A STUDY ON CHARACTERIZATION OF BEST APPROXIMATIONS IN NORMED SPACES IN TERMS OF SEMI-INNER PRODUCTS

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DECLARATION

I declare that the work presented in this thesis is based on the original work done by me under the guidance of Dr. M. S. Balasubramani, Professor, Department of Mathematics, University of Calicut, and has not been included in any other thesis submitted previously for the award of any other degree either to this university or to any other university / institution.

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CERTIFICATE

Certified that the work presented in this thesis is a bonafide work done by Sri. Gangadharan N. under my guidance in the Department of Mathematics, University of Calicut, and that this work has not been included in any other thesis submitted previously for the award of any degree.

Dr. M. S. Balasubramani
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Contents

CHAPTER 0

Introduction

The subject Best Approximation Theory in normed spaces has attracted the attention of several mathematicians since its introduction in 1920's by one of the founders of Functional Analysis, S. Banach. With the advent of computers, the research in this area has become even more vigorous. By now, the field has become so vast that it has significant intersection with every branch of analysis. Moreover, it plays an increasingly important role in application to many branches of applied sciences and engineering. The present work deals with one of the central themes in best approximation theory in normed spaces, namely the characterization of best approximations. Our endeavor here is the characterization of best approximations in normed spaces in terms of semi-inner products. The outcome of our attempt to characterize best approximations in normed spaces from convex sets, in particular from convex cones, subspaces, and their translates using the notion of semi-inner products is presented in the thesis.

The problem of best approximation in a normed space can be formulated as follows. Let $(X, \|\cdot\|)$ be a normed space over the real or complex number field $\mathbb{K}$, G a nonempty set in X, and $x \in X$. Then the *distance* of x from G, $d(x, G)$, is given by $d(x, G) := \inf \{\|x - g\| : g \in G\}$. The problem of best approximation consists of finding an element $g_0 \in G$ such that $\|x - g_0\| = d(x, G)$. Every element $g_0 \in G$ satisfying this property is called a *best approximation* of x from G. G is called the *approximating set*, and x the *approximated point*. If $g_0 \in G$ is a best approximation of x from G, then the number $d(x, G)$ is called the *error of approximation*. The set of all best approximations of x from G is denoted by $P_G(x)$. This defines a mapping $P_G : X \to \mathcal{P}(G)$, where $\mathcal{P}(G)$ is the power set of G. The set valued mapping P_G is called the *metric projection* onto G. The set G is called *proximinal* (respectively *semi Chebyshev*, *Chebyshev*) if $P_G(x)$ is nonempty (respectively $P_G(x)$ is either empty or a singleton, $P_G(x)$ is a singleton) for each $x \in X$.

The theory of best approximation in normed spaces is mainly concerned with the following fundamental problems: existence of best approximations, uniqueness of best approximations, characterization of best approximations, error of approximation, computation of best approximations, and continuity of best approximations. Among these problems, the one which we consider here is the problem of characterization of best approximation. We understand that the literature is rich with results characterizing best approximations, and that such results are separately available for general normed spaces and inner product spaces (e.g., H. N. Mhaskar and D. V. Pai [21], I. Singer [29, 30], H. Berens [4], F. Deutsch [14] and so on). Generally, characterizations of best approximations in a normed space are derived through the norm of the space, and those in an inner product space, through the inner product of the space. Here we take a different approach. Our endeavor is to characterize best approximations in a general normed space, not through the norm of the space, but through a *semi-inner product* that generates the norm of the space, a concept which was introduced by G. Lumer [17] in 1961 and modified by J. R. Giles [25] in 1967.

While trying to carry over a Hilbert space type argument to a general Banach space situation, G. Lumer [17] introduced the notion of a semi-inner product on a linear space with a more general axiom system than that of an inner product, and obtained some basic properties of this concept. The significance of this notion also was established by Lumer. After Lumer, many mathematicians have pursued the study of this concept. Among them, it was J. R. Giles [25] who put forward some decisive structural modifications of this notion. By a semi-inner product we mean a semi-inner product as introduced by G. Lumer [17] and modified by J. R. Giles [25].

On a normed space, we define a semi-inner product as follows. Let $(X, \|\cdot\|)$ be a normed space over the real or complex number field $\mathbb{K}$. A mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ is called a *semi-inner product* on X if the following properties hold for all $x, y, z \in X$ and all $\lambda,\ \mu \in \mathbb{K}$: $[\lambda x + \mu y|z] = \lambda\,[x|z] + \mu\,[y|z]$; $[x|\lambda y] = \overline{\lambda}\,[x|y]$; $[x|x] = \|x\|^2$; and $|[x|y]| \leq \|x\|\,\|y\|$.

In the setting of a normed space, a semi-inner product provides a sufficient structure as well as new techniques for obtaining some nontrivial general results. To us, this concept is important from the view point of best approximation theory. It is in terms of this notion of semi-inner product which generates the norm of the space that we characterize best approximations in normed spaces. Our main concern here is to derive some results characterizing best approximations in the framework of a general normed space through a semi-inner product that generates the norm of the space. Some consequences of these characterizations are also discussed in the thesis.

Our study is restricted to the setting of real normed spaces. If not mentioned otherwise, throughout our discussion on best approximation, by a normed space X we mean a real normed space $(X, \|\cdot\|)$ together with a semi-inner product $[\cdot|\cdot]$ which generates the norm $\|\cdot\|$. Since the theory of best approximation is the most well developed when the approximating set is a subspace, or more generally a convex set, we confine ourselves to the characterization of best approximation from convex sets. To make the thesis a self contained exposition, the definitions and results employed in our discussion are provided as and when necessary.

Apart from the Introduction, the thesis contains five chapters followed by an Epilogue, which are arranged as follows.

Chapter 1 is devoted for introducing the concept of a semi-inner product on a normed space, the tool with which the characterization of best approximations is carried out in the thesis. The chapter mainly contains our discussions on some facts about the existence and uniqueness of semi-inner products on normed spaces. As mentioned above, by a semi-inner product we mean a semi-inner product as introduced by G. Lumer [17] (Definition 1.2.1) and modified by J. R. Giles [25] (Definition 1.2.2). The significance of this concept is provided in Theorem 1.2.3, which in particular asserts that a linear space endowed with a semi-inner product is a normed space with the norm generated by the semi-inner product. Some properties of the *normalized duality mapping* of a normed space are discussed in Section 1.3. A natural connection between the normalized duality mapping of a normed space and semi-inner products on it is furnished by Theorem 1.4.1, which shows that on every normed space there always exists at least one (and, in general, infinitely many) semi-inner product which is consistent with the norm. The notion of a semi-inner product on a normed space is given by Definition 1.4.3. It is followed by some examples for semi-inner products on normed spaces. The fact that smoothness of the space is a condition which is necessary and sufficient for the existence of a unique semi-inner product on a normed space that generates the norm of the space is discussed in Section 1.5.

Our basic results on characterization of best approximations in normed spaces from convex sets, and in particular from convex cones, subspaces and their translates are the subject matter of Chapter 2. The chapter begins with a brief account on best approximation in normed spaces. In our attempt to characterize best approximations from convex sets, we first of all derive a sufficient condition for best approximations from arbitrary sets in Theorem 2.3.1. When the approximating set is in particular a convex set, a necessary condition for best approximations is given by Theorem 2.3.2. As a consequence of these two results, we have Theorem 2.3.3 which characterizes best approximations from convex sets. The notion of the *dual cone of a set*, which has been introduced in the framework of an inner product space in terms of the inner product of the space [14], is extended in Definition 2.3.4 to the setting of a normed space in terms of a semi-inner product that generates the norm of the space. Theorem 2.3.5 provides a reformulated version of Theorem 2.3.3 in terms of dual cones of sets. This result functions as the basis for every characterization theorem that we provide. It indicates that the characterization of best approximations requires, in essence, the determination of dual cones of sets.

Section 2.4 focusses on orthogonality in a normed space in terms of a semi-inner product that generates the norm of the space [15]. This notion is a generalization of the orthogonality concept in an inner product space. The orthogonality that we consider here is not generally symmetric, since a semi-inner product lacks the property of conjugate symmetry. The concepts of *orthogonality of elements* and *orthogonal complements of sets* are introduced in Definition 2.4.1 and Definitions 2.4.3 respectively, and some immediate consequences of these definitions are noted. Theorem 2.4.4 provides the exact relationship between the dual cone and the orthogonal complement of a given set. Some basic properties of dual cones and orthogonal complements are derived and presented in Theorems 2.4.5, 2.4.6 and 2.4.7. These results enable us to strengthen the characterization Theorem 2.3.5 for convex sets to the cases of convex cones and subspaces. A characterization of best approximations from convex cones is given in Theorem 2.5.1, and that from subspaces is provided in Theorem 2.5.3. Characterizing best approximations from translates of convex cones and subspaces are also considered in this chapter. We conclude this chapter with the Example 2.5.10 which illustrates Theorem 2.5.3 characterizing best approximations from subspaces.

Chapter 3 deals with a few direct applications of the characterization results which are already seen in the preceding chapter. In this chapter we first of all provide some new characterizations of best approximations from convex cones, subspaces and their translates in Theorems 3.2.1, 3.2.2, 3.2.3 and 3.2.4. In these results, best approximations are characterized in terms of errors of approximation. Following that, some properties of best approximations are presented. These include results such as Theorem 3.3.14, Corollary 3.3.15, Corollary 3.3.16, Theorem 3.3.17, Corollary 3.3.18 and Corollary 3.3.19, which assert that proximinality, semi Chebyshevity and Chebyshevity of convex sets, and in particular convex cones and subspaces, are invariant under translation as well as under scalar multiplication.

In Section 3.4 we introduce the novel concept of an *ordered orthogonal set* in the setting of a normed space in terms of a semi-inner product that generates the norm of the space, a concept which generalizes the notion of an orthogonal set in the framework of an inner product space. Because of the fact that semi-inner product orthogonality is not generally symmetric, we term the extended notion as an ordered orthogonal set. A nonempty, nonsingleton finite subset of a normed space is said to be ordered orthogonal, if its elements can be arranged as a sequence in which each element except the first is orthogonal to every element preceding it (see Definition 3.4.1). An arbitrary nonempty subset of a normed space is said to be ordered orthogonal if every nonempty nonsingleton finite subset of it is ordered orthogonal (Definition 3.4.2). An ordered orthogonal set whose every element is of norm 1 is called an *ordered orthonormal set* (Definition 3.4.3). Some examples of ordered orthogonal sets and ordered orthonormal sets are provided in Example 3.4.4 and Example 3.4.5. The connection between ordered orthogonality and linear independence of sets in a normed space is given by Theorem 3.4.6, where we prove that an ordered orthogonal set of nonzero elements in a normed space is linearly independent. This result, in particular, shows that every ordered orthonormal set in a normed space is linearly independent.

The converse of the above problem is considered in Section 3.5. Making use of the fact that every finite dimensional subspace of a normed space is proximinal [21], and our characterization Theorem 2.5.3, we arrive at an *ordered orthonormalization process* in Theorem 3.5.2. This result shows that, given a countable linearly independent set in a normed space, one can construct an ordered orthonormal set retaining the span of the elements at each step. The ordered orthonormal set thus constructed need not be unique. However, if the normed space is in particular strictly convex, employing the fact that every finite dimensional subspace of a strictly convex normed space is Chebyshev [21], we prove in Corollary 3.5.4 that the ordered orthonormalization process provided by Theorem 3.5.2 yields a unique ordered orthonormal set. Theorem 3.5.2 and Corollary 3.5.4 can be considered as analogues of the Gram-Schmidt orthonormalization process in a general normed space and in a strictly convex normed space respectively.

The main purpose of Chapter 4 is the characterization of proximinality and Chebyshevity of convex sets in normed spaces in terms of the decomposability of the space. Some consequences of this characterization are also considered here. In a sense, this chapter is a continuation of the preceding one, since what we present here are again consequences mainly of the characterization results of the second chapter. Our discussion in this chapter begins with some results characterizing proximinal, semi Chebyshev and Chebyshev convex sets, in particular convex cones, subspaces and their translates, which are arrived at using the characterization theorems of the second chapter. These results, which are provided in Section 4.2, suggest that proximinality and Chebyshevity of subspaces of a normed space can be characterized in terms of the decomposability of the space.

Section 4.3 concentrates on some decomposition theorems. In Corollary 4.3.3 we prove that a subspace of a normed space is proximinal (respectively Chebyshev) if and only if the normed space is the sum (respectively direct sum) of the subspace and its orthogonal complement. Employing the facts that every proximinal set in a normed space is closed, and every closed subspace of a reflexive normed space is proximinal [29], we deduce from Corollary 4.3.3 that a subspace of a reflexive normed space is closed if and only if the normed space is the sum of the subspace and its orthogonal complement. This result is presented in Theorem 4.3.7. In particular, when the reflexive normed space is strictly convex also, using the fact that every closed subspace of a strictly convex reflexive normed space is Chebyshev [29], we prove in Theorem 4.3.8 that a subspace of a strictly convex reflexive normed space is closed if and only if the normed space is the direct sum of the subspace and its orthogonal complement. Theorem 4.3.7 and Theorem 4.3.8 furnish analogues of the Projection theorem in the setting of a reflexive normed space and in the framework of a strictly convex reflexive normed space respectively. We show in Example 4.3.10 that these results do not hold for nonreflexive normed spaces.

Some consequences of the above decomposition results are provided in Section 4.4. We prove in Theorem 4.4.1 that a proximinal subspace of a normed space is dense in the normed space if and only if its orthogonal complement is $\{0\}$. A similar result in the framework of a reflexive normed space is given by Theorem 4.4.3 in which we show that a subspace of a reflexive normed space is dense in the normed space if and only if its orthogonal complement is $\{0\}$. Some sufficient conditions for orthogonal complements of subspaces of normed spaces to contain nonzero elements are presented in Theorem 4.4.4, Corollary 4.4.5 and Theorem 4.4.7.

Section 4.5 focusses on continuous linear functionals on normed spaces. Using Theorem 1.2.3 (b), we show in Theorem 4.5.1 that every element y belonging to a normed space X determines a continuous linear functional f_y on X defined by $f_y(x) = [x|y]$ with $\|f_y\| = \|y\|$. We also prove in Theorem 4.5.3 that if the normed space X is actually reflexive, then every continuous linear functional f on X is given by $f(x) = [x|y_f]$ for some suitable element y_f belonging to X with $\|f\| = \|y_f\|$. This result is derived with the help of our decomposition Theorem 4.3.7 and Theorem 4.4.3. In the setting of a reflexive normed space, Theorem 4.5.3 can be treated as an analogue of the Riesz Representation theorem. Example 4.5.5 shows that Theorem 4.5.3 does not hold for nonreflexive normed spaces.

In Chapter 5, using the concept of the orthogonality due to G. Birkhoff [5], we make a revisit to the characterization of best approximations seen so far in our discussion. First of all we provide a brief discussion on *Birkhoff orthogonality* (Definition 5.2.1). The notion of *Birkhoff orthogonal complement of a set* is introduced in Definitions 5.2.4, and some direct consequences of this definition are noted. The question of the equivalence of Birkhoff and semi-inner product orthogonalities is considered in Section 5.3. In Example 5.3.2 we show that these two orthogonalities are not generally equivalent. However, it has been shown that [15] Birkhoff orthogonality is equivalent to semi-inner product orthogonality for some suitable semi-inner product on the normed space that generates the norm of the space. This fact is discussed in Theorem 5.3.4. *It enables us to reformulate our results characterizing best approximations in terms of semi-inner products orthogonality into those in terms of Birkhoff orthogonality.* Here we concentrate on characterization results for subspaces and their translates only. A well known characterization of best approximations from subspaces due to I. Singer [29] is recaptured in Corollary 5.4.2 (see Remark 5.4.4).

Some decomposition theorems in terms of Birkhoff orthogonality are presented in Section 5.5. This includes results that characterize proximinality as well as Chebyshevity of translates of subspaces and subspaces in terms of Birkhoff orthogonality, which are presented respectively in Theorem 5.5.1 and Corollary 5.5.2. In the setting of a reflexive normed space, an analogue of the Projection theorem in terms of Birkhoff orthogonality is provided in Theorem 5.5.4. A similar result in the framework of a strictly convex reflexive normed space is given in Theorem 5.5.5. We note that these analogues do not hold for nonreflexive normed spaces (see Example 4.3.10). This section contains some consequences of these decomposition theorems too.

Some of the unexposed problems and possibilities that are closely related to our work, where further research is possible are briefly outlined in the Epilogue.

CHAPTER 1

Semi-Inner Products on Normed Spaces

1.1 Introduction

A Hilbert space can be thought of either as a complete inner product space, or as a Banach space whose norm satisfies the parallelogram law. In the theory of operators on a Hilbert space, it actually does not function as a particular Banach space, but rather as a particular inner product space. It is in terms of the inner product structure that most of the terminologies and techniques are developed. On the other hand, this type of Hilbert space considerations find no real parallel in the general Banach space setting.

While trying to carry over a Hilbert space type argument to a general Banach space situation, G. Lumer [17] was led to use a suitable mapping from a Banach space into its dual space in order to make up for the lack of an inner product. His procedure suggested the existence of a general theory which became very useful in the study of operator normed algebras by providing better insight on known facts, a more adequate language to classify special types of operators, as well as new techniques. It was these ideas which evolved into a theory of semi-inner products.

The purpose of this chapter is to introduce the concept of semi-inner product as defined by G. Lumer [17] and modified by J. R. Giles [25]. Our emphasis here is on the existence and uniqueness of semi-inner products on normed spaces. It has been shown that a linear space endowed with a semi-inner product is a normed space with the norm generated by the semi-inner product. Making use of some properties of the normalized duality mapping, it has been further shown that for every normed space one can construct at lest one (and, in general, infinitely many) semi-inner product which is consistent with the norm. Regarding the uniqueness of semi-inner products on normed spaces, it was proved that smoothness of the space is a condition which is necessary and sufficient for the existence of a unique semi-inner product on a normed space that generates the norm of the space. We mainly discuss these facts in this chapter.

Unless specified otherwise, all linear spaces, normed spaces and inner product spaces appearing in this chapter are always over the real or complex number field denoted by $\mathbb{K}$.

1.2 Semi-Inner Products

G. Lumer [17] has constructed on a linear space a particular type of mapping, which he called a semi-inner product, with a more general axiom system than that of an inner product. According to him, a semi-inner product, which we call a L-semi-inner product for the time being, is defined as follows.

Definition 1.2.1. Let X be a linear space. The mapping $[\cdot\,|\cdot] : X \times X \to \mathbb{K}$ is called a *L-semi-inner product* (or *semi-inner product in the sense of Lumer*) on X if the following properties are satisfied:

(L_1) $[x + y|z] = [x|z] + [y|z]$ for all $x, y, z \in X$;
(L_2) $[\lambda x|y] = \lambda\,[x|y]$ for all $x, y \in X$ and all $\lambda \in \mathbb{K}$;
(L_3) $[x|x] \geq 0$ for all $x \in X$, and $[x|x] = 0 \Rightarrow x = 0$;
(L_4) $|[x|y]|^2 \leq [x|x]\,[y|y]$ for all $x, y \in X$.

The significance of this notion was also established by G. Lumer [17] by showing that a L-semi-inner product $[\cdot|\cdot]$ on a linear space X always induces a norm on X by setting $\|x\| = [x|x]^{1/2}$, and for every normed space $(X, \|\cdot\|)$, one can construct at least one (and, in general, infinitely many) L-semi-inner product consistent with the norm in the sense $[x|x] = \|x\|^2$.

Though the notion of a L-semi-inner product on a linear space was first introduced and systematically studied by G. Lumer [17] in 1961, its history can be traced back to 1933 with the works of S. Mazur [19]. After G. Lumer, many mathematicians including J. R. Giles [25], P. M. Milicic [22], I. Rosca [27], B. Nath [23], R. Tapia [31], S. S. Dragomir [15] and J. Chmielinski [12] have pursued the study of this concept. Among them, it was J. R. Giles [25] who put forward some decisive structural modifications of this notion. In his attempt to determine what further developments can be made, J. R. Giles [25] has shown that a homogeneity property, $[x|\lambda y] = \overline{\lambda}\,[x|y]$ for all $\lambda \in \mathbb{K}$, where $\overline{\lambda}$ denotes the complex conjugate of λ, can be imposed on a L-semi-inner product (Definition 1.2.1). The imposition of this property adds much convenience without causing any significant restriction. Thus, according to J. R. Giles [25], a semi-inner product can be defined on a linear space as follows.

Definition 1.2.2. Let X be a linear space. The mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ is called a *semi-inner product* (or *semi-inner product in the sense of Lumer-Giles*) on X if the following properties are satisfied:

(LG_1) $[x + y|z] = [x|z] + [y|z]$ for all $x, y, z \in X$;

(LG_2) $[\lambda x|y] = \lambda\,[x|y]$, and $[x|\lambda y] = \overline{\lambda}\,[x|y]$ for all $x, y \in X$ and all $\lambda \in \mathbb{K}$;

(LG_3) $[x|x] \geq 0$ for all $x \in X$, and $[x|x] = 0 \Rightarrow x = 0$;

(LG_4) $|[x|y]|^2 \leq [x|x]\,[y|y]$ for all $x, y \in X$.

We notice that every inner product is a semi-inner product. However, the converse need not hold, since a semi-inner product lacks conjugate symmetry, a property which an inner product possesses. The examples given in Section 1.4 are all semi-inner products which are not inner products.

Now we aim at providing the concrete significance of the above concept of semi-inner product. As a first step, we have the following the result [15]. $\mathbb{R}$

denotes the set of all real numbers.

Theorem 1.2.3. *Let X be a linear space, and $[\cdot|\cdot]$ a semi-inner product on X. Then the following statements are true:*

(a) *The mapping $\|\cdot\| : X \to \mathbb{R}$ given by $\|x\| = [x|x]^{1/2}$ defines a norm on X;*

(b) *For every $y \in X$, the functional $f_y : X \to \mathbb{K}$ given by $f_y(x) = [x|y]$ is a continuous linear functional on X endowed with the norm generated by the semi-inner product $[\cdot|\cdot]$. Moreover, $\|f_y\| = \|y\|$.*

Proof. (a) Let $x \in X$. Then by (LG_3) of Definition 1.2.2, we have $\|x\| = [x|x]^{1/2} \geq 0$, and if $\|x\| = 0$, then $[x|x] = 0$ so that $x = 0$. If $x \in X$ and $\lambda \in \mathbb{K}$, by (LG_2) of the definition, we have

$$\begin{aligned} \|\lambda x\| &= [\lambda x|\lambda x]^{1/2} \\ &= |\lambda|\,\|x\|\,. \end{aligned}$$

Finally, for every $x, y \in X$, from (LG_1) and (LG_4) of the definition it follows that

$$\begin{aligned} \|x+y\|^2 &= [x+y|x+y] \\ &= [x|x+y] + [y|x+y] \\ &\leq \|x\|\,\|x+y\| + \|y\|\,\|x+y\|\,, \end{aligned}$$

and so $\|x+y\| \leq \|x\| + \|y\|\,.$

(b) Let $y \in X$. If y=0, the result is trivial, since in this case f_y is the zero functional on X, by (LG_2) of Definition 1.2.2. So suppose that $y \neq 0$. Then for all $x, z \in X$, and all λ, $\mu \in \mathbb{K}$, by (LG_1) and (LG_2) of the definition, we have

$$\begin{aligned} f_y(\lambda x + \mu z) &= [\lambda x + \mu z|y] \\ &= \lambda\,[x|y] + \mu\,[z|y] \\ &= \lambda f_y(x) + \mu f_y(z), \end{aligned}$$

so that f_y is linear on X.

Now, by (LG_4) of the definition, for all $x \in X$, we have

$$\begin{aligned} |f_y(x)| &= |[x|y]| \\ &\leq \|x\| \, \|y\| , \end{aligned}$$

which implies that f_y is bounded and $\|f_y\| \leq \|y\|$.
On the other hand,

$$\begin{aligned} \|f_y\| &\geq \frac{|f_y(y)|}{\|y\|} \\ &= \frac{[y|y]}{\|y\|} \\ &= \|y\| , \end{aligned}$$

and thus actually $\|f_y\| = \|y\|$. This completes the proof. $\square$

The above theorem shows, in particular, that a linear space endowed with a semi-inner product is a normed space with the norm generated by the semi-inner product. Later we will prove in Section 1.4 that for every normed space, one can construct at least one (and, in general, infinitely many) semi-inner product which is consistent with the norm. In this regard, we need some properties of the normalized duality mapping.

1.3 Normalized Duality Mappings

The normalized duality mapping of a normed space is defined as follows [13]. X^* denotes the dual space of a normed space X, and $\mathcal{P}(X^*)$, the power set of X^*.

Definition 1.3.1. Let $(X, \|\cdot\|)$ be a normed space. The mapping $\mathcal{J} : X \to \mathcal{P}(X^*)$ given by

$$\mathcal{J}(x) := \{f \in X^* : f(x) = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the *normalized duality mapping* of X.

The next result contains some fundamental properties of the set valued mapping $\mathcal{J}$ [15]. We include the proof here, since it is an interesting application of the Hahn-Banach extension theorem.

Theorem 1.3.2. *Let $(X, \|\cdot\|)$ be a normed space. Then the following statements are true:*

(a) *For each $x \in X$, the set $\mathcal{J}(x)$ is a nonempty convex subset of X^*;*
(b) *For all $x \in X$ and every $\lambda \in \mathbb{K}$, $\mathcal{J}(\lambda x) = \overline{\lambda}\mathcal{J}(x)$.*

Proof. (a) Let $x \in X$. By the very definition of the normalized duality mapping (Definition 1.3.1), $\mathcal{J}(x)$ is a subset of X^*.

If x=0, then clearly $\mathcal{J}(0) = \{0\}$. If $x \neq 0$, consider the subspace $S_x :=$ span$\{x\}$ of X. Define the functional $g : S_x \to \mathbb{K}$ by $g(u) = \lambda \|x\|^2$, where $u = \lambda x \in S_x$ $(\lambda \in \mathbb{K})$. Then for all $u = \lambda x$, $v = \mu x \in S_x$ $(\lambda, \mu \in \mathbb{K})$, and for all α, $\beta \in \mathbb{K}$, we have

$$\begin{aligned} g(\alpha u + \beta v) &= g((\alpha\lambda + \beta\mu)x) \\ &= (\alpha\lambda + \beta\mu) \|x\|^2 \\ &= \alpha\, g(u) + \beta\, g(v), \end{aligned}$$

so that g is linear on S_x. Further, for all $u = \lambda x \in S_x$ $(\lambda \in \mathbb{K})$, we have

$$\begin{aligned} |g(u)| &= \left|\lambda \|x\|^2\right| \\ &= \|\lambda x\| \|x\| \\ &= \|x\| \|u\|, \end{aligned}$$

so that g is also bounded on S_x, and $\|g\| = \|x\|$. Therefore, by virtue of the Hahn-Banach extension theorem, there exists a functional $f \in X^*$ which extends g to the whole of X such that $\|f\| = \|g\| = \|x\|$. Also, since $x \in S_x$, we have

$$f(x) = g(x) = g(1.x) = \|x\|^2 .$$

It follows that $f \in \mathcal{J}(x)$ so that $\mathcal{J}(x)$ is nonempty.

Now we will show that $\mathcal{J}(x)$ is convex. To this end, suppose that $x \neq 0$, and let f_1, $f_2 \in \mathcal{J}(x)$. Then for every $\lambda \in [0, 1]$, we have

$$\lambda f_1 + (1 - \lambda)\, f_2 \in X^*,$$

and

$$\begin{aligned}(\lambda f_1 + (1-\lambda) f_2)(x) &= \lambda f_1(x) + (1-\lambda) f_2(x) \\ &= \lambda \|x\|^2 + (1-\lambda) \|x\|^2 \\ &= \|x\|^2 .\end{aligned}$$

Hence for every $\lambda \in [0,1]$, we have

$$\begin{aligned}0 < \|x\| &= \left| (\lambda f_1 + (1-\lambda) f_2) \left(\frac{x}{\|x\|} \right) \right| \\ &\leq \sup_{0 \neq y \in X} \left| (\lambda f_1 + (1-\lambda) f_2) \left(\frac{y}{\|y\|} \right) \right| \\ &= \|\lambda f_1 + (1-\lambda) f_2\| ,\end{aligned}$$

which shows that $\|x\| \leq \|\lambda f_1 + (1-\lambda) f_2\|$. However, since $f_1, f_2 \in \mathcal{J}(x)$, we have $\|f_1\| = \|f_2\| = \|x\|$ so that

$$\|\lambda f_1 + (1-\lambda) f_2\| \leq \lambda \|f_1\| + (1-\lambda) \|f_2\| = \|x\|$$

for every $\lambda \in [0,1]$. Hence for every $\lambda \in [0,1]$, $\|\lambda f_1 + (1-\lambda) f_2\| = \|x\|$, and this completes the proof of (a).

(b) Let $x \in X$ and $\lambda \in \mathbb{K}$. If $\lambda = 0$, the statement is trivially true. Suppose $\lambda \neq 0$, and let $f \in \mathcal{J}(\lambda x)$. Then $f \in X^*$, $f(\lambda x) = \|\lambda x\|^2$ so that $f(x) = \overline{\lambda} \|x\|^2$, and $\|f\| = \|\lambda x\|$. Hence

$$\frac{1}{\overline{\lambda}} f \in X^*,$$

$$\left(\frac{1}{\overline{\lambda}} f \right)(x) = \frac{1}{\overline{\lambda}} f(x) = \|x\|^2 ,$$

and

$$\left\| \frac{1}{\overline{\lambda}} f \right\| = \frac{1}{|\overline{\lambda}|} \|f\| = \frac{1}{|\overline{\lambda}|} \|\lambda x\| = \|x\| .$$

This shows that

$$\frac{1}{\overline{\lambda}} f \in \mathcal{J}(x) \text{ so that } f \in \overline{\lambda} \mathcal{J}(x).$$

Thus we have the inclusion $\mathcal{J}(\lambda x) \subseteq \overline{\lambda}\mathcal{J}(x)$. A similar argument leads to the the reverse inclusion. Hence holds (b) also. $\square$

Remark 1.3.3. Proof of Theorem 1.3.2 (a) conveys the idea that for $x \in X$, in general, $\mathcal{J}(x)$ contains infinitely many different elements of X^*, since uniqueness of Hahn-Banach extensions is not generally guaranteed in the case of normed spaces, and there can even be infinitely many such extensions. For example, consider the normed space $X = (\mathbb{K}^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{2} |x_k|$ for $x = (x_1, x_2) \in X$, and the subspace

$$Y = \{x = (x_1, x_2) \in X : x_2 = 0\}$$

of X. Let $g : Y \to \mathbb{K}$ be defined by

$$g(x) = x_1, \quad x \in Y.$$

Then it is clear that $g \in Y^*$and $\|g\| = 1 = g(a)$, where $a = (1, 0)$. Since $Y = \text{span}\{\text{a}\}$, we see that a function f on X is a Hahn-Banach extension of g to X if and only if f is linear on X and $\|f\| = 1 = f(a)$. Now, if f is linear on X, then

$$f(x) = k_1 x_1 + k_2 x_2, \quad x \in X,$$

for some fixed k_1 and k_2 in $\mathbb{K}$. Then $\|f\| = \max\{|k_1|, |k_2|\}$, and $f(a) = 1$ if and only if $k_1 = 1$. Thus, for each $k_2 \in \mathbb{K}$ with $|k_2| \leq 1$, the function $f : X \to \mathbb{K}$ defined by

$$f(x) = x_1 + k_2 x_2, \quad x \in X$$

is a Hahn-Banach extension of g to X. For each such f, we have

$$f \in X^*,$$

$$f(a) = 1 = \|a\|^2$$

and

$$\|f\| = 1 = \|a\|,$$

so that

$$f \in \mathcal{J}(a).$$

Hence $\mathcal{J}(a)$ contains infinitely many different elements of X^*.

A section of the normalized duality mapping is defined as follows [15].

Definition 1.3.4. Let $\mathcal{J}$ be the normalized duality mapping of a normed space $(X, \|\cdot\|)$. A mapping $\widetilde{\mathcal{J}}: X \to X^*$ is called a *section* of $\mathcal{J}$, if $\widetilde{\mathcal{J}}(x) \in \mathcal{J}(x)$ for all $x \in X$.

Remark 1.3.5. Sections of $\mathcal{J}$ do exist, since $\mathcal{J}(x)$ is nonempty for each $x \in X$ (by Theorem 1.3.2 (a)). Indeed, as $\mathcal{J}(x)$ is nonempty for each $x \in X$, $\{\mathcal{J}(x)\}_{x \in X}$ is a nonempty class of nonempty sets. Hence, by the axiom of choice, a set can be formed which contains precisely one element, say f_x, taken from each set $\mathcal{J}(x)$. This determines a section $\widetilde{\mathcal{J}}: X \to X^*$ of $\mathcal{J}$ defined by $\widetilde{\mathcal{J}}(x) = f_x$ for all $x \in X$. The normalized duality mapping $\mathcal{J}$ of a normed space X can have, in general, infinitely many distinct sections, since for $x \in X$, generally $\mathcal{J}(x)$ contains infinitely many different elements of X^*. For example, consider the normalized duality mapping $\mathcal{J}$ of the normed space $X = (\mathbb{K}^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{2} |x_k|$ for $x = (x_1, x_2) \in X$. We have already seen (in Remark 1.3.3) that for $a = (1, 0) \in X$, $\mathcal{J}(a)$ contains the infinitely many different elements of X^* given by

$$f(x) = x_1 + k_2 x_2, \quad x \in X,$$

for each $k_2 \in \mathbb{K}$ with $|k_2| \leq 1$. Since $\mathcal{J}(x)$ is nonempty for every $x \neq a$ in X, each one of the infinitely many different elements in $\mathcal{J}(a)$ gives rise (by the axiom of choice) to a section $\widetilde{\mathcal{J}}$ of $\mathcal{J}$. Hence the normalized duality mapping of this normed space has infinitely many distinct sections. The normalized duality mapping $\mathcal{J}$ of a normed space X has a unique section if and only if $\mathcal{J}(x)$ is a singleton set for each $x \in X$.

1.4 Existence of Semi-Inner Products

In this section we consider the existence of semi-inner products on normed spaces. The following theorem due to I. Rosca [27] is a land mark result, as it establishes a natural connection between the normalized duality mapping of a normed space and semi-inner products on it.

Theorem 1.4.1. *Let $(X, \|\cdot\|)$ be a normed space. Then every semi-inner product on X which generates the norm $\|\cdot\|$ is of the form*

$$[x|y] = (\widetilde{\mathcal{J}}(y))(x) \text{ for all} x, y \in X,$$

where $\widetilde{\mathcal{J}}$ is a section of the normalized duality mapping $\mathcal{J}$ of X.

Proof. Let $\widetilde{\mathcal{J}}$ be a section of the normalized duality mapping $\mathcal{J}$ of X. Define the functional

$$[\cdot|\cdot] : X \times X \to \mathbb{K} \text{ by } [x|y] = (\widetilde{\mathcal{J}}(y))(x).$$

Then for every $x, y, z \in X$ and $\lambda \in \mathbb{K}$, by properties of elements in X^*, we have

$$\begin{aligned}
[x+y|z] &= (\widetilde{\mathcal{J}}(z))(x+y) \\
&= [x|z] + [y|z], \\
[\lambda x|y] &= (\widetilde{\mathcal{J}}(y))(\lambda x) \\
&= \lambda [x|y], \\
[x|\lambda y] &= (\widetilde{\mathcal{J}}(\lambda y))(x) \\
&= (\overline{\lambda}\widetilde{\mathcal{J}}(y))(x), \text{ by Theorem 1.3.2(b)} \\
&= \overline{\lambda} [x|y],
\end{aligned}$$

and

$$\begin{aligned}
|[x|y]|^2 &= \left|\widetilde{\mathcal{J}}(y)(x)\right|^2 \\
&\leq \left\|\widetilde{\mathcal{J}}(y)\right\|^2 \|x\|^2 \\
&= \|y\|^2 \|x\|^2, \text{ by definition of } \widetilde{\mathcal{J}}(y) \\
&= (\widetilde{\mathcal{J}}(y))(y) \cdot (\widetilde{\mathcal{J}}(x))(x) \\
&= [x|x] [y|y].
\end{aligned}$$

We also have, $[x|x] = (\widetilde{\mathcal{J}}(x))(x) = \|x\|^2 \geq 0$ for every $x \in X$, and $[x|x] = 0$, i.e., $\|x\| = 0$ implies $x = 0$. Hence the mapping $[\cdot|\cdot]$ is a semi-inner product on X which generates the norm $\|\cdot\|$ of X.

On the other hand, let $[\cdot|\cdot]$ be a semi-inner product on X which generates the norm $\|\cdot\|$ of X. Define the mapping $\widetilde{\mathcal{J}}:X \to X^*$ in such a way that, for each $y \in X$, the functional $\widetilde{\mathcal{J}}(y):X \to \mathbb{K}$ is given by

$$(\widetilde{\mathcal{J}}(y))(x) = [x|y] \quad \text{for all } x \in X.$$

Then for every $y \in X$, $(\widetilde{\mathcal{J}}(y))(y) = [y|y] = \|y\|^2$. Also, by Theorem 1.2.3(b), we have $\widetilde{\mathcal{J}}(y) \in X^*$ and $\left\|\widetilde{\mathcal{J}}(y)\right\| = \|y\|$ for all $y \in X$. Hence, for all $y \in X$, $\widetilde{\mathcal{J}}(y) \in \mathcal{J}(y)$, so that $\widetilde{\mathcal{J}}$ is a section of the normalized duality mapping $\mathcal{J}$ of X. □

Remark 1.4.2. Since sections of the normalized duality mapping $\mathcal{J}$ of a normed space X do exist (see Remark 1.3.5), the above theorem asserts in particular that on every normed space one can construct at least one semi-inner product which is consistent with the norm. Since, in general, $\mathcal{J}$ can have infinitely many distinct sections, there may exist infinitely many different semi-inner products on a given normed space. For example, consider the normed space $X = (\mathbb{K}^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{2} |x_k|$ for $x = (x_1, x_2) \in X$. As we have already seen (in Remark 1.3.5), the normalized duality mapping of this normed space has infinitely many distinct sections $\widetilde{\mathcal{J}}$. By the above theorem, each such section $\widetilde{\mathcal{J}}$ determines a semi-inner product on X defined by

$$[x|y] = (\widetilde{\mathcal{J}}(y))(x), \quad x, y \in X.$$

This shows the existence of infinitely many different semi-inner products on the given normed space. A discussion on the uniqueness of semi-inner products will be made in the next section.

It may be noticed that Theorem 1.4.1 actually has a four fold importance: In the setting of a normed space, the theorem throws light on the existence, uniqueness and characterization of semi-inner products that are consistent with the norm, and suggests a procedure for obtaining all semi-inner products which generate the norm in terms of the normalized duality mapping of the space. Here we concentrate on the existence of semi-inner products on normed spaces.

Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ on X need not come from an inner product. However, the above considerations tell us that there always exists a semi-inner product $[\cdot|\cdot]$ on X which generates the norm $\|\cdot\|$. Thus, in the framework of a normed space, a semi-inner product can be defined as follows.

Definition 1.4.3. Let $(X, \|\cdot\|)$ be a normed space. A mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ is called a *semi-inner product* on X if the following properties hold for all $x, y, z \in X$ and all $\lambda,\ \mu \in \mathbb{K}$:

(S$_1$) $[\lambda x + \mu y|z] = \lambda\,[x|z] + \mu\,[y|z]$;
(S$_2$) $[x|\lambda y] = \overline{\lambda}\,[x|y]$;
(S$_3$) $[x|x] = \|x\|^2$;
(S$_4$) $|[x|y]| \leq \|x\|\,\|y\|$.

It is time for some examples. An exhaustive supply of semi-inner products on a normed space is suggested by Theorem 1.4.1 : Let $(X, \|\cdot\|)$ be a normed space. For each section $\widetilde{\mathcal{J}}$ of the normalized duality mapping $\mathcal{J}$ of X, the functional defined by

$$[x|y] = (\widetilde{\mathcal{J}}(y))(x), \quad x, y \in X$$

is a semi-inner product on X which generates the norm $\|\cdot\|$. However, let us now look at some concrete examples.

Example 1.4.4. Consider the real normed space $(\mathbb{R}^3, \|\cdot\|_1)$, where $\|x\|_1 = \sum\limits_{k=1}^{3} |x_k|$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The functional given by

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{3} \frac{x_k y_k}{|y_k|}, \quad x, y \in \mathbb{R}^3$$

is a semi-inner product on $\mathbb{R}^3$ which generates the norm $\|\cdot\|_1$.

Example 1.4.5. Consider the complex normed space $(\ell_p, \|\cdot\|_p)$ of all p-summable sequences, with the norm

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p},$$

for $x = (x_1,\ x_2,\ x_3, ...) \in \ell_p$, where $1 \leq p < \infty, \quad p \neq 2$. Then the functional given by

$$[x|y] = \|y\|_p^{2-p} \sum_{k=1}^{\infty} x_k \overline{y_k} \, |y_k|^{p-2}, \quad x, y \in \ell_p,$$

where $\overline{y_k}$ denotes the complex conjugate of y_k, defines a semi-inner product on ℓ_p that generates the norm $\|\cdot\|_p$.

Example 1.4.6. If $1 < p < \infty$, consider the real normed space $L_p = L_p(X, \mathcal{A}, \mu)$ of all p-integrable functions on the measure space $(X, \mathcal{A}, \mu)$ with the norm

$$\|x\|_p = \left(\int_X |x|^p d\mu \right)^{\frac{1}{p}}, \quad x \in L_p.$$

The functional given by

$$[x|y] = \|y\|_p^{2-p} \int_X x \, |y|^{p-2} \, y \, d\mu, \quad x, y \in L_p$$

defines a semi-inner product on L_p which generates the $\|\cdot\|_p$.

Example 1.4.7. Consider the complex normed space $C\,[0,1]$ of all continuous complex valued functions on $[0,1]$ with the sup norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f\,(t)|\,, \quad f \in C\,[0,1]\,.$$

If $t \in [0,1]$, by Urysohn's lemma, there exists $f_t \in C\,[0,1]$ such that $f_t\,(t) = 1$ and $\|f_t\|_\infty = 1$. For such an f_t, we define $[g|f_t] = g(t)$ for all $g \in C\,[0,1]$. Now $[f_t|f_t] = f_t\,(t) = 1 = \|f_t\|_\infty^2$. If $f \in C\,[0,1] \setminus f_t$, choose $t \in [0,1]$ such that $f(t) = \|f\|_\infty$ and define $[g|f] = \overline{f(t)} g(t)$ for all $g \in C\,[0,1]$. Then the functional given by

$$[g|f] = \begin{cases} g(t) & \text{if } f = f_t \\ \overline{f(t)} g(t) & \text{if } f \in C\,[0,1] \setminus f_t \end{cases}$$

for all $g \in C\,[0,1]$ is a semi-inner product on $C\,[0,1]$ consistent with the sup norm $\|\cdot\|_\infty$.

1.5 Uniqueness of Semi-Inner Products

Having seen, as guaranteed by Theorem 1.4.1 and Theorem 1.3.2(a), the existence of a semi-inner product on a normed space that generates the norm, let us now turn to the question of its uniqueness.

Let $(X, \|\cdot\|)$ be a normed space. From Theorem 1.4.1 it follows that the functional $[\cdot|\cdot] : X \to \mathbb{K}$ given by

$$[x|y] = (\widetilde{\mathcal{J}}(y))(x) \quad \text{for all } \ x, y \in X$$

defines a semi-inner product on X that generates the norm $\|\cdot\|$ of X if and only if $\widetilde{\mathcal{J}}$ is a section of the normalized duality mapping $\mathcal{J}$ of X. Theorem 1.3.2 (a) shows that for each $x \in X$, $\mathcal{J}(x)$ is a nonempty subset of X^*. As a consequence of these facts, we notice that there is a one to one correspondence between the set of all semi-inner products on X that generate the norm $\|\cdot\|$ of X, and the set of all sections of the normalized duality mapping $\mathcal{J}$ of X. Hence the conditions for the existence of a unique semi-inner product on a normed space X are exactly the same as those required for making each $\mathcal{J}(x)$, $x \in X$ a singleton set. One such condition turns out [17] to be the smoothness of the space. Smoothness of a normed space [20] is a condition which is necessary and sufficient for the existence of a unique semi-inner product on the space that generates the norm of the space. $B(X)$ and $S(X)$ denote respectively the *closed unit ball* $\{x \in X : \|x\| \leq 1\}$ and *unit sphere* $\{x \in X : \|x\| = 1\}$ in the normed space $(X, \|\cdot\|)$.

Definition 1.5.1. Let $(X, \|\cdot\|)$ be a normed space. An element $x_0 \in S(X)$ is called a *point of smoothness* of $B(X)$, if there is exactly one element $f_0 \in S(X^*)$ satisfying $f_0(x_0)$=1. The space X is said to be *smooth*, if each point of $S(X)$ is a point of smoothness of $B(X)$, i.e., if for each $x \in S(X)$, there exists a unique $f \in S(X^*)$ such that $f(x)$=1.

The following result provides a characterization of points of smoothness in terms of the normalized duality mapping. The proof we provide here is a modified version of that given by S. S. Dragomir [15].

Theorem 1.5.2. *Let* $(X, \|\cdot\|)$ *be a normed space, and* $x_0 \in S(X)$*. Then the* *following are equivalent:*

(a) x_0 *is a point of smoothness of* $B(X)$*;*
(b) $\mathcal{J}(x_0)$ *is a singleton set in* X^**.*

Proof. (a) $\Rightarrow$ (b): For each $x_0 \in S(X)$, $\mathcal{J}(x_0)$ is a nonempty subset of X^*, by Theorem 1.3.2(a). Suppose that there exist distinct elements f, g in $\mathcal{J}(x_0)$. Since $x_0 \in S(X)$, we have

$$f(x_0) = \|x_0\|^2 = 1 \text{ and } \|f\| = \|x_0\| = 1,$$

and

$$g(x_0) = \|x_0\|^2 = 1 \text{ and } \|g\| = \|x_0\| = 1.$$

Consequently, $f(x_0) = 1 = g(x_0)$, *and* $f, g \in S(X^*)$, which contradicts (a).

(b) $\Rightarrow$ (a): Conversely, assume that x_0 is not a point of smoothness of $B(X)$. Then there exist distinct elements $f, g \in S(X^*)$ such that $f(x_0)$=1 and $g(x_0)$=1. Since $x_0 \in S(X)$, this yields that $f(x_0) = \|x_0\| = g(x_0)$. Put $f_1 = \|x_0\| f$ and $g_1 = \|x_0\| g$. Then $f_1,\ g_1 \in X^*$, $f_1 \neq g_1$, $f_1(x_0) = \|x_0\| f(x_0) = \|x_0\|^2$, $\|f_1\| = \|x_0\|$, $g_1(x_0) = \|x_0\| g(x_0) = \|x_0\|^2$ and $\|g_1\| = \|x_0\|$. Thus $f_1, g_1 \in \mathcal{J}(x_0)$ with $f_1 \neq g_1$, which contradicts (b).

Hence (a) and (b) are equivalent. □

As a direct consequence of the above theorem, we have the following corollary.

Corollary 1.5.3. *Let* $(X, \|\cdot\|)$ *be a normed space. Then the following are equivalent:*

(a) *The space* X *is smooth;*
(b) *For each* $x \in S(X)$*,* $\mathcal{J}(x)$ *is a singleton set in* X^**.*

Proof. The proof follows from Theorem 1.5.2 and Definition 1.5.1 of smoothness of a normed space. □

The next result provides a necessary and sufficient condition for the existence of a unique semi-inner product on a normed space that generates the norm. It is derived from Theorem 1.4.1 and the above corollary.

Theorem 1.5.4. *Let $(X, \|\cdot\|)$ be a normed space. Then the following statements are equivalent:*

(a) *The space X is smooth;*
(b) *There exists a unique semi-inner product on X that generates the norm $\|\cdot\|$ of X.*

Proof. $(a) \Rightarrow (b)$: Suppose that X is smooth. Then by Corollary 1.5.3, $\mathcal{J}(x)$ is a singleton set in X^* for each $x \in S(X)$, and hence for each $x \in X$ also. This implies the existence of a unique section of $\mathcal{J}$, and accordingly, by Theorem 1.4.1, there exists a unique semi-inner product on X that generates the norm $\|\cdot\|$.

$(b) \Rightarrow (a)$: Conversely, assume that there is a unique semi-inner product on X which generates the norm $\|\cdot\|$. Then, by Theorem 1.4.1, $\mathcal{J}$ has a unique section. This shows that $\mathcal{J}(x)$ is a singleton set in X^* for each $x \in X$, and in particular, for each $x \in S(X)$. Hence, by Corollary 1.5.3, the space X is smooth. □

Now, for the sake of completeness, let us briefly mention some more conditions which are necessary and sufficient for the existence of a unique semi-inner product on a normed space that generates the norm.

Just as there is a connection between the smoothness of the graph of a real valued function of a real variable and the differentiability of a function, so is there a connection between the smoothness of a normed space and the Gateaux differentiability of the norm [20].

Definition 1.5.5. Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ of X is said to be *Gateaux differentiable*, if

$$\lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x in the unit sphere $S(X)$ of X and each y in X.

The following well known result [20] establishes the exact connection between the smoothness of a normed space and the Gateaux differentiability of its norm.

Theorem 1.5.6. *Let $(X, \|\cdot\|)$ be a normed space. Then X is smooth if and only if its norm $\|\cdot\|$ is Gateaux differentiable.*

J. R. Giles [25] has proposed the notion of a continuous semi-inner product on a linear space by imposing a continuity property on the right hand member of the semi-inner product, using which the Gateaux differentiability of the norm can be characterized. $\mathrm{Re}\,[x|y]$ denotes the real part of $[x|y]$.

Definition 1.5.7. Let X be a linear space and $[\cdot|\cdot]$ a semi-inner product on X. The semi-inner product $[\cdot|\cdot]$ is said to be *continuous* if

$$\lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \mathrm{Re}\,[x|y+tx] = \mathrm{Re}\,[x|y]$$

for every $x, y \in S(X)$.

In the setting of a normed space, S. S. Dragomir [15] has provided a characterization of continuous semi-inner products in terms of Gateaux differentiability of the norm as follows.

Theorem 1.5.8. *Let $(X, \|\cdot\|)$ be a normed space, and let $[\cdot|\cdot]$ be a semi-inner product on X which generates the norm $\|\cdot\|$. Then the semi-inner product $[\cdot|\cdot]$ is continuous if and only if the norm $\|\cdot\|$ is Gateaux differentiable.*

As a result of consolidating Theorems 1.5.4, 1.5.6 and 1.5.8, we have the following characterization result on the uniqueness of semi-inner products on a normed space that generate the norm.

Theorem 1.5.9. *Let $(X, \|\cdot\|)$ be a normed space, and let $[\cdot|\cdot]$ be a semi-inner product on X which generates the norm $\|\cdot\|$. Then the following statements are equivalent:*

(a) *The semi-inner product $[\cdot|\cdot]$ is unique;*
(b) *The semi-inner product $[\cdot|\cdot]$ is continuous;*
(c) *The norm $\|\cdot\|$ is Gateaux differentiable;*
(d) *The space X is smooth.*

Having seen above some conditions for the existence of a unique semi-inner product on a normed space, let us now consider semi-inner products particularly on inner product spaces. It is clear that every inner product is a semi-inner product. G. Lumer [17] has shown that there exists a unique semi-inner product on a Hilbert space, and a semi-inner product is an inner product if and only if the norm it induces verifies the parallelogram law.

As mentioned at the beginning of this chapter, the concept of semi-inner product has made great progress since its introduction by G. Lumer [17]. In the setting of a normed space, a semi-inner product provides a sufficient structure as well as new techniques for obtaining some nontrivial general results. It plays an important role in the theory of accretive operators and dissipative operators, differential equations, linear and nonlinear semigroups in Banach spaces, and Banach space geometry (e.g., V. Barbu [1], B. Beauzamy [2], G. Lumer and R. S. Phillips [18] and so on). To us, this concept is important from the view point of best approximation theory. We employ this concept in the characterization of best approximations in normed spaces. Our main concern is to derive some results characterizing best approximations in the framework of a general normed space through a semi-inner product that generates the norm of the space. An attempt to this effect is made in the following chapters.

CHAPTER 2

Characterization of Best Approximations

2.1 Introduction

This chapter is devoted to characterizing best approximations in normed spaces. In a normed space, generally it is through the norm of the space that best approximations are characterized. Here we take a different approach. Instead of the norm, it is a semi-inner product which generates the norm of the space, that is employed in characterizing best approximations. This method enables us to derive some characterizations of best approximations in normed spaces which are entirely new.

We begin this chapter with a brief discussion on best approximation in normed spaces. In our attempt to characterize best approximations, we first of all derive a result characterizing best approximations from convex sets using the defining properties of semi-inner products and convexity of sets. The notions of dual cone and orthogonal complement of a set are then introduced, and some of their basic properties are discussed. From a reformulated version of the characterization result for convex sets in terms of dual cones, characterizations of best approximations from convex cones and subspaces are arrived at. Characterizing best approximations from translates of convex cones and subspaces are also considered in this chapter.

2.2 Best Approximation in Normed Spaces

It is well known that the problem of best approximation of a function consists of the determination of a function belonging to a fixed family such that its deviation from the given function is a minimum. This problem was first formulated in 1853 by P. L. Chebyshev, who investigated the approximation of continuous functions by algebraic polynomials of given degree. As a measure of the deviation between two functions, he used the maximum of the absolute values of their difference. Subsequently, a number of mathematicians have started studying other specialized problems of best approximation. With the development of the theory of normed spaces, it became clear that a wide range of problems of best approximation can be put into a general formulation in terms of normed spaces, if the norm of the space is taken as the measure of deviation. This formulation made possible the application of the methods and ideas of functional analysis and geometry to the problems of approximation theory.

The foundations of the theory of best approximation in normed spaces were established in 1920's by one of the founders of functional analysis, S. Banach. During 1930-1950, the ideas of Banach were developed and systematized by the mathematicians like S. M. Nicolescu, M. G. Krein, N. I. Achiezer, A. I. Markushevich, J. L. Walsh and A. N. Kolmogorov.

The problem of best approximation in a normed space can be formulated as follows: Let $(X, \|\cdot\|)$ be a normed space over the real or complex number field $\mathbb{K}$, G a nonempty set in X, and $x \in X$. Then the distance of x from G, $d(x, G)$, is given by

$$d(x, G) := \inf \{\|x - g\| : g \in G\}.$$

The problem of best approximation consists of finding an element $g_0 \in G$ such that

$$\|x - g_0\| = d(x, G).$$

Every element $g_0 \in G$ with this property is called a best approximation of x from

G. G is called the approximating set, and x the approximated point.

Definitions 2.2.1. Let $(X, \|\cdot\|)$ be a normed space over the real or complex number field $\mathbb{K}$, G a nonempty subset of X, and $x \in X$. An element $g_0 \in G$ is called a *best approximation* (or *element of best approximation* or *nearest point*) of x from G, if $\|x - g_0\| = d(x, G)$. In this case, the number $d(x, G)$ is called the *error of approximation* (or the *error in approximating* x by G).

An element $g_0 \in G$ is a best approximation of x from G if and only if $\|x - g_0\| \leq \|x - g\|$ for every $g \in G$. The set of all best approximations of x from G is denoted by $P_G(x)$ Thus

$$P_G(x) := \{g_0 \in G : \|x - g_0\| = d(x, G)\}.$$

This defines a mapping $P_G : X \to \mathcal{P}(G)$, where $\mathcal{P}(G)$ is the power set of G. The set valued mapping P_G is called the *metric projection* (or *nearest point mapping* or *proximity map*) onto G.

If each $x \in X$ has at least (respectively at most) one best approximation from G, then G is called a *proximinal* (respectively *semi Chebyshev*) *set*. If each $x \in X$ has exactly one best approximation from G, then G is called a *Chebyshev set*. Thus, G is proximinal (respectively semi Chebyshev, Chebyshev) if $P_G(x)$ is nonempty (respectively $P_G(x)$ is either empty or a singleton, $P_G(x)$ is a singleton) for each $x \in X$. It is obvious that a Chebyshev set is proximinal as well as semi Chebyshev. If G is a Chebyshev set in X, then the metric projection P_G is a single valued mapping of X onto G, and in this case, P_G is called the *Chebyshev map* (or *best approximation operator*) onto G.

The general theory of best approximation may be briefly outlined as follows: It is the mathematical study that is motivated by the desire to seek answers to the following basic questions, among others.

1. (Existence of best approximations) Which subsets are proximinal?
2. (Uniqueness of best approximations) Which subsets are Chebyshev?
3. (Characterization of best approximations) How does one recognize when a given element $g \in G$ is a best approximation of x from G?

4. (Error of approximation) How does one compute the error of approximation $d(x, G)$, or at least get sharp upper or lower bounds for it?

5. (Computation of best approximations) Can one describe some useful algorithms for actually computing best approximations?

6. (Continuity of best approximations) How does $P_G(x)$ vary as a function of x (or G)?

Many mathematicians were attracted by these questions and have made their contributions to the theory of best approximation. As a result, a pretty large collection of materials including Textbooks, Treaties, Monographs and Papers is there in the literature (e.g., A. L. Brown [6], P. L. Butzer and R. J. Nessel [7], W. Cheney and W. Light [10], E. W. Cheney and K. H. Price [8, 9], E. W. Cheney and P. D. Morris [11], F. Deutsch [14], R. A. DeVore [32], R. A. DeVore and G. G. Lorentz [33], W. O. G. Lewicki [16], H. N. Mhaskar and D. V. Pai [21], M. J. D. Powell [24], T. J. Rivlin [26], H. S. Shapiro [28], I. Singer [29, 30], G. A. Watson [34], R. Zielke [35] and so on).

Among the six questions mentioned above, the question which is of particular interest to us is the third one, that is, the characterization of best approximations. We understand that the literature is rich with results characterizing best approximations, and that such results are separately available for general normed spaces and inner product spaces (e.g., H. N. Mhaskar and D. V. Pai [21], I. Singer [29,30], H. Berens [4], F. Deutsch [14] and so on). Generally, characterizations of best approximations in a normed space are derived through the norm of the space, and those in an inner product space, through the inner product of the space. Here we take a different approach. Our endeavor is to characterize best approximations in a general normed space, not through the norm of the space, but through a semi-inner product that generates the norm of the space. Our attention here is to derive some results characterizing best approximations in the framework of a general normed space using some semi-inner product techniques.

We recall from Chapter 1 that given a normed space $(X, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$, there always exists a semi-inner product (see Definition 1.4.3) on it which generates the norm $\|\cdot\|$. This idea is employed here in characterizing best approximations in normed spaces from convex sets, and in particular, from convex cones and subspaces.

Since, for every nonempty subset G of a normed space X, we have

$$P_G(x) = \begin{cases} x & \text{if } x \in G \\ \emptyset & \text{if } x \in \overline{G} \setminus G, \end{cases}$$

it is sufficient to characterize the best approximations of the elements $x \in X \backslash \overline{G}$. In order to exclude the trivial case when such elements x do not exist, throughout the discussion, our approximating sets G, whether convex sets, convex cones or subspaces, as the case may be, are always assumed to be proper and nondense in the normed space X. However, we will not be making any special mention to these effects in the sequel.

Our further discussion is restricted to the setting of real normed spaces. Henceforth in this chapter, by a normed space X we mean a real normed space $(X, \|\cdot\|)$ together with a semi-inner product $[\cdot|\cdot]$ which generates the norm $\|\cdot\|$.

Since the theory of best approximation is the most well developed when the approximating set is a subspace, or more generally a convex set, we begin our study with the characterization of best approximations from convex sets.

2.3 Characterization from Convex Sets

The sole aim of this section is to present a characterization theorem for best approximations from convex sets, and to reformulate it in terms of dual cones of sets. This result will prove useful over and over again throughout our discussion. Indeed, it will be the basis for every characterization theorem that we provide.

We begin our discussion with a sufficient condition for best approximations from arbitrary sets. Recall that $P_G(x)$ denotes the set of all best approximations of x from G.

Theorem 2.3.1. *Let X be a normed space, G a subset of X, $x \in X$, and $y_0 \in G$. If*

$$[y - y_0 | x - y_0] \leq 0 \ \ \textit{for all} \ \ y \in G, \tag{2.1}$$

then $y_0 \in P_G(x)$.

Proof. Suppose that (2.1) holds. Then for all $y \in G$, we have

$$\begin{aligned} \|x - y_0\|^2 &= [x - y_0 | x - y_0]\,, \ \ \text{by } (\mathrm{S}_3) \\ &= [x - y | x - y_0] + [y - y_0 | x - y_0]\,, \ \ \text{by } (\mathrm{S}_1) \\ &\leq [x - y | x - y_0]\,, \ \ \text{by (2.1)} \\ &\leq \|x - y\| \, \|x - y_0\|\,, \ \ \text{by } (\mathrm{S}_4)\,. \end{aligned}$$

Hence $\|x - y_0\| \leq \|x - y\|$ for all $y \in G$, and so $y_0 \in P_G(x)$. □

When the approximating set is in particular a convex set, we have the following necessary condition for best approximations.

Theorem 2.3.2. *Let X be a normed space, K a convex subset of X, $x \in X$, and $y_0 \in K$. If $y_0 \in P_K(x)$, then*

$$[y - y_0 | x - y_0 - \lambda\,(y - y_0)] \leq 0 \ \ \textit{for all} \ \ y \in K \ \ \textit{and all} \ \lambda \in [0, 1]. \tag{2.2}$$

Proof. If (2.2) fails, then for some $y \in K$ and some $\lambda \in [0, 1]$, we have,

$$[y - y_0 | x - y_0 - \lambda\,(y - y_0)] > 0. \tag{2.3}$$

(Here $y \neq y_0$, since $y = y_0$ implies $[y - y_0 | x - y_0 - \lambda\,(y - y_0)] = 0$.) For these elements $y \in K$ and $\lambda \in [0, 1]$, the element $y_\lambda := \lambda y + (1 - \lambda)\, y_0 \in K$, by convexity of K. We have then

$$\begin{aligned} \|x - y_\lambda\| &= \|x - \lambda y - (1 - \lambda)\, y_0\| \\ &= \frac{[x - y_0 - \lambda\,(y - y_0)\,| x - y_0 - \lambda\,(y - y_0)]}{\|x - y_0 - \lambda\,(y - y_0)\|}, \ \ \text{by } (\mathrm{S}_3) \end{aligned}$$

$$= \frac{[x-y_0|x-y_0-\lambda(y-y_0)] - \lambda[(y-y_0)|x-y_0-\lambda(y-y_0)]}{\|x-y_0-\lambda(y-y_0)\|}, \quad \text{by } (\mathrm{S}_1)$$

$$< \frac{[x-y_0|x-y_0-\lambda(y-y_0)]}{\|x-y_0-\lambda(y-y_0)\|}, \quad \text{by } (2.3)$$

$$\leq \|x-y_0\|, \quad \text{by } (\mathrm{S}_4).$$

This shows that there is a $y_\lambda \in K$ such that $\|x-y_\lambda\| < \|x-y_0\|$, and so $y_0 \notin P_K(x)$. Hence (2.2) holds whenever $y_0 \in P_K(x)$. □

Combining Theorem 2.3.1 and Theorem 2.3.2, we have the following result which characterizes best approximations from convex sets.

Theorem 2.3.3. *Let X be a normed space, K a convex subset of X, $x \in X$, and $y_0 \in K$. Then the following statements are equivalent:*

(a) $y_0 \in P_K(x)$;
(b) $[(y-y_0)|x-y_0-\lambda(y-y_0)] \leq 0$ *for all* $y \in K$ *and all* $\lambda \in [0,1]$;
(c) $[y-y_0|x-y_0] \leq 0$ *for all* $y \in K$.

Proof. (a) $\Rightarrow$ (b) follows by Theorem 2.3.2, (b) $\Rightarrow$ (c) follows by taking $\lambda = 0$ in (b), and (c) $\Rightarrow$ (a) follows by Theorem 2.3.1. □

There is yet another way of stating the above characterization result. It involves the notion of the dual cone of a given set. This notion has been introduced in the framework of an inner product space in terms of the inner product of the space [14]. We extend this to the setting of a normed space in terms of a semi-inner product that generates the norm of the space.

Definition 2.3.4. Let X be a normed space, and G a nonempty subset of X. Then the set $\{x \in X : [y|x] \leq 0 \text{ for all } y \in G\}$ is called the *dual cone* (or *dual cone relative to the semi-inner product* $[\cdot|\cdot]$, or *negative polar relative to the semi-inner product* $[\cdot|\cdot]$) of G, denoted by G°.

By the definition, for every nonempty subset G of X, $0 \in G^\circ$ and $G \cap G^\circ$ is either empty or $\{0\}$.

Theorem 2.3.3 characterizing best approximations from convex sets can now be reformulated using dual cones as follows.

Theorem 2.3.5. *Let X be a normed space, K a convex subset of X, $x \in X$, and $y_0 \in K$. Then the following statements are equivalent:*

(i) $y_0 \in P_K(x)$;
(ii) $x - y_0 - \lambda (y - y_0) \in (K - y_0)^\circ$ *for all* $y \in K$ *and all* $\lambda \in [0, 1]$;
(iii) $x - y_0 \in (K - y_0)^\circ$.

Proof. From the equivalence of statements (a) and (b) of Theorem 2.3.3 and by the definition of dual cone of a set (Definition 2.3.4), we have

$$\begin{aligned} y_0 \in P_K(x) \;&\Leftrightarrow\; [y - y_0 | x - y_0 - \lambda (y - y_0)] \le 0 \;\text{ for all }\; y \in K \;\text{ and} \\ &\qquad \text{all } \lambda \in [0, 1] \\ &\Leftrightarrow\; x - y_0 - \lambda (y - y_0) \in \{y - y_0 : y \in K\}^\circ \;\text{ for all }\; y \in K \\ &\qquad \text{and all } \lambda \in [0, 1] \\ &\Leftrightarrow\; x - y_0 - \lambda (y - y_0) \in (K - y_0)^\circ \;\text{ for all }\; y \in K \;\text{ and} \\ &\qquad \text{all } \lambda \in [0, 1] \,. \end{aligned}$$

Hence (i) $\Leftrightarrow$ (ii).

Similarly, from the equivalence of statements (a) and (c) of Theorem 2.3.3, and by the definition of dual cone of a set, we have

$$\begin{aligned} y_0 \in P_K(x) \;&\Leftrightarrow\; [y - y_0 | x - y_0] \le 0 \quad \text{for all }\; y \in K \\ &\Leftrightarrow\; x - y_0 \in \{y - y_0 : y \in K\}^\circ \\ &\Leftrightarrow\; x - y_0 \in (K - y_0)^\circ \,. \end{aligned}$$

Hence (i) $\Leftrightarrow$ (iii), and this completes the proof. □

The above theorem shows that the characterization of best approximations requires, in essence, the determination of dual cones. For certain convex sets (e.g., convex cones and subspaces), substantial improvements in Theorem 2.3.5 are possible.

Definition 2.3.6. Let X be a normed space. A subset G of X is called a *convex cone* if $\alpha x + \beta y \in G$ whenever $x, y \in G$ and $\alpha, \beta \geq 0$.

We make the following observations:

1. If C is a convex cone, then $0 \in C$ and $C \cap C^\circ = \{0\}$.

2. Every subspace is a convex cone, but not conversely. Similarly, every convex cone is a convex set, but not conversely. For instance, in the real normed space $X = (\mathbb{R}^2, \|\cdot\|_2)$, where $\|x\|_2 = \left(\sum_{k=1}^{2} |x_k|^2\right)^{1/2}$ for $x = (x_1, x_2) \in X$, the set $\{x = (x_1, x_2) \in X : x_1 \geq 0,\ x_2 \geq 0\}$ is a convex cone which is not a subspace, and the closed unit ball $\{x \in X : \|x\|_2 \leq 1\}$ is a convex set which is not a convex cone.

3. Generally, the dual cone of a nonempty set is not a convex cone, and the same is the case with that of a convex cone also. For example, consider the real normed space $X = (R^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{2} |x_k|$ for $x = (x_1, x_2) \in X$, together with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{2} \frac{x_k y_k}{|y_k|}, \quad x, y \in X$$

that generates the norm $\|\cdot\|_1$. The set

$$C = \{x = (\lambda, \lambda) \in X : \lambda \geq 0\}$$

is a convex cone in X whose dual cone is given by

$$C^\circ = \left\{ x = (x_1, x_2) \in X : \lambda \|x\|_1 \sum_{\substack{k=1 \\ x_k \neq 0}}^{2} \frac{x_k}{|x_k|} \leq 0 \ \text{ for all } \lambda \geq 0 \right\}.$$

Then the elements $x = (5, -4)$ and $z = (-3, 5)$ are in C°. But $x + z = (2, 1) \notin C^\circ$. This shows that C° is not a convex cone.

2.4 Orthogonality relative to Semi-Inner Products

In this section we introduce the notion of orthogonality in a normed space in terms of a semi-inner product that generates the norm of the space [15]. This will enable us to derive some basic properties of dual cones that are needed for improving Theorem 2.3.5.

The notion of orthogonality in a normed space which we discuss here is a generalization of the orthogonality concept in an inner product space. Recall that two elements x, y in an inner product space $(X, (\cdot,\cdot))$ are said to be orthogonal if $(x, y) = 0$. Since a semi-inner product lacks conjugate symmetry, a property which an inner product possesses, and since it is in terms of a semi-inner product which generates the norm of the space that we introduce orthogonality here, this notion of orthogonality in a normed space is not generally symmetric. Unless specified otherwise, by orthogonality in a normed space we always mean this orthogonality, as defined below.

Definition 2.4.1. Let X be a normed space, and $x, y \in X$. Then x is said to be *orthogonal* (or *orthogonal in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$) to y, denoted by $x \perp y$, if $[y|x]=0$.

We observe that, if $x, y, z \in X$, and α is any scalar, then

(i) $0 \perp x$ and $x \perp 0$,
(ii) $x \perp x$ if and only if $x = 0$,
(iii) $x \perp y$ and $x \perp z$ imply that $x \perp (y+z)$, and
(iv) $x \perp y$ implies that $(\alpha x) \perp y$ and $x \perp (\alpha y)$.

However, as we have mentioned above, the main difference of this orthogonality concept in normed spaces in comparison with the orthogonality concept in inner product spaces is with regard to symmetry: If $x, y \in X$, then $x \perp y$ need not imply that $y \perp x$.

For example, consider the real normed space $X = (\mathbb{R}^3, \|\cdot\|_1)$, where $\|x\|_1 =$

$\sum_{k=1}^{3} |x_k|$ for $x = (x_1, x_2, x_3) \in X$, equipped with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{3} \frac{x_k y_k}{|y_k|}, \quad x, y \in X$$

that generates the norm $\|\cdot\|_1$. Consider the elements $x = (-2, 1, 0)$ and $y = (1, 1, 0)$ in X. Then $[y|x] = 0$ so that x is orthogonal to y, whereas $[x|y] = -2$ so that y is not orthogonal to x.

Definition 2.4.2. Let X be a normed space, G a nonempty subset of X, and $x \in X$. Then x is said to be *orthogonal* (or *orthogonal in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$) to G, denoted by $x \perp G$, if $x \perp y$ for all $y \in G$.

For every nonempty subset G of X, by the definition, $0 \perp G$.

Definitions 2.4.3. let X be a normed space, and G a nonempty subset of X. Then the set $\{x \in X : x \perp G\}$ is called the *orthogonal complement* (or *orthogonal complement in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$) of G, denoted by $G^{\perp}$.

If $y \in X$, the *orthogonal complement* (or *orthogonal complement in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$) of y, denoted by $y^{\perp}$, is the set $\{x \in X : x \perp y\}$.

We have

$$\begin{aligned} G^{\perp} &= \{x \in X : x \perp G\} \\ &= \{x \in X : x \perp y \ \text{ for all } y \in G\} \\ &= \bigcap_{y \in G} \{x \in X : x \perp y\} \\ &= \bigcap_{y \in G} y^{\perp}. \end{aligned}$$

The following are some easy consequences of these definitions:

1. $0^{\perp} = X$, and $X^{\perp} = \{0\}$.

2. If G is any nonempty subset of X, and α is any scalar, then,

   (a) $0 \in G^{\perp}$,
   (b) $x \in G^{\perp}$ implies that $\alpha x \in G^{\perp}$,
   (c) $G^{\perp} \subseteq G^{\circ}$, the dual cone of G, and
   (d) $G \cap G^{\perp}$ is either empty or $\{0\}$.

3. If C is a convex cone in X, we also have $C \cap C^{\perp}=\{0\}$. In particular, $M \cap M^{\perp}=\{0\}$ for every subspace M of X.

4. More importantly, even if M is a subspace of X, $M^{\perp}$ need not be a subspace of X. $M^{\perp}$ is not even a convex cone in X. For example, consider the real normed space $X = (R^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{2} |x_k|$ for $x = (x_1, x_2) \in X$, together with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{2} \frac{x_k y_k}{|y_k|}, \quad x, y \in X$$

   that generates the norm $\|\cdot\|_1$. Let $M = \operatorname{span}\{(1,1)\}$. The orthogonal complement of this subspace is given by

$$M^{\perp} = \left\{ x = (x_1, x_2) \in X : \lambda \|x\|_1 \sum_{\substack{k=1 \\ x_k \neq 0}}^{2} \frac{x_k}{|x_k|} = 0 \text{ for all } \lambda \in \mathbb{R} \right\}.$$

   Then the elements $x = (-3, 5)$ and $z = (4, -3)$ are in $M^{\perp}$. But $x + z = (1, 2) \notin M^{\perp}$. This shows that $M^{\perp}$ is not a convex cone.

Among these observations, the one which is mentioned last is the crucial difference in comparison with the usual orthogonal complements in inner product spaces.

The exact relationship between the dual cone and the orthogonal complement of a given set is provided by the following result.

Theorem 2.4.4. *Let X be a normed space, and G a nonempty subset of X. Then $G^{\perp} = G^{\circ} \cap (-G)^{\circ} = G^{\circ} \cap (-G^{\circ})$.*

Proof. We have

$$\begin{aligned}
G^{\perp} &= \{x \in X : x \perp G\} \\
&= \{x \in X : [y|x] = 0 \quad \text{for all } \ y \in G\} \\
&= G^{\circ} \cap \{x \in X : [y|x] \geq 0 \quad \text{for all } \ y \in G\} \\
&= G^{\circ} \cap \{x \in X : [-y|x] \geq 0 \quad \text{for all } \ -y \in G\} \\
&= G^{\circ} \cap \{x \in X : -[y|x] \geq 0 \quad \text{for all } \ y \in (-G)\} \\
&= G^{\circ} \cap \{x \in X : [y|x] \leq 0 \quad \text{for all } \ y \in (-G)\} \\
&= G^{\circ} \cap (-G)^{\circ}.
\end{aligned}$$

Further,

$$\begin{aligned}
(-G)^{\circ} &= \{x \in X : [y|x] \leq 0 \quad \text{for all } \ y \in (-G)\} \\
&= \{x \in X : [y|x] \leq 0 \quad \text{for all } \ -y \in G\} \\
&= \{x \in X : [-y|x] \leq 0 \quad \text{for all } \ y \in G\} \\
&= \{x \in X : [y| - x] \leq 0 \quad \text{for all } \ y \in G\} \\
&= -\{-x \in X : [y| - x] \leq 0 \quad \text{for all } \ y \in G\} \\
&= -G^{\circ}.
\end{aligned}$$

Hence $G^{\perp} = G^{\circ} \cap (-G)^{\circ} = G^{\circ} \cap (-G^{\circ})$. □

Next we consider some basic properties of dual cones and orthogonal complements. The following result will help us in improving Theorem 2.3.5.

Theorem 2.4.5. *Let X be a normed space.*

(a) *If C is a convex cone in X, then $(C - y)^{\circ} = C^{\circ} \cap y^{\perp}$ for every $y \in C$.*

(b) *If M is a subspace of X, then $M^{\circ} = M^{\perp}$.*

Proof. (a) Let $y \in C$ be arbitrary. If $x \in (C-y)^{\circ}$, then $[c - y|x] \leq 0$ for all $c \in C$. This implies, on letting $c = c + y$ that $[c|x] = [c + y - y|x] \leq 0$ for all $c \in C$, so that $x \in C^{\circ}$. Again, $[c - y|x] \leq 0$ for all $c \in C$ implies, on taking $c = 2y$ and $c = 0$, that $[y|x] \leq 0$ and $[y|x] \geq 0$ respectively. Thus $[y|x] = 0$, so that $x \in y^{\perp}$. Combining the above two conclusions we see that, $x \in (C - y)^{\circ}$ implies $x \in C^{\circ}$ and $x \in y^{\perp}$, so that $x \in C^{\circ} \cap y^{\perp}$.

On the other hand, if $x \in C^\circ \cap y^\perp$, then $[c|x] \leq 0$ for all $c \in C$ and $[y|x] = 0$. Consequently, $[c - y|x]=[c|x] - [y|x] \leq 0$ for all $c \in C$, so that $x \in (C - y)^\circ$. Hence $(C - y)^\circ = C^\circ \cap y^\perp$ for every $y \in C$.

(b) If M is a subspace of X, then $-M = M$, and hence by Theorem 2.4.4, $M^\perp = M^\circ \cap (-M)^\circ = M^\circ$. This completes the proof. $\square$

Another result which we will make use of in our further discussion is given below. By $\overline{G}$ we denote the closure of a set G under the norm.

Theorem 2.4.6. *Let X be a normed space, and G a nonempty subset of X. Then*

(a) $G^\circ = (\overline{G})^\circ$, *and*
(b) $G^\perp = (\overline{G})^\perp$.

Proof. (a) Let $x \in (\overline{G})^\circ$. Then for all $y \in \overline{G}$, we have $[y|x] \leq 0$. This shows, since $G \subseteq \overline{G}$, that $[y|x] \leq 0$ for all $y \in G$, so that $x \in G^\circ$. Thus $(\overline{G})^\circ \subseteq G^\circ$. Now let $x \in G^\circ$. If $y \in \overline{G}$, choose a sequence (y_n) in G such that $y_n \to y$. Then

$$|[y_n|x] - [y|x]| = |[y_n - y|x]| \leq \|y_n - y\| \, \|x\| \to 0 \text{ as } n \to \infty,$$

so that

$$[y|x] = \lim_{n\to\infty} [y_n|x] \leq 0.$$

Therefore $[y|x] \leq 0$ for all $y \in \overline{G}$, so that $x \in (\overline{G})^\circ$. Thus $G^\circ \subseteq (\overline{G})^\circ$. Hence $G^\circ = (\overline{G})^\circ$.

(b) An argument similar to the above shows that $G^\perp = (\overline{G})^\perp$. $\square$

The following result, which is of independent interest, contains some more properties of dual cones and orthogonal complements. We recall that the *sum* of a finite collection of nonempty sets $\{G_1, G_2, ..., G_n\}$ in a normed space X, denoted by $G_1+G_2+...+G_n$ or $\sum_{i=1}^{n} G_i$, is defined as the set $\left\{\sum_{i=1}^{n} x_i : x_i \in G_i \text{ for every } i\right\}$. Thus

$$\sum_{i=1}^{n} G_i := \left\{\sum_{i=1}^{n} x_i : x_i \in G_i \text{ for every } i\right\}.$$

Theorem 2.4.7. *Let X be a normed space, and $\{G_1, G_2, ..., G_n\}$ a finite collection of nonempty sets in X. Then*

(a) $\left(\bigcup\limits_{i=1}^{n} G_i\right)^{\circ} = \bigcap\limits_{i=1}^{n} G_i^{\circ}$ and $\left(\bigcup\limits_{i=1}^{n} G_i\right)^{\perp} = \bigcap\limits_{i=1}^{n} G_i^{\perp}$.

(b) If, in addition $0 \in \bigcap\limits_{i=1}^{n} G_i$, then
$\left(\bigcup\limits_{i=1}^{n} G_i\right)^{\circ} = \left(\sum\limits_{i=1}^{n} G_i\right)^{\circ}$ and $\left(\bigcup\limits_{i=1}^{n} G_i\right)^{\perp} = \left(\sum\limits_{i=1}^{n} G_i\right)^{\perp}$.

Proof. (a) We have

$$\begin{aligned} x \in \bigcap_{i=1}^{n} G_i^{\circ} &\Leftrightarrow x \in G_i^{\circ} \text{ for each } i \\ &\Leftrightarrow [y|x] \le 0 \text{ for each } y \in G_i \text{ and all } i \\ &\Leftrightarrow [y|x] \le 0 \text{ for all } y \in \bigcup_{i=1}^{n} G_i \\ &\Leftrightarrow x \in \left(\bigcup_{i=1}^{n} G_i\right)^{\circ}. \end{aligned}$$

Hence $\left(\bigcup\limits_{i=1}^{n} G_i\right)^{\circ} = \bigcap\limits_{i=1}^{n} G_i^{\circ}$.

Similarly, $\left(\bigcup\limits_{i=1}^{n} G_i\right)^{\perp} = \bigcap\limits_{i=1}^{n} G_i^{\perp}$.

(b) We have

$$\begin{aligned} x \in \left(\sum_{i=1}^{n} G_i\right)^{\circ} &\Leftrightarrow [y|x] \le 0 \text{ for } \textit{each } y \in \sum_{i=1}^{n} G_i \\ &\Leftrightarrow \left[\sum_{i=1}^{n} y_i|x\right] \le 0 \text{ whenever } y_i \in G_i \\ &\Leftrightarrow [y_i|x] \le 0 \text{ for each } y_i \in G_i \text{ and all } i, \text{ since } 0 \in \bigcap_{i=1}^{n} G_i \\ &\Leftrightarrow x \in G_i^{\circ} \text{ for each } i \end{aligned}$$

$$\Leftrightarrow \quad x \in \bigcap_{i=1}^{n} G_i^{\circ}$$

$$\Leftrightarrow \quad x \in \left(\bigcup_{i=1}^{n} G_i\right)^{\circ}, \quad \text{by (a) above.}$$

Hence $\left(\bigcup_{i=1}^{n} G_i\right)^{\circ} = \left(\sum_{i=1}^{n} G_i\right)^{\circ}$.

Similarly, $\left(\bigcup_{i=1}^{n} G_i\right)^{\perp} = \left(\sum_{i=1}^{n} G_i\right)^{\perp}$. □

2.5 Characterization from Convex Cones and Subspaces

Using the results of the above section, we now proceed to improve Theorem 2.3.5 characterizing best approximations from convex sets. In the particular case when the convex set is a convex cone, this result can be strengthened by using Theorem 2.4.5 (a) as follows.

Theorem 2.5.1. *Let X be a normed space, C a convex cone in X, $x \in X$, and $y_0 \in C$. Then the following statements are equivalent:*

(a) $y_0 \in P_C(x)$;
(b) $[y'|x - y_0 - \lambda(y - y_0)] \leq 0$ *and* $[y_0|x - y_0 - \lambda(y - y_0)] = 0$ *for all* $y', y \in C$ *and all* $\lambda \in [0, 1]$;
(c) $[y|x - y_0] \leq 0$ *and* $[y_0|x - y_0] = 0$ *for all* $y \in C$.

Proof. From the equivalence of statements (i) and (ii) of Theorem 2.3.5, we have

$$\begin{aligned} y_0 \in P_C(x) \quad &\Leftrightarrow \quad x - y_0 - \lambda(y - y_0) \in (C - y_\circ)^{\circ} \text{ for all } y \in C \text{ and all } \lambda \in [0, 1] \\ &\Leftrightarrow \quad x - y_0 - \lambda(y - y_0) \in C^{\circ} \cap y_0^{\perp} \text{ for all } y \in C \text{ and all } \lambda \in [0, 1], \\ &\qquad \text{by Theorem 2.4.5(a)} \\ &\Leftrightarrow \quad x - y_0 - \lambda(y - y_0) \in C^{\circ} \text{ and } x - y_0 - \lambda(y - y_0) \in y_0^{\perp} \text{ for all } y \in C \text{ and all } \lambda \in [0, 1] \end{aligned}$$

$$\Leftrightarrow \quad [y'|x - y_0 - \lambda(y - y_0)] \leq 0 \ \text{ and } \ [y_0|x - y_0 - \lambda(y - y_0)] = 0$$
$$\text{for all } y',\, y \in C \ \text{ and all } \lambda \in [0,1]\,.$$

Hence (a)⇔(b).

Similarly, from the equivalence of statements (i) and (iii) of Theorem 2.3.5, it follows that (a)⇔(c). Hence the theorem. □

Remark 2.5.2. The above theorem can be expressed in a purely set theoretic manner as follows. Under the hypothesis of Theorem 2.5.1, the statements

(a) $y_0 \in P_C(x)$,
(b) $x - y_0 - \lambda(y - y_0) \in C^\circ \cap y_0^\perp$ for all $y \in C$ and all $\lambda \in [0,1]$, and
(c) $x - y_0 \in C^\circ \cap y_0^\perp$.

are equivalent.

There is an even simpler characterization of best approximations when the convex set is actually a subspace. It is derived again from Theorem 2.3.5 with the aid of Theorem 2.4.5 (b).

Theorem 2.5.3. *Let X be a normed space, M a subspace of X, $x \in X$, and $y_0 \in M$. Then the following statements are equivalent:*

(a) $y_0 \in P_M(x)$;
(b) $[y'|x - y_0 - \lambda(y - y_0)] = 0$ *for all* $y',\, y \in M$ *and all* $\lambda \in [0,1]$;
(c) $[y|x - y_0] = 0$ *for all* $y \in M$.

Proof. From the equivalence (i)⇔(ii) of Theorem 2.3.5, we have

$$\begin{aligned} y_0 \in P_M(x) \quad &\Leftrightarrow \quad x - y_0 - \lambda(y - y_0) \in (M - y_0)^\circ \ \text{ for all } y \in M \text{ and} \\ &\qquad \text{all } \lambda \in [0,1] \\ &\Leftrightarrow \quad x - y_0 - \lambda(y - y_0) \in M^\circ \ \text{ for all } y \in M \text{ and all } \lambda \in [0,1], \\ &\qquad \text{since } y_0 \in M \\ &\Leftrightarrow \quad x - y_0 - \lambda(y - y_0) \in M^\perp \ \text{ for all } y \in M \text{ and all } \lambda \in [0,1]\,, \\ &\qquad \text{by Theorem 2.4.5(b)} \end{aligned}$$

$$\Leftrightarrow \quad [y'|x - y_0 - \lambda(y - y_0)] = 0 \quad \text{for all } y',\, y \in M \text{ and all } \lambda \in [0,1]\,.$$

Hence (a)$\Leftrightarrow$(b).

Similarly, from the equivalence of statements (i) and (iii) of Theorem 2.3.5, it follows that (a)$\Leftrightarrow$(c), and this completes the proof. □

Remark 2.5.4. The following is the restatement of the above theorem in terms of orthogonal complements. Under the hypothesis of Theorem 2.5.3, the statements

(a) $y_0 \in P_M(x)$,
(b) $x - y_0 - \lambda(y - y_0) \in M^\perp$ for all $y \in M$ and all $\lambda \in [0,1]$, and
(c) $x - y_0 \in M^\perp$

are equivalent.

Theorem 2.3.5 has some more consequences. It can be employed in deriving results characterizing best approximations from translates of convex cones and subspaces also.

The following result is for translates of convex cones.

Theorem 2.5.5. *Let X be a normed space, C a convex cone in X, $z \in X$, and $K = z + C$. Suppose also that $x \in X$ and $y_0 \in K$. Then the following statements are equivalent:*

(a) $y_0 \in P_K(x)$;
(b) $[y'|x - y_0 - \lambda(z + y - y_0)] \le 0$ *and* $[y_0 - z|x - y_0 - \lambda(z + y - y_0)] = 0$ *for all* $y',\, y \in C$ *and all* $\lambda \in [0,1]$;
(c) $[y|x - y_0] \le 0$ *and* $[y_0 - z|x - y_0] = 0$ *for all* $y \in C$.

Proof. Being a translate of the convex cone C in X, $K = z + C$ is a convex set in X. In fact, if $k_1 = z + c_1$ and $k_2 = z + c_2$ are in $K = z + C$, where $c_1, c_2 \in C$, then for every $t \in [0,1]$, we have

$$tk_1 + (1-t)k_2 = tz + (1-t)z + tc_1 + (1-t)c_2 \in z + C = K.$$

Hence by the equivalence of statements (i) and (ii) of Theorem 2.3.5, we have

$$\begin{aligned}
y_0 \in P_K(x) \quad &\Leftrightarrow \quad x - y_0 - \lambda(y' - y_0) \in (K - y_0)^\circ \text{ for all } y' \in K \text{ and} \\
&\qquad \text{all } \lambda \in [0,1] \\
&\Leftrightarrow \quad x - y_0 - \lambda(z + y - y_0) \in (z + C - y_0)^\circ \text{ for all } y \in C \text{ and} \\
&\qquad \text{all } \lambda \in [0,1] \\
&\Leftrightarrow \quad x - y_0 - \lambda(z + y - y_0) \in (C - (y_0 - z))^\circ \text{ for all } y \in C \text{ and} \\
&\qquad \text{all } \lambda \in [0,1] \\
&\Leftrightarrow \quad x - y_0 - \lambda(z + y - y_0) \in C^\circ \cap (y_0 - z)^\perp \text{ for all } y \in C \text{ and} \\
&\qquad \text{all } \lambda \in [0,1]\,, \text{by Theorem 2.4.5(a), since } y_0 - z \in C \\
&\Leftrightarrow \quad x - y_0 - \lambda(z + y - y_0) \in C^\circ \text{ and} \\
&\qquad x - y_0 - \lambda(z + y - y_0) \in (y_0 - z)^\perp \text{ for all } y \in C \text{ and} \\
&\qquad \text{all } \lambda \in [0,1] \\
&\Leftrightarrow \quad [y' | x - y_0 - \lambda(z + y - y_0)] \le 0 \text{ and} \\
&\qquad [y_0 - z | x - y_0 - \lambda(z + y - y_0)] = 0 \text{ for all } y',\, y \in C \text{ and} \\
&\qquad \text{all } \lambda \in [0,1]\,.
\end{aligned}$$

Hence (a)$\Leftrightarrow$(b).

Similarly, by the equivalence of statements (i) and (iii) of Theorem 2.3.5, it follows that (a)$\Leftrightarrow$(c). Hence a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). $\square$

Remark 2.5.6. Purely set theoretically, the above theorem can be stated as follows. Under the hypothesis of Theorem 2.5.5, the statements

(a) $y_0 \in P_K(x)$,
(b) $x - y_0 - \lambda(z + y - y_0) \in C^\circ \cap (y_0 - z)^\perp$ for all $y \in C$ and all $\lambda \in [0,1]$, and
(c) $x - y_0 \in C^\circ \cap (y_0 - z)^\perp$

are equivalent.

The next result characterizes best approximations from translates of subspaces.

Theorem 2.5.7. *Let X be a normed space, M a subspace of X, $z \in X$, and $K = z+M$. Suppose also that $x \in X$ and $y_0 \in K$. Then the following statements are equivalent.*

(a) $y_0 \in P_K(x)$;
(b) $[y'|x - y_0 - \lambda(z + y - y_0)] = 0$ *for all* $y', y \in M$ *and all* $\lambda \in [0, 1]$;
(c) $[y|x - y_0] = 0$ *for all* $y \in M$.

Proof. $K = z + M$, being a translate of the subspace M of X, is a convex set in X. Indeed, if $k_1 = z+m_1$ and $k_2 = z+m_2$ are in $K = z+M$, where $m_1, m_2 \in M$, then for every $t \in [0, 1]$, we have

$$tk_1 + (1 - t)k_2 = tz + (1 - t)z + tm_1 + (1 - t)m_2 \in z + M = K.$$

Hence by the equivalence of statements (i) and (ii) of Theorem 2.3.5, we have

$$\begin{aligned}
y_0 \in P_K(x) \;\; &\Leftrightarrow \;\; x - y_0 - \lambda(y' - y_0) \in (K - y_0)^\circ \;\text{ for all } y' \in K \text{ and} \\
&\quad\;\; \text{all } \lambda \in [0, 1] \\
&\Leftrightarrow \;\; x - y_0 - \lambda(z + y - y_0) \in (z + M - y_0)^\circ \;\text{ for all } y \in M \text{ and} \\
&\quad\;\; \text{all } \lambda \in [0, 1] \\
&\Leftrightarrow \;\; x - y_0 - \lambda(z + y - y_0) \in (M - (y_0 - z))^\circ \;\text{ for all } y \in M \text{ and} \\
&\quad\;\; \text{all } \lambda \in [0, 1] \\
&\Leftrightarrow \;\; x - y_0 - \lambda(z + y - y_0) \in M^\circ \;\text{ for all } y \in M \text{ and all } \lambda \in [0, 1], \\
&\quad\;\; \text{since } y_0 - z \in M \\
&\Leftrightarrow \;\; x - y_0 - \lambda(z + y - y_0) \in M^\perp \;\text{ for all } y \in M \text{ and all } \lambda \in [0, 1], \\
&\quad\;\; \text{by theorem 2.4.5}\,(b) \\
&\Leftrightarrow \;\; [y'|x - y_0 - \lambda(z + y - y_0)] = 0 \;\text{ for all } y', y \in M \text{ and} \\
&\quad\;\; \text{all } \lambda \in [0, 1].
\end{aligned}$$

Hence (a)⇔(b).

Similarly, by the equivalence of statements (i) and (iii) of Theorem 2.3.5, it follows that (a)⇔(c). Hence (a)⇔(b)⇔(c). □

Remark 2.5.8. A restatement of the above theorem in terms of orthogonal complements is given below. Under the hypothesis of Theorem 2.5.7, the statements

(a) $y_0 \in P_K(x)$,
(b) $x - y_0 - \lambda(z + y - y_0) \in M^{\perp}$ for all $y \in M$ and all $\lambda \in [0, 1]$, and
(c) $x - y_0 \in M^{\perp}$

are equivalent.

Remark 2.5.9. Among the four results characterizing best approximations seen so far in this section, namely Theorems 2.5.1, 2.5.3, 2.5.5 and 2.5.7, the one which is in the most general setting is Theorem 2.5.5 characterizing best approximations from translates K of convex cones C by z. We observe that, instead of proving each of these results separately, all of these can be deduced directly from an equivalent version of Theorem 2.5.5 in terms of dual cones, which is obtained from Theorem 2.3.5.

We have already seen some results characterizing best approximations from convex sets, in particular from convex cones, subspaces and their translates in this chapter. We conclude our discussions in this chapter with an illustration of one of those characterizations. As a typical case, we consider the illustration of Theorem 2.5.3 characterizing best approximations from subspaces in the following example.

Example 2.5.10. Consider the real normed space $X = (\mathbb{R}^3, \|\cdot\|_1)$, where

$$\|x\|_1 = \sum_{k=1}^{3} |x_k| \text{ for } x = (x_1, x_2, x_3) \in X,$$

equipped with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1\\ y_k \neq 0}}^{3} \frac{x_k y_k}{|y_k|}, \quad x, y \in X$$

that generates the norm $\|\cdot\|_1$. Consider the subspace

$$M = \{x = (x_1, 0, x_2) \in X : x_1, x_2 \in \mathbb{R}\}$$

of X. Let $x = (1, 2, 3) \in X$. Then

$$\begin{aligned} d(x, M) &= \inf \{\|x - m\|_1 : m \in M\} \\ &= \inf \{|1 - m_1| + |2 - 0| + |3 - m_2| : m_1, m_2 \in \mathbb{R}\} \\ &= 2 \text{ at } (1, 0, 3) \in M. \end{aligned}$$

Let $y_0 = (1, 0, 3)$. Then

$$\|x - y_0\|_1 = |1 - 1| + |2 - 0| + |3 - 3| = 2.$$

Thus $\|x - y_0\|_1 = d(x, M)$, and hence $y_0 = (1, 0, 3) \in P_M(x)$. Then for every $m = (m_1, 0, m_2) \in M$, we have

$$[m|x - y_0] = [(m_1, 0, m_2) \,|\, (0, 2, 0)] = \|(0, 2, 0)\|_1 \cdot 0 = 0,$$

so that $x - y_0 \in M^{\perp}$. Now let $y_0 = (y_1, 0, y_2) \in M$. Suppose that $x - y_0 \in M^{\perp}$. Then

$$\begin{aligned} 0 &= [m|x - y_0] \text{ for every } m \in M. \\ &= [(m_1, 0, m_2) \,|\, (1 - y_1, 2, 3 - y_2)] \text{ for every } m_1, m_2 \in \mathbb{R} \\ &= \|(1 - y_1, 2, 3 - y_2)\|_1 \left[\frac{m_1 (1 - y_1)}{|1 - y_1|} + \frac{m_2 (3 - y_2)}{|3 - y_2|}\right] \text{ for every } m_1, m_2 \in \mathbb{R}. \end{aligned}$$

This implies that $y_1 = 1$ and $y_2 = 3$, so that $y_0 = (1, 0, 3)$. Hence $y_0 \in P_M(x)$.

CHAPTER 3

Applications of the Characterization Results

3.1 Introduction

We have already seen some results characterizing best approximations in normed spaces from convex sets, and in particular from convex cones, subspaces and their translates, in the previous chapter. The present chapter deals with a few applications of those results. It illustrates how those results can be employed in deriving new characterizations and properties of best approximations, and in novel situations like ordered orthonormalization, a terminology which we have introduced.

Our discussion begins with some new characterizations of best approximations from convex cones, subspaces and their translates in terms of errors of approximation. These results also furnish methods for determining the error of approximation. Following that, some properties of best approximations are presented. These include results asserting that proximinality, semi Chebyshevity and Chebyshevity of convex sets, and in particular convex cones and subspaces, are invariant under translation as well as under scalar multiplication. The concepts of ordered orthogonal sets and ordered orthonormal sets are introduced in the setting of a normed space. It is shown that ordered orthogonal sets of nonzero elements are linearly independent. Some results on ordered orthonormalization are also provided in this chapter.

As in the previous chapter, here also our discussion is confined to the framework of real normed spaces. Thus, in this chapter too, by a normed space X we mean a real normed space $(X, \|\cdot\|)$ together with a semi-inner product $[\cdot|\cdot]$ which generates the norm $\|\cdot\|$.

3.2 Characterizations in terms of Errors of Approximation

This section contains some results characterizing best approximations from convex cones, subspaces and their translates in terms of errors of approximation, which we arrive at using the corresponding results of the previous chapter. We recall that, if G is a nonempty subset of a normed space X, $x \in X$ and $g_0 \in P_G(x)$, then

$$\|x - g_0\| = \inf \{\|x - g\| : g \in G\} = d(x, G),$$

and in this case, the number $d(x, G)$ is called the error of approximation (or the error in approximating x by G). We denote $(d(x, G))^2$ by $d^2(x, G)$.

We begin with the following result for convex cones. It is a consequence of Theorem 2.5.1.

Theorem 3.2.1. *Let X be a normed space, C a convex cone in X, $x \in X$, and $y_0 \in C$. Then the following statements are equivalent:*

(i) $y_0 \in P_C(x)$;
(ii) $[y_0|x - y_0] = 0$ *and* $[x|x - y_0] = d^2(x, C)$.

Proof. (i) $\Rightarrow$ (ii) : If $y_0 \in P_C(x)$, then by the implication (a) $\Rightarrow$ (c) of Theorem 2.5.1, we have in particular $[y_0|x - y_0] = 0$. Consequently,

$$\begin{aligned}
[x|x - y_0] &= [x|x - y_0] - [y_0|x - y_0] \\
&= [x - y_0|x - y_0] \\
&= \|x - y_0\|^2 = d^2(x, C).
\end{aligned}$$

(ii) $\Rightarrow$ (i) : Conversely, if $[y_0|x-y_0]=0$ and $[x|x-y_0]=d^2(x,C)$, then

$$\begin{aligned} d^2(x,C) &= [x|x-y_0] \\ &= [x|x-y_0]-[y_0|x-y_0] \\ &= [x-y_0|x-y_0] \\ &= \|x-y_0\|^2 . \end{aligned}$$

This shows, since $y \in C$, that $y_0 \in P_C(x)$. $\square$

When the convex cone is actually a subspace, our result is the following. It turns out that the criterion for best approximations from subspaces coincides with that for best approximations from convex cones. We derive it from Theorem 2.5.3.

Theorem 3.2.2. *Let X be a normed space, M a subspace of X, $x \in X$, and $y_0 \in M$. Then the following statements are equivalent:*

(i) $y_0 \in P_M(x)$;
(ii) $[y_0|x-y_0]=0$ *and* $[x|x-y_0]=d^2(x,M)$.

Proof. The proof is analogous to that of Theorem 3.2.1 with the only difference that instead of the implication (a) $\Rightarrow$ (c) of Theorem 2.5.1, here we make use of the implication (a) $\Rightarrow$ (c) of Theorem 2.5.3. $\square$

The next result, which is a consequence of Theorem 2.5.5, is for translates of convex cones.

Theorem 3.2.3. *Let X be a normed space, C a convex cone in X, $z \in X$, and $K=z+C$. Suppose also that $x \in X$ and $y_0 \in K$. Then the following statements are equivalent:*

(i) $y_0 \in P_K(x)$;
(ii) $[y_0-z|x-y_0]=0$ *and* $[x-z|x-y_0]=d^2(x,K)$.

Proof. (i) $\Rightarrow$ (ii): If $y_0 \in P_K(x)$, then by the implication (a) $\Rightarrow$ (c) of Theorem 2.5.5, in particular we have

$$[y_0-z|x-y_0]=0.$$

Consequently,

$$\begin{aligned}[x-z|x-y_0] &= [x-z|x-y_0]-[y_0-z|x-y_0] \\ &= [x-z-y_0+z|x-y_0] \\ &= [x-y_0|x-y_0] \\ &= \|x-y_0\|^2 \\ &= d^2(x,K).\end{aligned}$$

(ii) $\Rightarrow$ (i): Conversely, if $[y_0-z|x-y_0]=0$ and $[x-z|x-y_0]=d^2(x,K)$, then

$$\begin{aligned}d^2(x,K) &= [x-z|x-y_0] \\ &= [x-z|x-y_0]-[y_0-z|x-y_0] \\ &= [x-z-y_0+z|x-y_0] \\ &= [x-y_0|x-y_0] \\ &= \|x-y_0\|^2.\end{aligned}$$

Since $y_0 \in K$, this shows that $y_0 \in P_K(x)$. □

Our final result of this series is the following one for translates of subspaces. We notice that the criterion for best approximations from translates of subspaces coincides with that for best approximations from translates of convex cones. We derive the result from Theorem 2.5.7.

Theorem 3.2.4. *Let X be a normed space, M a subspace of X, $z \in X$, and $K = z+M$. Suppose also that $x \in X$ and $y_0 \in K$. Then the following statements are equivalent:*

(i) $y_0 \in P_K(x)$;
(ii) $[y_0-z|x-y_0]=0$ *and* $[x-z|x-y_0]=d^2(x,K)$.

Proof. The proof is similar to that of Theorem 3.2.3 except that instead of using the implication (a) $\Rightarrow$ (c) of Theorem 2.5.5, here we employ the implication (a) $\Rightarrow$ (c) of Theorem 2.5.7. □

Remark 3.2.5. (*Error of Approximation*) Apart from being characterizations of best approximations in their own right, Theorems 3.2.1, 3.2.2, 3.2.3 and 3.2.4 have the added advantage that they provide methods for computing the errors of approximation. Under the hypothesis of Theorem 3.2.3, we have $y_0 \in P_K(x)$ if and only if $[y_0 - z|x - y_0] = 0$ and $[x - z|x - y_0] = d^2(x, K)$. This shows that if y_0 is a best approximation of x from the translate K of a convex cone C by z, then the error of approximation, $d(x, K)$ can be computed using

$$d(x, K) = \, [x - z|x - y_0]^{1/2}.$$

The same formula holds good for translates of subspaces also. The above formula with $z = 0$ yields the error of approximation in the case of convex cones and subspaces.

3.3 Some Properties of Best Approximations

This section deals with some properties of best approximations that are derived from the characterization results of the previous chapter. We begin with some elementary results. As an immediate consequence of Theorem 2.3.5, we have the following result for convex sets.

Theorem 3.3.1. *Let X be a normed space, K a convex set in X, $x \in X$, and $y_0 \in K$. Then $y_0 \in P_K(x)$ if and only if $y_0 \in P_K\left(\lambda x + (1-\lambda)\, y_0\right)$ for all $\lambda \geq 0$.*

Proof. Since $\lambda x + (1-\lambda)\, y_0 \in X$ for all $\lambda \geq 0$, by (i) $\Leftrightarrow$ (iii) of Theorem 2.3.5, we have

$$\begin{aligned}
y_0 \in P_K\left(\lambda x + (1-\lambda)\, y_0\right) \;&\Leftrightarrow\; \lambda x + (1-\lambda)\, y_0 - y_0 \in (K - y_0)^\circ \\
&\Leftrightarrow\; \lambda(x - y_0) \in (K - y_0)^\circ \\
&\Leftrightarrow\; [y - y_0|\lambda(x - y_0)] \leq 0 \;\text{ for all }\; y \in K \\
&\Leftrightarrow\; \lambda\,[y - y_0|x - y_0] \leq 0 \;\text{ for all }\; y \in K \\
&\Leftrightarrow\; [y - y_0|x - y_0] \leq 0 \;\text{ for all }\; y \in K, \;\text{ since }\; \lambda \geq 0 \\
&\Leftrightarrow\; x - y_0 \in (K - y_0)^\circ \\
&\Leftrightarrow\; y_0 \in P_K(x).
\end{aligned}$$

Hence the theorem. □

Since convex cones and their translates are convex sets, we have the following two corollaries to the above theorem.

Corollary 3.3.2. *Let X be a normed space, C a convex cone in X, $x \in X$, and $y_0 \in C$. Then $y_0 \in P_C(x)$ if and only if $y_0 \in P_C\left(\lambda x + (1-\lambda)\, y_0\right)$ for all $\lambda \geq 0$.*

Proof. Since the convex cone C is a convex set, the proof follows from Theorem 3.3.1 on replacing K by C. □

Corollary 3.3.3. *Let X be a normed space, C a convex cone in X, $z \in X$, and $C' = z + C$. Suppose also that $x \in X$ and $y_0 \in C'$. Then $y_0 \in P_{C'}(x)$ if and only if $y_0 \in P_{C'}\left(\lambda x + (1-\lambda)\, y_0\right)$ for all $\lambda \geq 0$.*

Proof. $C' = z + C$, being the translate of a convex cone, is a convex set. So the proof follows from Theorem 3.3.1 on replacing K by C'. □

Similar results do hold for subspaces and their translates also, since they too are convex sets basically. However, we notice that actually something more happens in these cases. Here such results hold not merely for $\lambda \geq 0$, but for $\lambda < 0$ also. First we prove the result for translates of subspaces using Theorem 2.5.7, and then deduce the result for subspaces from it.

Theorem 3.3.4. *Let X be a normed space, M a subspace of X, $z \in X$, and $M' = z + M$. Suppose also that $x \in X$ and $y_0 \in M'$. Then $y_0 \in P_{M'}(x)$ if and only if $y_0 \in P_{M'}\left(\lambda x + (1-\lambda)\, y_0\right)$ for all $\lambda \in \mathbb{R}$.*

Proof. Since $\lambda x + (1-\lambda)\, y_0 \in X$ for all $\lambda \in \mathbb{R}$, by (a) $\Leftrightarrow$ (c) of Theorem 2.5.7, we have

$$
\begin{aligned}
y_0 \in P_{M'}\left(\lambda x + (1-\lambda)\, y_0\right) &\Leftrightarrow \lambda x + (1-\lambda)\, y_0 - y_0 \in M^{\perp} \\
&\Leftrightarrow \lambda(x - y_0) \in M^{\perp} \\
&\Leftrightarrow [m|\lambda(x - y_0)] = 0 \ \text{ for all } \ m \in M \\
&\Leftrightarrow \lambda\, [m|x - y_0] = 0 \ \text{ for all } \ m \in M \\
&\Leftrightarrow [m|x - y_0] = 0 \ \text{ for all } \ m \in M \\
&\Leftrightarrow x - y_0 \in M^{\perp} \\
&\Leftrightarrow y_0 \in P_{M'}(x).
\end{aligned}
$$

Hence the theorem. □

As an immediate consequence of the above theorem, we have the following result for subspaces.

Corollary 3.3.5. *Let X be a normed space, M a subspace of X, $x \in X$, and $y_0 \in M$. Then $y_0 \in P_M(x)$ if and only if $y_0 \in P_M\left(\lambda x + (1-\lambda)\, y_0\right)$ for all $\lambda \in \mathbb{R}$.*

Proof. The proof follows from Theorem 3.3.4 on letting $z = 0$. □

Remark 3.3.6. We observe that the above corollary can also be proved directly using Theorem 2.5.3 for subspaces.

The results which follow in this section allow us to replace the problems of approximating from convex sets, and in particular from convex cones and subspaces, with those of approximating from their translates as well as from their scalar multiples. Moreover, they also enable us to deduce that translates as well as scalar multiples of proximinal, semi Chebyshev and Chebyshev convex sets, and in particular convex cones and subspaces, are again sets of the same sort. Our discussion in this direction begins with the problem of approximating from translates of convex sets. As a consequence of Theorem 2.3.5, we have the following result for convex sets.

Theorem 3.3.7. *Let X be a normed space, K a convex set in X, $z \in X$, and $K' = z + K$. Then $P_{K'}(z+x) = z + P_K(x)$ for every $x \in X$.*

Proof. $K' = z + K$, being a translate of the convex set K, is a convex set in X. Indeed, if $k_1' = z + k_1$ and $k_2' = z + k_2$ are in $K' = z + K$, where $k_1, k_2 \in K$, then for every $t \in [0,1]$, by the convexity of K, we have

$$tk_1' + (1-t)k_2' = tz + (1-t)z + tk_1 + (1-t)k_2 \in z + K = K'.$$

Now let $y_0 \in K'$ so that $y_0 - z \in K$. Then for every $x \in X$, by (i)⇔(iii) of Theorem 2.3.5, we have

$$\begin{aligned}
y_0 \in P_{K'}(z+x) \quad &\Leftrightarrow \quad z + x - y_0 \in (K' - y_0)^\circ \\
&\Leftrightarrow \quad x - (y_0 - z) \in (z + K - y_0)^\circ\,, \ \text{ since } K' = z + K \\
&\Leftrightarrow \quad x - (y_0 - z) \in (K - (y_0 - z))^\circ \\
&\Leftrightarrow \quad y_0 - z \in P_K(x) \\
&\Leftrightarrow \quad y_0 \in z + P_K(x).
\end{aligned}$$

Hence $P_{K'}(z+x) = z + P_K(x)$ for every $x \in X$. □

We have the following corollary to the above theorem when the convex set is actually a convex cone.

Corollary 3.3.8. *Let X be a normed space, C a convex cone in X, $z \in X$, and $C' = z + C$. Then $P_{C'}(z+x) = z + P_C(x)$ for every $x \in X$.*

Proof. Since a convex cone is a convex set, the proof is a consequence of replacing K by C and K' by C' in Theorem 3.3.7. □

Remark 3.3.9. We notice that the above corollary can also be derived directly from Theorem 2.5.5 for translates of convex cones and Theorem 2.5.1 for convex cones as follows. If $y_0 \in C' = z + C$ so that $y_0 - z \in C$, then for every $x \in X$, we have

$$\begin{aligned} y_0 \in P_{C'}(z+x) &\Leftrightarrow z + x - y_0 \in C^\circ \cap (y_0 - z)^\perp, \text{ by (a)} \Leftrightarrow \text{(c) of} \\ &\qquad \text{Theorem 2.5.5} \\ &\Leftrightarrow x - (y_0 - z) \in C^\circ \cap (y_0 - z)^\perp \\ &\Leftrightarrow y_0 - z \in P_C(x), \text{ by (a)} \Leftrightarrow \text{(c) of Theorem 2.5.1} \\ &\Leftrightarrow y_0 \in z + P_C(x). \end{aligned}$$

As another corollary to Theorem 3.3.7, we have the following result in the particular case when the convex set is a subspace.

Corollary 3.3.10. *Let X be a normed space, M a subspace of X, $z \in X$, and $M' = z + M$. Then $P_{M'}(z+x) = z + P_M(x)$ for every $x \in X$. In particular, if $z \in M$, then $P_M(z+x) = z + P_M(x)$ for every $x \in X$.*

Proof. Since a subspace is a convex set, Theorem 3.3.7 on replacing K by M and K' by M' yields $P_{M'}(z+x) = z + P_M(x)$ for every $x \in X$. In particular, if $z \in M$, then $M' = z + M = M$ itself, and therefore $P_M(z+x) = z + P_M(x)$ for every $x \in X$. □

Next we turn our attention to the problem of approximating from scalar multiples of convex sets. As a consequence of Theorem 2.3.5, we have the following result for convex sets.

Theorem 3.3.11. *Let X be a normed space, K a convex set in X, $\alpha \in \mathbb{R}$, and $K' = \alpha K$. Then $P_{K'}(\alpha x) = \alpha P_K(x)$ for every $x \in X$.*

Proof. Being a scalar multiple of the convex set K, $K' = \alpha K$ is a convex set in X. In fact, if $k_1' = \alpha k_1$ and $k_2' = \alpha k_2$ are in $K' = \alpha K$, where $k_1, k_2 \in K$, then for every $t \in [0,1]$, by the convexity of K, we have

$$tk_1' + (1-t)k_2' = \alpha(tk_1 + (1-t)k_2) \in \alpha K = K'.$$

If $\alpha = 0$, the result is trivially true. Indeed, in this case $K' = \{0\}$ and then $P_{K'}(0 \cdot x) = P_{\{0\}}(0) = \{0\} = 0 \cdot P_K(x)$ for every $x \in X$. So assume that $\alpha \neq 0$. Let $y_0 \in K'$ so that $y_0/_\alpha \in K$. Then for every $x \in X$, by (i)⇔(iii) of Theorem 2.3.5, we have

$$\begin{aligned} y_0 \in P_{K'}(\alpha x) \quad &\Leftrightarrow \quad \alpha x - y_0 \in (K' - y_0)^\circ \\ &\Leftrightarrow \quad [k' - y_0 | \alpha x - y_0] \leq 0 \;\; \text{for all} \;\; k' \in K' \\ &\Leftrightarrow \quad \alpha^2 \left[k - \frac{y_0}{\alpha} \middle| x - \frac{y_0}{\alpha} \right] \leq 0 \;\; \text{for all} \;\; k \in K \\ &\Leftrightarrow \quad \left[k - \frac{y_0}{\alpha} \middle| x - \frac{y_0}{\alpha} \right] \leq 0 \;\; \text{for all} \;\; k \in K \\ &\Leftrightarrow \quad x - \frac{y_0}{\alpha} \in \left(K - \frac{y_0}{\alpha} \right)^\circ \\ &\Leftrightarrow \quad \frac{y_0}{\alpha} \in P_K(x) \\ &\Leftrightarrow \quad y_0 \in \alpha P_K(x). \end{aligned}$$

Hence $P_{K'}(\alpha x) = \alpha P_K(x)$ for every $x \in X$. □

The following two results are corollaries to Theorem 3.3.11.

Corollary 3.3.12. *Let X be a normed space, C a convex cone in X, $\alpha \in \mathbb{R}$, and $C' = \alpha C$. Then $P_{C'}(\alpha x) = \alpha P_C(x)$ for every $x \in X$. In particular, if $\alpha \geq 0$, then $P_C(\alpha x) = \alpha P_C(x)$ for every $x \in X$.*

Proof. Since a convex cone is a convex set, Theorem 3.3.11 on replacing K by C and K' by C' yields $P_{C'}(\alpha x) = \alpha P_C(x)$ for every $x \in X$. In particular, if $\alpha \geq 0$, then $C' = \alpha C = C$ itself, and hence $P_C(\alpha x) = \alpha P_C(x)$ for every $x \in X$. □

Corollary 3.3.13. *Let X be a normed space, M a subspace of X, and $\alpha \in \mathbb{R}$. Then $P_M(\alpha x) = \alpha P_M(x)$ for every $x \in X$.*

Proof. If $\alpha = 0$, the result is trivial. In fact, in this case $P_M(0 \cdot x) = P_M(0) = \{0\} = 0 \cdot P_M(x)$ for every $x \in X$. So assume that $\alpha \neq 0$. Then, since a subspace is a convex set, and since $\alpha M = M$ for every $\alpha \in \mathbb{R} \setminus \{0\}$, the required result follows from Theorem 3.3.11. $\square$

Making use of the above results on approximating from translates and scalar multiples of convex sets, now we show that proximinality, semi Chebyshevity and Chebyshevity of convex sets are invariant under translation as well as under scalar multiplication. We recall (from Definitions 2.2.1) that a nonempty subset G of a normed space X is said to be proximinal (respectively semi Chebyshev, Chebyshev), if each $x \in X$ has at least (respectively at most, exactly) one best approximation from G. The following result, which is an easy consequence of Theorem 3.3.7 shows that proximinality, semi Chebyshevity and Chebyshevity of convex sets are invariant under translation.

Theorem 3.3.14. *Let X be a normed space, and K a convex set in X. Then K is proximinal (respectively semi Chebyshev, Chebyshev) if and only if $z + K$ is proximinal (respectively semi Chebyshev, Chebyshev) for any given $z \in X$.*

Proof. Given $z \in X$. Let $y_0 \in K$ so that $z + y_0 \in z + K$. Then for every $x \in X$, $y_0 \in P_K(x)$ if and only if $z + y_0 \in z + P_K(x) = P_{z+K}(z + x)$, by Theorem 3.3.7. Hence, for every $x \in X$, $P_K(x)$ contains at least (respectively at most, exactly) one element of K if and only if $P_{z+K}(z + x)$ contains at least (respectively at most, exactly) one element of $z + K$. This completes the proof. $\square$

As consequences of Theorem 3.3.14, we have the following corollaries.

Corollary 3.3.15. *Let X be a normed space, and C a convex cone in X. Then C is proximinal (respectively semi Chebyshev, Chebyshev) if and only if $z + C$ is proximinal (respectively semi Chebyshev, Chebyshev) for any given $z \in X$.*

Proof. Since a convex cone is a convex set, the proof follows from Theorem 3.3.14 on replacing K by C. $\square$

Corollary 3.3.16. *Let X be a normed space, and M a subspace of X. Then M is proximinal (respectively semi Chebyshev, Chebyshev) if and only if $z + M$ is proximinal (respectively semi Chebyshev, Chebyshev) for any given $z \in X$.*

Proof. The proof follows from Theorem 3.3.14 on replacing K by M, since every subspace is a convex set. $\square$

The next result, which is a consequence of Theorem 3.3.11, shows that proximinality, semi Chebyshevity and Chebyshevity of convex sets are invariant under multiplication with nonzero real numbers.

Theorem 3.3.17. *Let X be a normed space, and K a convex set in X. Then K is proximinal (respectively semi Chebyshev, Chebyshev) if and only if αK is proximinal (respectively semi Chebyshev, Chebyshev) for any given $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. Given $\alpha \in \mathbb{R} \setminus \{0\}$. Let $y_0 \in K$ so that $\alpha y_0 \in \alpha K$. Then for every $x \in X$, $y_0 \in P_K(x)$ if and only if $\alpha y_0 \in \alpha P_K(x) = P_{\alpha K}(\alpha x)$, by Theorem 3.3.11. Hence, for every $x \in X$, $P_K(x)$ contains at least (respectively at most, exactly) one element of K if and only if $P_{\alpha K}(\alpha x)$ contains at least (respectively at most, exactly) one element of αK. This completes the proof. $\square$

We have the following corollaries to Theorem 3.3.17.

Corollary 3.3.18. *Let X be a normed space, and C a convex cone in X. Then C is proximinal (respectively semi Chebyshev, Chebyshev) if and only if αC is proximinal (respectively semi Chebyshev, Chebyshev) for any given $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. Since every convex cone is a convex set, the proof follows from Theorem 3.3.17 on replacing K by C. $\square$

Corollary 3.3.19. *Let X be a normed space, and M a subspace of X. Then M is proximinal (respectively semi Chebyshev, Chebyshev) if and only if αM is proximinal (respectively semi Chebyshev, Chebyshev) for any given $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. The proof is obvious, since for any $\alpha \in \mathbb{R} \setminus \{0\}$, $\alpha M = M$ itself as M is a subspace. $\square$

3.4 Ordered Orthogonal Sets

The main objective of this section is to introduce the concept of an ordered orthogonal set in the setting of a normed space, a concept which is a generalization of an orthogonal set in the framework of an inner product space. We recall that a nonempty subset G of an inner product space $(X, (\cdot,\cdot))$ is said to be orthogonal, if any two distinct elements in G are orthogonal, i.e., if $(x, y) = 0$ for all $x \neq y$ in G. This notion is extended here to the setting of a general normed space in terms of a semi-inner product that generates the norm of the space. The fact that inner product orthogonality is symmetric whereas semi-inner product orthogonality is not generally symmetric, suggests that certain ordering among the elements of the set is to be imposed while extending the notion. Because of the ordering involved, we term the resultant concept as an ordered orthogonal set. The notion of an ordered orthonormal set is also introduced in this section. We show that an ordered orthogonal set of nonzero elements in a normed space is linearly independent.

Let X be a normed space, and $x, y \in X$. We recall (from Definition 2.4.1) that x is orthogonal (or orthogonal in the sense of Lumer-Giles relative to the semi-inner product $[\cdot|\cdot]$) to y, denoted by $x \perp y$, if $[y|x] = 0$, and $x \perp y$ need not imply that $y \perp x$. In terms of the orthogonality of elements, below we introduce the totally new terminology of ordered orthogonal sets in a normed space.

Definition 3.4.1. Let X be a normed space, and G a nonempty nonsingleton finite subset of X. Then G is said to be *ordered orthogonal* (or *ordered orthogonal in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$), if elements of G can be arranged as a sequence in which each element except the first is orthogonal to every element preceding it, i.e., if elements of G can be arranged as $\{x_1, x_2, ..., x_n\}$, $n > 1$, where $x_j \perp x_i$ for every i, j satisfying $1 \leq i < j \leq n$.

Ordered orthogonality of an arbitrary subset is defined as follows.

Definition 3.4.2. Let X be a normed space. Then an arbitrary nonempty subset G of X is called an *ordered orthogonal set* (or *ordered orthogonal set in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$), if every nonempty nonsingleton finite subset of G is ordered orthogonal.

The following is our definition of an ordered orthonormal set.

Definition 3.4.3. Let X be a normed space, and G a nonempty subset of X. Then G is said to be *ordered orthonormal* (or *ordered orthonormal in the sense of Lumer-Giles relative to the semi-inner product* $[\cdot|\cdot]$), if G is ordered orthogonal and every element of G has norm 1. Thus, if G is an ordered orthonormal set, then every nonempty nonsingleton finite subset of G is ordered orthogonal and $\|x\| = [x|x]^{1/2} = 1$ for every $x \in G$.

Some examples of ordered orthogonal sets and ordered orthonormal sets are given below.

Example 3.4.4. In an inner product space, any orthogonal set is ordered orthogonal, and any orthonormal set is ordered orthonormal.

Example 3.4.5. Consider the real normed space $X = (\mathbb{R}^3, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{3} |x_k|$ for $x = (x_1, x_2, x_3) \in X$, together with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{3} \frac{x_k y_k}{|y_k|}, \quad x, y \in X.$$

Then the set $\{(1,1,0), (-1,1,2), (2,-3,1)\}$ is an ordered orthogonal set, and the set $\left\{\left(1/2, 1/2, 0\right), \left(-1/4, 1/4, 1/2\right), \left(1/3, -1/2, 1/6\right)\right\}$ is an ordered orthonormal set.

We have the following result which provides the connection between ordered orthogonality and linear independence of sets in a normed space.

Theorem 3.4.6. *Let X be a normed space, and G an ordered orthogonal subset of X such that $0 \notin G$. Then G is linearly independent.*

Proof. Let $G_n = \{x_1, x_2, ..., x_n\} \subseteq G$. Since G is ordered orthogonal, so is G_n also. Hence G_n can be expressed as $G_n = \{y_1, y_2, ..., y_n\}$, where each y_i is a unique x_j for every i, j such that $1 \leq i, j \leq n$, and $[y_i|y_j] = 0$ for every i, j satisfying $1 \leq i < j \leq n$. Since $0 \notin G$, $y_i \neq 0$ for every i such that $1 \leq i \leq n$.

Now suppose that $\sum_{i=1}^{n} \alpha_i y_i = 0$, where $\alpha_i \in \mathbb{R}$, $1 \leq i \leq n$. Then for each $j = 1, 2, ..., n$, we have

$$\sum_{i=1}^{n} \alpha_i [y_i|y_j] = \left[\sum_{i=1}^{n} \alpha_i y_i \middle| y_j \right] = 0.$$

Putting $j = n, n-1, ..., 1$ in succession, since $[y_i|y_j] = 0$ for $1 \leq i < j \leq n$, and $y_i \neq 0$ for $1 \leq i \leq n$, it follows that $\alpha_n = 0$, $\alpha_{n-1} = 0$,...,$\alpha_1 = 0$. This shows that $\{y_1, y_2, ..., y_n\}$ is linearly independent, and hence in turn $\{x_1, x_2, ..., x_n\}$ also is linearly independent. Therefore G is linearly independent. □

Remark 3.4.7. The above theorem shows, in particular, that every ordered orthonormal set in a normed space is linearly independent, since such sets are ordered orthogonal sets not containing 0.

3.5 Ordered Orthonormalization

Theorem 3.4.6 asserts that every ordered orthonormal set in a normed space is linearly independent. In this section, we consider the converse of the problem. Given a countable linearly independent set in a normed space, we show here using Theorem 2.5.3 that one can construct an ordered orthonormal set, retaining the span of the elements at each step.

We begin our discussion with an ordered orthonormalization result in the setting of a general normed space. The following result [21] on the existence of best approximations, the proof of which is omitted here, is made use of in our construction.

Theorem 3.5.1. *Let X be a normed space, and M a finite dimensional subspace of X. Then M is proximinal.*

Our ordered orthonormalization result in the framework of a general normed space is given below. It is a consequence of Theorem 2.5.3 characterizing best approximations from subspaces.

Theorem 3.5.2. *Let X be a normed space, and $\{x_1, x_2, ...\}$ a countable linearly independent subset of X. Let $M_0 = \{0\}$, $M_n =$ span $\{x_1, x_2, ..., x_n\}$ and $y_n \in P_{M_{n-1}}(x_n)$ for each $n \geq 1$. Define $z_n = x_n - y_n$ and $u_n = {}^{z_n}/_{\|z_n\|}$ for each $n \geq 1$. Then*

(a) $\{z_1, z_2, ...\}$ *is an ordered orthogonal set,*

(b) $\{u_1, u_2, ...\}$ *is an ordered orthonormal set, and*

(c) span $\{z_1, z_2, ..., z_n\} =$ span $\{u_1, u_2, ..., u_n\} = M_n$ *for each $n \geq 1$.*

Proof. Being a finite dimensional subspace of X, by Theorem 3.5.1, M_n is proximinal for each $n \geq 0$. So for each fixed $y_n \in P_{M_{n-1}}(x_n)$, the element $z_n = x_n - y_n$ is well defined for $n = 1, 2, \ldots$. Since $\{x_1, x_2, ..., x_n\}$ is linearly independent, $x_n \notin M_{n-1}$. Hence $z_n \neq 0$ so that the element $u_n = {}^{z_n}\!/\!_{\|z_n\|}$ is well defined for all $n \geq 1$. Since $y_j \in P_{M_{j-1}}(x_j)$, by the equivalence (a) $\Leftrightarrow$ (c) of Theorem 2.5.3, we get $z_j = x_j - y_j \in M_{j-1}^{\perp}$ for every j such that $1 < j \leq n$. Also, for any i such that $1 \leq i < j$, we obtain

$$z_i = x_i - y_i \in M_i - M_{i-1} \subseteq M_i \subseteq M_{j-1}.$$

Hence for every i, j satisfying $1 \leq i < j \leq n$, we have $[z_i|z_j] = 0$ so that $z_j \perp z_i$. This shows that, for each $n \geq 1$, $\{z_1, z_2, ..., z_n\}$ is an ordered orthogonal set. Hence so is $\{z_1, z_2, ...\}$. Thus (a) holds.

Since $u_n = {}^{z_n}\!/\!_{\|z_n\|}$ for each $n \geq 1$, and $[z_i|z_j] = 0$ for every i, j such that $1 \leq i < j \leq n$, we have

$$[u_i|u_j] = \left[{}^{z_i}\!/\!_{\|z_i\|} \,\middle|\, {}^{z_j}\!/\!_{\|z_j\|} \right] = \left({}^{1}\!/\!_{(\|z_i\| \, \|z_j\|)} \right) [z_i|z_j] = 0$$

for $1 \leq i < j \leq n$. Hence for each $n \geq 1$, $\{u_1, u_2, ..., u_n\}$ is ordered orthogonal. This together with the fact that $\|u_n\| = 1$ implies that $\{u_1, u_2, ...\}$ is an ordered orthonormal set. Hence holds (b) also.

Again, since $u_n = {}^{z_n}\!/\!_{\|z_n\|}$ for each $n \geq 1$, we have span$\{z_1, z_2, ..., z_n\}$ = span$\{u_1, u_2, ..., u_n\}$ for all $n \geq 1$. Further, since $\{u_1, u_2, ..., u_n\} \subseteq M_n$, M_n is n-dimensional, and ordered orthonormal sets are linearly independent by Theorem 3.4.6, we obtain that, for each $n \geq 1$, span$\{u_1, u_2, ..., u_n\} = M_n$. This completes the proof. □

We observe that the ordered orthogonal set $\{z_1, z_2, ...\}$, and hence in turn the ordered orthonormal set $\{u_1, u_2, ...\}$ also, constructed from the given countable linearly independent set $\{x_1, x_2, ...\}$ in the above theorem need not be unique. For, since M_{n-1} is proximinal, for each x_n, $P_{M_{n-1}}(x_n)$ may contain more than one element y_n for $n = 1, 2, \ldots$. As a consequence, for each $n \geq 1$, the element $z_n = x_n - y_n$, and hence the element $u_n = {}^{z_n}\!/\!_{\|z_n\|}$ also, may not be unique. This is the case of our ordered orthonormalization process (Theorem 3.5.2) in the setting of a general normed space. However, if the normed space is in particular strictly convex, then we do have uniqueness of such sets also. For showing this, we need the following result [21] on the uniqueness of best approximations, the proof of which we omit here. We recall that a normed space X is said to be *strictly convex*, if for $x, y \in X$, $\|x\| = 1 = \|y\|$ and $x \neq y$ imply $\left\| {}^{(x+y)}/_{2} \right\| < 1$.

Theorem 3.5.3. *Let X be a strictly convex normed space, and M a finite dimensional subspace of X. Then M is Chebyshev.*

If the normed space X in the hypothesis of Theorem 3.5.2 is strictly convex, then in view of the above result (Theorem 3.5.3), the $(n-1)$-dimensional subspace $M_{n-1} = \text{span}\{x_1, x_2, ..., x_{n-1}\}$ of X is Chebyshev. Hence for each x_n, the set $P_{M_{n-1}}(x_n)$ contains exactly one element for $n = 1, 2, \ldots$. Denoting this unique element in the set $P_{M_{n-1}}(x_n)$ as $P_{M_{n-1}}(x_n)$ itself, we see that for each $n = 1, 2, ...$, the element $z_n = x_n - P_{M_{n-1}}(x_n)$, and hence in turn the element $u_n = {}^{z_n}/_{\|z_n\|}$ also, are uniquely determined. Hence it follows from Theorem 3.5.2 that the ordered orthogonal set $\{z_1, z_2, ...\}$ as well as the ordered orthonormal set $\{u_1, u_2, ...\}$ is unique. Thus we have the following corollary to Theorem 3.5.2.

Corollary 3.5.4. *Let X be a strictly convex normed space, and $\{x_1, x_2, ...\}$ a countable linearly independent subset of X. Let $M_0 = \{0\}$ and $M_n =$ span $\{x_1, x_2, ..., x_n\}$ for each $n \geq 1$. Define $z_n = x_n - P_{M_{n-1}}(x_n)$ and $u_n = {}^{z_n}/_{\|z_n\|}$ for each $n \geq 1$. Then*

(a) $\{z_1, z_2, ...\}$ *is a unique ordered orthogonal set,*

(b) $\{u_1, u_2, ...\}$ *is a unique ordered orthonormal set, and*

(c) span $\{z_1, z_2, ..., z_n\} =$ span $\{u_1, u_2, ..., u_n\} = M_n$ *for each* $n \geq 1$.

Remark 3.5.5. We observe that Theorem 3.5.2 can be considered as an analogue of the Gram-Schmidt orthonormalization process in a general normed space. Similar is the case with Corollary 3.5.4 in the setting of a strictly convex normed space.

CHAPTER 4

Characterizations of Proximinality and Chebyshevity, and their Applications

4.1 Introduction

The main purpose of this chapter is the characterization of proximinality, semi Chebyshevity and Chebyshevity of convex sets in normed spaces in terms of the decomposability of the space. In doing so, we depend mainly on the results of the second chapter. Some consequences of the characterization are also considered in this chapter.

We begin this chapter with some results characterizing proximinal, semi Chebyshev and Chebyshev convex sets, and in particular convex cones and subspaces. These results enable us to derive some decomposition theorems which characterize proximinality and Chebyshevity of subspaces of normed spaces, closedness of subspaces of reflexive normed spaces, and that of strictly convex reflexive normed spaces. Following that, a few consequences of the decomposition theorems are provided. Finally, we make an attempt to study the analogue of Riesz Representation theorem for continuous linear functionals on normed spaces in this context.

As in the last two chapters, our discussion is restricted to the setting of real normed spaces. Thus, in this chapter also, by a normed space X we mean a real normed space $(X, \|\cdot\|)$ endowed with a semi-inner product $[\cdot|\cdot]$ that generates the norm $\|\cdot\|$.

4.2 Characterizations of Proximinality and Chebyshevity

In this section we present some results characterizing proximinal, semi Chebyshev and Chebyshev convex sets, in particular, convex cones, subspaces and their translates. These are derived from the characterization results of the second chapter. Our discussion begins with the following result for convex sets, which actually is a consequence of the characterization Theorem 2.3.5.

Theorem 4.2.1. *Let X be a normed space, and K a convex set in X. Then K is proximinal (respectively semi Chebyshev, Chebyshev) if and only if each $x \in X$ admits at least (respectively at most, exactly) one representation of the form $x = y + y'$ with $y \in K$ and $y' \in (K - y)^\circ$.*

Proof. From (i) $\Leftrightarrow$ (iii) of Theorem 2.3.5, it follows that K is proximinal (respectively semi Chebyshev, Chebyshev) if and only if for each $x \in X$, $P_K(x)$ contains at least (respectively at most, exactly)one element of K. That is,

$\Leftrightarrow$ For each $x \in X,$ there exists at least (respectively at most, exactly) one element $y \in K$ such that $y \in P_K(x)$

$\Leftrightarrow$ For each $x \in X,$ there exists at least (respectively at most, exactly)one element $y \in K$ such that $x - y \in (K - y)^\circ$

$\Leftrightarrow$ Each $x \in X$ has at least (respectively atmost, exactly) one representation of the form $x = y + y'$ where $y \in K$ and $y' = x - y \in (K - y)^\circ$.

Hence the result. $\square$

Similar characterization results hold for particular convex sets like convex cones, subspaces and their translates also. Our result for translates of convex cones is given below. We derive it directly from Theorem 2.5.5 characterizing best approximations from translates of convex cones.

Theorem 4.2.2. *Let X be a normed space, C a convex cone in X, $z \in X$, and $C' = z + C$. Then C' is proximinal (respectively semi Chebyshev, Chebyshev) if and only if each $x \in X$ admits at least (respectively at most, exactly) one representation of the form $x = y + y'$ with $y \in C'$ and $y' \in C^\circ \cap (y - z)^\perp$.*

Proof. The proof is analogous to that of Theorem 4.2.1 with the only difference that instead of using (i) $\Leftrightarrow$ (iii) of Theorem 2.3.5, here we employ (a) $\Leftrightarrow$ (c) of Theorem 2.5.5. □

As a consequence of the above theorem, we have the following characterization result for convex cones.

Corollary 4.2.3. *Let X be a normed space, and C a convex cone in X. Then C is proximinal (respectively semi Chebyshev, Chebyshev) if and only if each $x \in X$ admits at least (respectively at most, exactly) one representation of the form $x = y + y'$ with $y \in C$ and $y' \in C^\circ \cap y^\perp$.*

Proof. The proof follows from Theorem 4.2.2 on taking z=0. □

The characterization result for translates of subspaces is given below. It is a consequence of Theorem 2.5.7 characterizing best approximations from translates of subspaces.

Theorem 4.2.4. *Let X be a normed space, M a subspace of X, $z \in X$, and $M' = z + M$. Then M' is proximinal (respectively semi Chebyshev, Chebyshev) if and only if each $x \in X$ admits at least (respectively at most, exactly) one representation of the form $x = y + y'$ with $y \in M'$ and $y' \in M^\perp$.*

Proof. The proof is similar to that of Theorem 4.2.1 with the only difference that instead of using (i) $\Leftrightarrow$ (iii) of Theorem 2.3.5, here we employ (a) $\Leftrightarrow$ (c) of Theorem 2.5.7. □

The following characterization result for subspaces is a corollary to Theorem 4.2.4.

Corollary 4.2.5. *Let X be a normed space, and M a subspace of X. Then M is proximinal (respectively semi Chebyshev, Chebyshev) if and only if each $x \in X$ admits at least (respectively at most, exactly) one representation of the form $x = y + y'$ with $y \in M$ and $y' \in M^{\perp}$.*

Proof. The proof follows from Theorem 4.2.4 on letting z=0. □

4.3 Some Decomposition Theorems

Results of the last section suggest that proximinality and Chebyshevity of subspaces of a normed space can be characterized in terms of the decomposability of the space. We provide some decomposition theorems in this section. Our results characterize proximinal and Chebyshev subspaces of normed spaces, closed subspaces of reflexive normed spaces, and closed subspaces of strictly convex reflexive normed spaces. We begin this section with the following definitions.

Definitions 4.3.1. Let X be a normed space, and A, B be nonempty subsets of X. Then X is said to be the *sum* of A and B, denoted by $X = A + B$, if each $x \in X$ has at least one representation of the form $x = a + b$, where $a \in A$ and $b \in B$. We say that X is the *direct sum* of A and B, denoted by $X = A \oplus B$, if each $x \in X$ has a unique representation of the form $x = a + b$, where $a \in A$ and $b \in B$.

If A and B are actually subspaces of a normed space X, then $X = A \oplus B$ if and only if $X = A + B$ and $A \cap B = \{0\}$.

As an easy consequence of Theorem 4.2.4, we have the following decomposition result which characterizes proximinality as well as Chebyshevity of translates of subspaces of normed spaces.

Theorem 4.3.2. *Let X be a normed space, M a subspace of X, $z \in X$, and $M' = z + M$. Then M' is proximinal (respectively Chebyshev) if and only if $X = M' + M^{\perp}$ (respectively $X = M' \oplus M^{\perp}$).*

Proof. The proof follows from Theorem 4.2.4 and Definitions 4.3.1. □

The decomposition result given below is a corollary to Theorem 4.3.2, and it characterizes proximinality as well as Chebyshevity of subspaces.

Corollary 4.3.3. *Let X be a normed space, and M a subspace of X. Then M is proximinal (respectively Chebyshev) if and only if $X = M + M^{\perp}$ (respectively $X = M \oplus M^{\perp}$).*

Proof. The proof follows from Theorem 4.3.2 on letting z=0. □

Remark 4.3.4. We observe that the above corollary can also be proved directly using Corollary 4.2.5 and Definitions 4.3.1.

Our next result in this series is in the setting of a reflexive normed space. Let X^* and X^{**} denote respectively the dual space and the bidual space of a normed space X. We recall that a normed space X is said to be *reflexive* if the canonical embedding $J : X \to X^{**}$ defined by

$$(J(x))\,(f) = f(x) \quad \text{for all } \ x \in X \ \text{ and all } \ f \in X^*$$

is surjective. We need the following results [29], the proofs of which are omitted here, for proving our next decomposition theorem.

Theorem 4.3.5. *Let X be a reflexive normed space, and M a closed subspace of X. Then M is proximinal. If X is strictly convex also, then M is actually Chebyshev.*

Theorem 4.3.6. *Let X be a normed space, and G a proximinal set in X. Then G is closed. In particular, every Chebyshev set in a normed space is closed.*

The decomposition result in the framework of a reflexive normed space is given below. It provides a characterization of closed subspaces of reflexive normed spaces. We derive the result from Corollary 4.3.3.

Theorem 4.3.7. *Let X be a reflexive normed space, and M a subspace of X. Then M is closed if and only if $X = M + M^{\perp}$.*

Proof. Since X is reflexive, it follows from Theorem 4.3.5 and Theorem 4.3.6 that M is closed if and only if M is proximinal. By Corollary 4.3.3, M is proximinal if and only if $X = M + M^{\perp}$, and this completes the proof. □

The above result can be modified to have a direct sum decomposition of the reflexive normed space on imposing the additional condition of strict convexity on the space. The resultant decomposition result, which is given below, provides a characterization of closed subspaces of strictly convex reflexive normed spaces. We derive the result from Corollary 4.3.3 with the help of Theorem 4.3.5 and Theorem 4.3.6.

Theorem 4.3.8. *Let X be a strictly convex reflexive normed space, and M a subspace of X. Then M is closed if and only if $X = M \oplus M^{\perp}$.*

Proof. Since X is strictly convex and reflexive, Theorem 4.3.5 and Theorem 4.3.6 imply that M is closed if and only if M is Chebyshev. Further, by Corollary 4.3.3, M is Chebyshev if and only if $X = M \oplus M^{\perp}$. This completes the proof. $\square$

Remark 4.3.9. In the framework of a reflexive normed space, Theorem 4.3.7 can be treated as an analogue of the Projection theorem. Same is the case with Theorem 4.3.8 in the setting of a strictly convex reflexive normed space.

It may be noticed that Theorem 4.3.7 and Theorem 4.3.8 do not hold for nonreflexive normed spaces. To this effect, we have the following example.

Example 4.3.10. Consider the real normed space $X = (c_{00}, \|\cdot\|_1)$ of all real sequences having only a finite number of nonzero entries, where $\|x\|_1 = \sum\limits_{k=1}^{\infty} |x_k|$ for $x = (x_1, x_2, x_3, ...) \in X$, together with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{\infty} \frac{x_k y_k}{|y_k|}, \quad x, y \in X.$$

Define $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{x_k}{k}, \quad x \in X.$$

Then f is linear. By Holder's inequality, for every $x \in X$, we have

$$|f(x)|^2 \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) \left(\sum_{k=1}^{\infty} |x_k|^2 \right) = \frac{\pi^2}{6} \|x\|^2 ,$$

so that f is continuous and $\|f\| \leq \pi / \sqrt{6}$. Hence $f \in X^*$. Let $M = Z_f$, the zero space of f. Since $f \neq 0$, M is a proper closed subspace of X. Let $z \in M^{\perp}$.

Assume that $z \neq 0$. Then, since $z \in c_{00}$, we see that $z = (z_1, ..., z_m, 0, 0, ...)$ for some positive integer m. Consider $x \in X$ given by

$$x_k = \begin{cases} 1 & \text{if } k = 1, \\ -(m+1) & \text{if } k = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f(x) = \frac{1}{1} + \frac{-(m+1)}{m+1} = 0,$$

so that $x \in M$. Since $z \in M^{\perp}$, we have

$$0 = [x|z] = \|z\|_1 \sum_{\substack{k=1 \\ z_k \neq 0}}^{\infty} \frac{x_k z_k}{|z_k|} = \pm \|z\|_1 \,,$$

so that $\|z\|_1 = 0$, a contradiction. Thus $M^{\perp} = \{0\}$. This implies, since $M \neq X$, that $X \neq M + M^{\perp}$. This shows that Theorem 4.3.7 and Theorem 4.3.8 do not hold for X.

4.4 Consequences of the Decomposition Results

Let us now consider some consequences of the decomposition theorems which we have seen in the above section. Corollary 4.3.3 provides the following characterization of dense proximinal subspaces of a normed space in terms of the orthogonality condition.

Theorem 4.4.1. *Let X be a normed space, and M a proximinal subspace of X. Then M is dense in X if and only if $M^{\perp} = \{0\}$.*

Proof. Assume that $\overline{M} = X$, and let $x \in M^{\perp}$. Then $[y|x] = 0$ for every $y \in M$. Since M is proximinal, M is closed by Theorem 4.3.6, and then $M = \overline{M} = X$. Consequently, $[y|x] = 0$ for every $y \in X$. Hence, as $x \in X$, we get $\|x\|^2 = [x|x] = 0$ so that x=0. Thus $M^{\perp} = \{0\}$. Conversely, assume that $M^{\perp} = \{0\}$. Then, as M is proximinal and hence closed also, we have $\overline{M} = M = M + \{0\} = M + M^{\perp} = X$, by Corollary 4.3.3. This completes the proof. □

As an immediate consequence of the above result, we have the following characterization of dense Chebyshev subspaces of a normed space.

Corollary 4.4.2. *Let X be a normed space, and M a Chebyshev subspace of X. Then M is dense in X if and only if $M^{\perp} = \{0\}$.*

Proof. The proof follows from Theorem 4.4.1, since every Chebyshev subspace is proximinal. □

Remark 4.4.3. It may be noticed that the above corollary can also be proved directly using Theorem 4.3.6 and Corollary 4.3.3.

Our next result provides a similar characterization of dense subspaces of a reflexive normed space in terms of the orthogonality condition. It is a consequence of Theorem 4.3.7. We make use of Theorem 2.4.6 (b) in its proof.

Theorem 4.4.4. *Let X be a reflexive normed space, and M a subspace of X. Then M is dense in X if and only if $M^{\perp} = \{0\}$.*

Proof. Assume that $\overline{M} = X$, and let $x \in M^{\perp}$. By Theorem 2.4.6 (b), we have $\left(\overline{M}\right)^{\perp} = M^{\perp}$. Hence $x \in \left(\overline{M}\right)^{\perp}$ so that $[y|x] = 0$ for every $y \in \overline{M} = X$. Consequently, $\|x\|^2 = [x|x] = 0$ and so x=0. Thus $M^{\perp} = \{0\}$. Conversely, assume that $M^{\perp} = \{0\}$. Then, since $\left(\overline{M}\right)^{\perp} = M^{\perp}$ by Theorem 2.4.6 (b), we have $\overline{M} = \overline{M} + \{0\} = \overline{M} + \left(\overline{M}\right)^{\perp} = X$, by Theorem 4.3.7 as $\overline{M}$ is a closed subspace of X. This completes the proof. □

Another consequence of Corollary 4.3.3 is given below.

Theorem 4.4.5. *Let X be a normed space, and M a proper proximinal subspace of X. Then $M^{\perp}$ contains a nonzero element.*

Proof. Suppose on the contrary that $M^{\perp} = \{0\}$, and let $x \in X \setminus M$. Since M is proximinal, by Corollary 4.3.3, we have $x = y + 0$ with $y \in M$ and $0 \in M^{\perp}$, which is a contradiction. Hence the theorem. □

Theorem 4.4.5 has the following corollary.

Corollary 4.4.6. *Let X be a normed space, and M a proper Chebyshev subspace of X. Then $M^{\perp}$ contains a nonzero element.*

Proof. Since every Chebyshev subspace is proximinal, the proof follows from Theorem 4.4.5. □

Remark 4.4.7. We observe that the above result can be derived directly from Corollary 4.3.3 also

A result similar to the last two in the setting of a reflexive normed space is given below. It is a consequence of Theorem 4.3.7.

Theorem 4.4.8. *Let X be a reflexive normed space, and M a proper closed subspace of X. Then $M^{\perp}$ contains a nonzero element.*

Proof. Suppose on the contrary that $M^{\perp} = \{0\}$, and let $x \in X \setminus M$. Since M is closed, by Theorem 4.3.7, we have $x = y + 0$ with $y \in M$ and $0 \in M^{\perp}$, a contradiction. Hence the theorem. □

4.5 Continuous Linear Functionals on Normed Spaces

Our attempt in this section is to study continuous linear functionals on normed spaces. Using Theorem 1.2.3(b), we show here that every element y belonging to a normed space X determines a continuous linear functional f_y on X defined by $f_y(x) = [x|y]$ with $\|f_y\| = \|y\|$. Employing Theorem 4.3.7 and Theorem 4.4.4, it is further shown that, if the normed space X is actually reflexive, then every continuous linear functional f on X is given by $f(x) = [x|y_f]$ for some suitable $y_f \in X$ with $\|f\| = \|y_f\|$.

We begin our discussion with the following result, which is a direct consequence of Theorem 1.2.3(b).

Theorem 4.5.1. *Let X be a normed space and $y \in X$. Then there exists an element $f_y \in X^*$ such that*

$$f_y(x) = [x|y] \quad for\ all\ x \in X,$$

and $\|f_y\| = \|y\|$.

Proof. From Theorem 1.2.3(b), it follows that every $y \in X$ defines a continuous linear functional $f_y : X \to \mathbb{R}$ given by $f_y(x) = [x|y]$ such that $\|f_y\| = \|y\|$. $\square$

Remark 4.5.2. In the above theorem we observe that the element $f_y \in X^*$, which exists corresponding to a given $y \in X$, is unique also. Indeed, if there exists $f_y, f_y' \in X^*$ such that $f_y(x) = [x|y] = f_y'(x)$ for all $x \in X$, then $f_y = f_y'$.

Let X be a normed space. Theorem 4.5.1 shows that each $y \in X$ gives rise to a continuous linear functional f_y on X defined by $f_y(x) = [x|y]$ with $\|f_y\| = \|y\|$. We notice that the converse of this result is not true in general (see Example 4.5.5). However, as we shall see next, converse does hold for reflexive normed spaces. We show below that, if the normed space X is actually reflexive, then every continuous linear functional f on X is given by $f(x) = [x|y_f]$ for some suitable $y_f \in X$, and $\|f\| = \|y_f\|$. To establish this, we employ Theorem 4.3.7 and Theorem 4.4.4. Z_f denotes the zero space of an element $f \in X^*$.

Theorem 4.5.3. *Let X be a reflexive normed space, and $f \in X^*$. Then there exists an element $y_f \in X$ such that*

$$f(x) = [x|y_f] \quad for \ all \ x \in X,$$

and $\|f\| = \|y_f\|$. In fact, if z is a nonzero element of X such that $z \perp Z_f$, then

$$y_f = \frac{f(z)z}{[z|z]}.$$

Proof. If $f = 0$, then let $y_f = 0$, so that for all $x \in X$, we have

$$f(x) = 0 = [x|0] = [x|y_f],$$

and $\|f\| = 0 = \|y_f\|$.

Let $f \neq 0$. Then the zero space $Z_f = \{x \in X : f(x) = 0\}$ is a proper closed subspace of X. Hence, by Theorem 4.3.7, $X = Z_f + {Z_f}^{\perp}$, where ${Z_f}^{\perp} \neq \{0\}$, by Theorem 4.4.4. Consider a nonzero element $z \in {Z_f}^{\perp}$. Let $x \in X$. Being the zero space of a nonzero linear functional f on X, Z_f is a hyperspace in X. Hence

$$x = w + \alpha z,$$

for some $w \in Z_f$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} [x|z] &= [w + \alpha z|z] \\ &= [w|z] + \alpha [z|z] \\ &= \alpha [z|z], \end{aligned}$$

so that

$$\alpha = \frac{[x|z]}{[z|z]}.$$

Therefore

$$\begin{aligned} f(x) &= f(w + \alpha z) \\ &= f(w) + \alpha f(z) \\ &= \alpha f(z) \\ &= \frac{[x|z]}{[z|z]} f(z) \\ &= \left[x \left| \frac{f(z)z}{[z|z]} \right. \right]. \end{aligned}$$

Thus we let

$$y_f = \frac{f(z)z}{[z|z]},$$

so that

$$f(x) = [x|y_f] \quad \text{for all } \ x \in X.$$

Notice that, since $f \neq 0$ and $0 \neq z \in {Z_f}^{\perp}$, $y_f \neq 0$. Now, for all $x \in X$, we have

$$|f(x)| = |[x|y_f]| \leq \|x\| \, \|y_f\|,$$

so that $\|f\| \leq \|y_f\|$. On the other hand,

$$\|f\| \geq \frac{|f(y_f)|}{\|y_f\|} = \frac{[y_f|y_f]}{\|y_f\|} = \|y_f\|,$$

so that we actually have

$$\|f\| = \|y_f\|.$$

This completes the proof. □

Remark 4.5.4. In the setting of a reflexive normed space, Theorem 4.5.3 can be considered as an analogue of the Riesz Representation theorem. Further, in view of Corollary 4.3.3, the Riesz Representation theorem will be true in the case of general normed spaces whenever the continuous linear functional f on X is such that Z_f is proximinal, and hence in particular if Z_f is Chebyshev also.

We notice that Theorem 4.5.3 is not true in general. The following example shows that this result does not hold for nonreflexive normed spaces.

Example 4.5.5. Consider the real normed space $X = (c_{00}, \|\cdot\|_1)$ of all real sequences having only a finite number of nonzero entries, where $\|x\|_1 = \sum\limits_{k=1}^{\infty} |x_k|$ for $x = (x_1, x_2, x_3, ...) \in X$, together with the semi-inner product

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{\infty} \frac{x_k y_k}{|y_k|}, \ \ x, y \in X.$$

Define $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{x_k}{k}, \ \ x \in X.$$

Then f is linear. By Holder's inequality, for every $x \in X$, we have

$$|f(x)|^2 \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) \left(\sum_{k=1}^{\infty} |x_k|^2 \right) = \frac{\pi^2}{6} \|x\|^2,$$

so that f is continuous and $\|f\| \leq \pi/\sqrt{6}$. Hence $f \in X^*$. Let $y \in X$. Suppose that $f(x) = [x|y]$ for all $x \in X$. Then as $f \neq 0$, $y \neq 0$. Let $e_n = (0, ..., 0, 1, 0, 0, ...)$, where 1 occurs only in the n^{th} entry. Then $e_n \in X$, and for all $n = 1, 2, ...$, we have

$$\|y\|_1 \frac{y_n}{|y_n|} = [e_n|y] = f(e_n) = \frac{1}{n}.$$

Thus $\|y\|_1 = \pm 1/n$ for all $n = 1, 2, \ldots$, a contradiction. This shows that Theorem 4.5.3 does not hold for X.

CHAPTER 5

Birkhoff Orthogonality and a Revisit to the Characterization of Best Approximations

5.1 Introduction

The notion of orthogonality in an arbitrary normed space, with the norm not necessarily coming from an inner product, may be introduced in various ways as suggested by the mathematicians like B. D. Roberts, G. Birkhoff, S. O. Carlsson, C. R. Diminnie and R. C. James. Among these, the one which is frequently met with in the literature is the orthogonality due to G. Birkhoff [5] in 1935 (e.g., I. Singer [29], S. S. Dragomir [15], C. Benitez [3], J. Chmielinski [12] and so on). In this chapter, using the concept of Birkhoff orthogonality, we make a revisit to the characterization of best approximations seen so far in our discussion. Our main objective here is to derive some results on characterization of best approximations in normed spaces, especially from subspaces and their translates, in terms of Birkhoff orthogonality. We achieve this through the results which we have already seen in this regard in terms of semi-inner product orthogonality in the previous chapters.

The chapter begins with a brief discussion on Birkhoff orthogonality. Then the question of the equivalence of Birkhoff and semi-inner product orthogonalities is considered. It has been shown that [15] Birkhoff orthogonality is equivalent to semi-inner product orthogonality for some suitable semi-inner product on the normed space that generates the norm of the space. This enables us to reformulate our results characterizing best approximations in terms of semi-inner product orthogonality into those in terms of Birkhoff orthogonality, and in the process we recapture a well known characterization of best approximations due to I. Singer [29]. Some decomposition theorems in terms of Birkhoff orthogonality, and their consequences are also studied in this chapter.

As in the previous chapters, here also our discussion is limited to the case of real normed spaces.

5.2 Birkhoff Orthogonality

According to G. Birkhoff [5], the notion of orthogonality in an arbitrary normed space can be defined as follows.

Definition 5.2.1. Let $(X, \|\cdot\|)$ be a normed space, x, y be two given elements in X. Then x is said to be *Birkhoff orthogonal* to y, denoted by $x \perp y\,(\mathrm{B})$, if $\|x\| \leq \|x + \alpha y\|$ for all $\alpha \in \mathbb{R}$.

It is clear that if $(X, (\cdot, \cdot))$ is a real inner product space, then the usual orthogonality introduced by the inner product, i.e., $x \perp y$ if $(x, y) = 0$, is equivalent to Birkhoff orthogonality [29]. Indeed, if $(x, y) \neq 0$, then for

$$\alpha = -\frac{(x, y)}{(y, y)},$$

we have

$$\begin{aligned}
\|x + \alpha y\|^2 &= \left(x - \frac{(x, y)}{(y, y)} y,\, x - \frac{(x, y)}{(y, y)} y\right) \\
&= (x, x) - 2\frac{(x, y)^2}{(y, y)} + \frac{(x, y)^2}{(y, y)^2}(y, y) \\
&= (x, x) - \frac{(x, y)^2}{(y, y)}
\end{aligned}$$

$$\begin{aligned} &< (x, x) \\ &= \|x\|^2, \end{aligned}$$

so that x is not Birkhoff orthogonal to y. On the other hand, if $(x, y) = 0$, then for every $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|x + \alpha y\|^2 &= (x + \alpha y,\, x + \alpha y) \\ &= \|x\|^2 + |\alpha|^2 \|y\|^2 \\ &\geq \|x\|^2, \end{aligned}$$

so that x is Birkhoff orthogonal to y.

Let $(X, \|\cdot\|)$ be a normed space, $x, y, z \in X$ and $t \in \mathbb{R}$. The following are some easy consequences of the above definition of Birkhoff orthogonality.

(i) $0 \perp x\,(\mathrm{B})$ and $x \perp 0\,(\mathrm{B})$,
(ii) $x \perp x\,(\mathrm{B})$ if and only if $x = 0$, and
(iii) $x \perp y\,(\mathrm{B})$ implies that $x \perp (ty)\,(\mathrm{B})$.

However,

(iv) $x \perp y\,(\mathrm{B})$ need not imply that $y \perp x\,(\mathrm{B})$, and
(v) $x \perp y\,(\mathrm{B})$ and $x \perp z\,(\mathrm{B})$ need not imply that $x \perp (y + z)\,(\mathrm{B})$.

For example, consider the real normed space $X = (\mathbb{R}^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum_{k=1}^{2} |x_k|$ for $x = (x_1, x_2) \in X$. Let $x = (-2, 1)$, $y = (1, 1) \in X$. Then for all $\alpha \in \mathbb{R}$, we have

$$\|x + \alpha y\|_1 = |-2 + \alpha| + |1 + \alpha| \geq 3 = \|x\|_1,$$

so that $x \perp y\,(\mathrm{B})$. But for $\alpha = \frac{1}{2}$, we have

$$\|y + \alpha x\|_1 = \left\| \left(0, \frac{3}{2}\right) \right\|_1 = \frac{3}{2} \ngeq 2 = \|y\|_1,$$

so that y is not Birkhoff orthogonal to x. Also, if $x = (2, 2)$, $y = (5, -4)$ and

$z = (-3, 5)$ are in X, then for all $\alpha \in \mathbb{R}$, we have

$$\|x + \alpha y\|_1 = |2 + 5\alpha| + |2 - 4\alpha| \geq 4 = \|x\|_1 ,$$

and

$$\|x + \alpha z\|_1 = |2 - 3\alpha| + |2 + 5\alpha| \geq 4 = \|x\|_1 .$$

Hence $x \perp y$ (B) and $x \perp z$ (B). However, for $\alpha = -1$, we have

$$\|x + \alpha(y + z)\|_1 = \|(0, 1)\|_1 = 1 \not\geq 4 = \|x\|_1 ,$$

so that x is not Birkhoff orthogonal to $y + z$.

Another consequence of the above definition is given below [15]. For $x \in X$ and $G \subseteq X$, S_x denotes span $\{x\}$, and $d(x, G)$, the distance of x from G.

Theorem 5.2.2. *Let $(X, \|\cdot\|)$ be a normed space, and $x, y \in X$. Then $x \perp y$ (B) if and only if $\|x\| = d(x, S_y)$.*

Proof. If $x \perp y$ (B), then for every $\alpha \in \mathbb{R}$, we have

$$\|x\| \leq \|x + \alpha y\| \leq \|x\| + |\alpha| \, \|y\| ,$$

so that

$$\begin{aligned} \|x\| &= \inf \{\|x + \alpha y\| : \alpha \in \mathbb{R}\} \\ &= \inf \{\|x - z\| : z \in S_y\} \\ &= d(x, S_y). \end{aligned}$$

Conversely, if $\|x\| = d(x, S_y)$, then for every $z \in S_y$, we have

$$\|x\| \leq \|x - z\| ,$$

so that

$$\|x\| \leq \|x + \alpha y\| \quad \text{for all } \alpha \in \mathbb{R}.$$

Hence $x \perp y$ (B), and this completes the proof. □

Definition 5.2.3. Let $(X, \|\cdot\|)$ be a normed space, G a nonempty subset of X, and $x \in X$. We say that x is *Birkhoff orthogonal* to G, denoted by $x \perp G\,(\mathrm{B})$ if $x \perp y\,(\mathrm{B})$ for all $y \in G$.

By the definition, $0 \perp G\,(\mathrm{B})$ for every nonempty subset G of X.

Definitions 5.2.4. Let $(X, \|\cdot\|)$ be a normed space, and G a nonempty subset of X. Then the set $\{x \in X : x \perp G\,(\mathrm{B})\}$ is called the *Birkhoff orthogonal complement* of G, denoted by $G^{\perp}\,(\mathrm{B})$.

If $y \in X$, the Birkhoff orthogonal complement of y, denoted by $y^{\perp}\,(\mathrm{B})$, is the set $\{x \in X : x \perp y\,(\mathrm{B})\}$.

We have $G^{\perp}\,(\mathrm{B}) = \bigcap\limits_{y \in G} y^{\perp}\,(\mathrm{B})$, for

$$\begin{aligned} G^{\perp}\,(\mathrm{B}) &= \{x \in X : x \perp G\,(\mathrm{B})\} \\ &= \{x \in X : x \perp y\,(\mathrm{B}) \ \text{ for all } y \in G\} \\ &= \bigcap_{y \in G} \{x \in X : x \perp y\,(\mathrm{B})\} \\ &= \bigcap_{y \in G} y^{\perp}\,(\mathrm{B}). \end{aligned}$$

Some direct consequences of the above definitions are given below.

(a) $0^{\perp}\,(\mathrm{B}) = \mathrm{X}$, and $X^{\perp}(\mathrm{B}) = \{0\}$.

(b) If G is any nonempty subset of X, and t is any scalar, then,

- (i) $0 \in G^{\perp}(\mathrm{B})$,
- (ii) $x \in G^{\perp}(\mathrm{B})$ implies that $tx \in G^{\perp}(\mathrm{B})$, and
- (iii) generally, $G \cap G^{\perp}(\mathrm{B})$ is either empty or $\{0\}$.

(c) If C is a convex cone in X, we also have $C \cap C^{\perp}(\mathrm{B}) = \{0\}$. In particular, $M \cap M^{\perp}(\mathrm{B}) = \{0\}$ for any subspace M of X.

(d) More importantly, even if M is a subspace of X, $M^{\perp}(\mathrm{B})$ need not be a subspace of X. $M^{\perp}(\mathrm{B})$ is not even a convex cone in X. For example, consider the real normed space $X = (\mathbb{R}^2, \|\cdot\|_1)$, where $\|x\|_1 = \sum\limits_{k=1}^{2} |x_k|$ for $x = (x_1, x_2)$

$\in X$, and the subspace $M = \text{span}\{(1,1)\}$ of X. Then $M^{\perp}(\text{B})$ consists of all points $x = (x_1, x_2) \in X$ for which $|x_1| + |x_2| \leq |x_1 + \lambda| + |x_2 + \lambda|$ holds for all $\lambda \in \mathbb{R}$. Consider $x = (-3, 5)$ and $y = (4, -3)$ in X. Since for all $\lambda \in \mathbb{R}$, $|-3| + |5| = 8 \leq |-3+\lambda| + |5+\lambda|$, and $|4| + |-3| = 7 \leq |4+\lambda| + |-3+\lambda|$ hold, $(-3, 5)$ and $(4, -3)$ are in $M^{\perp}(\text{B})$. However the element $x + y = (1, 2)$ $\notin M^{\perp}(\text{B})$, since if $\lambda = -1$, we have $|(1+\lambda)| + |(2+\lambda)| = 1 \not\geq 3 = |1| + |2|$.

Among these observations, the one which is mentioned last is the crucial difference in comparison with the usual orthogonal complements in inner product spaces.

5.3 Equivalence of Birkhoff and Semi-Inner Product Orthogonalities

In this section we consider the equivalence of Birkhoff orthogonality and orthogonality relative to semi-inner products. We recall from second chapter (Definition 2.4.1) that if $(X\ \|\cdot\|)$ is a real normed space endowed with the semi-inner product $[\cdot|\cdot]$ which generates the norm $\|\cdot\|$, and if $x, y \in X$, then x is said to be orthogonal in the sense of Lumer-Giles relative to the semi-inner product $[\cdot|\cdot]$ to y, denoted by $x \perp y$, if $[y|x] = 0$. We also recall from first chapter that, in general, there may exist infinitely many distinct semi-inner products on a normed space which generate the norm of the space. Each of these semi-inner products gives rise to a semi-inner product orthogonality. Hence there may exist infinitely many different semi-inner product orthogonalities on a normed space. Consequently, in situations where semi-inner product orthogonalities are to be distinguished, we have to mention specifically the semi-inner product with respect to which each orthogonality is considered. Therefore we modify slightly our notations for orthogonality and orthogonal complement in the sense of Lumer-Giles relative to semi-inner products (see Definition 2.4.1, Definition 2.4.2 and Definitions 2.4.3) as follows.

Let $(X, \|\cdot\|)$ be a normed space, $[\cdot|\cdot]$ a semi-inner product on X that generates the norm $\|\cdot\|$, $x, y \in X$, and G a nonempty subset of X. The facts that x is orthogonal to y, and x is orthogonal to G are denoted respectively by $x \perp y\,([\cdot|\cdot])$

and $x \perp G\,([\cdot|\cdot])$. Thus

$$x \perp y\,([\cdot|\cdot]) \ \text{ if } \ [y|x] = 0$$

and

$$x \perp G\,([\cdot|\cdot]) \ \text{ if } \ [y|x] = 0 \ \text{ for all } \ y \in G.$$

The orthogonal complements of G and y are denoted by $G^{\perp}\,([\cdot|\cdot])$ and $y^{\perp}\,([\cdot|\cdot])$ respectively. Thus

$$G^{\perp}\ ([\cdot|\cdot]) := \{x \in X : x \perp G\ ([\cdot|\cdot])\}\,,$$

and

$$y^{\perp}\ ([\cdot|\cdot]) := \{x \in X : x \perp y\ ([\cdot|\cdot])\}\,.$$

We have $G^{\perp}\ ([\cdot|\cdot]) = \bigcap\limits_{y \in G} y^{\perp}\ ([\cdot|\cdot])\,.$

The following result [15] shows that semi-inner product orthogonality always implies Birkhoff orthogonality.

Theorem 5.3.1. *Let $(X, \|\cdot\|)$ be a normed space, $[\cdot|\cdot]$ a semi-inner product on X that generates the norm $\|\cdot\|$, and $x, y \in X$. If $x \perp y\ \ ([\cdot|\cdot])$, then $x \perp y\,(\mathrm{B})$.*

Proof. Assume that $x \perp y\ \ ([\cdot|\cdot])$. If $x = 0$, the result is trivial. If $x \neq 0$, then for all $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|x\|^2 &= [x|x] \\ &= [x + \alpha y|x]\,, \ \text{ since } \ [y|x] = 0 \\ &\leq \|x\|\,\|x + \alpha y\|\,, \end{aligned}$$

so that $\|x\| \leq \|x + \alpha y\|\,.$

Hence $x \perp y$ (B). □

We observe that the converse of the above result is not generally true, as is illustrated by the following example.

Example 5.3.2. Consider the real normed space $(\mathbb{R}^3, \|\cdot\|_1)\,,$ where $\|x\|_1 =$

$\sum_{k=1}^{3} |x_k|$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then

$$[x|y] = \|y\|_1 \sum_{\substack{k=1 \\ y_k \neq 0}}^{3} \frac{x_k y_k}{|y_k|}, \quad x, y \in \mathbb{R}^3$$

is a semi-inner product on $\mathbb{R}^3$ which generates the norm $\|\cdot\|_1$. Consider the elements $x = (1,1,0)$ and $y = (1,0,0)$ in $\mathbb{R}^3$. We have $\|y\|_1 = 1$, and $y + \alpha x = (1+\alpha, \alpha, 0)$ so that $\|y + \alpha x\|_1 = |1+\alpha| + |\alpha|$. Now

$$\|y\|_1 = 1 \leq |1+\alpha| + |\alpha| = \|y + \alpha x\|_1 \quad \text{for all} \quad \alpha \in \mathbb{R},$$

and hence $y \perp x$ (B). However,

$$[x|y] = 1 \neq 0,$$

which shows that y is not orthogonal to x in the sense of Lumer-Giles relative to the semi-inner product considered above.

Theorem 5.3.1 and Example 5.3.2 show that, in general, Birkhoff and semi-inner product orthogonalities are not equivalent on a normed space. However, as is suggested by the following result, there does exist at least one semi-inner product on a normed space which generates the norm, and for which the orthogonality relative to it is equivalent to Birkhoff orthogonality. The proof we provide here is a modified version of the one which is given by S. S. Dragomir [15].

Theorem 5.3.3. *Let $(X, \|\cdot\|)$ be a normed space, and $x, y \in X$. If $x \perp y$ (B), then there exists a semi-inner product $[\cdot|\cdot]$ on X which generates the norm $\|\cdot\|$ such that $x \perp y$ $([\cdot|\cdot])$.*

Proof. Assume that $x \perp y$ (B). If x=0, the result is true. If $x \neq 0$, consider the subspace $M := \text{span}\{y\} \oplus \text{span}\{x\}$, the direct sum of span $\{y\}$ and span $\{x\}$, of X. Define the functional $g : M \to \mathbb{R}$ by $g(m) = \lambda \|x\|^2$, where $m = z + \lambda x$ with $z \in \text{span}\{y\}$ and $\lambda \in \mathbb{R}$.

Let $m = z + \lambda x$ and $m' = z' + \lambda' x$ be in M, where $z, z' \in \text{span}\{y\}$ and $\lambda, \lambda' \in \mathbb{R}$. If $m = m'$, then we have $z - z' = (\lambda' - \lambda)x$ for all nonzero $x \in X$, where $z-z' \in \operatorname{span}\{y\}$ and $(\lambda' - \lambda)\, x \in \operatorname{span}\{x\}$. This implies, by the definition of M as the direct sum of two subspaces, that in particular $\lambda' - \lambda = 0$, so that $\lambda \|x\|^2 = \lambda' \|x\|^2$. Hence g is well defined on M. Further, for all $\mu, \mu' \in \mathbb{R}$, we have

$$\begin{aligned} g\left(\mu m + \mu' m'\right) &= g\left(\left(\mu z + \mu' z'\right) + \left(\mu\lambda + \mu'\lambda'\right) x\right) \\ &= \left(\mu\lambda + \mu'\lambda'\right) \|x\|^2, \quad \text{since } \mu z + \mu' z' \in \operatorname{span}\{y\} \\ &= \mu\lambda \|x\|^2 + \mu'\lambda' \|x\|^2 \\ &= \mu\, g(m) + \mu'\, g(m'). \end{aligned}$$

Hence g is also linear on M. Further, $g(x) = g(0 + 1 \cdot x) = \|x\|^2$, and $g(y) = g(y + 0 \cdot x) = 0$. Now for all $m = z + \lambda x \in M$, where $z \in \operatorname{span}\{y\}$ and $0 \neq \lambda \in \mathbb{R}$, we have,

$$\begin{aligned} \frac{|g(m)|}{\|m\|} &= \frac{|\lambda| \|x\|^2}{\|z + \lambda x\|} \\ &= \frac{\|x\|^2}{\left\|x + \frac{1}{\lambda} z\right\|} \\ &= \frac{\|x\|^2}{\left\|x + \frac{\mu}{\lambda} y\right\|}, \quad \text{for some } \mu \in \mathbb{R}, \quad \text{since } z \in \operatorname{span}\{y\} \\ &\leq \frac{\|x\|^2}{\|x\|}, \quad \text{since } x \perp y\, (\mathrm{B}) \\ &= \|x\|, \end{aligned}$$

so that $|g(m)| \leq \|x\| \|m\|$. Hence g is bounded on M and $\|g\| \leq \|x\|$. On the other hand,

$$\|g\| \geq \frac{|g(x)|}{\|x\|} = \frac{\|x\|^2}{\|x\|} = \|x\|,$$

and thus actually $\|g\| = \|x\|$. Consequently, by virtue of the Hahn-Banach extension theorem, there exists a functional $f \in X^*$ which extends g to the whole of X such that $\|f\| = \|g\| = \|x\|$. Then, since $x, y \in M$, $f(x) = g(x) = \|x\|^2$ and $f(y) = g(y) = 0$. Hence $f \in \mathcal{J}(x)$, where $\mathcal{J}$ is the normalized duality mapping of X. This shows, since $\mathcal{J}(0) = \{0\}$, that $\mathcal{J}(x)$ is a nonempty subset of X^* for every $x \in X$, so that $\{\mathcal{J}(x)\}_{x \in X}$ is a nonempty class of nonempty sets. Hence, by the axiom of choice, a set can be formed which contains precisely one element, say f_x, taken from each set $\mathcal{J}(x)$. This determines a section $\widetilde{\mathcal{J}}:X \to X^*$ of $\mathcal{J}$ defined by $\widetilde{\mathcal{J}}(x) = f_x$ for all $x \in X$. Since f_x is a Hahn-Banach extension of $g \in M^*$ to X, and since $x, y \in M$, we have $f_x(x) = g(x) = \|x\|^2$ and $f_x(y) = g(y) = 0$. Then, by Theorem 1.4.1,

$$[u|v] := (\widetilde{\mathcal{J}}(v))(u), \quad u,\, v \in X$$

is a semi-inner product on X that generates the norm $\|\cdot\|$. Consequently,

$$[y|x] = (\widetilde{\mathcal{J}}(x))(y) = f_x(y) = 0,$$

so that $x \perp y\,([\cdot|\cdot])$, and this completes the proof. □

Combining Theorem 5.3.1 and Theorem 5.3.3, we can formulate the exact connection between Birkhoff orthogonality and orthogonality relative to semi-inner products as given below.

Theorem 5.3.4. *Let $(X, \|\cdot\|)$ be a normed space, and $x, y \in X$. Then the following statements are equivalent:*

(a) $x \perp y\,(\mathrm{B})$*;*

(b) *There exists a semi-inner product $[\cdot|\cdot]$ on X which generates the norm $\|\cdot\|$ such that $x \perp y\,([\cdot|\cdot])$.*

Proof. (a) $\Rightarrow$ (b) follows by Theorem 5.3.3, and (b) $\Rightarrow$ (a) follows by Theorem 5.3.1. □

Remark 5.3.5. The above theorem shows that on any normed space $(X, \|\cdot\|)$, there always exist at least one semi-inner product that generates the norm $\|\cdot\|$, and for which Birkhoff orthogonality is equivalent to semi-inner product orthogonality. However, such semi-inner products need not be unique (see proof of Theorem 5.3.3), since uniqueness of Hahn-Banach extensions is not generally guaranteed in the case of normed spaces. In the case of normed spaces on which the existence of a unique semi-inner product is assured (e.g., smooth normed spaces), Birkhoff orthogonality is nothing other than semi-inner product orthogonality.

Theorem 5.3.4 has the following corollary. $\mathcal{J}(X)$ denotes the class of all semi-inner products on a normed space X that generate the norm of X.

Corollary 5.3.6. *Let $(X, \|\cdot\|)$ be a normed space, and $[\cdot|\cdot]$ a semi-inner product on X that generates the norm $\|\cdot\|$. Then the following statements are true:*

(a) If $y \in X$, then $y^{\perp}(\mathrm{B}) = \bigcup_{[\cdot|\cdot] \in \mathcal{J}(X)} y^{\perp}([.|.])$;

(b) If G is a nonempty subset of X, then $G^{\perp}(\mathrm{B}) = \bigcup_{[\cdot|\cdot] \in \mathcal{J}(X)} G^{\perp}([.|.])$.

Proof. (a) It follows from Theorem 5.3.4 that

$$\begin{aligned} x \in y^{\perp}(\mathrm{B}) &\Leftrightarrow x \perp y\,(\mathrm{B}) \\ &\Leftrightarrow x \perp y\,([\cdot|\cdot]) \quad \text{for some} \quad [\cdot|\cdot] \in \mathcal{J}(X) \\ &\Leftrightarrow x \in y^{\perp}([\cdot|\cdot]) \quad \text{for some} \quad [\cdot|\cdot] \in \mathcal{J}(X) \\ &\Leftrightarrow x \in \bigcup_{[\cdot|\cdot] \in \mathcal{J}(X)} y^{\perp}([\cdot|\cdot]). \end{aligned}$$

Hence (a) holds.

(b) We have

$$\begin{aligned} G^{\perp}(\mathrm{B}) &= \bigcap_{y \in G} y^{\perp}(\mathrm{B}) \\ &= \bigcap_{y \in G} \bigcup_{[\cdot|\cdot] \in \mathcal{J}(X)} y^{\perp}([\cdot|\cdot]), \quad \text{by (a) above} \\ &= \bigcup_{[\cdot|\cdot] \in \mathcal{J}(X)} \bigcap_{y \in G} y^{\perp}([\cdot|\cdot]) \\ &= \bigcup_{[\cdot|\cdot] \in \mathcal{J}(X)} G^{\perp}([\cdot|\cdot]). \end{aligned}$$

Hence holds (b) also. □

Remark 5.3.7. Let $(X, \|\cdot\|)$ be a normed space. If Birkhoff orthogonality on X is equivalent to the orthogonality relative to some semi-inner product $[\cdot|\cdot]$ on X that generates the norm $\|\cdot\|$, then

(i) $y^{\perp}(\mathrm{B}) = y^{\perp}([\cdot|\cdot])$ for every $y \in X$, and

(ii) $G^{\perp}(\mathrm{B}) = G^{\perp}([\cdot|\cdot])$ for every nonempty subset G of X.

5.4 Characterizations in terms of Birkhoff Orthogonality

Our intention in this section is to derive a few results characterizing best approximations in normed spaces in terms of Birkhoff orthogonality. Here we make a revisit to the characterizations of best approximations which we have already seen in our discussion, and reformulate some of them in terms of Birkhoff orthogonality using Theorem 5.3.4. One of the main benefits in doing so is that we can actually recapture a well known characterization result of best approximation due to I. Singer [29].

Observe that all the results characterizing best approximations in normed spaces which we have come across so far in our discussion are in terms of some, in fact any, semi-inner product on the normed space that generates the norm of the space. Hence Theorem 5.3.4 enables us to treat all those results actually in terms of any one of those particular semi-inner products on the normed space which generate the norm of the space and for which Birkhoff orthogonality and semi-inner product orthogonality are equivalent. When these two orthogonalities are equivalent, their orthogonal complements coincide. This shows that we can very well replace all the orthogonal complements relative to semi-inner product in the characterization results seen so far by Birkhoff orthogonal complement without causing any other modification. We employ this procedure in arriving at some results characterizing best approximations in normed spaces in terms of Birkhoff orthogonality. Though almost all the characterization results seen so far can be reformulated using this procedure, here we concentrate only on results for subspaces and their translates.

Our result which characterizes best approximations from translates of subspaces in terms of Birkhoff orthogonality is given below. It is deduced from Theorem 2.5.7.

Theorem 5.4.1. *Let $(X, \|\cdot\|)$ be a normed space, M a subspace of X, $z \in X$, and $K = z + M$. Suppose also that $x \in X$ and $y_0 \in K$. Then the following* *statements are equivalent:*

(a) $y_0 \in P_K(x)$;
(b) $x - y_0 - \lambda\,(z + y - y_0) \in M^{\perp}(\mathrm{B})$ for all $y \in M$ and all $\lambda \in [0,1]$;
(c) $x - y_0 \in M^{\perp}(\mathrm{B})$.

Proof. By virtue of Theorem 5.3.4, the proof follows from Theorem 2.5.7 on replacing $M^{\perp}$ by $M^{\perp}(\mathrm{B})$. □

As a corollary to the above theorem, we have the following result for subspaces.

Corollary 5.4.2. *Let $(X, \|\cdot\|)$ be a normed space, M a subspace of X, $x \in X$, and $y_0 \in M$. Then the following statements are equivalent:*

(a) $y_0 \in P_M(x)$;
(b) $x - y_0 - \lambda\,(y - y_0) \in M^{\perp}(\mathrm{B})$ for all $y \in M$ and all $\lambda \in [0,1]$;
(c) $x - y_0 \in M^{\perp}(\mathrm{B})$.

Proof. The proof is a consequence of letting $z = 0$ in Theorem 5.4.1. □

Remark 5.4.3. The above corollary can also be deduced directly from Theorem 2.5.3 for subspaces using Theorem 5.3.4.

Remark 5.4.4. At this juncture we notice that the following characterization of best approximations in normed spaces in terms of Birkhoff orthogonality due to I. Singer [29] is there in the literature.

Let $(X, \|\cdot\|)$ be a normed space, M a subspace of X, $x \in X$, and $y_0 \in M$. Then $y_0 \in P_M(x)$ if and only if $x - y_0 \perp M(B)$.

By our characterization result ((a) $\Leftrightarrow$ (c) of Corollary 5.4.2), we have

$$y_0 \in P_M(x) \Leftrightarrow x - y_0 \in M^{\perp}(\mathrm{B}).$$

However,

$$x - y_0 \in M^{\perp}(\mathrm{B}) \Leftrightarrow x - y_0 \perp y(\mathrm{B}) \text{ for all } y \in M \Leftrightarrow x - y_0 \perp M(\mathrm{B}).$$

This shows that through our characterization result, namely Corollary 5.4.2, we have actually recaptured the above mentioned result characterizing best approximations due to I. Singer [29].

5.5 Decomposition Theorems in terms of Birkhoff Orthogonality

In the above section we have seen how to reformulate results in terms of semi-inner product orthogonality into those in terms of Birkoff orthogonality using Theorem 5.3.4, and a few results thus obtained. The present section contains some more results in terms of Birkoff orthogonality which are arrived at through the corresponding results in terms of semi-inner product orthogonality using again Theorem 5.3.4. Our emphasis here is on decomposition theorems and their consequences.

Our discussion begins with some decomposition results which characterize proximinality as well as Chebyshevity of subspaces and their translates in terms of Birkoff orthogonality. The result below for translates of subspaces is a consequence of Theorem 4.3.2.

Theorem 5.5.1. *Let $(X, \|\cdot\|)$ be a normed space, M a subspace of X, $z \in X$, and $M' = z + M$. Then M' is proximinal (respectively Chebyshev) if and only if $X = M' + M^{\perp}\,(\mathrm{B})$ (respectively $X = M' \oplus M^{\perp}\,(\mathrm{B})$).*

Proof. Because of Theorem 5.3.4, the proof follows from Theorem 4.3.2 on replacing $M^{\perp}$ by $M^{\perp}\,(\mathrm{B})$. □

We have the following result for subspaces as a corollary to the above theorem.

Corollary 5.5.2. *Let $(X, \|\cdot\|)$ be a normed space, and M a subspace of X. Then M is proximinal (respectively Chebyshev) if and only if $X = M + M^{\perp}\,(\mathrm{B})$ (respectively $X = M \oplus M^{\perp}\,(\mathrm{B})$).*

Proof. The proof follows from Theorem 5.5.1 on letting $z = 0$. □

Remark 5.5.3. The above corollary can also be arrived at directly from Corollary 4.3.3 for subspaces using Theorem 5.3.4.

Our next result is in the setting of a reflexive normed space. It provides a characterization of closed subspaces of reflexive normed spaces in terms of Birkhoff orthogonality. We deduce the result from Theorem 4.3.7.

Theorem 5.5.4. *Let $(X, \|\cdot\|)$ be a reflexive normed space, and M a subspace of X. Then M is closed if and only if $X = M + M^{\perp}\,(\mathrm{B})$.*

Proof. By virtue of Theorem 5.3.4, the proof follows from Theorem 4.3.7 on replacing $M^{\perp}$ by $M^{\perp}(\mathrm{B})$. □

If the reflexive normed space in the above theorem is strictly convex also, then the decomposition of the space as a sum becomes actually a direct sum decomposition. The resultant decomposition result, which is given below, provides a characterization of closed subspaces of strictly convex reflexive normed spaces in terms of Birkhoff orthogonality. We derive the result from Theorem 4.3.8.

Theorem 5.5.5. *Let $(X, \|\cdot\|)$ be a strictly convex reflexive normed space, and M a subspace of X. Then M is closed if and only if $X = M \oplus M^{\perp}\,(\mathrm{B})$.*

Proof. Because of Theorem 5.3.4, the proof follows from Theorem 4.3.8 on replacing $M^{\perp}$ by $M^{\perp}\,(\mathrm{B})$. □

Remark 5.5.6. In the framework of a reflexive normed space, Theorem 5.5.4 can be treated as an analogue of the Projection theorem in terms of Birkhoff orthogonality. Same is the case with Theorem 5.5.5 in the setting of a strictly convex reflexive normed space.

It may be noticed that Theorem 5.5.4 and Theorem 5.5.5 do not hold for nonreflexive normed spaces (see Example 4.3.10).

Now let us consider some consequences of our present decomposition theorems in terms of Birkhoff orthogonality. As above, the results we present here also are deduced from the corresponding results in terms of semi-inner product orthogonality with the help of Theorem 5.3.4. Thus, the results that follow in this section, namely Theorem 5.5.7, Corollary 5.5.8, Theorem 5.5.9, Theorem 5.5.10, Corollary 5.5.11 and Theorem 5.5.12, are actually reformulated versions in terms of Birkhoff orthogonality of Theorem 4.4.1, Corollary 4.4.2, Theorem

4.4.4, Theorem 4.4.5, Corollary 4.4.6 and Theorem 4.4.8 respectively using Theorem 5.3.4. Hence no separate proof is given here for any of these results.

As a consequence of Corollary 5.5.2, we have the following characterization of dense proximinal subspaces of a normed space in terms of Birkhoff orthogonality condition.

Theorem 5.5.7. *Let $(X, \|\cdot\|)$ be a normed space, and M a proximinal subspace of X. Then M is dense in X if and only if $M^{\perp}(\mathrm{B}) = \{0\}$.*

The following result which characterizes denseness of Chebyshev subspaces of normed spaces in terms of Birkhoff orthogonality condition is also a consequence of Corollary 5.5.2. It actually becomes a corollary to the above theorem.

Corollary 5.5.8. *Let $(X, \|\cdot\|)$ be a normed space, and M a Chebyshev subspace of X. Then M is dense in X if and only if $M^{\perp}(\mathrm{B}) = \{0\}$.*

Our next result provides a similar characterization of dense subspaces of a reflexive normed space in terms of Birkhoff orthogonality condition. It is a consequence of Theorem 5.5.4.

Theorem 5.5.9. *Let $(X, \|\cdot\|)$ be a reflexive normed space, and M a subspace of X. Then M is dense in X if and only if $M^{\perp}(\mathrm{B}) = \{0\}$.*

Another consequence of Corollary 5.5.2 is given below.

Theorem 5.5.10. *Let $(X, \|\cdot\|)$ be a normed space, and M a proper proximinal subspace of X. Then $M^{\perp}(\mathrm{B})$ contains a nonzero element.*

Yet another consequence of Corollary 5.5.2 is given below. It becomes actually a corollary to the above theorem.

Corollary 5.5.11. *Let $(X, \|\cdot\|)$ be a normed space, and M a proper Chebyshev subspace of X. Then $M^{\perp}(\mathrm{B})$ contains a nonzero element.*

A result similar to the last two in the setting of a reflexive normed space is given below. It is a consequence of Theorem 5.5.4.

Theorem 5.5.12. *Let $(X, \|\cdot\|)$ be a reflexive normed space, and M a proper closed subspace of X. Then $M^{\perp}(\mathrm{B})$ contains a nonzero element.*

Epilogue

Some of the problems and possibilities that were thought about, and where further research work is possible, are discussed below briefly. The problems that we mention here are closely related in one way or another to the work which we have carried out in the thesis.

In Section 3.4 we have introduced the concepts of *ordered orthogonal sets* and *ordered orthonormal sets* in the setting of a normed space. It is shown in Theorem 3.4.6 that an ordered orthogonal set of nonzero elements in a normed space is linearly independent, and hence in particular, an ordered orthonormal set also is linearly independent. Regarding the converse of the problem, we have the *ordered orthonormalization* results in Theorem 3.5.2 for general normed spaces, and Corollary 3.5.4 for strictly convex normed spaces. In those results, we have constructed ordered orthonormal sets from given countable linearly independent sets. However, no algorithm for the actual construction of ordered orthonormal sets is provided there. Formulating an algorithm to this effect demands further research.

As a generalization of the notion of orthonormal basis for inner product spaces, one can think of introducing the concept of an *'ordered orthonormal basis'* for normed spaces in terms of semi-inner products that generate the norm of the space. Once it is done, one can very well attempt to find out some necessary and sufficient conditions under which an ordered orthonormal set is, in fact, an ordered orthonormal basis. This may allow us to speak of *'Fourier expansion'* in the setting of general normed spaces. This actually will result in the emergence of a vast and prolific area of research. This may even help us in the formulation of the algorithm mentioned above.

We have established the analogues of the Projection theorem for reflexive normed spaces and strictly convex reflexive normed spaces in Theorem 4.3.7 and Theorem 4.3.8 respectively. Explorations may be made to see whether one can introduce the concept of an *'ordered orthogonal projection'* in the setting of a general normed space as a generalization of the notion of orthogonal projection for inner product spaces, and if possible to proceed with further research in this area.

We have obtained an analogue of the Riesz Representation theorem in the setting of a reflexive normed space in Theorem 4.5.3. We noticed that (Remark 4.5.4) the Riesz Representation theorem holds in the case of general normed spaces whenever the continuous linear functional f on X is such that Z_f is proximinal, and hence in particular if Z_f is Chebyshev. It will be interesting to investigate those continuous linear functionals f on a general normed space X for which Z_f is proximinal, and in particular Z_f is Chebyshev.

By Theorem 5.3.4, we have seen that on any normed space, there always exists at least one semi-inner product that generates the norm of the space such that the orthogonality relative to this semi-inner product is equivalent to Birkhoff orthogonality. In general, infinitely many such semi-inner products are there on a general normed space, since uniqueness of Hahn-Banach extensions is not guaranteed generally. Investigations can be made to identify those semi-inner products on a normed space for which the two orthogonalities coincide. Attempts can also be made to characterize those normed spaces on which there exists a unique semi-inner product that generates the norm of the space such that the orthogonality relative to this semi-inner product is equivalent to Birkhoff orthogonality.

One important and interesting area which we have not dared to touch at all in this context is the study of linear transformations.

S. S. Dragomir has introduced the concept of a smooth normed space of (N)-type [15] as follows. Let $(X, \|\cdot\|)$ be a smooth normed space, and $[\cdot|\cdot]$ a semi-inner product on X that generates the norm $\|\cdot\|$ of X. Then X is said to be of (N)-*type*, if the semi-inner product $[\cdot|\cdot]$ satisfies the condition,

$$|[x|y+z]| \leq |[x|y]| + |[x|z]|$$

for all $x, y, z \in X$. It is obvious that any inner product space is a smooth normed space of (N)-type. It is an open problem whether the property (N) is characteristic for inner product spaces.

One can attempt a study on characterization of best approximations in smooth normed spaces of (N)-type in terms of semi-inner products. Analyzing the similarities and differences of the studies on characterization of best approximations in inner product spaces, smooth normed spaces of (N)-type in terms of semi-inner products, and normed spaces in terms of semi-inner products will be quite interesting.

These are just a few of the many problems that can be addressed in connection with our present work. Investigations of these problems itself will generate multitudes of problems and possibilities that demand further research.

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