1}^{\infty}\mathscr{V}_m$ be a (strict) countable union space. We say that $\phi \geq \psi$ for $\phi, \psi \in \mathscr{V}$ if $\phi - \psi \in C_m$ for any $m = 1, 2, \ldots$ where C_m denotes the positive cone in $\mathscr{V}_m$.

The dual of a (strict) countable union space is defined as follows:

DEFINITION 1.1.10. Let $\mathscr{V} = \cup_{m=1}^{\infty}\mathscr{V}_m$ be a (strict) countable union space. $\mathscr{V}'$, the dual of $\mathscr{V}$ is the linear space of all continuous linear functionals on $\mathscr{V}$, the topology on $\mathscr{V}'$ being $\beta(\mathscr{V}', \mathscr{V})$, the topology of bounded convergence.

We apply the notions of positive maps, strictly positive maps in Peressini [**29**] to linear maps on ordered multinormed spaces, ordered countable union spaces.

DEFINITION 1.1.11. Let $U, \mathscr{V}$ be ordered multinormed spaces or ordered countable union spaces with positive cones $C(U)$, $C(\mathscr{V})$ respectively. A linear map $T : U \to \mathscr{V}$ is
(i) positive if $T(C(U)) \subseteq C(\mathscr{V})$ i.e., if $T(\phi) \geq 0$ whenever $\phi > 0$, $\phi \in U$.
(ii) strictly positive if $T(\phi) > 0$ whenever $\phi > 0$, $\phi \in U$.

Note. Peressini [**29**] has observed that every strictly positive linear map is positive and every positive linear map is orderbounded.

The notion of the adjoint of a linear map defind on multinormed spaces, countable union spaces used by Zemanian [**50**] may be applied to linear maps on ordered multinormed spaces, ordered countable union spaces. If $T : U \to \mathscr{V}$ is a continuous linear map where

$U, \mathscr{V}$ are either ordered multinormed spaces or ordered (strict) countable union spaces the adjoint T' of T is $T' : \mathscr{V}' \to U'$ defined by

$$(T'f)(\phi) = f(T(\phi)), \quad f \in \mathscr{V}',\ \phi \in U.$$

THEOREM 1.1.2. *If $T : U \to \mathscr{V}$ is linear and continuous its adjoint $T' : \mathscr{V}' \to U'$ is also linear and continuous where $U, \mathscr{V}$ are ordered multinormed spaces or ordered (strict) countable union spaces with the topology of bounded convergence assigned to U', $\mathscr{V}'$.*

PROOF. For $\phi, \psi \in \mathscr{V}$, $\alpha, \beta \in \mathbb{C}$, $f \in \mathscr{V}'$,

$$\begin{aligned}(T'f)(\alpha\phi + \beta\psi) &= f(T(\alpha\phi + \beta\psi)) \\ &= f(\alpha T(\phi) + \beta T(\psi)) \\ &= f(\alpha T(\phi)) + f(\beta T(\psi)) \\ &= \alpha f(T(\phi)) + \beta T(\psi)) \\ &= \alpha(T'f)(\phi) + \beta(T'f)(\psi).\end{aligned}$$

so that $T'f$ is a linear functional on U.

Let $(\phi_\alpha)_{\alpha \in J}$ be a net converging to 0 in U.

Since $T : U \to \mathscr{V}$ is continuous $T(\phi_\alpha) \to 0$ as $\alpha \to \infty$ with respect to the topology induced by the multinorm on $\mathscr{V}$. Then $(T'f)(\phi_\alpha) = f(T(\phi_\alpha)) \to 0$ as $\alpha \to \infty$, since $f \in \mathscr{V}'$ is continuous with respect to the topology of bounded convergence on $\mathscr{V}'$. This implies that

$T'f$ is a continuous linear functional on U, i.e., T' is a mapping of $\mathscr{V}'$ to U'. Also

$$\begin{aligned} T'(\alpha f + \beta g)(\phi) &= (\alpha f + \beta g)T(\phi), \\ &= \alpha f(T(\phi)) + \beta g(T(\phi)) \\ &= \alpha (T'f)(\phi) + \beta (T'g)(\phi) \\ &= (\alpha (T'f) + \beta (T'g))(\phi), \end{aligned}$$

$f, g \in \mathscr{V}'$, $\phi \in U$, $\alpha, \beta \in \mathbb{C}$. Thus, $T' : \mathscr{V}' \to U'$ is linear.

Let $(f_\alpha)_{\alpha \in J}$ be a net converging to 0 in $\mathscr{V}'$ with respect to the topology of bounded convergence. We prove that $(T'f_\alpha)_{\alpha \in J}$ converges to 0 in U' with respect to the topology of bounded convergence in U'. We observe that if B_U is a $\sigma(U, U')$-bounded subset of U, since $T : U \to \mathscr{V}$ is linear and continuous, $T(B_U) = B_{\mathscr{V}}$ is a $\sigma(\mathscr{V}, \mathscr{V}')$-bounded subset of $\mathscr{V}$. To prove the convergence of $(T'f_\alpha)_{\alpha \in J}$ to 0 in U' we assume that B_U^0 is a neighbourhood of 0 in U' for the topology of bounded convergence. Since $T(B_U) = B_{\mathscr{V}}$ is a $\sigma(\mathscr{V}, \mathscr{V}')$-bounded subset of $\mathscr{V}$, $B_{\mathscr{V}}^0$ is a basis element for the topology of bounded convergence in $\mathscr{V}'$ containing 0. Since $f_\alpha \to 0$ in $\mathscr{V}'$ with respect to the above topology $f_\alpha \in B_{\mathscr{V}}^0$ for $\alpha \geq \beta$, $\beta \in J$.

i.e., $|f_\alpha(\psi)| < 1$, for all $\psi \in B_{\mathscr{V}}$, $\alpha \geq \beta$, $\beta \in J$.

For $\phi \in B_U$, $|T'(f_\alpha)(\phi)| = |f_\alpha(T(\phi))| < 1$ for $\alpha \geq \beta$, $\beta \in J$, since $T(\phi) \in B_{\mathscr{V}}$. We conclude that $(T'f_\alpha)_{\alpha \in J}$ converges to 0 with respect to the topology of bounded convergence on U' and hence $T' : \mathscr{V}' \to U'$ is continuous. $\square$

Remark. The above theorem assures that though the usual weak topology (topology of pointwise convergence) used on duals of multinormed spaces (see [**50**]) is replaced by the topology of bounded convergence, the basic properties like continuity of adjoint maps on duals of multinormed spaces are preserved.

The following theorem pertains to the order properties of the adjoint of a linear map. The theorem proves that the order properties of a linear map are followed by its adjoint also.

THEOREM 1.1.3. *Let $U, \mathscr{V}$ be ordered multinormed spaces or ordered counable union spaces. If $T : U \to \mathscr{V}$ is a strictly positive map, its adjoint $T' : \mathscr{V}' \to U'$ is also strictly positive.*

PROOF. Let $f > 0$, $f \in \mathscr{V}'$. For $\phi \in U$, $\phi > 0$, $T(\phi) > 0$ since T is strictly positive. $T(\phi) \in \mathscr{V}$, $T(\phi) > 0 \Rightarrow f(T(\phi)) > 0$ since $f > 0$. But $f(T(\phi)) = T'(f(\phi))$. Thus $\phi > 0$, $\phi \in U \Rightarrow (T'f)(\phi) > 0$.

Thus $f > 0 \Rightarrow T'f > 0$, i.e., T' is strictly positive. It follows that T' is positive and order bounded (see Note following Definition 1.1.11.) $\square$

Remark. The ideas developed in section 1.1, we subsequently apply to particular countable union spaces $\mathscr{E}$, ζ, $\mathscr{L}(a, b)$, $M^m_{a,b}$ and their dual spaces $\mathscr{E}'$, ζ', $\mathscr{L}'(a, b)$, $(M^m_{a,b})'$ etc.

1.2. The test function space $\mathcal{D}$ and the space of distributions $\mathcal{D}'$

In the preliminary chapter we have mentioned the test function space $\mathcal{D}$, the linear space of smooth complex-valued functions with compact support in $\mathbb{R}^p$ with the Schwartz topology (Definition 0.1.21, 0.1.22 and 0.1.23). The dual $\mathcal{D}'$ of D is called the space of distributions [**33**]. Anthony L. Peressini in his book 'Ordered Topological Vector Spaces' [**29**] has defined an order relation on $\mathcal{D}$ by identifying the positive cone C to be the set of all non-negative functions in $\mathcal{D}$. $\mathcal{D}$ and $\mathcal{D}'$ are actually examples for the notion of ordered multinormed spaces and their ordered duals introduced in 1.1. Peressini has observed that the cone of $\mathcal{D}$ is a strict b-cone (Example 1.17(c), Chapter 2, [**29**]). The order dual of $\mathcal{D}$, $\mathcal{D}^+ = C^* - C^*$ is the linear hull of C^* where C^* is the wedge $C(\mathcal{D}, \mathbb{R})$ of non-negative elements in $\mathcal{L}^b(\mathcal{D}, \mathbb{R})$, the subspace of orderbounded linear functionals (see page 24, [**29**]). Peressini has also observed that the topological dual $\mathcal{D}'$ of $\mathcal{D}$ contains the order dual $\mathcal{D}^+$ of $\mathcal{D}$ (Example 2.20(c), Chapter 2, [**29**]). The space of distributions $\mathcal{D}'$ equipped with the strong topology $\beta(\mathcal{D}', \mathcal{D})$ (also called the topology of bounded convergence) and ordered by the dual cone C' of the cone C of non-negative functions in $\mathcal{D}$ is a reflexive space ordered by a closed normal cone (Example 3.8, Chapter 2, [**29**]). The normality of C' for $\beta(\mathcal{D}', \mathcal{D})$ follows from corollary 1.26, Chapter 2, [**29**] and the fact that C is a strict b-cone (Example 1.17(c), Chapter 2, [**29**]). By proposition following Example 3.8, Chapter 2, [**29**], it follows that $\mathcal{D}'$ equipped with the topology of bounded convergence $\beta(\mathcal{D}', \mathcal{D})$ is order complete and hence coincides with the order dual $D^+ = C^* - C^*$. Also it follows that the cone in D' is generating, since $D' = D^+ = C^* - C^* = C' - C'$. Thus D' is both order complete and topologically complete with respect to $\beta(\mathcal{D}', \mathcal{D})$.

Though order has been introduced on D' and it has been observed that the order dual $\mathcal{D}^+$ and the topological dual $\mathcal{D}'$ coincide when the topology of bounded convergence is assigned to $\mathcal{D}'$ [**29**], a comparitive study of the order topology and the topology of bounded convergence on $\mathcal{D}'$ has not been made. From the properties of the ordered topological vector spaces we make the following observations.

(1) D' is regularly ordered. (An ordered vector space E is regularly ordered if the order dual E^+ separates points of E, i.e., if there exists an order bounded linear functional f on E such that if $\phi \neq 0$, $\phi \in E$, $f(\phi) \neq 0$.) By proposition 1.29, Chapter 2, [**29**] it follows that the order dual of $\mathcal{D}'$ separates points of $\mathcal{D}'$.

(2) The order topology and the topology of bounded convergence on $\mathcal{D}'$ are the same.

By proposition 1.16, Chapter 3, [**29**] if E is a regularly ordered vector space with the decomposition property and τ is a locally convex topology on E then τ is the order topology on E if and only if τ is the finest locally convex topology on E for which the cone K in E

is normal. By the above paragraph D' is regularly ordered. Being a vector lattice $\mathcal{D}'$ has the decomposition property. (see page 8, [**29**]). Since the topology of bounded convergence $\beta(\mathcal{D}', \mathcal{D})$ is the finest locally convex topology for which the cone C' is normal it follows that the order topology and the topology of bounded convergence on $\mathcal{D}'$ are the same.

Vladimirov [**44**] has studied the notions of direct product and convolution of elements on $\mathcal{D}'$ with the topology of pointwise convergence assigned to $\mathcal{D}'$. We next study the above notions applied to the ordered topological vector space $\mathcal{D}'$ with the topology on $\mathcal{D}'$ changed to the topology of bounded convergence. We observe that the direct product and convolution which are continuous with respect to the topology of pointwise convergence continue to remain so with respect to the topology of bounded convergence. These operations are proved to be order preserving and compatible with the lattice properties in $\mathcal{D}'$.

Direct product and convolution in $\mathcal{D}'$. The following definitions and notations have been taken from Vladimirov [**44**].

Let $f(x)$, $g(y)$ be locally integrable functions defined on open sets $I_1 \subseteq \mathbb{R}^n$, $I_2 \subseteq \mathbb{R}^m$ respectively. The function $f(x) \times g(y)$ is locally integrable on $I_1 \times I_2$. It defines a regular generalized function $f(x)g(y) = g(y)f(x)$ in $\mathcal{D}'(I_1 \times I_2)$ operating on test functions $\phi(x, y)$ in $\mathcal{D}(I_1 \times I_2)$ via the formula

$$\begin{aligned}\langle f(x)g(y), \phi\rangle &= \int_{I_1\times I_2} f(x)g(y)\phi(x,y)dxdy \\ &= \int_{I_1} f(x)\int_{I_2} g(y)\phi(x,y)dydx \\ &= \int_{I_2} g(y)\int_{I_1} f(x)\phi(x,y)dxdy \\ &= \int_{I_1\times I_2} g(y)f(x)\phi(x,y)dxdy\end{aligned}$$

i.e., $\langle f(x)g(y), \phi\rangle = \langle f(x), \langle g(y), \phi(x,y)\rangle\rangle$
$\langle g(y)f(x), \phi\rangle = \langle g(y), \langle f(x), \phi(x,y)\rangle\rangle$

Vladimirov [**44**] takes the above two equations as starting inequalities for defining the direct product $f(x)\times g(y)$ and $g(y)\times f(x)$ of the generalized functions $f \in \mathcal{D}'(I_1)$ and $g \in \mathcal{D}'(I_2)$.

$$\langle f(x)\times g(y), \phi\rangle = \langle f(x), \langle g(y), \phi(x,y)\rangle\rangle \tag{2}$$

$$\langle g(y)\times f(x), \phi\rangle = \langle g(y), \langle f(x), \phi(x,y)\rangle\rangle \tag{3}$$

where $\phi \in \mathcal{D}(I_1 \times I_2)$. Vladimirov [**44**] proves that the right hand side of (2) (and hence of (3) also) defines a continuous linear functional on $\mathcal{D}(I_1\times I_2)$ by proving that the operation $\phi(x,y) \to \psi(x) = \langle g(y), \phi(x,y)\rangle$ is linear and continuous from $\mathcal{D}(I_1 \times I_2)$ into $\mathcal{D}(I_1)$. Thus the right hand side of (2) which is equal to $\langle f, \psi\rangle$ defines a continuous linear functional on $\mathcal{D}(I_1\times I_2)$ so that $f(x)\times g(y) \in \mathcal{D}'(I_1\times I_2)$. In a similar manner it follows that $g(y) \times f(x) \in \mathcal{D}'(I_2 \times I_1)$.

THEOREM 1.2.1. *If $g \in \mathcal{D}'(I_2)$ the operation $f \to f \times g$ is linear, continuous and order preserving operation from $\mathcal{D}'(I_1)$ into $\mathcal{D}'(I_1 \times I_2)$ with respect to the topology of bounded convergence.*

PROOF. The linearity of the operation is obvious. Vladimirov [**44**] has proved the following result. The operation $\phi(x, y) \to \psi(x) = \langle g(y), \phi(x, y)\rangle$ is linear and continuous from $\mathcal{D}'(I_1 \times I_2)$ into $\mathcal{D}(I_1)$ (Corollary, sec 3.1, [**44**]). We prove that the adjoint of the above operator is $f \to f \times g$.

Let $T : \mathcal{D}(I_1 \times I_2) \to \mathcal{D}(I_1)$ be defined by

$$T(\phi(x, y)) = \psi(x) = \langle g(y), \phi(x, y)\rangle.$$

The adjoint operator $T' : \mathcal{D}'(I_1) \to \mathcal{D}'(I_1 \times I_2)$ is defined by

$$\begin{aligned}(T'f)(\phi) &= f(T(\phi))\\ &= \langle f(x), \langle g(y), \phi(x, y)\rangle\rangle\\ &= \langle f \times g, \phi\rangle\end{aligned}$$

Being the adjoint of a continuous operator $f \to f \times g$ is continuous from $\mathcal{D}'(I_1)$ to $\mathcal{D}'(I_1 \times I_2)$ with the topology of bounded convergence assigned to $\mathcal{D}'(I_1)$, $\mathcal{D}'(I_1 \times I_2)$ by Theorem 1.1.2.
Now, let $\phi(x, y) \geq 0$, $f \geq 0$, $g \geq 0$.
Since $\langle g(y), \phi(x, y)\rangle \geq 0$, $\langle f(x) \times g(y), \phi(x, y)\rangle \geq 0$, it follows that $f \times g \geq 0$. Hence the theorem. $\square$

Remark By proving the above theorem we have obtained the additional feature that direct product is order preserving without losing the linearity and continuity of the operation in the new situation when the topology on $\mathcal{D}'(I_1 \times I_2)$ is changed to the topology of bounded convergence.

The following definitions of convolution of locally integrable functions, Notes 1, 2 and definition of convolution of elements in $\mathcal{D}'$ have been taken from Vladimirov [**44**].

Let f, g be locally integrable functions defined on $\mathbb{R}^n$. If the integral $\int f(y)g(x-y)dy$ exists for almost all $x \in \mathbb{R}^n$ and defines a locally integrable function on $\mathbb{R}^n$ then it is called the convolution of the functions f and g and is represented as $f * g$. Thus

$$
\begin{aligned}
(f * g)(x) &= \int f(y)g(x-y)dy \\
&= \int g(y)f(x-y)dy = (g * f)(x).
\end{aligned}
$$

Note 1. [**44**] The convolution $f * g$ defines a regular functional on $\mathcal{D}(R^n)$ via the rule

$$
\begin{aligned}
\langle f * g, \phi > &= \int (f * g)(x)\phi(x)dx \\
&= \int \phi(x) \int f(y)g(x-y)dydx \\
&= \int f(y) \int g(x-y)\phi(x)dxdy \\
&= \int f(y) \int g(\xi)\phi(y+\xi)d\xi dy
\end{aligned}
$$

Thus

$$\langle f * g, \phi \rangle = \int f(x)g(y)\phi(x+y)dxdy, \quad \phi \in \mathcal{D} \tag{4}$$

Note 2. [**44**] A sequence (η_k) of functions in $\mathcal{D}(\mathbb{R}^n)$ converges to 1 in $\mathbb{R}^n$ if

(i) for any compact set K there is a number $n = n(K)$ such that $\eta_k(x) = 1, x \in K, k \geq n$ and

(ii) the functions (η_k) are uniformly bounded together with all their derivatives, i.e.,

$|\partial^\alpha \eta_k(x)| < M_\alpha, x \in \mathbb{R}^n, k = 1, 2, \ldots, \alpha = 0, 1, 2, \ldots$.

Vladimirov [**44**] has proved that equation (4) can be rewritten as

$$\langle f * g, \phi \rangle = \lim_{k \to \infty} \langle f(x) \times g(y), \eta_k(x, y)\phi(x+y) \rangle, \quad \phi \in \mathcal{D}$$

where η_k is any sequence of functions in $\mathcal{D}(\mathbb{R}^n)$ that converges to 1 in $\mathbb{R}^n$.

Let $f, g \in \mathcal{D}'(\mathbb{R}^n)$ be such that their direct product admits of an extension $\langle f(x) \times g(y), \phi(x+y) \rangle$ to functions of the form $\phi(x+y)$ where ϕ is any function in $\mathcal{D}(\mathbb{R}^n)$ in the following sense: if (η_k) is any sequence of functions in $\mathcal{D}(\mathbb{R}^{2n})$ which converges to 1 in $\mathbb{R}^{2n}$ there exists a limit to the numerical sequence,

$$\lim_{k \to \infty} \langle f(x) \times g(y), \eta_k(x; y)\phi(x+y) \rangle = \langle f(x) \times g(y), \phi(x+y) \rangle$$

the convolution $f * g$ is the functional defined by

$$\langle f(x) * g(y), \phi\rangle = \langle f(x) \times g(y), \phi(x+y)\rangle$$
$$= \lim_{k\to\infty} \langle f(x) \times g(y), \eta_k(x;y)\phi(x+y)\rangle, \quad \phi \in \mathcal{D}(\mathbb{R}^n).$$

We now prove that if $f, g \in \mathcal{D}'$, $f(x)*g(y)$ is a continuous operation on $\mathcal{D}(\mathbb{R}^n)$ with the topology of bounded convergence assigned to $\mathcal{D}'$.

THEOREM 1.2.2. *If f, g are elements of $\mathcal{D}(\mathbb{R}^n)$ as above, $f * g \in \mathcal{D}'(\mathbb{R}^n)$ with the topology of bounded convergence assigned to $\mathcal{D}'(\mathbb{R}^n)$.*

PROOF. Vladimirov [**44**] has proved that $f * g \in \mathcal{D}'(\mathbb{R}^n)$ with the topology of pointwise convergence assigned to $\mathcal{D}'(\mathbb{R}^n)$ in the following manner: If (ϕ_i) is any sequence of functions in $\mathcal{D}(\mathbb{R}^n)$ such that $\phi_i \to 0$ as $i \to \infty$, $\eta_k(x,y)\phi_i(x+y) \to 0$ as $i \to \infty$ in $\mathcal{D}(\mathbb{R}^n)$. Since the functional $f(x) \times g(y)$ is continuous on $\mathcal{D}(\mathbb{R}^{2n})$, the numerical sequence

$$\langle f(x) \times g(y), \eta_k(x;y)\phi(x+y)\rangle \to 0 \text{ as } i \to \infty.$$

i.e., $\langle f * g, \phi_i\rangle \to 0$ as $i \to \infty$.

This implies that $f * g$ is a member of $\mathcal{D}'(\mathbb{R}^n)$ with the topology of pointwise convergence assigned to $\mathcal{D}'(\mathbb{R}^n)$. Since the topology of pointwise convergence is weaker than the topology of bounded convergence by Theorem 1.1.1 it follows that $f * g \in \mathcal{D}'(\mathbb{R}^n)$ with the topology of bounded convergence assigned to $\mathcal{D}'(\mathbb{R}^n)$. Hence the theorem. $\square$

THEOREM 1.2.3. *The operation* $f \to f * g$ *is order preserving on* $\mathcal{D}'(\mathbb{R}^n)$.

PROOF. Let $f, g \in \mathcal{D}'(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, $f \geq 0$, $g \geq 0$, $\phi \geq 0$. Then $\langle f(x) \times g(y), \eta_k(x;y)\phi(x+y)\rangle \geq 0$ for each k, so that $\lim_{k\to\infty}\langle f(x) \times g(y), \eta_k(x;y)\phi(x+y)\rangle \geq 0$.

By definition (see Vladimirov [**44**], page 52) the above limit, it it exists is independent of the sequence (η_k). So we conclude that $f * g \geq 0$. $\square$

Remark. As in the case of direct product we have obtained that convolution of elements in $\mathcal{D}'(\mathbb{R}^n)$ is order preserving, without losing the continuity of the operation when the topology on $\mathcal{D}'(\mathbb{R}^n)$ is changed to the topology of bounded convergence. We also prove the following theorem which involves the order property of direct product and convolution in $\mathcal{D}'(\mathbb{R}^n)$.

THEOREM 1.2.4. *The operations of direct product and convolution of comparable elements in* $\mathcal{D}'$ *are compatible with lattice operations i.e., if* f_1, f_2 *and* g_1, g_2 *are comparable elements in* $\mathcal{D}'$ *then* $f_i \times g_i$, $i = 1, 2$ *and* $f_i * g_i$, $i = 1, 2$ *are comparable with*

$$(f_1 \times g_1) \vee (f_2 \times g_2) = (f_1 \vee f_2) \times (g_1 \vee g_2)$$

$$(f_1 \times g_1) \wedge (f_2 \times g_2) = (f_1 \wedge f_2) \times (g_1 \wedge g_2)$$

and

$$(f_1 * g_1) \vee (f_2 * g_2) = (f_1 \vee f_2) * (g_1 \vee g_2)$$

$$(f_1 * g_1) \wedge (f_2 * g_2) = (f_1 \wedge f_2) * (g_1 \wedge g_2)$$

PROOF. By definition,

$$\begin{aligned}\langle f_1 \times g_1, \phi\rangle &= \langle f_1(x), \langle g_1(y), \phi(x,y)\rangle\rangle \\ &= \int f_1(x) \int g_1(y)\phi(x,y)dydx \\ &= \int f_1(x)g_1(y)\phi(x,y)dydx\end{aligned}$$

so that

$$\begin{aligned}(f_1 \times g_1) \vee (f_2 \times g_2) = (&\int f_1(x)g_1(y)\phi(x,y)dydx) \\ &\vee (\int f_2(x)g_2(y)\phi(x,y)dydx)\end{aligned}$$

$$(f_1 \times f_2) \vee (g_1 \times g_2) = \int (f_1 \vee f_2)(x)(g_1 \vee g_2)(y)\phi(x,y)dydx.$$

Since

$$\int f_1(x)g_1(y)\phi(x,y)dydx \vee \int f_2(x)g_2(y)\phi(x,y)dydx$$
$$= \int (f_1 \vee f_2)(x)(g_1 \vee g_2)(y)\phi(x,y)dydx$$

for any $\phi \in \mathcal{D}$, it follows that $(f_1 \times g_1) \vee (f_2 \times g_2) = (f_1 \vee f_2) \times (g_1 \vee g_2)$. Since

$$\int f_1(x)g_1(y)\phi(x,y)dydx \wedge \int f_2(x)g_2(y)\phi(x,y)dydx$$
$$= \int (f_1 \wedge f_2)(x)(g_1 \wedge g_2)(y)dydx$$

for any $\phi \in \mathcal{D}$, it follows that $(f_1 \times g_1) \wedge (f_2 \times g_2) = (f_1 \wedge f_2) \times (g_1 \wedge g_2)$.

By definition, $\langle f*g, \phi\rangle = \lim_{k\to\infty}\langle f(x)\times g(y), \eta_k(x;y)\phi(x+y)\rangle$ we obtain $(f_1 * g_1) \vee (f_2 * g_2) = (f_1 \vee f_2) * (g_1 \vee g_2)$.

Since

$$\lim_{k\to\infty} (\langle f_1(x)g_1(y), \eta_k(x;y)\phi(x+y)\rangle) \vee (\langle f_2(x)g_2(y), \eta_k(x;y)\phi(x+y)\rangle)$$
$$= \lim_{k\to\infty} \langle (f_1 \vee f_2)(x)(g_1 \vee g_2)(y), \eta_k(x;y)\phi(x+y)\rangle$$

Similarly it follows that $(f_1 * g_1) \wedge (f_2 * g_2) = (f_1 \wedge f_2) * (g_1 \wedge g_2)$. $\square$

CHAPTER 2

The Fourier Transformation

In this chapter we apply the notion of ordered countable union spaces and their ordered dual spaces developed in Chapter 1, sect 1 to the spaces $\mathscr{E}$, ζ and there dual spaces $\mathscr{E}'$, ζ' respectively called the distributions with compact support and the tempered distributions. By changing the usual weak topology on $\mathscr{E}'$, ζ' and assigning the topology of bounded convergence on these spaces we observe that the order dual and the topological dual of $\mathscr{E}$ and ζ coincide. We observe that the direct product and convolution of elements in $\mathscr{E}'$, ζ' are continuous and order preserving with respect to the new topology. The Fourier transform and its inverse applied to the elements of ζ' by Vladimirov [**44**] are applicable in the present situation also to the elements of ζ' which can be used to solve differential equations involving tempered distributions. As the solutions belong to an ordered vector space comparison of solutions is also possible. This is an additional feature achieved by the introduction of order relation on ζ'. Illustrations of the comparison of solutions of differential equations solved by the application of Fourier transforms are also done in the chapter.

2.1. The space $\mathscr{E}$ of smooth complex valued functions and its dual $\mathscr{E}'$, the distributions with compact support

Zemanian [**50**] has treated $\mathscr{E}$ as a multinormed space and studied its dual $\mathscr{E}'$, the distributions with compact support. We have assigned an order relation on $\mathscr{E}$ by identifying a positive cone in it. The dual space $\mathscr{E}'$ is also ordered by the dual cone C' of the cone C of non-negative functions in $\mathscr{E}$. Using the fact that $\mathcal{D}'$ is a dense subspace of $\mathscr{E}$ and using the properties of ordered topological vector spaces we observe that the positive cone C of $\mathscr{E}$ is not normal. As proved by Peressini [**29**] in the case of the positive cone in $\mathcal{D}$, it can be proved that the positive cone in $\mathscr{E}$ is a strict b-cone.

The topology of bounded convergence is assigned to $\mathscr{E}'$. Applying some properties of ordered topological vector spaces we make some useful observations about $\mathscr{E}'$. We also compare the order and topology on $\mathscr{E}'$ and $\mathcal{D}'$.

Consider $\mathscr{E}$, the linear space of all smooth complex-valued functions defined on $\mathbb{R}^n$. Let (K_m) be a sequence of compact subsets of $\mathbb{R}^n$ such that $K_1 \subseteq K_2 \subseteq \cdots$ and such that each compact subset of $\mathbb{R}^n$ is contained in some K_m. Let

$$\gamma_{K_{m,k}}(\phi) = \sup_{t \in K_m} |D^k \phi(t)|,\ \phi \in \mathscr{E},\ k = 0, 1, 2, \ldots$$

$\{\gamma_{K_{m,k}}\}_{m,k}$ is a multinorm and generates a topology $\tau_{\mathscr{E}}$ on $\mathscr{E}$ with respect to which $\mathscr{E}$ is complete. (The completeness of $\mathscr{E}$ with respect to $\tau_{\mathscr{E}}$ follows in the same method as Zemanian [**50**] proves that $\mathcal{D}_K$ is complete in Example 1.6.1, Chapter 1, [**50**]).

Note. $\mathcal{D}$ is dense in $\mathscr{E}$ [**33**].

DEFINITION 2.1.1. The positive cone C of $\mathscr{E}$ when $\mathscr{E}$ is restricted to real-valued functions is the set of all non-negative functions in $\mathscr{E}$. When the elements of $\mathscr{E}$ are allowed to take complex values also, $C + iC$ is the positive cone in $\mathscr{E}$ which is also denoted as $\mathscr{E}$.

THEOREM 2.1.1. *The cone C of $\mathscr{E}$ is not normal.*

PROOF. Anthony L. Peressini [**29**] has proved that the cone C of the non-negative functions in $\mathcal{D}$ is not normal (Example 1.9.c, p.66, [**29**]). He has also observed that if the cone C of E an ordered topological vector space is normal and if M is a linear subspace of E then $K \cap M$ is a normal cone in M for the subspace topology (Proposition 1.8, Chapter 1, [**29**]). From this result it follows that the cone of $\mathscr{E}$ is not normal. □

THEOREM 2.1.2. *The cone C is a strict b-cone in $\mathscr{E}$.*

PROOF. Let $\mathscr{E}$ be restricted to real valued functions. Let $\mathcal{B}$ be the saturated class of all bounded, circled subsets $\mathscr{E}$ for $\tau_{\mathscr{E}}$. Then $\mathscr{E} = \cup_{B \in \mathcal{B}} B$. The collection
$\mathcal{B}_C = \{(B \cap C) - (B \cap C) : B \in \mathcal{B}\}$ is a fundamental system for $\mathcal{B}$ and by Definition 1.1.4 it follows that C is a strict b-cone since $\mathcal{B}$ is the class of all $\tau_{\mathscr{E}}$-bounded subsets of $\mathscr{E}$. □

Order and topology on the dual of $\mathscr{E}$. Let $\mathscr{E}'$ denote the linear space of all continuous linear functionals defined on $\mathscr{E}$, which is referred to as functions with compact support. An order relation is defined on $\mathscr{E}'$ by identifying the positive cone of $\mathscr{E}'$ to be the dual cone C' of the cone C of non-negative functions in $\mathscr{E}$. The class of all B^0, the polars of B as B varies over all $\sigma(\mathscr{E}, \mathscr{E}')$-bounded subsets of $\mathscr{E}$ is a neighbourhood basis of 0 in $\mathscr{E}'$ for the locally convex topology $\beta(\mathscr{E}', \mathscr{E})$. When $\mathscr{E}'$ is ordered by the dual cone C' and is equipped with the topology $\beta(\mathscr{E}', \mathscr{E})$, it follows that C' is a normal cone since C is a strict b-cone in $\mathscr{E}$ for the topology $\tau_{\mathscr{E}}$ of $\mathscr{E}$. (Proposition 1.2.7, Chapter 2, [**29**]). Since $\mathscr{E}'$ is an ordered topological vector space we note that $\mathscr{E}'$ has the following properties. (An ordered vector space E is regularly ordered if the order dual E^+ separates points of E, *i.e.*, there exists an orderbounded linear functional f on E such that if $\phi \neq 0$, $\phi \in E$, $f(\phi) \neq 0$)

(1) $\mathscr{E}'$ is regularly ordered (See Proposition 1.29, Chapter 2, [**29**]).

(2) The order topology and the topology of bounded convergence on $\mathscr{E}'$ are the same.

By Proposition 1.16, Chapter 3, [**29**], if E is a regularly ordered vector space with the decomposition property and if τ is a locally convex topology on E, τ is the order topology on E, if and only if τ is the finest locally convex topology on E for which the cone K in E is normal. By (1), $\mathscr{E}'$ is regularly ordered. Being a vector lattice, $\mathscr{E}'$ has the decomposition property (see page 8, [**29**]). Since the topology of bounded convergence $\beta(\mathscr{E}', \mathscr{E})$ is the finest locally convex topology for which the cone C' is normal, it follows that the order topology and the topology of bounded convergence on $\mathscr{E}'$ are the same.

(3) The order dual $\mathscr{E}^+$ and the topological dual $\mathscr{E}'$ (with respect to the topology of bounded convergence) of $\mathscr{E}$ are the same.

Since $\mathcal{D}$ is a subspace of $\mathscr{E}$ their dual spaces D' and $\mathscr{E}'$ are such that $\mathscr{E}' \subseteq \mathcal{D}'$. In the following theorem we make a comparative study of the order and topology on $\mathscr{E}'$, $\mathcal{D}'$.

THEOREM 2.1.3. *$\mathscr{E}'$ is a subspace of the space of distributions $\mathcal{D}'$, the topology on $\mathscr{E}'$ being the same as the topology of bounded convergence. Also, the order induced on $\mathscr{E}'$ by $\mathcal{D}'$ is the same as the order on $\mathscr{E}'$.*

PROOF. Clearly $\mathscr{E}'$ is a subspace of $\mathcal{D}'$. We prove that $B^o_{\mathscr{E}} = B^o_{\mathcal{D}} \cap \mathscr{E}'$ where $B^o_{\mathscr{E}}$, $B^o_{\mathcal{D}}$ denote respectively the polar of $B_{\mathscr{E}}$ in $\mathscr{E}'$ and the polar of $B_{\mathcal{D}}$ in $\mathcal{D}'$.

$B_{\mathscr{E}} = \{\psi \in \mathscr{E} : |f(\psi)| < t\epsilon$ for some $f \in \mathscr{E}'\}$, $\epsilon > 0$, $\forall t > s$, $t, s \in \mathbb{R}$

$B_{\mathcal{D}} = \{\psi \in \mathcal{D} : |f(\psi)| < t\epsilon$ for some $f \in \mathcal{D}'\}$, $\epsilon > 0$, $\forall t > s$, $t, s \in \mathbb{R}$.

The elements of $\mathscr{E}'$ are the elements of $\mathcal{D}'$ having compact support. So if $f_1 \in B^o_{\mathscr{E}}$, $|f_1(\psi)| < 1$, for all $\psi \in B_{\mathscr{E}}$, $f_1 \in \mathscr{E}'$.

Since $B_{\mathcal{D}} = B_{\mathscr{E}} \cap \mathcal{D}$ it follows that $f_1 \in B^o_{\mathcal{D}} \cap \mathscr{E}'$. Conversely if $f_1 \in B^o_{\mathcal{D}} \cap \mathscr{E}'$ then $f_1 \in \mathscr{E}'$ and $|f_1(\psi)| < 1, \forall \psi \in B_{\mathcal{D}}$.
If $\psi \in B_{\mathcal{D}}$, $|f(\psi)| < t\epsilon$ for some $f \in \mathcal{D}'$, $\epsilon > 0$, for all $t > s$, $t, s \in \mathbb{R}$. Since $\mathscr{E}' \subseteq \mathcal{D}'$, $\psi \in B_{\mathcal{D}}$ it follows that $\psi \in B_{\mathscr{E}}$, $f_1 \in B^o_{\mathscr{E}}$ and we conclude that $B^o_{\mathscr{E}} = B^o_{\mathcal{D}} \cap \mathscr{E}'$.

Since the order on $\mathscr{E}'$ is defined by the dual cone C' of the cone C of non negative functions in $\mathscr{E}$ it follows that the order of $\mathscr{E}'$ and the order induced on $\mathscr{E}'$ by $\mathcal{D}'$ are the same. (See Proposition 1.8, Chapter 1, [**29**]) □

Since $\mathscr{E}'$ is a subspace of D', the order and topology on $\mathscr{E}'$ being the same as the order and topology induced by $\mathcal{D}'$, the notions of direct product and convolution defined on $\mathcal{D}'$ are applicable to the elements of $\mathscr{E}'$ also. We make the following observations.

(1) For $g \in \mathscr{E}'$ the operation $f \to f \times g$ is linear, continuous (with respect to the topology of bounded convergence) and order preserving from $\mathscr{E}'$ to $\mathscr{E}'$.

(2) For $f, g \in \mathscr{E}'$ the operation $f \to f * g$ is continuous (with respect to the topology of bounded convergence) and order preserving from $\mathscr{E}'$ to $\mathscr{E}'$.

(3) The operations of convolution and direct product of comparable elements in $\mathscr{E}'$ are compatible with lattice operations *i.e.* if f_1, f_2 and g_1, g_2 are comparable elements in $\mathscr{E}'$ then $f_i * g_i$, $i = 1, 2$ and $f_i \times g_i$, $i = 1, 2$ are comparable and

$$(f_1 * g_1) \vee (f_2 * g_2) = (f_1 \vee f_2) * (g_1 \vee g_2)$$

$$(f_1 * g_1) \wedge (f_2 * g_2) = (f_1 \wedge f_2) * (g_1 \wedge g_2)$$

and

$$(f_1 \times g_1) \vee (f_2 \times g_2) = (f_1 \vee f_2) \times (g_1 \vee g_2)$$

$$(f_1 \times g_1) \wedge (f_2 \times g_2) = (f_1 \wedge f_2) \times (g_1 \wedge g_2).$$

2.2. The space ζ of rapidly decreasing test functions and its dual ζ', the tempered distributions

Rudin [**33**] and Vladimirov [**44**] have studied the test function space ζ and its dual ζ', the tempered distributions with the topology of pointwise convergence assigned to ζ'. We study ζ as an ordered strict countable union space and define an order relation on ζ' via the dual cone C' of the cone of non negative functions in ζ and assign the topology of bounded convergence to ζ'.

Functions of slow growth. [**30**] A function $f(x) = f(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ is of slow growth if $f(x)$ together with all its derivatives grows at ∞ more slowly than some polynomial. This means that there exists constants C, m and A such that $|D^k f(x)| \leq C|x|^m$, $|x| > A$.

Test functions of rapid decay. [**30**] The space ζ of test functions of rapid decay contains the complex valued functions

$\phi(x) = \phi(x_1, \ldots, x_n)$ having the following properties

(i) $\phi(x)$ is infinitely differentiable

(ii) $\phi(x)$ as well as its derivatives of all orders vanishes at ∞ faster than the reciprocal of any polynomial. Property (2) may be expressed as $|x^p D^k \phi(x)| < C_\rho k$, where $p = (p_1, \ldots, p_n)$ and $k = (k_1, \ldots, k_n)$ are n tuples of non-negative integers and $C_\rho k$ is a constant depending on p, k and $\phi(x)$.

Let (K_m) denote a sequence of compact subsets of $\mathbb{R}^n$ such that $K_1 \subseteq K_2 \subseteq \cdots$ and such that each compact subset of $\mathbb{R}^n$ is contained in some K_j, $j = 1, 2, \ldots$. On each K_j define

$$\|\phi\|_p = \sup_{\substack{x \in K_j \\ |\alpha| \leq p}} (1 + |x|^2)^{p/2} |\partial^\alpha \phi(x)|, \ \phi \in \tau, \ p = 0, 1, 2, \ldots$$

$\{\| \ \|_p\}$ is a multinorm on ζ_{K_j} where ζ_{K_j} is the linear subspace of ζ consisting of functions with support in K_j. The above multi norm generates a topology τ_{K_j} on ζ_{K_j}. If $m < p$, $\zeta_{K_j} \subseteq \zeta_{K_p}$ and the topology of ζ_{K_m} is the same as the topology induced on it by τ_{K_p}. Then $\zeta = \cup_{m=1}^{\infty} \zeta_{K_m}$ is the strict countable union space. By definition 0.1.14, a sequence (ϕ_i) in ζ converges to ϕ in ζ if all $\phi_i, \phi \in \zeta_{K_m}$ for some m and (ϕ_i) converges to ϕ with respect to topology τ_{K_m}. Since ζ_{K_m} is complete, ζ is also complete. (That ζ_{K_m} is complete can be proved in the same method as Zemanian proves that $\mathcal{D}_K$ is complete in Example 1.6.1, Chapter 1, [**50**]. Being a strict countable union space it follows that ζ is also complete.).

DEFINITION 2.2.1. Restricting ζ to real-valued functions an order relation is defined on ζ by identifying the positive cone to be the set of all non-negative functions in ζ. When the elements of ζ are allowed to take complex-values also, $C + iC$ is a positive cone in ζ which is also denoted as C.

Note. C defines an order relation on ζ, $\phi \leq \psi$ if $\psi - \phi \in C$ where $\psi, \phi \in \zeta$.

Vladimirov [**44**] has observed (see pages 74, 75 [**44**]) that $\mathcal{D}$ is a dense proper subspace of ζ. This result along with the comparison of the order relation in $\mathcal{D}$ and ζ we state as a theorem.

THEOREM 2.2.1. *$\mathcal{D}$ is a dense proper subspace of ζ, the order on $\mathcal{D}$ being the same as the order induced on $\mathcal{D}$ by ζ.*

PROOF. For $\phi \in \zeta$, the sequence of functions (ϕ_k) in $\mathcal{D}$ defined by

$$\phi_k(x) = \phi(x)\eta(\frac{x}{k}), \quad k = 1, 2, 3, \ldots$$

where $\eta(x) = 1$, $|x| < 1$, $\eta \in \mathcal{D}$ is such that $\phi_k \to \phi$. So $\mathcal{D}$ is dense in ζ. However $e^{-|x|^2}$ belongs to ζ but not to $\mathcal{D}$. Since $\mathcal{D}$ is a subspace of ζ, the order on $\mathcal{D}$ induced by ζ is determined by $C_\zeta \cap \mathcal{D}$ where C_ζ is the cone in ζ. Clearly $C_\zeta \cap \mathcal{D}$ is the positive cone in $\mathcal{D}$. □

Note. As in the case of $\mathscr{E}$ it can be proved that the cone of ζ is not normal but is a strict b-cone.

Order and topology on the dual of ζ. Let ζ' denote the linear space of all continuous linear functionals (continuous with respect to the topology of pointwise convergence) defined on ζ. An order relation is defined on ζ' by identifying the positive cone of ζ' to be the dual cone C' of the cone C of non-negative functions in ζ. The class of all B^0, the polars of B as B varies over all $\sigma(\zeta, \zeta')$-bounded subsets of ζ is a neighbourhood basis of 0 in ζ' for the locally convex topology $\beta(\zeta', \zeta)$. When ζ' is ordered by the dual cone C' and is assigned the topology of bounded convergence $\beta(\zeta', \zeta)$ it follows that C' is a normal cone for $\beta(\zeta', \zeta)$, since C is a strict b-cone in ζ for the topology defined on ζ.

As in the case of $\mathcal{D}'$, $\mathscr{E}'$ we observe that ζ' has the following properties

(1) ζ' is regularly ordered.

(2) The order topology and the topology of bounded convergence on ζ' are the same.

(3) The order dual ζ^+ and the topological dual ζ' (with respect to the topology of bounded convergence) of ζ are the same.

We observe in the following theorem that as a subspace of $\mathcal{D}'$, the topology induced on ζ' by $\mathscr{D}'$ is the same as the topology of bounded convergence defined on ζ'. Also the order relation induced on ζ' by $\mathcal{D}'$ is the same as the order relation ζ' has.

THEOREM 2.2.2. *The space of tempered distributions ζ' is a subspace of the space of distributions $\mathcal{D}'$, the topology on ζ' being the* *same as the topology induced on ζ' by $\mathcal{D}'$. Also the order relation on ζ' is the same as the order relation induced on ζ' by $\mathcal{D}'$.*

PROOF. In Theorem 2.1.3 we have proved that $B^0_{\mathscr{E}} = B_{\mathcal{D}} \cap \mathscr{E}'$ and $C'_{\mathscr{E}'} = C_{\mathcal{D}'} \cap \mathscr{E}'$. Similarly we can prove that $B^0_{\zeta} = B^0_{\mathcal{D}} \cap \zeta'$ and $C_{\zeta'} = C_{\mathcal{D}'} \cap \zeta'$ from which the theorem follows $\square$

In the next theorem we observe that certain operations defined on ζ' are continuous with respect to the topology of bounded convergence defined on ζ'. Vladimirov [**44**] has proved that these operations on ζ' are continuous with respect to the topology of pointwise convergence (see page 79, [**44**]).

THEOREM 2.2.3. *(i) If $f \in \mathscr{E}'$ then $f \in \zeta'$ and $\langle f, \phi \rangle = \langle f, \eta\phi \rangle$ $\phi \in \zeta$, where $\eta \in \mathcal{D}$ and $\eta = 1$ in the support of f.*
(ii) If $f \in \zeta'$ then every derivative $\partial^\alpha f \in \zeta'$ and the operation $f \to \partial^\alpha f$ is continuous and linear on ζ' (when ζ' is assigned the topology of bounded convergence).
(iii) If $f \in \zeta'$ and det $A \neq 0$ then $f(Ax + b) \in \zeta'$ and the operation $f(x) \to f(Ax + b)$ is linear and continuous on ζ' (with respect to the topology of bounded convergence).
(iv) If $f \in \zeta'$ and $a \in \theta_M$ then $af \in \zeta'$ and the operation $f \to af$ is linear and continuous on ζ' (with respect to the topology of bounded convergence).

(Suppose that the function $a \in C^\infty$ grows at infinity together with all its derivatives not faster than the polynomial $|\partial^\alpha a(x)| \leq$

$C_\alpha(1+|x|)^{m_\alpha}$. Denote by θ_M the set of all such functions. This set is called the set of all multipliers in ζ. See page 76, [**44**]).

PROOF. Vladimirov [**44**] has proved the above results with the topology of pointwise convergence assigned to ζ'. Since the topology of bounded convergence is finer than the topology of pointwise convergence (Theorem 1.1.1) the required results follow. □

2.3. Direct product and convolution of tempered generalized functions

Since ζ' is a subspace of $\mathcal{D}'$ the notions of direct product and convolution defined on $\mathcal{D}'$ are applicable to the elements of ζ' also. Vladimirov [**44**] has proved that since $\zeta' \subseteq \mathcal{D}'$, for $f(x) \in \zeta'(\mathbb{R}^n)$ and $g(y) \in \zeta'(\mathbb{R}^m)$, $f(x) \times g(y) \in \zeta'(\mathbb{R}^{m+n})$ (see section 5.5, chapter 1, [**44**]), Vladimirov [**44**] has also defined the convolution of $f, g \in \zeta'$ to be the limit $\lim_{k\to\infty}(f\eta_k) * g$ in ζ' if this limit exists for any sequence (η_k) converging to 1 in $\mathbb{R}^n$. The above limit, if it exists, is independent of (η_k). Thus in this case $f * g = \lim_{k\to\infty}(f\eta_k) * g$. Also $f * g = g * f$. For $f \in \zeta'$, $g \in \mathscr{E}'$ the convolution $f * g$ belongs to ζ' and can be represented as

$$\langle f * g, \phi\rangle = \langle f(x) \times g(y), \eta(y)\phi(x+y)\rangle, \phi \in \zeta$$

where η is any function from $\mathcal{D}$ equal to 1 in a neighbourhood of the support of g; here the operation $f \to f * g$ is continuous from

ζ' to ζ' and the operation $g \to f * g$ is continuous from $\mathscr{E}'$ to ζ' (see 5.6.1., chapter 1, [**44**]).

Though the topology of pointwise convergence assigned to ζ' and $\mathscr{E}'$ has been replaced by the topology of bounded convergence by us, the above definitions of direct product and convolution of tempered generalized functions hold good and their continuity properties are retained by virtue of Theorem 1.1.1 in the new situation when ζ', $\mathscr{E}'$ are treated as ordered duals of ordered (strict) countable union spaces, ordered multinormed spaces respectively, with the topology of bounded convergence assigned to them. The relevant results we state and prove as follows:

THEOREM 2.3.1. *Let $f \in \zeta'$, $g \in \mathscr{E}'$. The convolution $f * g \in \zeta'$ and can be represented as*

$$\langle f * g, \phi \rangle = \langle f(x) \times g(y), \eta(y)\phi(x+y) \rangle, \quad \phi \in \zeta$$

where η is any function in $\mathcal{D}$ equal to 1 in a neighbourhood of the support of g. The operation $f \to f * g$ is orderpreserving and continuous from ζ' to ζ' and $g \to f * g$ is orderpreserving and continuous from $\mathscr{E}'$ to ζ' when the topology of bounded convergence is assigned to ζ' and $\mathscr{E}'$.

PROOF. Vladimirov [**44**] has proved that the operations $f \to f * g$ from ζ' to ζ' is continuous and $g \to f * g$ from $\mathscr{E}'$ to ζ' is continuous when $\mathscr{E}'$, ζ' are assigned the topology of pointwise convergence (5.6.1, chapter 1, [**44**]). By Theorem 1.1.1 it follows that the above operations are continuous when $\mathscr{E}'$, ζ' are assigned the topology of bounded convergence.

Let $f \geq 0$, $g \geq 0$, $\phi \geq 0$. Since

$$\langle f(x) \times g(y), \eta_k(x;y)\phi(x+y)\rangle \geq 0 \text{ for } \eta_k(x) = \eta(\frac{x}{k}).$$

$k = 1, 2, 3, \ldots$ where $\eta \in \mathcal{D}$, $\eta(x) = 1$, $|x| < 1$ and since $\lim_{k\to\infty}\langle f(x) \times g(y), \eta_k(x;y)\phi(x+y)\rangle$, if it exists, is independent of (η_k) it follows that $f * g \geq 0$. □

THEOREM 2.3.2. *The operations of convolution and direct product of comparable elements in ζ' are compatible with lattice operations i.e. if f_1, f_2 and g_1, g_2 are comparable elements in ζ' then $f_i * g_i$, $i = 1, 2$ and $f_i \times g_i$, $i = 1, 2$ are comparable and*

$$(f_1 * g_1) \vee (f_2 * g_2) = (f_1 \vee f_2) * (g_1 \vee g_2)$$

$$(f_1 * g_1) \wedge (f_2 * g_2) = (f_1 \wedge f_2) * (g_1 \wedge g_2)$$

and

$$(f_1 \times g_1) \vee (f_2 \times g_2) = (f_1 \vee f_2) \times (g_1 \vee g_2)$$

$$(f_1 \times g_1) \wedge (f_2 \times g_2) = (f_1 \wedge f_2) \times (g_1 \wedge g_2).$$

PROOF. Similar to the proof of Theorem 1.2.4. □

2.4. Fourier transform of tempered generalized functions

Vladimirov [**44**] has studied the integral transforms of generalized functions in detail. We have used the definition and properties of Fourier transform of tempered generalized functions in Vladimirov [**44**] to illustrate our idea that comparison of solutions of differential equations involving generalized functions can be done.

DEFINITION 2.4.1. [**44**] If f is an integrable function on $\mathbb{R}^n$ its Fourier transform is defined as

$$F(f)(\xi) = \int_{\mathbb{R}^n} f(x)e^{i(\xi\cdot x)}dx$$

which is a continuous function on $\mathbb{R}^n$ and hence determines a regular tempered generalized function by the formula

$$\langle F(f), \phi\rangle = \int F(f)(\xi)\phi(\xi),\ \phi \in \zeta$$

The inverse of the Fourier transform $F^{-1} : \zeta' \to \zeta'$ is defined as $F^{-1}(f) = \frac{1}{(2\pi)^n}F[f(-x)]$ where $f(-x)$ is the reflexion of $f(x)$.

$$\begin{aligned}
F^{-1}(\psi)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \psi(\xi)e^{-i(x\cdot\xi)}d\xi \\
&= \frac{1}{(2\pi)^n}F(\psi)(-x) \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \psi(-\xi)e^{i(x\cdot\xi)}d\xi \\
&= \frac{1}{(2\pi)^n}F(\psi)(-\xi).
\end{aligned}$$

The following are some observations made on the basis that the topology of pointwise convergence is weaker than the topology of bounded convergence on ζ'. The following results with the topology of pointwise convergence assigned to ζ' have been proved in detail by Vladimirov (section 6.2, chapter 2, [**44**]).

(1) Since the operation $\phi \rightarrow F(\phi)$ is linear and continuous on ζ, being the adjoint of the operation, $f \rightarrow F(f)$ is linear and continuous on ζ' with the topology of bounded convergence assigned to ζ'.

(2) The operation $f \rightarrow F(f)$ is an isomorphism on ζ' with the topology of bounded convergence assigned to ζ'.

(3) For $f(x,t) \in \zeta'(\mathbb{R}^{n+m})$ where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ the restricted Fourier transform $F_x(f)$ with respect to x acts on $\phi(\xi, y) \in \zeta(\mathbb{R}^{n+m})$ by the equation $\langle F_x(f), \phi\rangle = \langle f, F_\xi(\phi)\rangle$.

(4) The operation $f \rightarrow F_x(f)$ is an isomorphism on $\zeta'(\mathbb{R}^{n+m})$ with the topology of bounded convergence assigned to $\zeta'(\mathbb{R}^{n+m})$.

2.5. Application of Fourier transforms to solve non-zero linear differential equations with constant coefficients and comparison of solutions

Rudin [**33**] and Vladimirov [**44**] have worked out in detail the method of solving non-zero linear differential equations with constant coefficients by applying Fourier transform and its inverse to tempered distributions. We give a brief outline of obtaining tempered fundamental solutions to such differential equations. Our main objective in this section is to illustrate that comparison of such solutions of differential equations is possible since we have assigned an order relation to the linear space of tempered distributions.

Fundamental solutions in $\mathcal{D}'$ [44]

If $P(\partial) = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha$, $\sum_{|\alpha|=m} a_\alpha \neq 0$ is a partial differential operator of the mth order with constant coefficients there exist $\epsilon(x)$ in $\mathcal{D}'$ such that

$$P(\partial)\epsilon(x) = \delta(x) \tag{5}$$

where δ is the delta function of Dirac which operates via the rule

$$\langle \delta, \phi \rangle = \phi(0), \quad \phi \in \mathcal{D}$$

$\epsilon(x)$ is called the fundamental solution of $P(\partial)$ in $\mathcal{D}'$. Every differential operator with constant coefficients $P(\partial) \not\equiv 0$ has a fundamental solution in $\mathcal{D}'$. Having a fundamental solution ϵ of the operator $P(\partial)$ we can construct a solution $u \in \mathcal{D}'$ of the equation

$$P(\partial)\, u = f, \quad f \in \mathcal{D}' \tag{6}$$

in the form of the convolution $u = \epsilon * f$ for those $f \in \mathcal{D}'$ for which the convolution exists in $\mathcal{D}'$. The solution of (6) is unique in the class of generalized functions from $\mathcal{D}'$ for which the convolution with ϵ exists.

Fundamental solution in ζ' [44]

The equation (6) in the class ζ' is equivalent to the algebraic equation

$$P(-i\,\xi)\tilde{\epsilon}(\xi) = 1, \tag{7}$$

with respect to the Fourier transform $F(\epsilon) = \tilde{\epsilon}$. Thus the problem of seeking a tempered fundamental solution turns out to be the problem of finding a solution u in ζ' of the equation

$$P(\xi)u = f \tag{8}$$

where $P \not\equiv 0$ is a polynomial and f is a specified member of ζ'. It has been proved that every equation of the form (8) has a tempered fundamental solution.

Illustration

(1) Vladimirov [**44**] has derived a fundamental solution of the heat conduction operator

$$\frac{\partial \epsilon}{\partial t} - a^2 \Delta \epsilon = \delta(x,t), \quad \text{where } \Delta = \frac{\partial^2}{\partial x^2}$$

as

$$\epsilon(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4a^2 t}}$$

(15.4.5, chapter 3, [**44**]). As $\epsilon(x,t)$ is a member of ζ' it is possible to compare different solutions obtained corresponding to different values of x, t.

Corresponding to $x = x_1$, $t = t_1$, $t = t_2$ we get two different solutions

$$\epsilon(x_1, t_1) = \frac{\theta(t_1)}{(2a\sqrt{\pi t_1})^n} e^{-\frac{|x_1|^2}{4a^2 t_1}}$$

$$\epsilon(x_1, t_2) = \frac{\theta(t_2)}{(2a\sqrt{\pi t_2})^n} e^{-\frac{|x_1|^2}{4a^2 t_2}}$$

The two solutions obtained corresponding to two different values of t, say $t = t_1$ and $t = t_2$ are comparable since $\epsilon(x_1, t_1)$ and $\epsilon(x_1, t_2)$ as elements of ζ' satisfy the relation $\epsilon(x_1, t_1) \geq \epsilon(x_1, t_2)$ with respect to the order relation defined on ζ' if $t_1 \leq t_2$. On the other hand the two solutions obtained corresponding to two different values of x, say $x = x_1$, $x = x_2$, $t = t_1$ satisfy the relation $\epsilon(x_1, t_1) \leq \epsilon(x_2, t_1)$ if $x_1 \leq x_2$.

(2) Fundamental solution of the Schrodinger operator

$$i\frac{\partial \epsilon}{\partial t} + \frac{1}{2m}\Delta\epsilon = \delta(x, t) \text{ is}$$

$$\epsilon(x, t) = -\frac{1+i}{\sqrt{2}}\theta(t)\sqrt{\frac{m}{2\pi t}} e^{i\frac{m}{2t}x^2}$$

(15.4.10, Chapter 2, [**44**]) For a fixed value of x, say $x = x_1$, the two solutions $\epsilon(x_1, t_1)$ and $\epsilon(x_1, t_2)$ are comparable as elements of ζ' since $t_1 \geq t_2 \Rightarrow \epsilon(x_1, t_1) \leq \epsilon(x_1, t_2)$.

Also for $t = t_1$, $x = x_1$, $x = x_2$ the solutions $\epsilon(x_1, t_1)$ and $\epsilon(x_2, t_1)$ satisfy $x_1 \leq x_2 \Rightarrow \epsilon(x_1, t_1) \leq \epsilon(x_2, t_1)$.

CHAPTER 3

The Laplace Transformation

Zemanian [**50**] has studied the two-sided Laplace transform of generalized functions. In chapter 1 we have introduced the notions of ordered multinormed spaces, ordered (strict) countable union spaces and their ordered dual spaces with the topology of bounded convergence assigned to the dual spaces. We have adapted the techniques applied by Zemanian [**50**] to the present situation by applying the two-sided Laplace transformation to generalized functions which are elements of ordered topological vector spaces. Without losing any of the original properties of the Laplace transformation and its inverse by the changes we have made, we observe that comparison of solutions of differential equations involving generalized functions is possible in the present situation.

3.1. The spaces $\mathcal{L}_{a,b}$, $\mathcal{L}(w, z)$ and their duals

We begin by defining the testing function spaces $\mathcal{L}_{a,b}$ and $\mathcal{L}(w, z)$. Though our definitions are based on the definitions of $\mathcal{L}_{a,b}$ and $\mathcal{L}(w, z)$ by Zemanian (see section 3.2, chapter 3, [**50**]), our definitions differ from that of Zemanian since we treat $\mathcal{L}_{a,b}$ as the (strict) countable union space of $\mathcal{L}_{a,b,K_m}$. Let $a, b, c, d, t \in \mathbb{R}$, $s \in \mathbb{C}$ and let

$$L_{a,b}(t) = e^{at}, \ 0 \le t < \infty$$
$$= e^{bt}, \ -\infty < t < 0.$$

Let (K_m) be a sequence of compact subsets of $\mathbb{R}$ such that $K_1 \subseteq K_2 \subseteq \cdots$ and such that each compact subset of $\mathbb{R}$ is contained in some K_m, $m = 1, 2, 3, \ldots$. Let $\mathcal{L}_{a,b,K_m}$ denote the linear space of smooth complex-valued functions defined on $\mathbb{R}$ having support in K_m. Define

$$\gamma_{K_m,k}(\phi) = \sup_{t \in K_m} |L_{a,b}(t) D^k \phi(t)|, \ k = 0, 1, \ldots, \phi \in \mathcal{L}_{a,b,K_m}$$

$\{\gamma_{K_m,k}\}_{k=0}^{\infty}$ is a multinorm on $\mathcal{L}_{a,b,K_m}$ and generates a topology τ_{a,b,K_m} on $\mathcal{L}_{a,b,K_m}$. Each $\mathcal{L}_{a,b,K_m}$ is complete with respect to the topology $\mathcal{L}_{a,b,K_m}$. Let $\mathcal{L}_{a,b} = \cup_{m=1}^{\infty} \mathcal{L}_{a,b,K_m}$ be the strict countable union space. Since each $\mathcal{L}_{a,b,K_m}$ is complete, $\mathcal{L}_{a,b}$ is also complete. (Proof follows as in the case of Example 1.17, chapter 1, [**50**]).

We now define an order relation on $\mathcal{L}_{a,b}$ by identifying a positive cone in $\mathcal{L}_{a,b}$.

DEFINITION 3.1.1. The positive cone C of $\mathcal{L}_{a,b}$ when $\mathcal{L}_{a,b}$ is restricted to real-valued functions is the set of all non-negative functions in $\mathcal{L}_{a,b}$. In the general case when the field of scalars is $\mathbb{C}$ the complex numbers $C + iC$ is the positive cone of $\mathcal{L}_{a,b}$ which is also denoted as C.

Note. The positive cone defines an order relation on $\mathcal{L}_{a,b}$ by setting $\phi \le \psi$ if and only if $\psi - \phi \in C$.

As done in Chapter 2 in the case of $\mathcal{D}$, $\mathscr{E}$, ζ it can be proved that the cone C of $\mathcal{L}_{a,b}$ is not normal but is a strict b-cone.

Order and topology on the dual $\mathcal{L}'_{a,b}$ of $\mathcal{L}_{a,b}$. An order relation is defined on the dual $\mathcal{L}'_{a,b}$, the linear space of all continuous linear functionals on $\mathcal{L}_{a,b}$ by identifying the positive cone in $\mathcal{L}'_{a,b}$ to be the dual cone C' of the cone C of $\mathcal{L}_{a,b}$. The class of all B^0, the polars of B as B varies over all $\sigma(\mathcal{L}_{a,b}, \mathcal{L}'_{a,b})$-bounded subsets of $\mathcal{L}_{a,b}$ is a neighbourhood basis of 0 in $\mathcal{L}'_{a,b}$ for the locally convex topology $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$. When $\mathcal{L}'_{a,b}$ is ordered by the dual cone C' and is equipped with the topology $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$ it follows that C' is a normal cone since C is a strict b-cone by corollary 1.26, chapter 2, [**29**].

As in case of $\mathcal{D}'$, $\mathscr{E}'$, ζ' we observe that when the topology on $\mathcal{L}'_{a,b}$ is changed to the topology of bounded convergence, $\mathcal{L}'_{a,b}$ is order complete and topologically complete, the order dual and the topological dual of $\mathcal{L}_{a,b}$ coincide and the order topology and the topology of bounded convergence on $\mathcal{L}'_{a,b}$ coincide. Also the cones of $\mathcal{L}_{a,b}$, $\mathcal{L}'_{a,b}$ are generating.

We make the following observations from the results proved by Zemanian [**50**].

(1) Let $a \le c$, $d \le b$. The restriction of any $f \in \mathcal{L}'_{a,b}$ to $\mathcal{L}_{c,d}$ is in $\mathcal{L}'_{c,d}$ when the topology of bounded convergence is assigned to $\mathcal{L}'_{a,b}$, $\mathcal{L}'_{c,d}$.

When $a \le c$, $d \le b$, $\mathcal{L}_{c,d} \subseteq \mathcal{L}_{a,b}$ and the topology $\tau_{c,d}$ on $\mathcal{L}_{c,d}$ is stronger than the topology induced on $\mathcal{L}_{c,d}$ by $\mathcal{L}_{a,b}$. Also the restriction of any $f \in \mathcal{L}'_{a,b}$ to $\mathcal{L}_{c,d}$ is in $\mathcal{L}'_{c,d}$ when $\mathcal{L}'_{a,b}$, $\mathcal{L}'_{c,d}$ are assigned the topology of pointwise convergence (see pages 49, 50, [**50**]). The above results remain true when the topology of bounded convergence is assigned to $\mathcal{L}'_{a,b}$, $\mathcal{L}'_{c,d}$ by Theorem 1.1.1.

(2) If $a < b$ or $d < b$, $\mathcal{L}'_{a,b}$ cannot be identified in one-to-one correspondence with a subspace of $\mathcal{L}'_{c,d}$.

Zemanian [**50**] has illustrated that two different members of $\mathcal{L}'_{a,b}$ have the same restriction to $\mathcal{L}_{c,d}$ if $a < c$ or $d < b$ when $\mathcal{L}'_{a,b}$, $\mathcal{L}'_{c,d}$ are assigned the topology of pointwise convergence (see Example 3.2.1, Chapter III, [**50**]). By Theorem 1.1.1 it follows that the above result is true when $\mathcal{L}'_{a,b}$, $\mathcal{L}'_{c,d}$ are assigned the topology of bounded convergence.

The ordered countable union space $\mathcal{L}(w, z)$ and its dual $\mathcal{L}'(w, z)$. Zemanian [**50**] has defined the space $\mathcal{L}(w, z)$ (see page 50, [**50**]) as follows.

DEFINITION 3.1.2. [**50**] Let $w \in \mathbb{R}$ or $w = -\infty$, $z \in \mathbb{R}$ or $z = +\infty$. Let (a_i), (b_i) be sequence of real numbers such that

$a_i \to w^+$, $b_i \to z^-$. The strict countable union space $\cup_{i=1}^{\infty}\mathcal{L}_{a_i,b_i}$ is denoted as $\mathcal{L}(w,z)$.

Each $\mathcal{L}_{a_i,b_i}$ is an ordered topological linear space, the order being assigned by identifying a positive cone in each $\mathcal{L}_{a_i,b_i}$ and the topology of each $\mathcal{L}_{a_i,b_i}$ being τ_{a_i,b_i} as described in the beginning of Section 3.1. Being a strict countable union space $\mathcal{L}(w,z)$ is also an ordered topological vector space, the order relation on $\mathcal{L}(w,z)$ being determined as follows: for $\phi, \psi \in \mathcal{L}(w,z)$ we say that $\phi \leq \psi$ if $\phi, \psi \in \mathcal{L}_{a_i,b_i}$ for some a_i, b_i and $\phi \leq \psi$ with respect to the order relation defined on $\mathcal{L}_{a_i,b_i}$. The topology on $\mathcal{L}(w,z)$ is determined by the notion of convergence defined on $\mathcal{L}(w,z)$ as follows: a sequence (ϕ_n) in $\mathcal{L}(w,z)$ converges to $\phi \in \mathcal{L}(w,z)$ if all $\phi_n, \phi \in \mathcal{L}_{a_i,b_i}$ for some a_i, b_i and (ϕ_n) converges to ϕ with respect to the topology τ_{a_i,b_i}. Since each $\mathcal{L}_{a_i,b_i}$ is complete it follows that $\mathcal{L}(w,z)$ is complete.

From the result I and II (pages 51, 52, Zemanian [**50**]) we make the following observation: $\mathcal{D} \subseteq \mathcal{L}_{a,b} \subseteq \mathscr{E}$ and $\mathcal{D}$ is not dense in $\mathcal{L}_{a,b}$ but $\mathcal{D}$ is dense in $\mathcal{L}(w,z)$ for every w, z. In particular, $\mathcal{D}$ is dense in $\mathcal{L}(a,b)$ though not in $\mathcal{L}_{a,b}$.

The dual $\mathcal{L}'(w,z)$ is ordered by the dual cone $C'(\mathcal{L}'(w,z))$ of the cone $C(\mathcal{L}(w,z))$ of $\mathcal{L}(w,z)$ and we assign the topology of bounded convergence $\beta(\mathcal{L}'(w,z), \mathcal{L}(w,z))$ on $\mathcal{L}'(w,z)$. Since $C(\mathcal{L}(w,z))$ is a strict b-cone it follows that $C'(\mathcal{L}'(w,z))$ is a normal cone with respect to $\beta(\mathcal{L}'(w,z), \mathcal{L}(w,z))$ by corollary 1.26, chapter 2, [**29**]. As in the cases of $\mathcal{D}'$, $\mathscr{E}'$, ζ' it follows that the order dual and the topological dual of $\mathcal{L}(w,z)$ coincide and since $\mathcal{L}'(w,z) = \mathcal{L}^+(w,z) = C' - C'$ it follows that the cone in $\mathcal{L}'(w,z)$ is generating. Also, since $\beta(\mathcal{L}'(w,z), \mathcal{L}(w,z))$ is the finest locally convex topology on $\mathcal{L}'(w,z)$ for which the cone C' is normal, it follows that the order topology and the topology of bounded convergence on $\mathcal{L}'(w,z)$ coincide.

3.2. Linear maps on $\mathcal{L}_{a,b}$, $\mathcal{L}(w,z)$ and their adjoints on $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w,z)$

Zemanian [**50**] has studied some linear maps and their adjoints (see pages 42, 43, 61, 62, 63, [**50**]). We make some observations on these maps defined on the ordered multinormed spaces $\mathcal{L}_{a,b}$, ordered countable union spaces $\mathcal{L}(w,z)$ and their adjoints on the ordered dual spaces $\mathcal{L}'_{a,b}$ and $\mathcal{L}'(w,z)$ with the topology of bounded convergence assigned to the dual spaces. Without losing any of the properties of the linear maps and their adjoints mentioned by Zemanian we obtain some additional order properties for the adjoints of these maps.

A linear partial differential operator and its adjoint on generalized functions

Zemanian [**50**] has discussed a type of operator that may be applied to generalized functions under certain conditions. The operator in question is defined as the adjoint of a linear partial differential operator acting on testing function spaces.

Let I be an open set either in $\mathbb{R}^n$ or $\mathbb{C}^n$ and let for $t = (t_1, \ldots, t_n) \in I$, $\theta_v(t)$ $(v = 0, 1, \ldots, m)$ denote complex-valued smooth functions on I. Considers the partial differential operator

$$R = (-1)^{|k|}\theta_0 D^{k_1}\theta_1 D^{k_2} \cdots \theta_{m-1} D^{k_m}\theta_m$$

where k_v now denote non negative integers in $\mathbb{R}^n$ and $|k|$ is the one-dimensional integer $|k_1| + \cdots + |k_m|$. The order of R is $|k|$. (The above symbol R denotes the sequence of operations: multiply by θ_m, differentiate according to D^{k_m}, multiply by θ_{m-1} etc. Moreover when I is an open set in $\mathbb{C}^n$, D has the customary definition when the limit is required to be independent of the direction in which the complex increment goes to zero.) Finally let $U(I)$ and $V(I)$ be testing function spaces on I. If $R : U(I) \to V(I)$ is linear and continuous its adjoint $R' : V'(I) \to U'(I)$ defined by $\langle R'f, \phi\rangle = \langle f, R\phi\rangle$ is also linear and continuous. Note that the order of differentiation in each D^{k_v}, $|k_\nu| > 1$ can be changed in any fashion without altering $R\phi$ because ϕ and the θ_v are smooth functions. Therefore $R'f$ also does not depend on this order. On the other hand, $R\phi$ and therefore $R'f$ do depend on the order in which multiplication by θ_v and the differential operators D^{k_v} are applied.

When I is an open set in $\mathbb{R}^n$ and when f is a smooth function whose support is a compact subset of I (i.e., $f \in \mathcal{D}(I)$) f and each of its derivatives define regular distributions in $\mathscr{E}'(I)$. For $f \in \mathcal{D}(I)$,

$\phi \in U(I)$

$$\langle R'f, \phi\rangle = \langle f, R\phi\rangle = \int_I f(t)R\phi(t)dt.$$

By successive integration by parts this becomes

$$\int_I \phi[\theta_m D^{k_m} \cdots \theta_1 D^{k_1}\theta_0 f]dt.$$

Thus in this case we can identify R' as the operator

$$\theta_m D^{k_m} \cdots \theta_1 D^{k_1}\theta_0$$

where D denotes the conventional differentiation. When R' acts upon $f \in \nu'(I)$, D denotes the generalized differentiation.

The following theorem proves that generalized differentiation is a continuous operation on $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w, z)$ with respect to the topology of bounded convergence.

THEOREM 3.2.1. *The generalized differentiation is a continuous mapping of $\mathcal{L}'_{a,b}$ into itself and of $\mathcal{L}'(w, z)$ into itself with the topology of bounded convergence assigned to the dual spaces.*

PROOF. For $\phi \in C(\mathcal{L}_{a,b})$, $\phi \in C(\mathcal{L}'_{a,b,K_m})$ for some compact set K_m in $\mathbb{R}$. By the definition of seminorms $\{\gamma_{K_m,k}\}_{k=0}^{\infty}$ on $\mathcal{L}_{a,b,K_m}$,

$\gamma_{K_m,k}(-D\phi) = \gamma_{K_m,k+1}(\phi)$. By Lemma 1.10.1, [**50**]

$(-D)$ is a continuous linear mapping of $C(\mathcal{L}_{a,b,K_m})$ into itself and hence of $C(\mathcal{L}_{a,b})$ into itself. It follows that $(-D)$ is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into itself, since the cone in $\mathcal{L}_{a,b}$ is generating. By Theorem 1.1.2 its adjoint operator, the generalized differentiation is a continuous linear mapping of $\mathcal{L}'(w, z)$ into itself. Since $\mathcal{L}(w, z)$ is the countable union of the spaces $\mathcal{L}_{a,b}$, it follows that $(-D)$ is a continuous linear mapping of $\mathcal{L}(w, z)$ into itself and its adjoint operator D, the generalized differentiation is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into itself with respect to the topology of bounded convergence. □

Remark. As the cone in $\mathcal{L}_{a,b}$ is generating it is enough if we prove the continuity of the operator $(-D)$ on the elements of the cone of $\mathcal{L}_{a,b}$.

Zemanian [**50**] has made the following observation: θ_M is the space of smooth functions defined as follows: $\theta(t)$ is in θ_M if and only if it is smooth on $-\infty < t < \infty$ and for each non-negative integer k there exists an integer N_k for which $(1 + t^2)^{-N_k} D^k \theta(t)$ is bounded on $-\infty < t < \infty$. For $\phi \in \mathcal{L}_{c,d}$ and for arbitrary real numbers a, b with $a < c$, $d < b$, for any $\theta \in \theta_M$ the operation $\phi \to \theta\phi$ is a continuous linear mapping of $\mathcal{L}_{c,d}$ into $\mathcal{L}_{a,b}$. Let (ϕ_i) be a sequence in $\mathcal{L}(w, z)$ that converges in $\mathcal{L}(w, z)$. Then there exist real numbers a, b, c, d such that $w < a < c$, $d < b < z$ such that (ϕ_i) converges in $\mathcal{L}_{c,d}$. By what proved above, $(\theta\phi_i)$ converges in $\mathcal{L}_{a,b}$

and hence in $\mathcal{L}(w,z)$. Thus $\phi \to \theta\phi$ is a continuous linear mapping of $\mathcal{L}(w,z)$ into itself.

The following theorem makes use of the above result to prove that $f \to \theta f$, $f \in \mathcal{L}'(w,z)$, $\theta \in \theta_M$ is a continuous linear operator on $\mathcal{L}'(w,z)$ with respect to the topology of bounded convergence.

THEOREM 3.2.2. *For $\theta \in \theta_M$, $f \to \theta f$ is a continuous linear operator on $\mathcal{L}'(w,z)$ with respect to the topology of bounded convergence.*

PROOF. For $\theta \in \theta_M$, $\phi \in \mathcal{L}(w,z)$, $\phi \to \theta\phi$ is a continuous linear mapping of $\mathcal{L}(w,z)$ into itself. The adjoint of this map is $f \to \theta f$, $f \in \mathcal{L}'(w,z)$. Being the adjoint of a continuous linear map on $\mathcal{L}(w,z)$, $f \to \theta f$ is a continuous linear map on $\mathcal{L}'(w,z)$ with respect to the topology of pointwise convergence and hence with respect to the topology of bounded convergence on $\mathcal{L}'(w,z)$ by Theorem 1.1.1. □

THEOREM 3.2.3. *Let α be a fixed complex number and $r = Re \cdot \alpha$. The mapping $f \to e^{-\alpha t} f$ from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{a-r,b-r}$ is linear, continuous, strictly positive and hence orderbounded. The map is a strictly positive, orderbounded isomorphism from $\mathcal{L}'(w,z)$ onto $\mathcal{L}'(w-r,z-r)$.*

PROOF. The map $\phi(t) \to e^{-\alpha t}\phi(t)$ and its inverse are linear and strictly positive. Zemanian [**50**] has proved that the map from $\mathcal{L}_{a-r,b-r}$ onto $\mathcal{L}_{a,b}$ and its inverse are continuous. By Theorem 1.1.2

its adjoint map $f \to e^{-\alpha t} f$, $f \in \mathcal{L}'_{a-r,b-r}$ is linear and continuous with respect to the topology of bounded convergence. By Theorem 1.1.3 it follows that the adjoint map and its inverse are strictly positive and hence are orderbounded. We conclude that the map is a strictly positive, orderbounded isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{a-r,b-r}$ with respect to the topology of bounded convergence. By the definition of $\mathcal{L}(w,z)$ corresponding results follow in the case of the map from $\mathcal{L}'(w,z)$ onto $\mathcal{L}'(w-r,z-r)$ with the topology of bounded convergence assigned to these dual spaces. □

THEOREM 3.2.4. *Let λ be a fixed positive real number. For every a,b,w,z, $f(t) \to f(t-\lambda)$ from $\mathcal{L}'_{a,b}$ to itself is linear, continuous, strictly positive and orderbounded. Hence the map is an orderbounded automorphism on $\mathcal{L}'_{a,b}$ with respect to the topology of bounded convergence. The map is an orderbounded automorphism on $\mathcal{L}'(w,z)$ with respect to the topology of bounded convergence.*

PROOF. Zemanian [**50**] has observed that for a fixed real number λ, $\phi(t) \to \phi(t+\lambda)$ is linear and continuous $\mathcal{L}(w,z)$. We observe that for $\lambda > 0$, $\phi(t) \to \phi(t+\lambda)$ is strictly positive on $\mathcal{L}_{a,b}$ and hence on $\mathcal{L}(w,z)$. The adjoint of this map $f(t) \to f(t+\lambda)$ on $\mathcal{L}'_{a,b}$ is continuous with respect to the topology of bounded convergence by Theorem 1.1.2 and strictly positive and hence orderbounded by Theorem 1.1.3. The unique inverse mapping of $\phi(t) \to \phi(t+\lambda)$ is $\phi(t) \to \phi(t-\lambda)$ and it maps $\mathcal{L}_{a,b}$ into itself and $\mathcal{L}(w,z)$ into itself. It follows that $f(t) \to f(t-\lambda)$ is continuous with respect to the topology of bounded convergence, strictly positive and orderbounded on $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w,z)$. $\square$

3.3. The two-sided Laplace transformation: Definition and some basic properties

For defining the two-sided Laplace transformation and its inverse, we follow the methods of Zemanian [**50**]. But the dual space on which the Laplace transform is applied is an ordered vector space on which the topology of bounded convergence is applied. By making these changes on the dual space we obtain the additional feature that comparison of solutions of differential equations which can be solved by applying the Laplace transform is possible.

DEFINITION 3.3.1. Let f be a linear functional defined on L_f, a linear space of conventional functions which satisfies the following properties.
(i) $\mathcal{L}_{a,b} \subseteq L_f$ for at least one pair of real numbers a, b with $a < b$.
(ii) For every $\mathcal{L}_{a,b} \subseteq L_f$ the restriction of f to $\mathcal{L}_{a,b}$ is in $\mathcal{L}'_{a,b}$.
We call f a Laplace transformable generalized function.

DEFINITION 3.3.2. Let f be a Laplace transformable generalized function. We define a set Λ_f to be the union of all open intervals (a,b) such that $\mathcal{L}_{a,b} \subseteq L_f$, i.e., $\Lambda_f = \cup\{(a,b) : \mathcal{L}_{a,b} \subseteq L_f\}$.

Note. Λ_f is an open set.

THEOREM 3.3.1. *The collection $\mathcal{W}$ of all Laplace transformable functions is a linear space.*

PROOF. Let $f, g \in \mathcal{W}$. Then there exists real numbers a, b and c, d such that $\mathcal{L}_{a,b} \subseteq L_f$, $\mathcal{L}_{c,d} \subseteq L_g$. Then we can find real numbers l, m such that $\mathcal{L}_{l,m} \subseteq \mathcal{L}_{a,b}$ and $\mathcal{L}_{l,m} \subseteq \mathcal{L}_{c,d}$ so that $f + g$ can be defined on $L_f \cap L_g$ by

$$\langle f + g, \phi \rangle = \langle f, \phi \rangle + \langle g, \phi \rangle.$$

Also for $\lambda \in \mathbb{C}$, $\lambda f \in \mathcal{W}$ if $f \in \mathcal{W}$ so that $\mathcal{W}$ is a linear space. $\square$

As a prelude to defining the Laplace transformation we state and prove the following theorem.

THEOREM 3.3.2. *Let $\sigma_1 = \inf \Lambda_f$, $\sigma_2 = \sup \Lambda_f$. Given a functional f defined on a linear space L_f of conventional functions f can be extended to a functional f_1 on $L_f \cup \mathscr{L}(\sigma_1, \sigma_2)$ such that*
(i) the restriction of f_1 to $\mathcal{L}(\sigma_1, \sigma_2)$ is a member of $\mathcal{L}'(\sigma_1, \sigma_2)$
(ii) the restriction of f_1 to L_f coincide with f.

PROOF. Since $\sigma_1 = \inf \Lambda_f$, $\sigma_2 = \sup \Lambda_f$, there exists two sequences (c_i), (d_i) of real numbers such that $c_i \to \sigma_1^+$ and $d_i \to \sigma_2 -$, $c_i, d_i \in \Lambda_f$, $c_i < d_i$, $\forall i$. Then $f \in \mathcal{L}'_{c_i,c_i}$, $f \in \mathcal{L}'_{d_i,d_i}$, $\forall i$. (see pages 53, 54, [**50**]). Let $\lambda(t)$ be a fixed smooth function on $\mathbb{R}$ such that

$$\lambda(t) = 0 \quad \text{for } t < -1$$

$$= 1 \quad \text{for } t > 1$$

f can be extended to $\mathcal{L}_{c_i,d_i}$ as follows:
Let $\psi \in \mathcal{L}_{c_i,d_i}$. Define $f(\psi) = f(\lambda\psi)+f((1-\lambda)\psi)$. f is continuous and linear on $\mathcal{L}_{c_i,d_i}$. Using the above method f may be extended to $\mathcal{L}(\sigma_1, \sigma_2)$. This extension of f is unique. □

For a given Laplace transformable function f let Ω_f denote the open strip in the complex s-plane:

$$\Omega_f = \{s : \sigma_1 < \text{Re}\, s < \sigma_2\},\ \text{where } \sigma_1 = \text{inf}\Lambda_f.$$

$\sigma_2 = \sup \Lambda_f$. Then the Laplace transform $\mathcal{L}(f)$ is defined by

$$F(s) = (\mathcal{L}f)(s) = \langle f(t), e^{-st}\rangle,\ s \in \Omega_f$$

Note.

(1) By the notation $\langle f(t), e^{-st}\rangle$, $s \in \Omega_f$ we mean

$$\langle f(t), e^{-st}\rangle = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

(2) For any fixed $s \in \Omega_f$, the right hand side has a meaning as the application of $f \in \mathcal{L}'(\sigma_1, \sigma_2)$ to $e^{-st} \in \mathcal{L}(\sigma_1, \sigma_2)$.

The following two theorems deal with the order relation introduced by us on $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w, z)$, $\mathcal{L}'(\sigma_1, \sigma_2)$. The proof of Theorem 3.3.3 is based on the results proved by Zemanian [**50**] in section 3.2 (v), Chapter 2.

THEOREM 3.3.3. *If $f(t)$ is a positive locally integrable function such that $\frac{f(t)}{L_{a,b}(t)}$ is absolutely integrable on $-\infty < t < \infty$ then*

$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)dt$ *is positive and so* $f \in C'(\mathcal{L}'_{a,b})$. *Also, if* $f(t)$ *is a positive locally integrable function such that* $\frac{f(t)}{L_{a,b}(t)}$ *is absolutely integrable on* $-\infty < t < \infty$ *for every choice of* a *and* b *satisfying* $w < a$ *and* $b < z$, f *generates a positive regular member of* $C'(\mathcal{L}'_{w,z})$.

PROOF. Zemanian [**50**] has proved that if $f(t)$ is a locally integrable function such that $\frac{f(t)}{L_{a,b}(t)}$ is absolutely integrable on $-\infty < t < \infty$, the functional defined by $\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)dt$, $\phi \in \mathcal{L}_{a,b}$ is continuous and linear on $\mathcal{L}_{a,b}$ (see pages 53, 54, [**50**]). Since the topology of pointwise convergence is weaker than the topology of bounded convergence it follows that $f \in \mathcal{L}'_{a,b}$ with the topology on $\mathcal{L}'_{a,b}$ replaced by the topology of bounded convergence. A linear functional f on a linear space X of scalar valued functions on a set T is said to be positive if $f(x) \geq 0$, for all $x \in X$ such that $x(t) \geq 0$, for all $t \in T$ ([**18**]). If $\phi(t) \geq 0$, $\int_{-\infty}^{\infty} f(t)\phi(t)dt \geq 0$ and it follows that $f \in C'(\mathcal{L}'_{a,b})$. i.e., f is a member of the dual cone defined on $\mathcal{L}'_{a,b}$.

Zemanian [**50**] has also proved that if $f(t)$ is a locally integrable function such that $\frac{f(t)}{L_{a,b}(t)}$ is absolutely integrable on $-\infty < t < \infty$, for every choice of a and b satisfying $w < a$ and $b < z$ then f generates a regular member of $\mathcal{L}'(w, z)$ by the definition

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)dt, \quad \phi \in \mathcal{L}(w, z).$$

As in the above case, it follows that f is a linear functional on

$\mathcal{L}(w, z)$, continuous with respect to the topology of bounded convergence. Also, it follows that $f \in C'(\mathcal{L}'(w, z))$. □

THEOREM 3.3.4. *The Laplace transform is continuous, strictly positive and orderbounded map on* $\mathcal{L}'(\sigma_1, \sigma_2)$.

PROOF. For $f \in \mathcal{L}'(\sigma_1, \sigma_2)$, $\mathcal{L}f \in \mathcal{L}'(\sigma_1, \sigma_2)$ and acts upon elements of $\mathcal{L}(\sigma_1, \sigma_2)$ in the following way:

$$\langle \mathcal{L}f, \phi \rangle = \int_{-\infty}^{\infty} f(t) e^{-st} \phi(t) dt = \langle f, \mathcal{L}\phi \rangle.$$

The mapping $\phi \rightarrow \mathcal{L}\phi$ is the conventional Laplace transform on $\mathcal{L}(\sigma_1, \sigma_2)$ and is continuous on $\mathcal{L}(\sigma_1, \sigma_2)$. The adjoint of this map is $f \rightarrow \mathcal{L}f$ which is continuous on $\mathcal{L}'(\sigma_1, \sigma_2)$ with respect to the topology of bounded convergence (by Theorem 1.1.2). Defining an order relation on the field of complex numbers by identifying the positive cone to be the set of complex numbers $\alpha + i\beta$, $\alpha > 0$, $\beta > 0$, it follows that if $f > 0$, $f \in \mathcal{L}'(\sigma_1, \sigma_2)$, then $\mathcal{L}f > 0$ so that the Laplace transformation is strictly positive and hence is orderbounded. □

3.4. Inversion and Uniqueness

The results on the inversion of the Laplace transform and Uniqueness Theorems [**50**] hold good in the ordered dual spaces when they are assigned the topology of bounded convergence. The theorems are stated without proof.

LEMMA 3.4.1. *Let* $\mathcal{L}(f) = F(s)$ *for* $\sigma_1 < Re\ s < \sigma_2$, *let* $\phi \in C(\mathcal{D})$ *and set* $\psi(s) = \int_{-\infty}^{\infty} \phi(t)e^{st}dt$. *Then for any fixed real number* r *with* $0 < r < \infty$, $\int_{-\tau}^{\tau} \langle f(t), e^{s\tau}\rangle \psi(s)d\omega = \langle f(\tau), \int_{-\tau}^{\tau} e^{-s\tau}\psi(s)d\omega\rangle$ *where* $s = \sigma + i\omega$ *and* σ *is fixed with* $\sigma_1 < \sigma < \sigma_2$.

LEMMA 3.4.2. *Let* a, b, σ *and* τ *be real numbers with* $a < \sigma < b$, $\phi \in C(\mathcal{D})$. *Then* $\frac{1}{\pi}\int_{-\infty}^{\infty} \phi(t+\tau)e^{\sigma t}\frac{\sin rt}{t}dt$ *converges in* $C(\mathcal{L}_{a,b})$ *to* $\phi(\tau)$ *and hence in* $\mathcal{L}_{a,b}$ *as* $r \to \infty$.

THEOREM 3.4.1. *Let* $\mathcal{L}(f) = F(s)$ *for* $s \in \Omega_f$ *and* $\mathcal{L}(h) = H(s)$ *for* $s \in \Omega_h$ *and if* $\Omega_f \cap \Omega_h \neq \phi$ *and if* $F(s) = H(s)$ *for* $s \in \Omega_f \cap \Omega_h$ *then* $f = h$ *in* $\mathcal{L}'(w, z)$ *where the interval* $w < \sigma < z$ *is the intersection of* $\Omega_f \cap \Omega_h$ *with the real axis.*

3.5. Operational calculus and solution and comparison of solutions of differential equations

The Laplace transform and its inverse may be applied to differential equations involving generalized functions to find solutions to such differential equations. We give below a brief sketch of the method applied to solve differential equations involving Laplace transformable functions. Zemanian [**50**] has proved a few results required to establish the operational calculus. We state below these results with out proof.

RESULT 3.5.1. [**50**] *Let* $F(s)$ *be a strictly positive function. The necessary and sufficient condition for* $F(s)$ *to be the Laplace transform of a positive generalized function* f *and for the corresponding* *strip of definition to be* $\Omega_f = \{s : \sigma_1 < \textit{Re } s < \sigma_2\}$ *is that* $F(s)$ *be analytic on* Ω_f *and for each closed substrip* $\{s : a \leq \textit{Re } \cdot s \leq b\}$ *of* Ω_f *where* $\sigma_1 < a < b < \sigma_2$ *is that there be a polynomial* P *such that* $F(s) \leq P(|s|)$ *for* $a < \textit{Re } \cdot s < b$*. The polynomial will in general depend on the choices of* a *and* b*.*

RESULT 3.5.2. **[50]** *Let* $F(s)$ *be a strictly positive function and let* $\mathcal{L}(f) = F(s)$ *for* $s \in \Omega_f$*. Choose three fixed real numbers* a, σ *and* b *such that* $a < \sigma < b$ *and choose a polynomial* $Q(s)$ *that has no zeros for* $a \leq \textit{Re} \cdot s \leq b$ *and such that*

$$\frac{F(s)}{Q(|s|)} \leq \frac{k}{|s|^2},\ a < \textit{Re} \cdot s < b,\ k \textit{ a constant.}$$

Then in the sense of equality in $\mathcal{L}'(a, b)$

$$F(t) = Q(D_t)\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{F(s)}{Q(s)} e^{st} ds,\ a < \sigma < b.$$

where D_t *denotes the generalized differentiation in* $\mathcal{L}'(a, b)$ *and the integral converges in the conventional sense to a continuous function that generates a regular member of* $\mathcal{L}'(a, b)$*.*

Operational Calculus. Consider the linear differential equation

$$L(u(t)) = (a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0)u(t) = g(t)$$

where the a_i's are constants, $a_n \neq 0$ and $g(t)$ is a given Laplace transformable generalized function. Applying the Laplace transform to both sides,

$$B(s)U(s) = G(s)$$

where $B(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$.

$$U(s) = \mathcal{L}(u),\ G(s) = \mathcal{L}(g),\ s \in \Omega_g = \{s : \sigma_{g_1} < \text{Re } \cdot s < \sigma_{g_2}\}.$$

If $B(s)$ has no zeros in Ω_g then by Theorem 3.5.1 there exists a generalized function $u(t)$ whose Laplace transform is $\frac{G(s)}{B(s)}$ on Ω_g. $u(t)$ is a unique member of $\mathcal{L}'(\sigma_{g_1}, \sigma_{g_2})$ and satisfies the given equation. If $B(s)$ has a finite number of zeros in Ω_g there exists a set of m adjoint substrips

$$\sigma_{g_1} = \sigma_0 < \text{Re } \cdot s < \sigma_1, \sigma_1 < \text{Re } \cdot s < \sigma_2, \ldots, \sigma_{m-1} < \text{Re } \cdot s < \sigma_m = \sigma_{g_2}$$

on which $\frac{G(s)}{B(s)}$ is analytic and satisfies the growth condition of Result 3.5.2. For a given substrip, say, $\sigma_i < \text{Re } \cdot s < \sigma_{i+1}$ there exists a unique member $u(t)$ of $\mathcal{L}'(\sigma_i, \sigma_{i+1})$ and whose Laplace transform is $\frac{G(s)}{B(s)}$ on $\sigma_i < \text{Re} \cdot s < \sigma_{i+1}$. For any other choice of the substrip there will be a different solution.

Given below is an illustration of the fact that comparison of different solutions of a differential equation which can be solved by the application of Laplace transform since we have introduced an order on the linear space of Laplace transformable functions.

ILLUSTRATION 3.5.1. In Zemanian [**50**] the solution of the non-homogeneous wave equation is given as follows.

$$(D_x^2 - c^{-2}D_t^2)u(x,t) = g(x,t) \tag{9}$$

where $x, t \in \mathbb{R}$, $g(x,t)$ is a given Laplace transformable function, c is a + ve real number, the speed of the wave

$$h(x,t) = -\frac{c}{2}(1 + ct - |x|)$$

is an elementary solution to the wave equation in one dimensional space. The solution to (9) is now given by

$$u(x,t) = h(x,t) * g(x,t).$$

If $x = x_1$, $t = t_1$, $t = t_2$ are such that $t_1 \leq t_2$

$$h(x_1,t_1) \leq h(x_1,t_2).$$

Since the operation of convolution is orderpreserving, it follows that

$$u(x_1,t_1) \leq u(x_1,t_2).$$

If $|x_1| \leq |x_2|$ we get $h(x_1,t_1) \geq h(x_2,t_1)$ and accordingly

$$u(x_1,t_1) \geq u(x_2,t_1).$$

CHAPTER 4

The Stieltjes Transformation

Arora [2] has studied the Stieltjes transformation by defining the test function space $M^m_{a,b}$ and its dual $(M^m_{a,b})'$ with the weak topology and has applied the Stieltjes transformation of the form $\hat{f}(x) = \int_0^\infty \frac{f(t)}{(x^m+t^m)^\rho}dt$, $m, \rho > 0$ to the elements of $(M^m_{a,b})'$. We treat $M^m_{a,b}$ as a countable union space and assign an order relation on $M^m_{a,b}$ by identifying a positive cone in it. The dual space $(M^m_{a,b})'$ is ordered by the dual cone of the cone of $M^m_{a,b}$ and the topology of $(M^m_{a,b})'$ is changed to the topology of bounded convergence so that the order dual and the topological dual of $M^m_{a,b}$ coincide. The Stieltjes transform of the above form is applied to the ordered vector space $(M^m_{a,b})'$ with the topology of bounded convergence assigned to $(M^m_{a,b})'$. Without losing any of the properties the transformation originally had, we get some additional order properties satisfied by it. John J. K. [13] has studied the asymptotic behaviour of a distribution $f \in \zeta'_+$ at ∞ with respect to a regularly varying function $v(k)$ using the techniques used by Troger [42]. In our study since the topology of bounded convergence which is also called the strong topology is assigned to ζ' we call the above behaviour the strong asymptotic behaviour of $f \in \zeta'_+$. Results connecting the strong asymptotic behaviour of $f \in (M^m_{a,b})'$ at ∞ with respect to $v(k)$ and that of $S^m_\rho(f) \in (M^m_{a,b})'$ at ∞ with respect to $k^{1-m\rho}v(k)$ are proved. Using the fact that $(M^m_{a,b})'$ and $(M^m_{\alpha,\beta})'$ are order complete we apply the above results to monotone nets in these spaces.

4.1. The testing function space $M^m_{a,b}$ and its dual $(M^m_{a,b})'$

Let (K_m) be a sequence of compact subsets of R_+ such that $K_1 \subseteq K_2 \subseteq \ldots$ and such that each compact subset of $(0, \infty)$ is contained in one K_j, $j = 1, 2, 3, \ldots$. Let M_{a,b,K_j} denote the linear space of all (infinitely) smooth complex-valued functions defined on $\mathbb{R}$ having compact support in K_j, on which is defined

$$\mu^m_{a,b,K_j,k}(\phi) = \sup_{t \in k_j} t^{m(1-a+k)}(1-t^m)^{a-b}|(t^{1-m}\frac{d}{dt})^k \phi(t)|,$$

$a, b \in \mathbb{R}$, $k = 0, 1, 2, \ldots, m \in (0, \infty)$. $\{\mu^m_{a,b,K_j,k}\}^\infty_{k=0}$ is a multinorm on M^m_{a,b,K_j} and generates a topology τ^m_{a,b,K_j} on M^m_{a,b,K_j}. M^m_{a,b,K_j} is complete with respect to τ^m_{a,b,K_j}. Denote the strict countable union space $\cup^\infty_{j=1} M^m_{a,b,K_j}$ as $M^m_{a,b}$. Since M^m_{a,b,K_j} is complete with respect to τ^m_{a,b,K_j}, as in the case of $\mathcal{L}_{a,b}$, it follows that $M^m_{a,b}$ is complete. On each M^m_{a,b,K_j} an equivalent multinorm is given by

$$\bar{\mu}^m_{a,b,K_j,k}(\phi) = \sup_{0 \le k' \le k} \mu^m_{a,b,K_j,k'}(\phi), \quad \text{see [13]}.$$

We define an order relation on $M^m_{a,b}$ by identifying a positive cone in it.

DEFINITION 4.1.1. The cone C of $M^m_{a,b}$ when $M^m_{a,b}$ is restricted to real-valued functions is the set of all non-negative functions in

$M_{a,b}^m$. When the field of scalars is $\mathbb{C}$, the complex numbers. $C + iC$ is the positive cone in $M_{a,b}^m$ which is also denoted as C.

Note We say that $\phi \leq \psi$ in $M_{a,b}^m$ when $\psi - \phi \in M_{a,b}^m$.

As in the case of previous examples it can be proved that the cone C in $M_{a,b}^m$ is not normal but is a strict b-cone.

Order and topology on the dual of $M_{a,b}^m$. An order relation is defined on the dual $(M_{a,b}^m)'$ the linear space of all continuous linear functionals on $M_{a,b}^m$ by identifying the positive cone in $(M_{a,b}^m)'$ to be the dual cone C' of the cone of $M_{a,b}^m$. The class of all B^0, the polars of B as B varies over all $\sigma(M_{a,b}^m, (M_{a,b}^m)')$-bounded subsets of $M_{a,b}^m$ is a neighbourhood basis of 0 in $(M_{a,b}^m)'$ for a locally convex topology $\beta((M_{a,b}^m)', M_{a,b}^m)$. When $(M_{a,b}^m)'$ is ordered by the dual cone C' and is equipped with the topology of bounded convergence $\beta((M_{a,b}^m)', M_{a,b}^m)$ it follows that C' is a normal cone since C is a strict b-cone, by Corollary 1.26, Chapter 2. [**29**].

As in the case of previous examples we observe that when the topology on $(M_{a,b}^m)'$ is changed to the topology of bounded convergence, $(M_{a,b}^m)'$ is order complete and topologically complete, the order dual and the topological dual of $M_{a,b}^m$ coincide and the order topology and the topology of bounded convergence on $(M_{a,b}^m)'$ coincide. Also the cones of $M_{a,b}^m$, $(M_{a,b}^m)'$ are generating.

4.2. The Stieltjes transformation

For $\phi \in M^m_{\alpha,\beta}$ the Stieltjes transformation is defined as

$$S^m_\rho(\phi) = \hat{\phi}(x) = \int_0^\infty \frac{\phi(t)}{(x^m + t^m)^\rho} dt \quad \text{for a fixed } m > 0,\ \rho \geq 1.$$

John J. K. [**13**] has proved the following result:
Let $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$. Then the Stieltjes transform maps $M^m_{\alpha,\beta}$ continuously into $M^m_{a,b}$ if $a \leq 1$, $a \leq \frac{1}{m} + \alpha - \rho$ and $a < 1$ if $\alpha = \rho + 1 - \frac{1}{m}$; $b \geq 1 - \rho$, $b \geq \frac{1}{m} + \beta - \rho$ and $b > 1 - \rho$, if $\beta = 1 - \frac{1}{m}$. We use this result in a modified form where we claim that the Stieltjes transform maps the positive cone of $M^m_{\alpha,\beta}$ continuously to the positive cone of $M^m_{a,b}$.

THEOREM 4.2.1. *Let $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$. Then the Stieltjes transformation maps $C \cap M^m_{\alpha,\beta}$ continuously into $C \cap M^m_{a,b}$ if $a < 1$, $a \leq \frac{1}{m} + \alpha - \rho$ and $a < 1$ if $\alpha = \rho + 1 - \frac{1}{m}$, $b \geq 1 - \rho$, $b \geq \frac{1}{m} + \beta - \rho$ and $b > 1 - \rho$ if $\beta = 1 + \frac{1}{m}$.*

PROOF. By definition, the Stieltjes transform is order preserving. From the result quoted above it follows that the transform maps the positive cone of $M^m_{\alpha,\beta}$ to the positive cone of $M^m_{a,b}$ under the specified conditions. Since the cones of $M^m_{\alpha,\beta}$ and $M^m_{a,b}$ are generating it follows that the Stieltjes transform maps $M^m_{\alpha,\beta}$ continuously into $M^m_{a,b}$. $\square$

With suitable integrability properties the double integral

$$\int_0^\infty \int_0^\infty \frac{f(x)\phi(t)}{(x^m + t^m)^\rho} dt\, dx$$

can be evaluated in two different ways so that

$$\langle S_\rho^m(f), \phi \rangle = \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle = \langle f, S_\rho^m(\phi) \rangle, \ \phi \in M_{\alpha,\beta}^m, \ f \in (M_{\alpha,\beta}^m)'.$$

Thus the Stieltjes transform of a generalized function $f \in (M_{\alpha,\beta}^m)'$ is defined to be adjoint of the map $\phi \to S_\rho^m(\phi)$, $\phi \in M_{\alpha,\beta}^m$.

THEOREM 4.2.2. *The Stieltjes transform is strictly positive, orderbounded and continuous with respect to the topology of bounded convergence.*

PROOF. Being the adjoint of strictly positive map, $f \to S_\rho^m(f)$ is strictly positive and hence is orderbounded. By the same reason, $f \to S_\rho^m(f)$ is continuous with respect to the topology of bounded convergence. □

Note. John J. K. [**13**] has observed that the Stieltjes transform of $\phi \in M_{\alpha,\beta}^m$ may be inverted by the application of a differential operator L_n where n is a non-negative integer, defined by

$$L_n = L_n(\phi(x)) = \frac{(-1)^n m^{1-2n}\Gamma(\rho)}{\Gamma(n+\frac{1}{m})\Gamma(\rho+n-\frac{1}{m})} \frac{d}{dx}(x^{1-m}\frac{d}{dx})^{n-1} x^{2mn+m\rho-m}(x^{1-m}\frac{d}{dx})^n \phi(x)$$

for $\rho + n > \frac{1}{m}$. The formal adjoint of this operator is itself.

We make the following conclusions from the results proved by John J. K. [**13**], modified to suit the present situation.

RESULT 4.2.1. *If* $\rho + n > \frac{1}{m}$, $\int_0^\infty L_{n,x}(x^m + t^m)^{-\rho} dt = 1$.

RESULT 4.2.2. $L_{n,t}$ *maps* $C \cap M^m_{a,b}$ *continuously into* $C \cap M^m_{\alpha,\beta}$ *provided* $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$.

RESULT 4.2.3. *If* L_n *is the differential operator and* S *is the Stieltjes transform operator then either* $x^{1-m\rho}L_n$ *and* $Sx^{m\rho-1}$ *or* $L_n x^{1-m\rho}$ *and* $x^{m\rho-1}S$ *commute on* $M^m_{a,b}$ *where* $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$.

$$\begin{aligned} \textit{i.e. } x^{m\rho-1}S^m_\rho(L_{n,t}(\phi)) &= x^{m\rho-1}(L_{n,t}(\phi))^\wedge \\ &= L_{n,x}\int_0^\infty (x^m + t^m)^{-\rho} t^{m\rho-1}\phi(t)dt \\ &= L_{n,x}S^m_\rho(t^{m\rho-1}\phi). \end{aligned}$$

RESULT 4.2.4. *If* $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, *the sequence* $(L_{n,x}\hat{\phi}(x))$ *converges in* $C \cap M^m_{\alpha,\beta}$ *to* $\phi(x)$.

RESULT 4.2.5. *Let* $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$. *Then* $(L_n(\phi)^\wedge)$ *converges to* ϕ *in* $M^m_{a,b}$ *as* $n \to \infty$.

RESULT 4.2.6. *Let* $f \in (M^m_{a,b})'$. *Then* $f \in C'$ *if and only if for every non-negative integer* n, $L_n S^m_\rho(f) \in C'$ *where* C' *is the positive cone in* $(M^m_{a,b})'$. *It follows that* L_n *is strictly positive and hence is orderbounded.*

RESULT 4.2.7. *For* $f \in (M^m_{\alpha,\beta})'$, $\phi \in M^m_{\alpha,\beta}$

$$\langle S^m_\rho(L_n(f)), \phi\rangle = \langle f, L_n S^m_\rho(\phi)\rangle \to \langle f, \phi\rangle \text{ as } n \to \infty.$$

For $f \in (M^m_{a,b})'$, $\phi \in M^m_{a,b}$

$$\langle L_n(S^m_\rho(f), \phi)\rangle = \langle f, S^m_\rho L_n(\phi)\rangle \to \langle f, \phi\rangle.$$

4.3. Abelian and Tauberian theorems

We have studied ζ', linear space of tempered distributions as an ordered vector space with the topology of bounded convergence defined on it in Chapter 2. The elements f of ζ' whose support is contained in $[0, \infty)$ form a subspace of ζ' denoted as ζ'_+. ζ'_+ is ordered by the order derived from ζ' and has the topology derived from the topology of bounded convergence assigned to ζ'. Using the techniques applied by Troger [**42**],John J. K. [**13**] has studied the asymptotic behaviour of a distribution $f \in \zeta'_+$ at ∞ with respect to a regularly varying function, the topology on ζ' being the topology of pointwise convergence. We define a similar notion on ζ'_+ with the topology of bounded convergence assigned to ζ'_+ and call it the strong asymptotic behaviour of $f \in \zeta'_+$. Besides obtaining results similar to the abelian and Tauberian theorems proved in [**13**], since $(M^m_{\alpha,\beta})'$ and $(M^m_{a,b})'$ are ordercomplete we have extended the above results to monotone nets in these space.

A function $v(k)$ which is positive and continuous on $(0, \infty)$ is said to be regularly varying of order r, $r \in \mathbb{R}$ if for any $a > 0$ the limit

$$\lim_{k\to\infty} \frac{v(ak)}{v(k)} = a^r \text{ exists.}$$

DEFINITION 4.3.1. A distribution $f \in \zeta'_+$ is said to have the strong asymptotic behaviour at ∞ with respect to a regularly varying function $v(k)$ if the limit $\lim_{k\to\infty} \frac{f(kt)}{v(k)} = F(t)$ exists, in the sense of convergence in ζ' with respect to the topology of bounded convergence, provided $F \neq 0$.

Note.

(1) F is a homogeneous function of order r and hence $F \in \zeta'$ and support of $F \subseteq (0, \infty)$.

(2) By saying that $\lim_{k\to\infty} \frac{f(kt)}{v(k)} = F(t)$ exists in ζ' with respect to the topology of bounded convergence what we mean is that if $F(t) \in B^0$ where B^0 is a basis element for the topology of bounded convergence on ζ', $\frac{f(kt)}{v(k)} \in B^0$ for $k \geq k_0$,

$$\text{i.e. } \left|\langle \frac{f(kt)}{v(k)}, \psi \rangle\right| < 1, \ \forall \psi \in B, k \geq k_0$$

whenever $|\langle F(t), \psi \rangle| < 1$, $\forall \psi \in B$, where $B = \{\psi \in \zeta' : |g(\psi)| < s'\epsilon$ for some $g \in \zeta'\}$, $\epsilon > 0$, for all $s' > s$, $s', s \in \mathbb{R}$.

LEMMA 4.3.1. *Let* $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$, *then the set* $A = \{S^m_\rho(\phi) : \phi \in M^m_{\alpha,\beta}\}$ *is dense in* $M^m_{a,b}$.

PROOF. Let $\phi \in M^m_{a,b}$, $\phi_n = S^m_\rho[L_{n,t}(\phi)]$. Then $\phi_n \in A$ and $\phi_n \to \phi$ in $M^m_{a,b}$. □

THEOREM 4.3.1. *Let $f \in (M^m_{a,b})'$ and $v(k)$ be a regularly varying function of order $r > (-ma)$. Then the following statements are equivalent*

(i) f has in $C' \cap (M^m_{a,b})'$ strong asymptotic behaviour at ∞ with respect to $v(k)$

(ii) $S^m_\rho(f)$ has in $C' \cap (M^m_{\alpha,\beta})'$ strong asymptotic behaviour at ∞ with respect to $k^{1-m\rho}v(k)$ and $\frac{1}{k^{1-m\rho}v(k)}L_{n,x}S^m_\rho(f)(kx)$, for $k > 0$ is uniformly continuous with respect to the topology of bounded convergence in $C' \cap (M^m_{a,b})'$ for all values of n.

PROOF. (i) $\Rightarrow$ (ii)

Let $\phi \in M^m_{\alpha,\beta}$. Also let

$$\lim_{k\to\infty} \frac{f(kt)}{v(k)} = g(t),\ f \in C' \cap (M^m_{a,b})'.$$

Then we have

$$\lim_{k\to\infty} \frac{1}{v(k)}\langle f(kt), \phi(t)\rangle = \langle g(t), \phi(t)\rangle.$$

For $\phi_1 \in M^m_{\alpha,\beta}$, $S^m_\rho(\phi_1) \in M^m_{a,b}$.

$$\text{Hence} \quad \langle g(t), S^m_\rho(\phi_1)(t)\rangle = \lim_{k\to\infty} \frac{1}{v(k)}\langle f(kt), S^m_\rho(\phi_1)(t)\rangle$$
$$= \lim_{k\to\infty} \frac{1}{k^{1-m\rho}v(k)}\langle S^m_\rho(f)(kx), \phi_1(x)\rangle.$$

But $\langle g(t), S^m_\rho(\phi_1)(t)\rangle = \langle S^m_\rho(g)(x), \phi_1(x)\rangle$.

So we conclude that

$$\lim_{k\to\infty}\frac{1}{k^{1-m\rho}v(k)}S_\rho^m(f)(kx)=S_\rho^m(g)(x).$$

This means that $S_\rho^m(f)$ has in $C'\cap(M_{\alpha,\beta}^m)'$ strong asymptotic behaviour at ∞ with respect to $k^{1-m\rho}v(k)$.

$$|\frac{1}{k^{1-m\rho}v(k)}\langle L_nS_\rho^m(f)(kx),\phi(x)\rangle|=|\langle\frac{f(kt)}{v(k)},S_\rho^mL_n(\phi)(t)\rangle|.$$

Since $\lim_{k\to\infty}\frac{f(kt)}{v(k)}=F(t)$ in ζ',

$$|\langle\frac{f(kt)}{v(k)},\psi(t)\rangle|<1$$

for all $\psi\in B$, $k\geq k_0$ whenever

$$|\langle F(t),\psi(t)\rangle|<1$$

for all $\psi\in B$ where $B=\{\psi\in\zeta':|f(\psi)|<s'\epsilon$ for some $f\in\zeta'\}$, $\epsilon>0$ for all $s'>s$, $s',s\in\mathbb{R}$. Since $\frac{f(kt)}{v(k)}$ is a continuous linear functional there exists a positive constant C_1, and integers j, q such that

$$\Big|\frac{1}{k^{1-m\rho}v(k)}\langle L_nS_\rho^m(f)(kx),\phi(x)\rangle\Big|\leq C_1\bar{\mu}_{a,b,K_j,q}^m(S_\rho^mL_n(\phi))$$

$$\bar{\mu}_{a,b,K_j,q}^m(S_\rho^mL_n(\phi))\leq\bar{\mu}_{a,b,K_j,q}^m(S_\rho^mL_n(\phi)-\phi)+\bar{\mu}_{a,b,K_j,q}^m(\phi)$$

Using Results 4.2.3, 4.2.4, 4.2.5.

$$\bar{\mu}^m_{a,b,K_j,q}(S^m_\rho L_n(\phi) - \phi) = \bar{\mu}^m_{a,b,K_j,q}(x^{1-m\rho} L_n S^m_\rho(t^{m\rho-1}\phi(t))) - \phi(x))$$
$$\leq C_2 \bar{\mu}^m_{\alpha,\beta,K_j,q}(L_n S^m_\rho(\phi_0) - \phi_0)$$

where

$$\phi_0(t) = t^{m\rho-1}\phi(t) \in M^m_{\alpha,\beta,K_j} \leq \epsilon_n C_3 \bar{\mu}^m_{\alpha,\beta,K_j,q+1}(\phi).$$

Consequently $\bar{\mu}^m_{a,b,K_j,q}(S^m_\rho L_n(\phi)) \leq C_4 \bar{\mu}^m_{a,b,k_j,q+1}(\phi)$, for all n, uniformly for $k \geq k_0$ where C_4 is a constant depending only on f, j and q. Hence we conclude that $\frac{1}{k^{1-m\rho}\upsilon(k)} L_{n,x} S^m_\rho(f)(kx)$, for $k > 0$ is uniformly continuous with respect to the topology of bounded convergence in $C' \cap (M^m_{\alpha,\beta})'$ for all values of n.

(ii) $\Rightarrow$ (i)

Let $\lim\limits_{k\to\infty} \dfrac{S^m_\rho(f)(kt)}{k^{1-m\rho}\upsilon(k)} = S^m_\rho(g)(t)$, say.

Then

$$\lim_{k\to\infty} \frac{1}{k^{1-m\rho}\upsilon(k)} \langle S^m_\rho(f)(kx), \phi(x)\rangle = \langle S^m_\rho(g)(x), \phi_1(x)\rangle$$
$$= \langle g(t), S^m_\rho(\phi_1)(t)\rangle.$$

But

$$\lim_{k\to\infty} \frac{1}{k^{1-m\rho}\upsilon(k)} \langle S^m_\rho(f)(kx), \phi_1(x)\rangle = \lim_{k\to\infty} \frac{1}{\upsilon(k)} \langle f(kt), S^m_\rho(\phi_1)(t)\rangle.$$

It follows that $\lim_{k\to\infty} \frac{f(kt)}{v(k)} = g(t)$ on a dense set of elements of the space $M^m_{a,b}$. Now we prove that $\{\frac{f(kt)}{v(k)} : k \geq k_0\}$ is bounded in $(M^m_{a,b})'$. By the theorem of uniform convergence it follows that f has strong asymptotic behaviour at ∞ with respect to $v(k)$.

Let $\phi \in M^m_{a,b}$. Since $\frac{1}{k^{1-m\rho}v(k)} L_{n,x} S^m_\rho(f)(kx)$, $k \geq 0$ is uniformly continuous in $(M^m_{a,b})'$ there exists a constant $C_1 > 0$ and integers j, q such that

$$\Big|\frac{1}{k^{1-m\rho}v(k)}\langle L_n S^m_\rho(f)(kx), \phi(x)\rangle\Big| \leq C_1 \bar{\mu}^m_{a,b,K_j,q}(\phi)$$

for $k \geq k_0$, for all $n \in N$. Since

$$\Big|\frac{1}{k^{1-m\rho}v(k)}\langle L_n S^m_\rho(f)(kx), \phi(x)\rangle\Big| = \Big|\frac{1}{v(k)}\langle f(kt), S^m_\rho(L_n\phi)(t)\rangle\Big|,$$

it follows that

$$\Big|\frac{1}{v(k)}\langle f(kt), \phi(t)\rangle\Big| \leq C_2 \bar{\mu}^m_{a,b,K_j,q+1}(\phi)$$

Hence the theorem. □

REMARK. As the dual cone C' in $(M^m_{a,b})'$ is generating it is enough if we state and prove equivalence of (i) and (ii) for elements of the dual cone and the results for elements of the whole space $(M^m_{a,b})'$ follow automatically.

COROLLARY 4.3.1. *If $(f_\alpha)_{\alpha\in J}$ is a monotone net in $C' \cap (M^m_{a,b})'$ having strong asymptotic behaviour at ∞ with respect to a regularly varying function $v(k)$ and if $(f_\alpha)_{\alpha\in J}$ converges to f in $C' \cap$*

$(M^m_{a,b})'$ with respect to the topology of bounded convergence then $(S^m_\rho(f_\alpha))_{\alpha\in J}$ converges to $S^m_\rho(f)$ in $C' \cap (M^m_{\alpha,\beta})'$ where $S^m_\rho(f)$ has strong asymptotic behaviour at ∞ with respect to $k^{1-m\rho}v(k)$. Also $\frac{1}{k^{1-m\rho}v(k)} L_{n,x} S^m_\rho(f)(kx)$, for $k \geq k_0$ is uniformly continuous with respect to the topology of bounded convergence in $C' \cap (M^m_{a,b})'$ for all values of n.

PROOF. Follows from Theorem 4.3.1 since $(M^m_{a,b})'$, $(M^m_{\alpha,\beta})'$ are order complete. □

REMARK. Studies have been made earlier about the asymptotic behaviour at ∞ of generalized functions and their integral transforms. (see for example [**13, 44**]). The relative merits of our method which studies the strong asymptotic behaviour at ∞ of elements of $(M^m_{a,b})'$ and their Stieltjes transforms are the following:

(i) As the elements of $(M^m_{a,b})'$, $(M^m_{\alpha,\beta})'$ are ordered and since these spaces are ordercomplete the results proved regarding the strong asymptotic behaviour at ∞ could be extended to monotone nets in these spaces.

(ii) As the cones in $(M^m_{a,b})'$ and $(M^m_{\alpha,\beta})'$ are generating it was enough if we verify the results for the elements of the cones of these spaces.

From the proof of (i) $\Rightarrow$ (ii) of Theorem 4.3.1 we obtain the following result.

Let $f \in (M^m_{a,b})'$ and $v(k)$ be a regularly varying function of order $r > (-ma)$. If $S^m_\rho(f)$ has strong asymptotic behaviour at ∞ with respect to $k^{1-m\rho}v(k)$ in $C' \cap (M^m_{\alpha,\beta})'$ then

$$\frac{1}{k^{1-m\rho}v(k)} L_{n,x} S^m_\rho(f)(kx) \text{ for } k \geq k_0$$

is uniformly continuous with respect to the topology of bounded convergence in $C' \cap (M^m_{a,b})'$ for all values of n. Since $(M^m_{\alpha,\beta})'$, $(M^m_{a,b})'$ are ordered vector spaces the above results may be extended to monotone nets in $(M^m_{a,b})'$.

THEOREM 4.3.2. *Let $(f_\alpha)_{\alpha \in J}$ be a monotone net in $C' \cap (M^m_{a,b})'$ such that $(S^m_\rho(f_\alpha))_{\alpha \in J}$ has strong asymptotic behaviour at ∞ with respect to $k^{1-m\rho}v(k)$ in $C' \cap (M^m_{\alpha,\beta})'$ where $v(k)$ is a regularly varying function of order $r > (-ma)$. Then $(S^m_\rho(f_\alpha))_{\alpha \in J}$ converges to a function f in $C' \cap (M^m_{\alpha,\beta})'$ and*

$$\frac{1}{k^{1-m\rho}v(k)} L_{n,x} S^m_\rho(f)(kx), \text{ for } k \geq k_0$$

is uniformly continuous with respect to the topology of bounded convergence in $C' \cap (M^m_{a,b})'$ for all values of n.

PROOF. Follows from the proof of (i) $\Rightarrow$ (ii) of Theorem 4.3.1, extending the result to monotone nets since $(M^m_{\alpha,\beta})'$ is order complete and the cone in $(M^m_{\alpha,\beta})'$ is generating. $\square$

CHAPTER 5

The Laplace-Stieltjes transformation

Geetha K. V. and John J. K. [**11**] in a paper titled 'The Laplace-Stieltjes Transform' has combined the Laplace transformation and the Stieltjes transform of the form

$$\hat{f}(x) = \int_0^\infty \frac{f(t)}{(x^m + t^m)^\rho} dt, \ m, \rho > 0$$

and applied it the elements of the dual space $(M^m_{a,b,c})'$ of a testing function space $M^m_{a,b,c}$. The application of the combination transform was done in the conventional way based on the methods adopted by Zemanian [**50**]. In this chapter, we apply the same combination transform to $(M^m_{a,b,c})'$, the dual of the testing function space $M^m_{a,b,c}$. The difference from the earlier method and the present one are the following

(1) We treat the testing function space $M^m_{a,b,c}$ as a strict countable union space.

(2) $M^m_{a,b,c}$ and its dual $(M^m_{a,b,c})'$ are ordered topological vector spaces.

(3) The pointwise convergence topology on $(M^m_{a,b,c})'$ is replaced by the topology of bounded convergence.

We observe that without losing any of the original properties of the combination transform some additional features like order properties of the transform can be studied. The features of operational calculus and solution of initial value problems are retained. Comparison of initial value problems is possible in the present situation.

5.1. The testing function space $M^m_{a,b,c}$ and its dual $(M^m_{a,b,c})'$ as ordered vector spaces

Let (K_m) be a sequence of compact subsets of $R_+ \times R_+$ such that $K_1 \subseteq K_2 \subseteq \ldots$ and such that each compact subset of $R_+ \times R_+$ is contained in one K_j, $j = 1, 2, \ldots$. Let M^m_{a,b,c,K_j} denote the linear space of all smooth complex-valued functions defined on $R_+ \times R_+$ whose support is contained in K_j on which is defined

$$\mu^m_{a,b,c,K_j,}(\phi)$$
$$= \sup_{(u,t)\in K_j} |e^u u^{q+1} D^q_u t^{m(1-a+l)}(1+t^m)^{a-b}(t^{1-m}D^l_t)\phi(u,t)|$$

where $a, b, c \in \mathbb{R}$, $q, l = 0, 1, 2, \ldots$, $m \in (0, \infty)$, $D_u \equiv \frac{\partial}{\partial u}$, $D_t \equiv \frac{\partial}{\partial t}$. $\{\mu^m_{a,b,c.K_j,q,l}\}^{\infty}_{q,l=0}$ is a multinorm on M^m_{a,b,c,K_j} and generates topology τ^m_{a,b,c,K_j} on M^m_{a,b,c,K_j}. M^m_{a,b,c,K_j} is complete with respect to τ^m_{a,b,c,K_j}. $M^m_{a,b,c} = \cup^{\infty}_{j=1} M^m_{a,b,c,K_j}$ is a (strict) countable union space. Since each M^m_{a,b,c,K_j} is complete with respect with respect to τ^m_{a,b,c,K_j} it follows that $M^m_{a,b,c}$ is complete. On each M^m_{a,b,c,K_j} an equivalent multinorm is given by

$$\overline{\mu}^m_{a,b,c,K_j,q,l}(\phi) = \sup_{\substack{0 \le q' \le q \\ 0 \le l' \le l}} \mu^m_{a,b,c,K_j,q',l'}(\phi).$$

We define an order relation on $M^m_{a,b,c}$ by identifying a positive cone in it.

DEFINITION 5.1.1. The positive cone of $M^m_{a,b,c}$ when $M^m_{a,b,c}$ is restricted to real valued functions is the set of all non-negative functions in $M^m_{a,b,c}$. When the field of scalars is $\mathbb{C}$, the complex numbers, the positive cone in $M^m_{a,b,c}$ is $C + iC$ which is also denoted as C.

Note. We say that $\phi \leq \psi$ in $M^m_{a,b,c}$ when $\psi - \phi \in C$ in $M^m_{a,b,c}$.

As in the previous cases it can be proved that the positive cone in $M^m_{a,b,c}$ is not normal but is a strict b-cone.

Order and topology on the dual of $M^m_{a,b,c}$. An order relation is defined on the dual $(M^m_{a,b,c})'$, the linear space of all continuous linear functionals on $M^m_{a,b,c}$, by identifying the positive cone in $(M^m_{a,b,c})'$ to be the dual cone C' of the cone C in $M^m_{a,b,c}$. The class of all B^0 , the polars of B as B varies over all $\sigma(M^m_{a,b,c}, (M^m_{a,b,c})')$-bounded subsets of $M^m_{a,b,c}$ is a neighbourhood basis of 0 in $(M^m_{a,b,c})'$ for the locally convex topology $\beta((M^m_{a,b,c})', M^m_{a,b,c})$. When $(M^m_{a,b,c})'$ is ordered by the dual cone C' and is equipped with the topology of bounded convergence $\beta((M^m_{a,b,c})', M^m_{a,b,c})$ it follows that C' is a normal cone since C is a strict b-cone by Corollary 1.2.6, Chapter 2, [**29**].

As in the case of the previous examples we observe that when the topology on $(M^m_{a,b,c})'$ is changed to the topology of bounded convergence, $(M^m_{a,b,c})'$ is order complete and topologically complete, the order dual and the topological dual of $M^m_{a,b,c}$ coincide and the order topology and the topology of bounded convergence on $(M^m_{a,b,c})'$ coincide. Also the cones of $M^m_{a,b,c}$ and $(M^m_{a,b,c})'$ are generating.

5.2. The Laplace-Stieltjes transformation

For $\phi(u,t) \in M^m_{\alpha,\beta,\gamma}$ the Laplace-Stieltjes transformation is defined as

$$\mathrm{SL}^m_\rho \phi(u,t) = \hat{\phi}(y,x) = \int_0^\infty \int_0^\infty e^{-yu}(x^m + t^m)^{-\rho}\phi(u,t)dudt$$

for a fixed $m > 0$, $\rho \geq 1$. With suitable integrability conditions the multiple integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-yu}(x^m + t^m)^{-\rho} f(x,y)\phi(u,t)dudtdxdy$$

for $f \in (M^m_{\alpha,\beta,\gamma})'$, $\phi \in M^m_{\alpha,\beta,\gamma}$ can be evaluated in two different ways so that

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle.$$

Geetha K. V. and John J. K. [**11**] has proved that for $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, the Laplace-Stieltjes transform maps $M^m_{\alpha,\beta,\gamma}$ continuously into $M^m_{a,b,c}$ if

$a \leq 1$, $a \leq \frac{1}{m} + \alpha - \rho$ and $a < 1$ if $\alpha = \rho + 1 - \frac{1}{m}$

$b \geq 1 - \rho$, $b \geq \frac{1}{m} + \beta - \rho$ and $b > 1 - \rho$ if $\beta = 1 - 1/m$.

Now, let $f \in (M^m_{a,b,c})'$. For each $\phi \in M^m_{\alpha,\beta,\gamma}$ we have $\mathrm{SL}^m_\rho(\phi) \in M^m_{a,b,c}$. Then the adjoint mapping

$$\langle \mathrm{SL}_\rho^m(f), \phi \rangle = \langle f, \mathrm{SL}_\rho^m(\phi) \rangle$$

defines the Laplace-Stieltjes transform
$\mathrm{SL}_\rho^m(f) \in (M^m_{\alpha,\beta,\gamma})'$ of $f \in (M^m_{a,b,c})'$.

THEOREM 5.2.1. *The Laplace-Stieltjes transform is strictly positive and orderbounded.*

PROOF. $(M^m_{a,b,c})'$, $(M^m_{\alpha,\beta,\gamma})'$, $M^m_{a,b,c}$, $M^m_{\alpha,\beta,\gamma}$ are ordered vector spaces. Let $f > 0$, $f \in (M^m_{a,b,c})'$. For $\phi \in M^m_{\alpha,\beta,\gamma}$, $\phi > 0$, $\mathrm{SL}_\rho^m(\phi) > 0$, $\mathrm{SL}_\rho^m(f) \in M^m_{a,b,c}$. Thus $\phi \to \mathrm{SL}_\rho^m(\phi)$ is a strictly positive map. Being the adjoint of this map, $f \to \mathrm{SL}_\rho^m(f)$ is a strictly positive map from $(M^m_{a,b,c})'$ to $(M^m_{\alpha,\beta,\gamma})'$. Since every strictly positive map is order bounded, the theorem follows. □

5.3. Inversion

For a non-negative integer n for $\rho+n > \frac{1}{m}$ a differential operator can be defined by

$$L_{n,y,x}\phi(y,x)$$
$$= My^{2n+1}(D_yD_x)(D_yx^{1-m}D_x)^{n-1}x^{2mn+m\rho-m}(D_yx^{1-m}D_x)^n\phi(y,x)$$

where $M = \dfrac{m^{1-2n}\Gamma(\rho)}{\Gamma(n+\frac{1}{m})\Gamma(\rho+n-1/m)\Gamma(2n+1)}$.

The Laplace-Stieltjes transform can be inverted by the application of this differential operator. The formal adjoint of this operator is itself.

Geetha K. V. and John J. K. [**11**] have proved the following results which are true in the present situation also.

RESULT 5.3.1. *If* $\rho + n > \frac{1}{m}$,

$$\int_0^\infty \int_0^\infty L_{n,y,x} e^{-yx} (x^m + t^m)^{-\rho} dudt = 1.$$

RESULT 5.3.2. $L_{n,u,t}$ *maps* $M^m_{a,b,c}$ *continuously into* $M^m_{\alpha,\beta,\gamma}$ *provided* $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$.

RESULT 5.3.3. *If* L_n *is the differential operator and* SL^m_ρ *is the Laplace-Stieltjes transform operator then either* $y^{-2n} x^{1-m\rho} L_n$ *and* $SL^m_\rho x^{m\rho-1} y^{-2n}$ *or* $L_n x^{1-m\rho} y^{-2n}$ *and* $y^{-2n} x^{m\rho-1} SL^m_\rho$ *commute on* $M^m_{a,b,c}$ *where* $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$

$$\begin{aligned} \textit{i.e.}, y^{-2n} x^{m\rho-1} SL^m_\rho (L_{n,u,t}(\phi)) &= y^{-2n} x^{m\rho-1} (L_{n,u,t}(\phi))^\wedge \\ &= L_{n,y,x} \int_0^\infty \int_0^\infty e^{-yu} (x^m + t^m)^{-\rho} u^{-2n} t^{m\rho-1} \phi(u,t) dudt \\ &= L_{n,y,x} SL^m_\rho (u^{-2n} t^{m\rho-1} \phi) \text{ for } \phi \in M^m_{a,b,c}. \end{aligned}$$

RESULT 5.3.4. *Let* $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, *then the sequence* $\{L_{n,y,x} \hat{\phi}(y,x)\}$ *converges in* $M^m_{\alpha,\beta,\gamma}$ *to* $\phi(y,x)$.

RESULT 5.3.5. *Let* $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$ *then* $(L_n(\hat{\phi}))$ *converges to* ϕ *in* $M^m_{a,b,c}$ *as* $n \to \infty$.

The following result proved in [**11**] has been suitably modified to suit the present situation.

RESULT 5.3.6. *Let $f \in (M^m_{a,b,c})'$. Then $f \in C'$ if and only if for every non-negative integer n, $L_{n,y,x}SL^m_\rho(f) \in C'$ where C' is the positive cone in $(M^m_{a,b,c})'$. It follows that $L_{n,y,x}$ is strictly positive and hence is orderbounded.*

We summarize the above results as follows:

For $f \in (M^m_{\alpha,\beta,\gamma})'$, $\phi \in M^m_{\alpha,\beta,\gamma}$

$$\langle \mathrm{SL}^m_\rho L_{n,y,x}(f), \phi\rangle = \langle f, L_{n,y,x}\mathrm{SL}^m_\rho(\phi)\rangle \to \langle f, \phi\rangle \text{ as } n \to \infty.$$

For $f \in (M^m_{a,b,c})'$, $\phi \in M^m_{a,b,c}$

$$\langle L_{n,y,x}\mathrm{SL}^m_\rho(f), \phi\rangle = \langle f, \mathrm{SL}^m_\rho L_{n,y,x}(\phi)\rangle \to \langle f, \phi\rangle.$$

5.4. Operational calculus

$$\mathrm{SL}^m_\rho[D_u D_t(\phi)] = (m\rho)y\mathrm{SL}^m_{\rho+1}[t^{m-1}\phi(u,t)]$$

provided

$$\lim_{t\to\infty} D_u(\phi(u,t)) = 0 = \lim_{t\to 0} D_u(\phi(u,t))$$
$$\lim_{u\to\infty} \phi(u,t) = 0 = \lim_{u\to 0} \phi(u,t).$$

Consider the differential equation

$$(D_u D_t)\phi(u,t) = f(u,t),\ u > 0,\ t > 0$$

where $f(u,t)$ is a generalized function upon which the

Laplace-Stieltjes transform can be applied.

Let $F_1(u,t) = \int f(u,t)dt$ be such that

$$\lim_{t\to\infty} F_1(u,t) = 0 = \lim_{t\to 0} F_1(u,t) \text{ and}$$

$$F_2(u,t) = \int F_1(u,t)du \text{ be such that}$$

$$\lim_{u\to\infty} F_2(u,t) = 0 = \lim_{u\to 0} F_2(u,t)$$

Applying the operational calculus

$$(m\rho)y\mathrm{SL}^m_{\rho+1}[t^{m-1}\phi(u,t)] = (m\rho)y\mathrm{SL}^m_{\rho+1}[t^{m-1}F_2(u,t)].$$

Inverting using the differential operator $L_{n,y,x}$ for $\rho + 1 + n > \frac{1}{m}$,

$$t^{m-1}\phi(u,t) = t^{m-1}F_2(u,t)$$

so that $\phi(u,t) = F_2(u,t)$ where $F_2(u,t) = \int\int f(u,t)dtdu$. Comparison between different solutions arising out of different initial value conditions is possible since they belong to an ordered vector space.

CHAPTER 6

Abelian and Tauberian theorems

In this chapter we study distributions which are bounded on the sides of a wedge W in $\mathbb{R}^n$, tempered distributions having their support in a wedge W in $\mathbb{R}^n$ and holomorphic generalized functions defined on the tube region T^V. The notion of distributions having asymptotic, strongasymptotic of order α is defined and the compatibility of these notions with the lattice properties in $\mathcal{D}'(W)$, $\zeta'(W)$ respectively is proved. Those functions which are holomorphic in T^V form a convolution algebra $H(W)$ which is isomorphic to $\zeta'(W)$ via the Laplace transformation. We define an order relation on $H(W)$ by identifying a cone in $H(W)$ with respect to which the above cone is normal. The notion of elements in $H(W)$ having strongasymptotic is defined and is observed to be compatible with lattice properties in $H(W)$. The Tauberian and Abelian theorems in this new background for the Laplace transform are proved. Two corollaries extending the results of the theorem to monotone nets are also stated. A special case of the Tauberian theorem applied to the one-dimensional case is also stated.

The basic definitions have been taken from Vladimirov [**44**].

6.1. Basic definitions

DEFINITION 6.1.1. [**44**] A wedge in $\mathbb{R}^n$ with vertex at 0 is a subset W of $\mathbb{R}^n$ containing 0 and satisfying the conditions $W+W \subseteq W$, $\alpha W \subseteq W$ for $\alpha > 0$, $\alpha \in \mathbb{R}$.

DEFINITION 6.1.2. [**44**] The intersection of W with the unit sphere with center at 0 is denoted as pr W and a wedge W' is said to be compact in the wedge W if pr $W' \subseteq$ pr W.

DEFINITION 6.1.3. [**44**] The wedge $W^* = \{\xi : \langle \xi, x \rangle \geq 0, x \in W\}$ is said to be conjugate to the wedge W.

Note. W^* is a closed convex wedge with vertex at 0.

DEFINITION 6.1.4. [**44**] A set $A \subseteq \mathbb{R}^n$ is said to be bounded on the side of the wedge W if $A \subseteq W + K$ where K is a compact set in $\mathbb{R}^n$.

DEFINITION 6.1.5. [**44**] The collection of all distributions in $\mathcal{D}'$ whose supports are bounded on the side of the wedge W is denoted as $\mathcal{D}'(W_+)$.

DEFINITION 6.1.6. Let $(f_\alpha)_{\alpha \in J}$ be a net of functions in $\mathcal{D}'(W_+)$, with the topology of bounded convergence defined on $\mathcal{D}'$. We say that $f_\alpha \to 0$ in $\mathcal{D}'(W)$ if $f_\alpha \to 0$ in $\mathcal{D}'$ with respect to the topology of bounded convergence and supp$f_\alpha \subseteq W + K$ for each $\alpha \in J$ where the compact set K does not depend on $\alpha \in J$.

DEFINITION 6.1.7. [**44**] A wedge W is said to be acute if there exists a plane of support for $\overline{\text{ch}W}$ that has unique point in common with the $\overline{\text{ch}W}$ where ch$W$ is the convex hull of W.

DEFINITION 6.1.8. [**44**] Let W be a closed convex acute cone in $\mathbb{R}^n$ with vertex at 0 and let $V = \text{int}\, W^*$. Then T^V denote the tubular domain in $\mathbb{C}^n$ with base V *i.e.*, $T^V = \mathbb{R}^n + iV = \{z = x + iy : x \in \mathbb{R}^n, y \in V\}$

DEFINITION 6.1.9. [**44**] Let W be a connected open wedge in $\mathbb{R}^n$ with vertex at 0 and let W^* be the conjugate wedge. Then

$$\mathcal{K}_W(z) = \int_{W^*} e^{i\langle z,\xi\rangle} d\xi \equiv \mathcal{L}(\chi_{W^*}) = F(\chi_{W^*} e^{-\langle y,\xi\rangle})$$

where $\mathcal{L}$ represents the Laplace transformation and F represents the Fourier transformation is called the Cauchy kernel of the tubular region T^V. Here χ_{W^*} is the characteristic function of W^*.

DEFINITION 6.1.10. [**44**] Let W be an acute convex open wedge such that the Cauchy kernel $\kappa_W(z) \neq 0$ in the tube $T^V = \mathbb{R}^n + iV$. Such wedges are said to be regular.

In what follows W represents a closed convex acute solid wedge and $V = \text{int}\, W^*$ is a regular wedge with vertex at 0. $\kappa_W(z)$ is the Cauchy kernel of the tube domain $T^V = \mathbb{R}^n + iV$ and $\chi_W^\alpha = \mathcal{L}^{-1}(\kappa_W^\alpha)$, $-\infty < \alpha < \infty$.

Some properties of χ_W^α.

(1) $\chi_W^\alpha * \chi_W^\beta = \chi_W^{\alpha+\beta}$, $-\infty < \alpha, \beta < \infty$.

(2) $\chi_W^\alpha(t\xi) = t^{n(\alpha-1)}\chi_W^\alpha(\xi)$, $t > 0$

(3) For any $m = 0, 1, \ldots$ there exists N such that

$$\chi_W^\alpha \in \mathcal{C}^m(\mathbb{R}^n),\ \alpha > N$$

where $\mathcal{C}^m(\mathbb{R}^n)$ denote the set of all functions that are continuous in $\mathbb{R}^n$ together with all derivatives $\partial^\alpha f(x)$, $|\alpha| \le m$.

(4) $|\partial^m \chi_W^\alpha(\xi)| \le M|\xi|^{n(\alpha-1)-m}$, $\xi \in \mathbb{R}^n$, $\alpha > N$.

DEFINITION 6.1.11. [**44**] Let $f \in \zeta'(W)$. The convolution $\chi_W^\alpha * f$ is called the primitive of f of order α with respect to the wedge W and is denoted as $f^{(-\alpha)}(\xi)$, i.e.,

$$f^{(-\alpha)}(\xi) = \chi_W^\alpha * f \tag{10}$$

Note. From Vladimirov [**44**] it follows that for $f \in \zeta'(W)$ the primitive $f^{(-\alpha)}$ for all sufficiently large $\alpha > N$ is continuous in $\mathbb{R}^n$ and

$$f^{(-\alpha)}(\xi) = \langle f(\xi'), \eta(\xi')\chi_W^\alpha(\xi - \xi')\rangle \tag{11}$$

and

$$|f^{(-\alpha)}(\xi)| \le M\|f\|_{-m}|\xi|^r, \tag{12}$$

$M > 0$, $r \ge 0$ where m is the order of f.

6.2. Asymptotic, strongasymptotic

DEFINITION 6.2.1. [**44**] A generalized function $f(\xi)$ is said to have an asymptotic $g(\xi)$ of order α in the wedge W as $|\xi| \to \alpha$ if for any $\xi \in \text{int}\, W$

$$\lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|^\alpha} = g(\frac{\xi}{|\xi|})$$

and there exists constants M and r such that

$$\frac{|f(\xi)|}{|\xi|^\alpha} \leq M,\ |\xi| > r,\ \xi \in \text{int}\, W.$$

EXAMPLE 6.2.1. Let $n = 1, W = [0, \infty]$, $\text{int}\, W = (0, \infty)$. Consider the generalized function $P\frac{1}{x}$ which coincides with the regular generalized function $\frac{1}{x}$ for $x \neq 0$. $P\frac{1}{x} \in \mathcal{D}'(W)$. For any $\xi \in \text{int}\, W$

$$\lim_{|\xi|\to\infty} |\xi| P\frac{1}{\xi} = \pm 1 = g(\frac{\xi}{|\xi|})$$

and there exists constants M and r such that

$$|\xi||P\frac{1}{\xi}| \leq M,\ |\xi| > r,\ \xi \in (0, \infty)$$

so that $P\frac{1}{x}$ has an asymptotic of order (-1).

DEFINITION 6.2.2. A generalized function $f \in \zeta'(W)$ is said to have a strongasymptotic g of order α at ∞ if

$$\lim_{k\to\infty} \frac{f(k\xi)}{k^\alpha} = g(\xi) \text{ in } \zeta'$$

with respect to the topology of bounded convergence in ζ' or equivalently, DEFINITION 6.2.3. A generalized function f in ζ' has a strongasymptotic g of order α at 0 if

$$\lim_{\rho \to 0^+} \rho^\alpha f(\rho x) = g(x)$$

with respect to the topology of bounded convergence in ζ'.

Note. Definition 6.2.2 implies that if $g(\xi) \in B^0$, a basis element for the topology of bounded convergence in ζ'

$$\frac{f(k\xi)}{k^\alpha} \in B^0 \text{ for } k > k_0, k_0 \in N$$

i.e., if $|\langle g, \psi \rangle| < 1, \forall \psi \in B$,
where $B = \{\psi \in \zeta : |f(\psi)| < t\epsilon$ for some $f \in \zeta'$, $\epsilon > 0$, $t > s$, $t, s \in \mathbb{R}\}$ then

$$|\langle \frac{f(k\xi)}{k^\alpha}, \psi \rangle| < 1,\ \forall \psi \in B,\ k \geq k_0, k_0 \in N.$$

Note. From Definition 6.2.2 it follows that the strongasymptotic g of order α at ∞ if it exists is a homogeneous generalized function of degree α which belongs to $\zeta'(W)$ of homogeneity degree $\alpha + N$.

THEOREM 6.2.1. *For $f \in \zeta'(W)$ to have the strongasymptotic g of order α at ∞ it is necessary and sufficient that the Fourier transform $F(f)$ has the strongasymptotic $F(g)$ of order $\alpha + n$ at 0.*

PROOF. We know that

$$F(f(Ax)(\xi)) = \frac{1}{|\det A|} F(f)((A^{-1})^t(\xi)),\ \det A \neq 0$$

$$F(k^\alpha f(k\xi)) = \rho^{\alpha+n} F(f)(\rho x), \quad \rho = \frac{1}{k} > 0.$$

From the definitions 6.2.2, 6.2.3 and from the fact that the Fourier transform is a continuous operation on ζ' with respect to the topology of bounded convergence the required result follows. □

COROLLARY 6.2.1. *If $f \in \zeta'(W)$ has the strongasymptotic g of order α at ∞ then $f^{(-N)}$, $-\infty < N < \infty$ has the strongasymptotic $g^{(-N)}$ of order $\alpha + N$ at ∞.*

LEMMA 6.2.1. *If a function $f(\xi)$ in $\zeta'(W)$ has the asymptotic $g(\xi)$ of order $\alpha > -n$ in the wedge W as $|\xi| \to \infty$ then f has the strongasymptotic g of the same order α at ∞.*

PROOF. From Definition 6.2.1 it follows that

$$k^{-\infty} f(k\xi) \to |\xi|^\alpha g(\frac{\xi}{|\xi|}) = g(\xi),\ k \to \infty$$

almost everywhere in $\mathbb{R}^n$ (assume that g is continued by zero onto the whole of $\mathbb{R}^n$) and

$$|k^{-\alpha} f(k\xi)| \leq M|\xi|^\alpha, \quad |\xi| > \frac{R}{k},\ \xi \in \mathbb{R}^n.$$

Let $\phi \in \zeta$. Then

$$\begin{aligned}\langle k^{-\alpha} f(k\xi), \phi\rangle &= k^{-\alpha} \int f(k\xi)\phi(\xi)d\xi \\ &= \int_{|\xi|>\frac{R}{k}} k^{-\alpha} f(k\xi)\phi(\xi)d\xi + k^{-\alpha} \int_{|\xi|<\frac{R}{k}} f(k\xi)\phi(\xi)d\xi \\ &\to \int g(\xi)\phi(\xi)d\xi = \langle g, \phi\rangle, \quad k \to \infty\end{aligned}$$

since one can pass to the limit under the integral sign in the first term and the second which is equal to

$$k^{-n-\alpha} \int_{|\xi|<1} f(\xi)\phi(\frac{\xi}{k})d\xi \to 0 \text{ as } k \to \infty \text{ if } n + \alpha > 0.$$

It follows that f has the strongasymptotic g of order α at ∞. □

THEOREM 6.2.2. *For $f \in \zeta'(W)$ to have the strongasymptotic g of order α at ∞ it is necessary and sufficient that there exists $N > -1 - \frac{\alpha}{n}$ such that the function $f^{(-N)}(\xi)$ has the asymptotic $g^{(-N)}(\xi)$ of order $\alpha + nN$ in the wedge W and*

$$|\xi|^{-\alpha-nN} f^{(-N)}(\xi) \to g^{(-N)}(\frac{\xi}{|\xi|}), \quad \frac{\xi}{|\xi|} \in pr\, W,\ |\xi| \to \infty$$

and the function $g^{(-N)}(\xi)$ is continuous in $\mathbb{R}^n$ with support in W, $g^{(-N)} \in C_0(W)$.

PROOF. That the conditions are sufficient follows from corollary 6.2.1 and theorem 6.2.1.

Now we prove the necessity. By assumption $\frac{f(k\xi)}{k^\alpha}$ converges in ζ' as $k \to \infty$ with respect to the topology of bounded convergence. The set of functions

$$\{\xi' \to \eta(\xi')\chi_W^N(e-\xi') : |e| = 1\}$$

is bounded in ζ for sufficiently large $N > -1 - \frac{\alpha}{n}$. Also,

$$\eta(k\xi)f(k\xi) \to \eta(\xi)f(k\xi) \text{ as } k \to \infty.$$

Using the above results,

$$\begin{aligned}
k^{-\alpha-nN}f^{(-N)}(ke) &= \langle k^{-\alpha-nN}f(\xi'), \eta(\xi')\chi_W^N(ke-\xi')\rangle \\
&= \langle k^{-\alpha-n}f(\xi'), \eta(\xi')\chi_W^N(e-\frac{\xi'}{k})\rangle \\
&= \langle k^{-\alpha}f(\xi'), \eta(\xi)\chi_W^N(e-\xi)\rangle \\
&\to \langle g(\xi), \eta(\xi)\chi_W^N(e-\xi)\rangle,\ |e| = 1 \\
&= g^{(-N)}(e), \qquad (13)
\end{aligned}$$

By (12) $|f^{(-\alpha)}(\xi)| \leq M\|f\|_{-m}|\xi|^r$, so that

$$|\xi|^{-\alpha-nN}|f^{(-N)}(\xi)| \leq M \sup_{k\geq 1} \|k^{-\alpha}f(k\xi)\|_{-m} \leq M',\ |\xi| \geq 1$$

Thus by Definition 6.2.1, $f^{(-N)}(\xi)$ has the asymptotic $g^{(-N)}(\xi)$ of order $\alpha + nN$ in the wedge W. Supp $g^{(-N)} \in W$, hence $g^{(-N)} \in C_0(W)$. □

THEOREM 6.2.3. *The property of comparable generalized functions having asymptotic is compatible with lattice operations i.e., if f_1, f_2 are comparable elements in ζ' with asymptotics g_1, g_2 respectively of order α in W then $f_1 \vee f_2$, $f_1 \wedge f_2$ have the asymptotics $g_1 \vee g_2$, $g_1 \wedge g_2$ respectively of order α.*

PROOF. Let $\frac{f_1(\xi)}{|\xi|^\alpha} \to g_1(\frac{\xi}{|\xi|})$ as $|\xi| \to \infty$.

$$\frac{f_2(\xi)}{|\xi|^\alpha} \to g_2(\frac{\xi}{|\xi|}) \text{ as } |\xi| \to \infty, \text{ for } \xi \in \text{int}\, W.$$

with respect to the topology of bounded convergence and let there exist constants M_1, M_2, r_1, r_2 such that

$$\frac{|f_1(\xi)|}{|\xi|^\alpha} \leq M_1, \quad \frac{|f_2(\xi)|}{|\xi|^\alpha} \leq M_2$$

$$|\xi| > r_1,\ |\xi| > r_2,\ \xi_1, \xi_2 \in \text{int}\, W.$$

Then

$$\frac{(f_1 \vee f_2)(\xi)}{|\xi|^\alpha} \to g(\frac{\xi}{|\xi|}) \text{ as } |\xi| \to \infty$$

where $g(\frac{\xi}{|\xi|})$ is either $g_1(\frac{\xi}{|\xi|})$ or $g_2(\frac{\xi}{|\xi|})$. Also,

$$\frac{|(f_1 \vee f_2)(\xi)|}{|\xi|^\alpha} \leq M \text{ for } |\xi| > r, \xi \in \text{int}\, W.$$

It follows that $f_1 \vee f_2$ has an asymptotic in ζ' of order α in the wedge W. Similarly it can be proved that $f_1 \wedge f_2$ has an asymptotic in ζ' of order α in the wedge W. □

COROLLARY 6.2.2. *The property of comparable generalized functions in $\zeta'(W)$ having strongasymptotic are compatible with the lattice operations in ζ'.*

PROOF. Follows from Theorem 6.2.2, 6.2.3. □

LEMMA 6.2.2. *Let $f \in \mathcal{D}'(W)$ and $k^{-\alpha} f(k\xi) \to g(\xi)$ as $k \to \infty$ in $\mathcal{D}'$ with respect to the topology of bounded convergence. Then $f \in \zeta'(W)$ and f has the strongasymptotic $g \in \zeta'(W)$ of order α at ∞.*

PROOF. The support of all functions

$$\{\xi' \to \eta(\xi')\chi_W^N(e - \xi') : |e| = 1\}$$

are contained in a ball U. Let m be the order of $f \in \mathcal{D}'(W)$ in U and $N > -1 - \frac{\alpha}{n}$ be such that the above functions belong to $\mathcal{C}^m(\bar{U})$ and are bounded and continuous with respect to e in this space. It follows from (13) that the sequence of continuous functions

$$k^{-\alpha - nN} f^{(-N)}(ke),\ k \to \infty$$

converges uniformly with respect to e, $|e| = 1$ to the continuous function $g^{(-N)}(e)$.

This fact together with the following inequality

$$|\xi|^{-\alpha - nN} |f^{(-N)}(\xi)| \le \sup_{k \ge 1, |e| = 1} \|k^{-\alpha} f(k\xi)\|_{\mathcal{C}^{m'}(\bar{U})} \|\eta(\xi)\chi_W^N(e - \xi)\|_{\mathcal{C}^m(\bar{U})}$$

for $|\xi| > 1$ shows that $f^{(-N)}$ has the asymptotic $g^{(-N)}$ of order $\alpha + nN$ in the wedge W and $f^{(-N)} \in \zeta'(W)$. Hence $f \in \zeta'(W)$. By Theorem 6.2.2, $f^{(-N)}$ has the strongasymptotic $g^{(-N)}$ of order $\alpha + nN$. Thus $f = f^{(-N)N}$ has the strongasymptotic g of order α at ∞. □

COROLLARY 6.2.3. *Let* $f \in \mathcal{D}'(W)$. *If the set of generalized functions*
$\{k^{-\alpha} f(k\xi) : k \geq 1\}$ *is bounded in* $\mathcal{D}'$ *then it is bounded in* $\zeta'(W)$ *also.*

6.3. The algebras $H_+(W)$ and $H(W)$ as ordered topological vector spaces

The following definitions of the spaces $H_a(W)$ and $H_+(W)$ have been taken from Vladimirov [**44**].

Let W be an open connected wedge with vertex at 0. Denote by $H^{(\alpha,\beta)}(W)$, $\alpha \geq \beta$, $\beta \geq 0$ the set of all functions that are holomorphic in T^V and that satisfy the following growth condition

$$|f(z)| \leq Me^{a|y|}(1 + |z|^2)^{\alpha/2}[1 + \Delta^{-\beta}(y)], \quad z \in T^V \tag{14}$$

A topology is introduced on $H_a^{(\alpha,\beta)}(W)$ via the norm

$$\|f\|_a^{(\alpha,\beta)} = \sup_{z \in T^V} \frac{|f(z)|e^{-a|y|}}{(1 + |z|^2)^{\alpha/2}[1 + \Delta^{-\beta}(y)]} \tag{15}$$

The spaces $H_a^{(\alpha,\beta)}(W)$ are Banach spaces and

$$H_a^{(\alpha,\beta)}(W) \subseteq H_{a'}^{(\alpha',\beta')}(W),\ \alpha' \geq \alpha,\ \beta' \geq \beta,\ a' \geq a \tag{16}$$

with the inclusion (16) to be understood together with the appropriate topology by virtue of the obvious inequality

$$\|f\|_{a'}^{(\alpha',\beta')} \leq 2\|f\|_a^{(\alpha,\beta)}.$$

Define $H_a(W) = \cup_{\alpha\geq 0,\beta\geq 0} H_a^{(\alpha,\beta)}$, $H_+(W) = \cup_{a\geq 0} H_a(W)$.

The set $H_+(W)$ form an algebra of functions that are holomorphic in T^V and satisfy the estimate (14) for certain $a \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ relative to the operation of ordinary multiplication. This algebra is associative, commutative, contains a unit element but does not contain divisors of zero. The spaces $H_a(W)$ and $H_+(W)$ are endowed with the inductive limit topology.

RESULT 6.3.1. *From Vladimirov* [**44**] *it follows that the algebra $H_+(W)$ and $\zeta'(W_+)$ and also the subalgebra $H(W)$ and $\zeta'(W)$ are isomorphic and that the isomorphism is accomplished via the Laplace transformation.*

We prove the next theorem that the algebra $H_+(W)$ is a complete ordered topological vector space.

THEOREM 6.3.1. *The algebra $H_+(W)$ is an ordered topological vector space, complete with respect to the topology τ_β.*

PROOF. We have seen in chapter 3 that ζ' is an ordered topological vector space, the order relation being determined by the dual cone C' and the topology being the the topology of bounded convergence with respect to which C' is a normal cone, $\zeta'(W_+^*)$ is a subspace of ζ' and is ordered by $C' \cap \zeta'(W_+^*)$. The topology on $\zeta'(W_+^*)$ is the subspace topology whose basis elements are $B^0 \cap \zeta'(W_+^*)$ where B^0 is the polar of B consisting of elements $\psi \in \zeta$ such that $|f(\psi)| < t\epsilon$ for some $f \in \zeta'$, $\epsilon > 0$, $t > s$, $t, s \in \mathbb{R}$. The cone $C' \cap \zeta'(W_+^*)$ is normal with respect to this subspace topology. The isomorphism from $\zeta'(W_+^*)$ to $H_+(W)$ is derived via the Laplace transformation, the isomorphism being denoted as $\mathcal{L}_*$. $\{\mathcal{L}_*(B^0 \cap \zeta'(W_+^*)) : B^0$ is a basis element for the topology of bounded convergence in $\zeta'\}$ generates a topology τ_β on $H_+(W)$. Since $\mathcal{L}_*$ is orderpreserving and continuous it follows that $\{\mathcal{L}_*(f) : f \in C' \cap \zeta(W_+^*)\}$ is a cone in $H_+(W)$ which is normal with respect to τ_β. It also follows that the order topology and the topology τ_β are identical and $H_+(W)$ is order complete and topologically complete. $\square$

COROLLARY 6.3.1. *The subalgebra $H(W)$ is ordercomplete and topologically complete with respect to the subspace topology derived from τ_β.*

6.4. Abelian and Tauberian theorems

DEFINITION 6.4.1. A function $f(z)$ holomorphic in T^V is said to have an asymptotic $h(z)$ of order α in T^V if

(i)

$$\lim_{\rho\to 0^+} \rho^\alpha f(\rho z) = h(z),\ z \in T^V \tag{17}$$

with respect to the topology of bounded convergence defined on $H(W)$ and

(ii) there exist numbers M, a and b such that

$$\rho^\alpha |f(\rho z)| \leq M \frac{1+|z|^a}{\Delta_W^b(y)},\ 0<\rho\leq 1,\ z\in T^V \tag{18}$$

THEOREM 6.4.1. *If f_1 and f_2 are holomorphic functions in T^V having asymptotics h_1, h_2 respectively of order α at 0 in T^V then $f_1 \vee f_2$ and $f_1 \wedge f_2$ also have the asymptotic $h_1 \vee h_2$ and $h_1 \wedge h_2$ respectively of order α at 0 in T^V.*

PROOF. Follows as in the proof of Theorem 6.2.3. □

Note. Let $f \in \zeta'(W)$. Then $\hat{f}(z) = \mathcal{L}(f)(z) = \langle f(\xi), \eta(\xi)e^{i\langle z,\xi\rangle}\rangle$ belongs to the $H(W)$-algebra of functions that are holomorphic in T^V and satisfy the growth condition

$$|\tilde{f}(z)| \leq M\frac{1+|z|^c}{\Delta_W^d(y)},\ z = x+iy \in T^V \text{ for some } M, c, d.$$

THEOREM 6.4.2. *In order that $f \in \zeta'(W)$ has the strongasymptotic g of order α at ∞ it is necessary that $\tilde{f}(z)$ has the asymptotic $h(z)$ of order $\alpha + n$ at 0 in T^V. i.e.,*

(1)

$$\lim_{\rho\to 0^+} \rho^{\alpha+n}\tilde{f}(\rho z) = h(z),\ z\in T^V \tag{19}$$

with respect to the topology of bounded convergence defined on $H(W)$.

(2)

$$\rho^{\alpha+n}|f(\rho z)| \le M\frac{1+|z|^a}{\Delta_W^b(y)}, 0<\rho\le 1,\ z\in T^V \tag{20}$$

and it is sufficient that the following conditions hold:

(a) there exists a solid subwedge $W'\subseteq W$ such that $\tilde{f}(iy)$ has an asymptotic $h(iy)$ of order $\alpha+n$ at 0 in the wedge W', i.e.,

$$\lim_{\rho\to 0} \rho^{\alpha+n}\tilde{f}(i\rho y) = h(iy),\ y\in W' \tag{21}$$

(b) there exist numbers M, q and $\beta\in[0,1)$ and a vector $e\in W$ such that

$$\rho^{\alpha+n}|\tilde{f}(\rho x + i\rho\lambda e)| \le M\lambda^{-q}, 0<\rho\le 1, 0<\lambda\le 1, |x|\in\lambda^{\beta} \tag{22}$$

In this case the equalities

$$h(z) = \mathcal{L}(g)(z) = \tilde{g}(z),\ z\in T^V \tag{23}$$

$$\mathcal{K}_W^N(z)h(z) = \Gamma(\alpha+n+nN)\int_{pr\,W}\frac{g^{(-N)}(\sigma)d\sigma}{\langle -iz,\sigma\rangle^{\alpha+n+nN}},\ z\in T^V \tag{24}$$

hold for all sufficiently large N.

PROOF. If $f(\xi)$ has the strongasymptotic of order α at ∞

$$\begin{aligned}\rho^{\alpha+n} f(\rho z) &= \rho^{\alpha+n}\langle f(\xi), \eta(\xi)e^{i(\xi,\rho z)}\rangle \\ &= \langle \rho^{\alpha} f(\frac{\xi'}{\rho}), \eta(\xi')e^{i(\xi',z)}\rangle \\ &\to \langle g(\xi'), \eta(\xi')e^{i\langle \xi',z\rangle}\rangle = \tilde{g}(z), \quad \rho \to 0,\ z \in T^V\end{aligned}$$

Also,

$$\rho^{\alpha+n}|\tilde{f}(\rho z)| \leq M\frac{1+|z|^a}{\Delta_W^b(y)},\ 0<\rho\leq 1,\ z\in T^V \tag{25}$$

Conversely assume that (a) and (b) hold. First we prove that

$$\{\rho^{\alpha} f(\xi/\rho) : 0<\rho\leq 1\} = \{k^{-\alpha} f(k\xi) : k>1\}$$

is bounded in ζ'. By Corollary 6.2.3 it is enough if we prove that the above set is bounded on $\mathcal{D}$. It can be proved that

$$|k^{-\alpha-n}\tilde{f}(\frac{x}{k}+i\frac{e}{k})| \leq K(1+|x|^s),\ k\geq 1,\ x\in\mathbb{R}^n \tag{26}$$

where K and s do not depend on k. Since

$$\rho^{\alpha+n}|\tilde{f}(\rho z)| \leq M\frac{1+|z|^a}{\Delta_W^b(y)},\ 0<\rho\leq 1,\ z\in T^V$$

and since

$$\{\rho^{\alpha} f(\frac{\xi}{\rho}) : 0<\rho\leq 1\} = \{k^{-\alpha} f(k\xi) : k\geq 1\} \tag{27}$$

it follows that the set of numbers

$$\{k^{-\alpha}\langle f(k\xi),\phi\rangle : k\geq 1\} \text{ is bounded for any } \phi\in\mathcal{D}.$$

Condition (a) implies that the sequence (27) converges as $k\to\infty$ on the functions $\{\eta(\xi)e^{-\langle y,\xi\rangle}, y\in W'\}$ to the function $h(iy)$, by virtue of

$$\rho^{\alpha+n}\tilde{f}(i\rho y)=\langle k^{-\alpha}f(k\xi),\eta(\xi)e^{-\langle y,\xi\rangle}\rangle,\ \rho\to 0,\ y\in W' \tag{28}$$

We now prove that the linear hull of functions $\{\eta(\xi)e^{-(y,\xi)}, y\in W'\}$ is dense in the set of functions $\{\psi=\eta\phi,\phi\in\zeta\}$. If $g_1\in\zeta'$ vanishes on these functions then $\tilde{g}_1(iy)=\langle g_1(\xi),\eta(\xi)e^{-\langle y,\xi\rangle}\rangle=0$, $y\in W'$.

By the Uniqueness Theorem for holomorphic functions we deduce that $\tilde{g}_1(z)=0$, $z\in T^V$, hence $g_1=0$. This fact and the Hahn-Banach theorem imply that the sequence of functionals

$$\{\eta(\xi)k^{-\alpha}f(k\xi)=k^{-\alpha}f(k\xi),1\leq k<\infty\}$$

is bounded and hence converges in ζ' to a function $g\in\zeta'(W)$. From (28) we conclude that

$$h(iy)=\langle g(\xi),\eta(\xi)e^{-\langle y,\xi\rangle}\rangle=\tilde{g}(iy).$$

Thus $f\in\zeta'(W)$ has the strongasymptotic g of order α at ∞ and $h(z)=\mathcal{L}(g)(z)=\tilde{g}(z)$, $z\in T^V$. $\square$

COROLLARY 6.4.1. *If* $(f_\alpha)_{\alpha\in J}$ *is a monotone net in* $C'\cap\zeta'(W)$ *having strongasymptotic* g_α *of order* α *at* ∞ *with respect to the topology of bounded convergence and if* $f_\alpha\to f$, $g_\alpha\to g$ *in* $\zeta'(W)$ *the net* $(\mathcal{L}(f_\alpha))_{\alpha\in J}$ *converges to* $\mathcal{L}(f)$ *where* $\mathcal{L}(f)$ *has the asymptotic* $\mathcal{L}(g)$ *of order* $\alpha+n$ *at 0 in* T^V.

PROOF. Follows from Theorem 6.4.2 since $\zeta'(W)$, $H(W)$ are ordercomplete. □

COROLLARY 6.4.2. *If there exists a solid subwedge* $W'\subseteq W$ *such that the monotone net* $(\mathcal{L}(f_\alpha(iy)))_{\alpha\in J}$ *in* $H_+(W)$ *has an asymptotic* $h(iy)$ *of order* $\alpha+n$ *at 0 in the wedge* W' *and there exist numbers* M, q *and* $\beta\in[0,1]$ *and a vector* $e\in W$ *such that*

$$\rho^{\alpha+n}|\mathcal{L}(f_\alpha)(\rho x+i\rho\lambda e)|\le M\lambda^{-q}$$

$$0<\rho\le 1,\quad 0<\lambda\le 1,\ |x|\le\lambda^\beta$$

and if $\mathcal{L}(f_\alpha(iy))$ *converges to* $(\alpha f)(iy)$ *in* $H_+(W)$ *then* $(f_\alpha)_{\alpha\in J}$ *in* $\zeta'(W)$ *converges to* f *in* $\zeta'(W)$ *with respect to the topology of bounded convergence where* f *has the strongasymptotic* g *of order* α *at* ∞ *where* $h=\mathcal{L}(g)$.

6.5. Abelian and Tauberian theorems for one dimension

The algebra $\mathcal{D}'(\bar{R}^1_+)$ is denoted as $\mathcal{D}'_+$. For $-\infty<\alpha<\infty$ define $f_\alpha\in\mathcal{D}'_+$ as

$$f_\alpha(x) = \begin{cases} \frac{\theta(x)x^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0 \\ f'_{\alpha+1}, & \alpha \leq 0 \end{cases}$$

where $\theta(x)$ is the Heaviside unit function defined by

$$\theta(x) = 0, \quad x < 0$$
$$= 1, \quad x \geq 0.$$

It can be proved that $f_\alpha * f_\beta = f_{\alpha+\beta}$.

Consider the convolution operator $f_\alpha *$ in the algebra $\mathcal{D}'_+$. Since $f_0 = \theta' = \delta$ it follows that the fundamental solution f_α^{-1} of the operator $f_\alpha *$ exists and is equal to $f_{-\alpha}$ i.e., $f_\alpha^{-1} = f_{-\alpha}$. For integers $n < 0$, $f_n = \delta^{(-n)}$ and so, $f_n * u = \delta^{(n)} * u = u^{(n)}$ which means that $f_n *$ is the operator of n-fold differentiation. For integers $n > 0$

$$(f_n * u)^{(n)} = f_{-n} * (f_n * u) = (f_{-n} * f_n) * u = \delta * u = u$$

so that $f_n * u$ is the antiderivative of order n of the generalized function u.

In what follows we take $n = 1$, $W = [0, \infty)$, $V = \text{ int } W^* = (0, \infty)$, $T^V = T^1$, $\chi_v(z) = \frac{1}{iz}$.

LEMMA 6.5.1. *If $g \in \zeta'_+$ is a homogeneous generalized function of degree α, $g(\xi) = M' f_{\alpha+1}(\xi)$ where M' is a constant.*

THEOREM 6.5.1. *For $f \in \zeta'_+$ to have a strongasymptotic of order α at ∞ the following conditions are necessary*

(1)

$$\lim_{\rho\to 0^+} \rho^{\alpha+1}\tilde{f}(\phi z) = M'(iz)^{-\alpha-1},\ z\in T^1$$

(2)

$$\rho^{\alpha+1}|\tilde{f}(\rho z)| \le M(\frac{1+|z|^\alpha}{y^b}),\quad 0<\rho\le 1,\ z\in T^1$$

and the following conditions are sufficient

(a)

$$\lim_{y\to 0^+} y^{1+\alpha}\tilde{f}(iy) = M'$$

there exist numbers M, q, r_0 and $\beta \in [0,1)$ such that

(b)

$$r^{\alpha+1}|\tilde{f}(re^{i\phi})| \le M\sin^{-q}(\phi), 0<r\le r_0, |x|\le y^\beta.$$

In this case for all sufficiently large q the function $f^{(-q)}(\xi)$ is continuous with respect to $\xi > 0$ and has the asymptotic

$$\lim_{\xi\to\infty} \frac{f^{(-q)}(\xi)}{\xi^{q+\alpha+1}} = \frac{M'}{\Gamma(\alpha+1+q)}$$

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(1) Geetha K. V. and John J. K., The Laplace-Stieltjes Transformation. The Journal of the Indian Academy of Mathematics, No. 2, Vol. 20, 1998, p. 135–145.

(2) Geetha K. V. and Mangalambal N. R., On dual spaces of ordered multinormed spaces and countable union spaces, Bull. KMA, No. 2, Vol. 4, Dec 2007, p. 63–74.

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(4) Geetha K. V. and Mangalambal N. R., Stieltjes Transformation on ordered vector space of Generalized Functions and Abelian and Tauberian Theorems, Int. J. Contemp. Math. Sciences, No. 33, Vol. 3, 2008, p. 1619–1628.

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(6) Geetha K. V. and Mangalambal N. R., The Convolution Transformation, Proceedings of the International Seminar on Recent Trends in Topology and its Applications, March 19–21, 2009, p. 25–31.

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(8) Geetha K. V. and Mangalambal N. R., Tempered generalized functions ζ' as an ordered topological vector space and Fourier transform of elements of ζ' (communicated).

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