NEIGHBOURS IN THE LATTICE OF TOPOLOGIES

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CERTIFICATE

This is to certify that the Thesis entitled "**Neighbours in the Lattice of Topologies**" submitted by Sri. Joymon Joseph P., Department of Mathematics, University of Calicut, is the bonafide record of the research work undertaken by him under my supervision and that no part thereof has been submitted for any other degree elsewhere.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis

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CONTENTS

INTRODUCTION

The collection $\Sigma(X)$ of all topologies on a fixed non-empty set X is a partially ordered set under the natural partial order of set inclusion. The discrete and indiscrete topologies are the smallest and largest elements in $\Sigma(X)$ respectively.

The intersection of an arbitrary collection of topologies on the set X is a topology on X. So under the partial ordering, any subset of the collection of all topologies on the set X has an infimum, namely their set theoretic intersection. Hence the collection $\Sigma(X)$ of all topologies on the set X is a complete lattice under the partial order mentioned above ([6] Theorem 3, page 112).

The two lattice operations join and meet of the lattice Σ(X) can be described as follows. The meet of a collection of topologies on X is simply their set theoretic intersection. The join of a collection of topologies is the topology generated by the union of all the topologies in the collection.

The study of lattice of topologies and its sublattices is one of the interesting areas in point set topology. As mentioned earlier, G. Birkhoff [6] and R. Vaidynathaswamy [32] initiated the study of the lattice theoretic properties of Σ(X)and some of its sublattices. Following them many authors studied the lattice properties of the lattice of topologies from different perspectives.

If L is a lattice, and a and b are elements of L, then b is said to be an upper neighbour of a if $a < b$, and $a \leq c \leq b$, $c \in L$ implies that $a = c$ or $c = b$. An element a is said to be a lower neighbour of an element b in a lattice, when the element b is an upper neighbour of a.

An atom in a lattice is any element 'a' in the lattice which is an upper neighbour of the smallest element. Dually a dual atom in a lattice is any element in the lattice which is lower neighbour of the largest element.

A lattice such that every element other than the smallest can be written as the join of atoms smaller than or equal to the element is called an atomic lattice. A dually atomic lattice is defined dually.

There are atoms in the lattice $\Sigma(X)$. The atoms of the lattice $\Sigma(X)$ are precisely the topologies of the form $\{\phi, X, A\}$ where A is a proper nonempty subset of the set X. In the lattice $\Sigma(X)$ every element other than the indiscrete topology can be written as a join of atoms smaller than or equal to it. Hence the lattice of topologies $\Sigma(X)$ is an atomic lattice [32].

If τ is a topology on X and if A is a nonempty subset of X, then the simple expansion of the topology τ by the set A is the join $\tau \vee \{\varnothing, X, A\}$ in the lattice of topologies. It is denoted by $\tau(A)$. The topology $\tau(A)$ can be represented in the following form

$\tau(A) = \{ U \cup (V \cap A)$; where U and V are open in the topology $\tau \}$ [12].

N. Levine [17] initiated the investigation of the properties of $\tau(A)$ in relation with the properties of the topology τ and the set A.

In the lattice $\Sigma(X)$, there are dual atoms also. The dual atoms of $\Sigma(X)$, which are the lower neighbours of the discrete topology, are called ultra topologies. The ultra topologies in $\Sigma(X)$ are described below using ultra filters on X.

A filter on a set X is a nonempty collection of nonempty subsets of X such that the intersection of two elements in it is again a member of the collection and a superset of a member of the collection is also a member of the collection

There are two types of filters. When the intersection of all the members in a filter is non-empty, it is called a fixed filter. When this intersection is empty, it is called free filter.

The collection of all filters on a fixed set is ordered under the natural order of set inclusion. There are maximal members in this collection and these maximal members are called ultrafilters.

Fixed ultra filters are also called principal ultra filters. The principal ultrafilters are of the form $\mathcal{U}(a) = \{ A \subseteq X : a \in A\}$ for an element a in X.

The ultratopologies are topologies of the form $\wp(X \setminus \{x\}) \cup \mathcal{U}$, for $x \in X$ where $\mathcal{U}$ is an ultrafilter on X such that $\{x\}$ is not a member of $\mathcal{U}$. This ultratopology is denoted as $\Im (x , \mathcal{U})$. This ultra filter $\mathcal{U}$ is called the associated ultra filter of the ultra topology $\Im (x , \mathcal{U})$. In the lattice $\Sigma(X)$ every non discrete topology can be written as the greatest lower bound of the ultratopologies finer than it. So the lattice of topologies is dually atomic [11].

Corresponding to simple expansion of a topology, the simple reduction of topology τ is defined as the greatest lower bound with an ultra topology[22].

Also the collection of all T_1 topologies forms a sublattice of $\Sigma(X)$[7]. This sub lattice is denoted by $\Lambda(X)$. The sublattice $\Lambda(X)$ has properties very similar to that of $\Sigma(X)$. It has a least element. It is called the cofinite topology on X. The non-empty closed sets of this topology are the finite subsets of X. This topology is sometimes called the minimum T_1 topology on X. The greatest element of the lattice of T_1 topologies on X is also the discrete topology on X.

There are atoms and dual atoms in the lattice $\Lambda(X)$. The atoms of Λ are unions of the singletons of the set X with the co-finite topology. Since there are topologies on infinite sets without singleton open sets, this lattice is not atomic.

There are dual atoms in this lattice when X is infinite. The dual atoms of $\Lambda(X)$ are of the form $\Im(x, \mathcal{U}) = \wp(X \setminus \{x\}) \cup \mathcal{U}$, for $x \in X$ where $\mathcal{U}$ is a non-principal ultrafilter on X. This lattice is also dually atomic.

There are many interesting sublattices of the lattice of the topologies. As noted earlier, the lattice of all T_1 topologies is one of the important sublattices. Other important sublattices of the lattice of the topologies are the lattice of all principal topologies, the lattice of all regular topologies, the lattice of all completely regular topologies, the lattice of all countably accessible topologies etc. (See [13], [14], [15]).

The properties upper semimodularity and lower semimodularity are defined in terms of the neighbours of an element. Specifically a lattice L is called upper semimodular if for all elements a, b in L such that a is a lower neighbour of $a \wedge b$ implies $a \vee b$ is an upper neighbour of b. The lattice L is lower semimodular if its dual lattice is upper semimodular.

Most of the works appeared on the lattice of topologies on a fixed set discuss properties like complementation, distributivity, modularity, lattice morphisms etc. But authors like R. E. Larson, W. J. Thron, R. Valent and others. (See [35] , [31], [34], [21] etc.) deviated from this main stream of study and they attempted to study properties like upper modularity, lower modularity, upper and lower semi modularity, embedding different types of lattices in the lattice of topologies, existence of upper and lower neighbours etc.

The study of upper or lower neighbours is closely connected with study of properties like upper semimodularity, lower semimodularity. Hence a study of upper or lower neighbours a topology are important.

In their survey article R. E. Larson and Susan J. Andima [15] observed the following. "If X is an infinite set and P is any topological property, then the collection of topologies in the lattice of the topologies possessing the property P may be identified simply from the lattice structure of the lattice of the topologies. This follows from the theorem that for an infinite X, the group of lattice automorphisms of the lattice of the topologies is isomorphic to the symmetric group on X.

Therefore the only automorphisms of the lattice of the topologies for infinite X are those which simply permute the elements of X. Therefore any automorphism of the lattice of the topologies must map all the topologies in the lattice of the topologies onto homeomorphic images. *Thus the topological properties of the elements the lattice of the topologies must be determined by the position of topologies in the lattice of the topologies.*" This means study of the neighbours of topologies is important and this thesis is an attempt in this direction.

We can view this problem from another perspective also. The collection of subsets of a topological space containing a point in their interior is a filter. We call this filter as the neighbourhood filter of that point with respect to the topology. The topology of a set can be charecterised using the neighbourhood filters of its points. So two different topologies have different neighbourhood filters atleast one point. In other words altering neighbourhood filters means altering topologies. This thesis attempts to study how to alter the neighbourhood filter of a point or minimum number of points. A typical case is the neighbourhood filters of all points except one point are the same. Here our intention is to study how to alter neighbourhoods of minimum number of points.

In the first chapter we discuss upper or lower neighbours for general lattices. A theorem and its dual about neighbours are proved in general lattices. Using the theorem it is proved that in an atomic modular lattice every element has an upper neighbour and in a dually atomic modular lattice every element has a lower neighbour. The results are in such a general fashion that they are also applicable to some of the sublattices of the lattice of topologies. Using the above results we charecterise the lower neighbours of ultra topologies as the intersection of two ultra topologies.

In the second chapter we study upper neighbours in the Lattice of Topologies. An equivalence relation on the non open sets is defined on the collection of all non open subsets. Then an order relation is defined among the equivalence classes. Using this order relation the upper neighbours in Lattice of Topologies are charecterised. Using these results we determine that a special interval in the lattice of topologies is a Boolean Algebra generated by a subset.

In the third chapter we conduct a similar study of lower neighbours in the lattice of topologies. The results in this chapter are generalisations of results given by Larson and Thron from the lattice of T_1 topologies to Lattice of Topologies.

In the fourth chapter we consider topologies without upper neighbours. We generalise some existing results and prove that no countably accessible topology possesses upper neighbours in the Lattice of Topologies. We also compare the Lattice of Topologies with the lattice of Cech Closure Operators in this context.

CHAPTER 1

PRELIMINARIES

1.1 Covering relations in lattices

A partially ordered set in which every pair of elements has an infimum and a supremum is called a lattice. A partially ordered set in which an arbitrary set of elements has an infimum and a supremum is called a complete lattice. If a lattice has a least element, then it is unique. The least element of a lattice is usually denoted by 0. Any element of a lattice which is an upper neighbour of the least element is called an atom. A lattice is called an atomic lattice if every element of the lattice except the least element can be written as the least upper bound of a collection of atoms.

If a lattice has a greatest element then this greatest element is unique. The greatest element of a lattice is usually denoted by 1. Any lower neighbour of the greatest element is called an dual –atom or anti atom. A lattice is called dually-atomic or anti atomic if every element other than the greatest element can be written as the greatest lower bound of dual-atoms.

In an arbitrary lattice the following three identities are equivalent [6].

1. $(x\wedge y)\vee (x\wedge z)\vee (y\wedge z) = (x\vee y) \wedge (x\vee z) \wedge(y \vee z)$ for all x, y, and z in the lattice

2. $x\wedge (y \vee z) = (x\wedge y)\vee (x\wedge z)$, for all x, y, and z in the lattice

3. $x\vee (y\wedge z) = (x\vee y) \wedge (x\vee z)$. for all x, y, and z in the lattice

A lattice in which any three elements x, y, z satisfy one (and hence all) of the above identities is called a distributive lattice.

Similarly in an arbitrary lattice the following identities are equivalent [6].

1. $(x\vee y) \wedge ((x\wedge y) \vee z)) = (x\wedge y)\vee((x\vee y)\wedge z)$ for all x, y, and z in the lattice

2. $x \geq y$ implies $x\wedge (y \vee z) = y\vee (x\wedge z)$, for all x, y, and z in the lattice

A lattice in which any three elements x, y, z satisfy one (and hence all) of the above identities is called a modular lattice.

A distributive lattice is always modular. There are modular lattices which are not distributive.

First we obtain a convenient method to determine covers in lattices. In atomic modular lattices this methods becomes a handy one.

1.1.1 Theorem

Let L be a lattice and a and b be any two elements of L such that a is a lower neighbour of b. Let c be any element of the lattice such that the interval $[a\wedge c, b\vee c]$ in L is a modular interval and let L_1 be a subset of L such that every element of the interval $[a\wedge c, b\vee c]$ in L can be written as join of some subset of L_1. Then the element $a \wedge c$ is a lower neighbour of $b\wedge c$.

Proof

Let the element a be a lower neighbour of the element b in L and let c be any element of the lattice such that the interval $[a\wedge c, b\vee c]$ in L is a modular interval.

Let d be an element of the lattice such that $a \wedge c < d \leq b \wedge c$. Since $a \wedge c < d$, by the property of the set L_1 that every element of the interval

$[a \wedge c, b \vee c]$ is a join of some subset of L_1, there is an element e of the set L_1 such that e neither less than nor equal to $a \wedge c$ and e is less than or equal to d. If $e \leq a$ also then $e \leq d \leq b \wedge c \leq c$ and this means $e \leq a \wedge c$, a contradiction to the choice of e. Hence e is neither less than nor equal to a.

So we have $a < a \vee d \leq b$,since $e < a \vee d$, e not less than a. Since a is a lower neighbour of b we have $a \vee d = b$. Also we have $d \leq c$. So the elements a,b and d all belong to the interval $[a \wedge c, b \vee c]$. By assumption this interval is modular and we have $d \leq c$.

Hence by the modular law we have

$$d \vee (a \wedge c) = (d \vee a) \wedge c$$

So we have $b \wedge c = (a \vee d) \wedge c$

$$= (d \vee a) \wedge c$$

$$= d \vee (a \wedge c)$$

$$= d \ , \quad \text{since } (a \wedge c) < d.$$

Hence $b \wedge c = d$.

So $a \wedge c$ is a lower neighbour of $b \wedge c$.

Dually we have the following theorem

1.1.2 Theorem

Let L be a lattice and let a and b be any two elements of L such that a is a lower neighbour of b. Let c be any element of the lattice such that the interval $[a \wedge c, b \vee c]$ in L is a modular interval and let L_1 be a subset of L such that every element of the interval $[a \wedge c, b \vee c]$ in L can be written as the meet of some subset of L_1. Then the element $a \vee c$ is a lower neighbour of $b \vee c$.

1.1.3 Remarks

a) In an atomic lattice every element is a join of atoms. Hence in theorem 1.1.1 the set L_1 can be taken as the collection of all atoms in the lattice. There fore we have the following corollary.

Let L be an atomic lattice. Let a and b be any two elements of L such that a is a lower neighbour of b. Let c be any element of the lattice such that the interval $[a \wedge c, b \vee c]$ in L is a modular interval. Then the element $a \wedge c$ is a lower neighbour of $b \wedge c$.

b) In a modular lattice every sub interval is modular and hence in a modular lattice we have the following result.

Let L be a lattice and let a and b be any two elements of L such that a is a lower neighbour of b. Let L_1 be a subset of L such that every element of the interval $[a \wedge c, b \vee c]$ in L can be written as join of some subset of L_1. If the lattice is modular ,then the element $a \wedge c$ is a lower neighbour of $b \wedge c$.

c) In modular atomic lattices the above results takes a simple form.

Let L be an atomic modular lattice. If a is a lower neighbour of b then $a \wedge c$ is a lower neighbour of $b \wedge c$ for all elements c in the lattice L.

Because of the importance we list the duals of the above results.

d) Let L be an dually atomic lattice. Let a and b be any two elements of L such that a is a lower neighbour of b. Let c be any element of the lattice such that the interval $[a \wedge c, b \vee c]$ in L is a modular interval. Then the element $a \vee c$ is a lower neighbour of $b \vee c$.

e) Let L be a lattice and let a and b be any two elements of L such that a is a lower neighbour of b.

Let L_1 be a subset of L such that every element of the interval $[a \wedge c, b \vee c]$ in L can be written as meet of some subset of L_1. If the lattice is modular ,then the element $a \vee c$ is a lower neighbour of $b \vee c$.

As in the earlier case in dually atomic modular lattices the above results takes very simple form.

f) Let L be an dually atomic modular lattice. Then if a is a lower neighbour of b then $a \vee c$ is a lower neighbour of $b \vee c$ for all elements c in the lattice L.

Next we show that in atomic modular lattices and dually atomic modular lattices every element other than the smallest and largest has upper neighbours and lower neighbours respectively

1.1.4 Theorem

In an atomic modular lattice every element other than the greatest element has an upper neighbour.

Proof

Let a be an arbitrary element of an atomic modular lattice which is not the greatest element. Since the lattice is atomic it has atoms and the element a is join of atoms. Since a is not the greatest element, there is an atom b in the lattice which is not greater than a.

Since b is an atom it is an upper neighbour of the smallest element 0 of the lattice . Since the lattice is modular the join of a and the smallest element 0 i.e. a is a lower neighbour of the element $a \vee b$ in the lattice. In particular the element a has an upper neighbour.

Hence every element of a modular atomic lattice has an upper neighbour.

Dually we have the following theorem.

1.1.5 Theorem

In a dually atomic modular lattice every element other than the smallest element has a lower neighbour.

Next theorem enables us to charecterise covers in atomic and dually atomic lattices.

1.1.6 Theorem

Let L be a lattice. Let L_1 be a collection of elements of L such that every element of L can be written as a join of a collection of elements of L_1. Let a and b be two elements of L such that $a < b$. Let L_2 be the collection of all elements of L_1 less than or equal to b but not less than a. Then a is a lower neighbour of b if and only if all c in the set L_2, the relation $a \vee c = b$ holds in L.

Proof

Let $c \in L_2$. Then $c \leq b$ but c is not less than a. So $a < a \vee c \leq b$ and if $a \neq c$ and since b is an upper neighbour of a, we must have $a \vee c = b$.

Conversely let $a \vee c = b$ for all $c \in L_2$. Let $d \in L$ be such that $a < d \leq b$. Hence there is an element $e \in L_2$ such that $e \leq d$ but e is not less than a. So $e \in L_2$ and hence by assumption we have $a \vee e = b$. We have $e \leq d$, $a \leq d$ implies $a \vee e \leq d$ i.e., $b \leq d$. Hence $b = d$. So a is a lower neighbour of b.

Dually we have the following theorem

1.1.7 Theorem

Let L be a lattice. Let L_1 be a collection of elements of L such that every element of L can be written as a meet of a collection of elements of L_1. Let a and b be two elements of L such that $a < b$. Let L_2 be the collection of all elements of L_1 neither greater than nor equal to b but not greater than a. Then a is a lower neighbour of b if and only if all c in the set L_2, the relation $a = b \wedge c$ holds in L.

1.1.8 Remarks

In an atomic lattice we can take L_1 to be the collection of all atoms. Similarly in an dually atomic lattice we can take L_1 to be the collection of all dual atoms. So we have the following theorems.

a) Let L be an atomic lattice. Let a and b be two elements of L such that $a < b$. Let L_1 be the collection of all atoms less than or equal to b but neither less than nor equal to a. Then a is a lower neighbour of b if and only if all c in the set L_1, $a \vee c = b$.

b) Let L be a dually atomic lattice. Let a and b be two elements of L such that $a < b$. Let L_2 be the collection of all dual atoms greater than or equel to a but neither less than nor equal to b. Then a is a lower neighbour of b if and only if $a = b \wedge c$ for all c in L_2.

1.2 Applications to lattice of topologies

We apply theorems 1.1.12 and 1.1.13 developed in the previous section to the particular case of the lattice of all topologies on a fixed set.

When we apply theorem 1.1.12 to the lattice of topologies we get the following result first proved by Larson and Thron[16].

1.2.1 Lemma

Let τ and τ^1 are topologies on X . For τ^1 to be an upper neighbour of τ it is necessary and sufficient that $\tau^1 = \tau$ (A) for all A in the topology τ^1 and not in the topology τ.

Applying theorem 1.1.13 to the lattice of topologies we get the following result.

1.2.2 Lemma

Let τ and τ^1 are topologies on X. For τ^1 to be an upper neighbour of τ it is necessary and sufficient that for all ultratopologies $\mathfrak{I}(b, \mathcal{U})$ finer than τ, not finer than τ we must have $\tau = \tau^1 \cap \mathfrak{I}(b, \mathcal{U})$.

As a corollary we get the following result proved by Larson and Thron[16].

If τ and τ^1 are topologies on X such that τ^1 is an upper neighbour of τ .Then there is an ultratopology $\Im(b, \mathcal{U})$ such that $\tau = \tau^1 \cap \Im(b, \mathcal{U})$.

These results together with the results we have developed in the previous section enable us to charecterise the lower neighbours of the ultra topologies.

1.3 Lower neighbours of ultra topologies

In their paper entitled "Basic Intervals in the Lattice of Topologies", R. Valent and R.E. Larson introduced the concept of a basic interval in the Lattice of Topologies as an interval $[\varsigma, \vartheta]$ of Lattice of Topologies such that there is an element 'a' in the set X such that for all elements G in the topology ϑ, we have $G \setminus \{a\}$ is an element of the topology ς [34] . This element a is unique.

Later the first author R. Valent [35] modified this definition of the basic interval to include a wider class of intervals in the lattice of topologies. Following B. Banaschewsky, R. Valent defined an expansion of a topology by an element 'a' in X as topology in which the neighbourhood filters of all points except a single point are the same. An interval $[\varsigma ,\vartheta]$ of Lattice of Topologies is basic if the topology ϑ is an expansion of the topology ς by some element in X. Valent made this modification of the definition of the basic interval in such a way that all the results of the original paper remains true.

The most important thing about basic intervals is that they have a very rich lattice structure. In fact Valent and Larson proved that a basic interval is infinitely meet distributive, compactly generated and complete. An interval of the Lattice of Topologies is called a very basic interval if the lower end point of the interval can be obtained by intersecting the upper end point of the interval by an ultra topology [34]. A very basic interval is always a basic interval.

Next we charecterise the lower neighbours of the ultra topologies as those topologies which can be written as the intersection of two ultra topologies.

1.3.1 Theorem

Every ultra topology has a lower neighbour and the lower neighbours of the ultra topologies are precisely those topologies which can be written as the intersection of two ultra topologies.

Proof

We apply the theorem 1.1.3 to the lattice of topologies. We take the element b to be the discrete topology, the element a and c to be two distinct ultra topologies. Then the interval [$a \wedge c$, $b \vee c$] is a basic interval and hence infinitely meet distributive. So this interval is modular. There fore the requirments of the theorem 1.1.3 are satisfied and so the element $a \wedge c$ is a lower neigbour of the element $b \wedge c$. since the element b is the discrete topology, the element $b \wedge c$ is the element c itself. Hence the ultra topology c has an lower neighbour $a \wedge c$.

By lemma 1.2.3, every lower neighbour of every topology can be obtained by taking meet with a suitable an ultra topology. There fore every lower neighbour of an ultra topology is the intersection of two distinct ultra topologies.

Hence the theorem.

CHAPTER 2

UPPER NEIGHBOURS IN THE LATTICE OF TOPOLOGIES

R. E. Larson and W. J. Thron[16] investigated the upper and lower nighbours of a T_1 topology. They determined conditions under which a T_1 topology is covered by a simple expansion by a non empty open set. N. Levine [17] presented necessary and sufficient conditions for the simple expansion τ (A) to be an upper neighbour of the topology τ. The results of Larson and Thron were extended by F.Plastria.[22]

P. Agashe and N. Levine [1] discussed the existence of neighbours for topologies in the lattice of topologies. They showed that every regular and non T_1 topology has an upper neighbour.

2.1 Upper neighbours in the lattice of topologies

R. E. Larson and W. J. Thron[16] obtained necessary and sufficient conditions the simple expansion $\tau(A)$ of a T_1 topology τ by the set A to be an upper neighbour of the topology τ in the lattice of T_1 topologies. They showed that the every upper neighbour of T_1 topology τ is of the form $\tau(U \cup \{x\})$ for an open set U and a point x in X such that $\{x\}$ is not an element of the topology τ. Later R. Valent [35] showed that the above result is true for T_0 topologies as well.

We extend the above results to the lattice of all topologies. We start this by representing the simple expansion in another form.

2.1.1 Lemma

Let τ be a topology on the set X. Then the simple expansion $\tau(A)$ of τ by the set A can be represented in the following form.

$\tau(A) = \{U \cap (V \cup A)$ for sets U and V open in $\tau\}$

Proof

Let $\tau' = \{U \cap (V \cup A)$ for sets U and V open in $\tau\}$

Let B be an arbitrary element of topology τ'. Then the set B can be written in the form $U \cap (V \cup A)$ for sets U and V open in τ. This set B can also be written in the form

$(U \cap V) \cup (U \cap A)$. Since U and V are opensets in τ, $U \cap V$ is also an element of τ. Hence B belongs to $\tau(A)$. Hence $\tau` \subseteq \tau(A)$.

On the other hand $\tau(A)$ is the smallest topology containing the elements of τ and the set A. τ' is such a topology between τ and τ (A). If A is open in τ then $\tau = \tau` = \tau$ (A). Otherwise $\tau` \neq \tau$ So $\tau' = \tau$ (A).

There fore $\tau(A) = \{U \cap (V \cup A)$for $U, V \in \tau \}$.

If A is a subset of X ; $Cl_\tau A$,$Int_\tau A$ denote respectively the closure and interior of the subset A with respect to the topology τ.

2.1.2 Theorem

Let τ be a topology on X and let A be a subset of X, not open in τ. Then the following are equivalent.

1. $A \setminus int_{\tau} A$ is indiscrete

2. $A \setminus int_{\tau} A = B \setminus int_{\tau} B$ for all B in $\tau(A) \setminus \tau$

3. $B \setminus int_{\tau} B$ is indiscrete for all B in $\tau(A) \setminus \tau$

4. For all points in the set $A \setminus int_{\tau} A$ have the same τ closure.

Proof

$$(1) \Rightarrow (2)$$

Suppose $A \setminus int_{\tau} A$ be indiscrete and B be an arbitrary element of $\tau(A) \setminus \tau$.

First we show that the set B can be taken as $O \cup (A \setminus int_{\tau} A)$ where O is a τ -open set disjoint with $A \setminus int_{\tau} A$.

So let $B \in \tau(A) \setminus \tau$. Then $B = U \cup (V \cap A)$ for some open sets U and V in τ.

We write B as

$U \cup (V \cap (int_{\tau} A \cup (A \setminus int_{\tau} A))$

$= U \cup (V \cap int_{\tau} A) \cup (V \cap (A \setminus int_{\tau} A))$

If $V \cap (A \setminus int_{\tau} A) = \varphi$ then $B = U \cup (V \cap int_{\tau} A)$ is an element of τ. But this is not true since $B \notin \tau$. Hence $V \cap (A \setminus int_{\tau} A) \neq \varphi$. Since $(A \setminus int_{\tau} A)$ is indiscrete we have $V \cap (A \setminus int_{\tau} A) = (A \setminus int_{\tau} A)$. Thus B is of the form

$B = U \cup (V \cap int_{\tau} A) \cup (A \setminus int_{\tau} A)$

We take the open set $U \cup (V \cap int_{\tau} A)$ as O. Thus

$B = O \cup (A \setminus \text{int}_\tau A)$. If the open set O is not disjoint with $(A \setminus \text{int}_\tau A)$, then since $(A \setminus \text{int}_\tau A)$ is indiscrete, $(A \setminus \text{int}_\tau A)$ will be a subset of O and hence their union will be the τ open set O. But this can not be the case since the above union is B which is not in the topology τ.

So the set B can be taken as $O \cup (A \setminus \text{int}_\tau A)$ where O is a τ -open set and O is disjoint with $A \setminus \text{int}_\tau A$. Hence $A \setminus \text{int}_\tau A$ must be a subset of $X \setminus O$. Also, since $A \setminus \text{int}_\tau A$ cannot contain any open set, the τ-interior of B must be O.

Hence $B \setminus \text{int}_\tau B$

$= [O \cup (A \setminus \text{int}_\tau A)] \cap (X \setminus O)$

$= (A \setminus \text{int}_\tau A) \cap (X \setminus O)$

$= A \setminus \text{int}_\tau A$

Hence, $B \setminus int_{\tau} B$ = $A \setminus int_{\tau} A$ for all $B \in \tau(A) \setminus \tau$

So (2) holds.

(2) $\Rightarrow$(3)

Let $A \setminus int_{\tau} A$ be equal to $B \setminus int_{\tau} B$ for all B in $\tau(A) - \tau$.

Let O be a τ -openset intersecting $A \setminus int_{\tau} A$. So there is an element b in $O \cap (A \setminus int_{\tau} A)$.

If $O \cap A$ is in τ, then b is an element of the τ open set $O \cap A$ which is therefore must be a subset of $int_{\tau} A$ and hence O is disjoint with $A \setminus int_{\tau} A$, which is not so.

Hence $O \cap A$ is not in the topology τ,. but it is an element of the topology $\tau(A)$.

So by assumption, since $A \in \tau(A) \setminus \tau$ we have

$$A \setminus \text{int}_{\tau} A = (O \cap A) - (O \cap \text{int}_{\tau} A)$$

$$= O \cap A \cap (X \setminus (O \cap \text{int}_{\tau} A))$$

$$= O \cap A \cap (X \setminus \text{int}_{\tau} A)$$

$$= O \cap (A \setminus \text{int}_{\tau} A)$$

Hence $A \setminus \text{int}_{\tau} A \subset O$, which means $A \setminus \text{int}_{\tau} A$ is indiscrete. Since $A \setminus \text{int}_{\tau} A = B \setminus \text{int}_{\tau} B$ for all B in $\tau(A)$ not in τ, $B \setminus \text{int}_{\tau} B$ is indiscrete for all B in $\tau(A)$ not in τ. Hence (3) holds.

(3) $\Rightarrow$ (1)

Suppose $B \setminus int_\tau B$ be indiscrete for all B in τ (A) not in τ. Then since A is such an element, we have $A \setminus int_\tau A$ is indiscrete. Hence (1) holds.

(1) $\Rightarrow$ (4)

Let $A \setminus int_\tau A$ be indiscrete and let a and b be any two elements of $A \setminus int_\tau A$. If O is any open subset of X having a nonempty intersection with $(A \setminus int_\tau A)$, then since $A \setminus int_\tau A$ is indiscrete we have $A \setminus int_\tau A$ is a subset of the open set O.

Every open set containing the element a, has a non-empty intersection with $A \setminus int_\tau A$ as a subset and hence this open set contains b. This means $b \in Cl_\tau(\{a\})$.

Hence $Cl_\tau(\{b\}) \subset Cl_\tau(\{a\})$.

Similarly $Cl_\tau(\{a\}) \subset Cl_\tau(\{b\})$

So we have the equality $Cl_\tau(\{a\}) = Cl_\tau(\{b\})$.

(4) $\Rightarrow$ (1) Let $Cl_\tau(\{a\}) = Cl_\tau(\{b\})$ for all elements a, b in $A \setminus int_\tau A$. If $A \setminus int_\tau A$ is not indiscrete, then there is an open set O having a non empty intersection with $A \setminus int_\tau A$, yet $A \setminus int_\tau A$ is not completely contained in the set A.

Let $a \in O \cap (A \setminus int_\tau A)$ and

$b \in (A \setminus int_\tau A) \setminus O$.

Here $a, b \in A \setminus int_\tau A$. Hence by assumption we have $Cl_\tau(\{a\}) = Cl_\tau(\{b\})$, but here there is an open set O containing the element a but not the element b. So $Cl_\tau(\{a\})$

must be different from Cl ({b}), a contradiction to our assumption.

This contradiction proves that A \ int_τ A must be indiscrete.

Hence, the theorem.

F. Plastria [22] defined that a set A has the property P, if A satisfies the following equivalent conditions

1) Every open set intersecting A contains A.

2) A is contained in the closure of each of its points.

3) All points of A have the same neighbourhood filter.

Combining these properties with our theorem we have the

2.1.3 Theorem

Let τ be a topology on X and let A be a subset of X, not open in τ. Then the following are equivalent.

1. $A \setminus int_\tau A$ is indiscrete

2. $A \setminus int_\tau A = B \setminus int_\tau B$ for all B in $\tau(A) \setminus \tau$

3. $B \setminus int_\tau B$ is indiscrete for all B in $\tau(A)\setminus\tau$

4. For all points in the set $A \setminus int_\tau A$ have the same τ closure.

5. $A \setminus int_\tau A$ is contained in the closure of each of its points.

6. All points of A \ int_{τ} A have the same neighbourhood filter.

Now we extend the results of Larson and Thron [16] from the lattice of T_1 topologies to the lattice of all topologies. We do this by extending the results of Larson and Thron from the lattice of T_1 topologies to the lattice of all topologies.

2.1.4 Lemma

Let $\tau \in \Sigma(X)$, A and B be any two subsets of X. If the simple expansion topology τ (B) is finer then the simple expansion topology τ (A) then the set A \ int_{τ}A is a subset of the set B and the intersection $A \cap B$ can be represented as the intersection of an open set in τ with B.

Proof

Let the simple expansion topology τ (B) be finer then the simple expansion topology τ (A). Since τ (A) always contains the set A, the set A is a member of the topology τ (B). This means the set A is of the form $U \cup (V \cap B)$ for some open sets U, V in τ. Hence we have the inclusion $U \subseteq A$ and therefore, since U is open we have $U \subseteq \text{int}_\tau A$.

Now we show subset relation $A \setminus \text{int}_\tau A \subseteq B$. If $t \in A \setminus \text{int}_\tau A$ then $t \notin \text{int}_\tau A$. So $t \notin U$. Since $t \in A$ we have $t \in U \cup (V \cap B)$. So $t \in V \cap B$ and consequently $t \in B$. So we have the relation $A \setminus \text{int}_\tau A \subseteq B$.

For the other relation we have $A = U \cup (V \cap B)$ and hence $A \cap B =$ $(U \cup (V \cap B)) \cap B = (U \cap B) \cup (V \cap B) =$

$(U \cup V) \cap B$. So the set $A \cap B$ is the intersection of an open set in τ with B.

2.1.5 Lemma

Let $\tau \in \Sigma(X)$, A and B be any two subsets of X such that the set $A \setminus int_{\tau}A$ is a subset of the set B. Then the simple expansion topology $\tau(A \cap B)$ is finer than the simple expansion topology $\tau(A)$.

Proof

Suppose A and B are two subsets of X such that $A \setminus int_{\tau}A \subseteq B$.

In this case we have $A \setminus int_{\tau}A \subseteq A \cap B$ and hence we have the following inclusions

$A \subseteq (A \cap B) \cup int_\tau A =$

$A \cap (int_\tau A \cup B) \subseteq A$ and this means $(A \cap B) \cup int_\tau A = A$. So A is an element of the topology $\tau(A \cap B)$.

So $\tau(A)$, which is the join of the topology τ and the atom $\{\phi, A, X\}$ in the lattice of topologies, is contained in the topology $\tau(A \cap B)$.

2.1.6 Lemma

Let $\tau \in \Sigma(X)$, A and B be any two subsets of X such that the set $A \cap B$ is the intersection of an open set in τ with the set B. Then the simple expansion topology $\tau(B)$ is finer than the simple expansion topology $\tau(A \cap B)$.

Proof

Since the set $A \cap B$ is the intersection of an open set in τ with the set B, there is an open set U such that $A \cap B = U \cap B$. An element in the topology $\tau(A \cap B)$ is of the form $V \cup (W \cap A \cap B)$ for elements V, W in τ. This set can be rewritten as $V \cup (W \cap (U \cap B))$. Clearly this set belongs to the topology $\tau(B)$. Therefore the simple expansion topology $\tau(B)$ is finer than the simple expansion topology $\tau(A \cap B)$.

2.1.7 Theorem

Let $\tau \in \Sigma(X)$, A and B be any two subsets of X. Then the simple expansion topology $\tau(B)$ is finer then the simple expansion topology $\tau(A)$ if and only if the set $A \setminus int_\tau A$ is a subset of the set B and the set $A \cap B$ can be represented as the intersection of an open set in τ with B.

Proof

The only if part is the lemma 2.1.4

If A and B are two sets of X such that the set $A \setminus int_{\tau}A$ is a subset of the set B and the set $A \cap B$ can be represented as the intersection of an open set in τ with B ,then by Lemma 2.1.5.the simple expansion topology $\tau(A \cap B)$ is finer than the simple expansion topology $\tau(A)$. Again by lemma 2.1.6 we have, the topology $\tau(B)$ is finer than the topology $\tau(A \cap B)$. Hence the result.

2.1.8 Corollary

Let τ be a topology on X. Let A and B be any two subsets of X. Then $\tau(A) \subseteq \tau(B)$ if and only if $\tau(A) \subseteq \tau(A \cap B) \subseteq \tau(B)$.

Proof

One implication is trivial and for the other implication, let $\tau(A) \subseteq \tau(B)$. Then by the lemma 2.1.4 we have the set $A \setminus \mathrm{int}_\tau A$ is a subset of the set B and the set $A \cap B$ can be represented as the intersection of an open set in τ with B. So the lemma 2.1.5 implies the relation $\tau(A) \subseteq \tau(A \cap B)$ and the lemma 2.1.6 implies the relation $\tau(A \cap B) \subseteq \tau(B)$. Hence we have the inclusions

$$\tau(A) \subseteq \tau(A \cap B) \subseteq \tau(B).$$

The following corollary follows from the theorem 2.1.8

2.1.9 Corollary

Let $(X, \tau) \in \Sigma(X)$, A and B be any two subsets of X. Then $\tau(A) = \tau(B)$ if and only if $A \setminus int_{\tau}A \subseteq B$, $B \setminus int_{\tau}B \subseteq A$ and $A \cap B$ is relatively open in both A and B.

Also by the previous corollary we have $\tau(A) = \tau(B)$ implies $\tau(A) = \tau(A \cap B) = \tau(B)$.

2.1.10 Remark

The inclusion $\tau(A) \subseteq \tau(A \setminus int_{\tau}A)$ between the topologies $\tau(A)$ and $\tau(A \setminus int_{\tau}A)$ always holds. For an element of $\tau(A)$ is of the form $U \quad (V \cap A)$ and this set can be written as $U \cup (V \cap (int_{\tau}A \cup (A \setminus int_{\tau}A))) =$

$U \cup (V \cap int_\tau A) \cup (V \cap (A \setminus int_\tau A))$. Clearly this set belongs to $\tau(A \setminus int_\tau A)$. Hence every element of $\tau(A)$ is an element of $\tau(A \setminus int_\tau A)$. Thus we have always $\tau(A) \subseteq \tau(A \setminus int_\tau A)$.

The above remark shows that the condition $A \setminus int_\tau A \subseteq B$ of the theorem 2.1.7 cannot be weakened further.

2.1.11 Proposition

Let τ be a topology on X. Let A and B be any two subsets of X. If $A \setminus int_\tau A$ is relatively open in B, then $\tau(A) \subseteq \tau(B)$

Proof

The condition $A \setminus int_\tau A$ is relatively open in B implies that the set $A \setminus int_\tau A$ is of the form $V \cap B$ for some open set U in X. There fore

A is of the form $int_\tau A \cup (V \cap B)$ and hence A belongs to $\tau(B)$. Hence $\dot{\tau}(A) = \tau \vee \{\phi, A, X\} \subseteq \tau(B)$.

2.1.12 Example

Now we show that the conditions i) $A \setminus int_\tau A \subseteq B$ and ii) $A \cap B$ is relatively open in B of theorem 2.1.7 are independent of one another.

We take $X = \{a, b, c\}$. Let the non trivial open subsets of the topology τ be $\{b,c\}$, $\{b\}$ and $\{c\}$. If we take $A = \{a\}$, $B = \{a,b\}$, then $int_\tau A = \phi$ So $A \setminus int_\tau A \subseteq B$, but there is no open set U such that $A \cap B = U \cap B$.

On the other hand if we take $A_1 = \{a, b\}$, $B_1 = \{b\}$ then $A_1 \setminus int_\tau A_1$ is not contained in B_1, but $A_1 \cap B_1 = \{b\} = U \cap B_1$, where the set $U = \{b\}$ is open in τ.

Now we define an equivalence relation $\equiv_\tau$ on the subsets of X which are not open in τ.

2.1.13 Definition

For subsets A, B of X which are not open in τ, define $A \equiv_\tau B$ if $A \setminus \mathrm{int}_\tau A \subseteq B$, $B \setminus \mathrm{int}_\tau B \subseteq A$ and $A \cap B$ is relatively open in both A and B.

2.1.14 Lemma

The relation $\equiv_\tau$ defined above is an equivalence relation.

Proof

We have by theorem2.1.9, $A \equiv_\tau B$ if and only if $\tau(A) = \tau(B)$. In this form, the properties of the equivalence relation follows easily.

2.1.15 Lemma

For subsets A, B of X which are not open in τ, $A \equiv_\tau B$ if and only if $A \equiv_\tau B \equiv_\tau A \cap B$.

Proof

This is clear since from the corollary 2.1.9 we have $\tau(A) = \tau(B)$ if and only if $\tau(A) = \tau(A \cap B) = \tau(B)$.

2.1.16 Definition

Let the equivalence class containing the subset A of X be denoted by [A]. We now define a partial order $\leq_\tau$ among the equivalence classes as follows.

$[A] \leq_\tau [B]$ if $\tau(A) \subseteq \tau(B)$. This partial order is well defined for if $[A] = [C]$ and $[B] = [D]$ then,

$$\tau(A) = \tau(C) \text{ and } \tau(B) = \tau(D).$$

So $[A] \leq_\tau [B]$ i.e. $\tau(A) \subseteq \tau(B)$ implies $\tau(C) \subseteq \tau(D)$ and therefore we have $[C] \leq_\tau [D]$.

2.1.17 Theorem

Let A be a subset of X, which is not open in a topology τ on X. Then $\tau(A)$ is an upper neighbor of τ if and only if $[A]$ is a minimal element in the partial order $\leq_\tau$ defined above.

Proof

If τ is a lower neighbour of $\tau(A)$ and if $[B] \leq_{\tau} [A]$ then we have $\tau(A) = \tau(B)$ and so $[A] = [B]$ which implies that [A] is a minimal element.

Now if $\tau(A)$ is not an upper neighbor of τ for some topology τ^1 we have $\tau \subset \tau^1 \subset \tau(A)$ there fore for any $B \in \tau^1 \setminus \tau$ we have $\tau(B) \subseteq \tau(A)$. There fore $[B] \leq_{\tau} [A]$. So [A] is not a minimal element.

P. Agashe and N. Levine [1]proved that if τ is a T_1 topology and if A isa non open subset such that $\tau(A)$ is an upper neighbour of τ then $A \setminus int_{\tau}A$ is an singleton set. So in the lattice of T_1 topologies $A \setminus int_{\tau}A$ is always indiscrete whenever $\tau(A)$ is an upper neighbour of τ. Also the condition $A \setminus int_{\tau}A \subseteq B$ of definition 2.1.14

becomes a trivial condition. So Theorem 2.1.17 generalises result 2.6, page 104 of [16].

2.1.18 Theorem

Let τ be a topology on X and A be a subset of X be such that

1. If O is an τ open set disjoint with $(A \setminus int_{\tau} A)$ then $Cl_{\tau} (O)$ is disjoint with $(A \setminus int_{\tau} A)$,

2. $(A \setminus int_{\tau} A)$ is indiscrete.

then τ is an immediate predecessor of $\tau(A)$.

Proof

Let B be an element in $\tau(A)$ not in τ. Then B must be of the form

$$B = U \cup (U \cap A)$$

$$= U \cup [V \cap ((A \setminus \text{int}_\tau A) \cup \text{int}_\tau A)]$$

$$= U \cup (V \cap \text{int}_\tau A) \cup (V \cap (A \setminus \text{int}_\tau A))$$

If V is disjoint with $(A \setminus \text{int}_\tau A)$, then B $= U \cup (V \cap \text{int}_\tau A)$ will be an element of τ, which is not true. Hence V must have a non-empty intersection with $(A \setminus \text{int}_\tau A)$. Since $(A \setminus \text{int}_\tau A)$ is indiscrete, this non-empty intersection must be $(A \setminus \text{int}_\tau A)$. Denoting the τ-open set $U \cup (V \cap \text{int}_\tau A)$ by W we see that $B = W \cup (A \setminus \text{int}_\tau A)$. Again W must be disjoint with $(A \setminus \text{int}_\tau A)$, otherwise, since $A \setminus \text{int}_\tau A$ is indiscrete, $A \setminus int_{\tau} A$ will be a subset of W and hence its union with W is the same as the τ-open set W which cannot happen.

Hence, $B = W \cup (A \setminus int_{\tau} A)$ where W is τ open and W is disjoint with $(A \setminus int_{\tau} A)$. So its τ-closure $Cl_{\tau} W$ is also disjoint with $(A \setminus int_{\tau} A)$. So we have either $Cl_{\tau} W \subset X \setminus A$ or $Cl_{\tau} W \subset int_{\tau} A$.

Now the set $int_{\tau} A \cup (X \setminus Cl_{\tau} W) \cap B)$ is an element of τ (B) and

$$int_{\tau} A \cup ((X \setminus Cl_{\tau} W) \cap B)$$

$$= int_{\tau} A \cup (X \setminus Cl_{\tau} W) \cap (W \cup (A \setminus int_{\tau} A)))$$

$$= int_{\tau} A \cup (X \setminus Cl_{\tau} W) \cap (A \setminus int_{\tau} A))$$

$$= int_{\tau} A \cup (X \setminus Cl_{\tau} W) \cap A \cap (X \setminus int_{\tau} A))$$

If $W \subset \text{int}_\tau A$ then $X \setminus \text{int}_\tau A \subset X \setminus Cl_\tau W$. In this case the above set is

$$\text{int}_\tau A \cup ((X \setminus \text{int}_\tau A) \cap A) = \text{int}_\tau A \cup (A \setminus \text{int}_\tau A) = A$$

If $W \subset X \setminus A$ then $A \subset X \setminus Cl_\tau W$. In this case also we have

$$\text{int}_\tau A \cup ((X \setminus Cl_\tau W) \cap B)$$

$$= \text{int}_\tau A \cup ((X \setminus Cl_\tau W) \cap A \cap (X \setminus \text{int}_\tau A))$$

$$= \text{int}_\tau A \cup (A \setminus \text{int}_\tau A) = A.$$

Hence in both the cases the set A is an element of the topology $\tau(B)$. So the join $\tau \vee \{X. \phi, A\}$ is again contained in $\tau(B)$. i.e. we have $\tau(A) \subset \tau(B)$. Since $B \in \tau(A) \setminus \tau$, we have also τ (B) $\subset$ τ (A). Hence by theorem 1.2.1 τ is an immediate predecessor of τ (A).

2.2 Topologies containing an ultra filter

The results developed so far can be used to describe an interval in the lattice of topologies completely.

2.2.1 Theorem

Let τ be a topology containing an ultrafilter. Then τ has upper neighbours and they are of the form τ ({x}) , where {x} $\notin$ τ.

Proof

Let τ be a topology containing the ultrafilter $\mathcal{U}$. We have to prove that τ has upper neighbours and every upper neighbor of τ is of the form τ {x} for some $\{x\} \notin \tau$.

By the last theorem it is enough to prove that the equivalence class $[\{x\}]$ containing the singleton set {x}, is a minimal element for $\{x\} \notin \tau$ and every minimal element is of this form.

So let $A \notin \tau$, $\{x\} \notin \tau$ and $[A] \leq_\tau [\{x\}]$. Therefore by definition we have $A \setminus int_\tau A \subseteq \{x\}$. Since $A \notin \tau$, $\{x\} \notin \tau$ we have $A \setminus int_\tau A = \{x\}$. So $A = int_\tau A \cup \{x\}$, since $int_\tau A \subseteq A$. We have $\{x\} \setminus int_\tau \{x\} \subseteq A$ and $A \notin \tau \Rightarrow A \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we have $int_\tau A \notin \mathcal{U}$ and $X \setminus int_\tau A \in \mathcal{U}$ and there fore $X \setminus int_\tau A$ is open and $\{x\} = A \cap (X \setminus int_\tau A)$. So $[A] \leq_\tau [\{x\}]$ and therefore [A] = [{x}]. Therefore [{x}] is a minimal element.

Now if τ' is an upper neighbour of τ, then $\tau' = \tau(A)$ for $A \in \tau' \setminus \tau$. Let $x \in A$ be such that $\{x\} \notin \tau$. Now since A is not in τ, it is not in the ultrafilter $\mathcal{U}$. Therefore its complement belongs to $\mathcal{U}$, and hence $(X \setminus A) \cup \{x\} \in \mathcal{U}$. Therefore $\{x\} = A \cap [(X \setminus A) \cup \{x\}]$, is the intersection of an open sets in τ with A. Hence it belongs to $\tau(A)$. Therefore the upper neighbour $\tau(A)$ must be of the form $\tau(\{x\})$. Hence $\tau(\{x\}) = \tau(A)$.

Hence every upper neighbour of τ is a simple expansion by a non-open singleton.

2.2.2 Lemma

Let $\mathcal{U}$ be an ultra filter on the set X. Let τ' be a topology finer than τ. Let A be a subset of X open in the topology τ' but not open in the topology τ. Then every singleton subset of A is open in the topology τ'.

Proof

Let $A \in \tau' \setminus \tau$ and let $a \in A$. We have to prove that $\{a\}$ is open in τ'. A is not a member of the topology τ and hence A is not a member of the ultrafilter $\mathcal{U}$. Hence its component $X \setminus A$ belongs to the ultrafilter $\mathcal{U}$. So its superset $(X \setminus A) \cup \{a\}$ is an element of $\mathcal{U}$. So this set belongs to τ and hence it belongs to the finer topology τ'. Hence the sets A and $(X \setminus A) \cup \{a\}$

belongs to the topology τ'. Hence their intersection $\{a\}$ is also in the topology τ'.

Since $a \in A$ was arbitrary, it follows that every singleton subset of A is open in the topology τ'.

2.2.3 Theorem

Let $\mathcal{U}$ be an ultra filter on X. Let τ be the topology $\mathcal{U} \cup \{\phi\}$ on X. Then every topology τ' finer than τ is of the form $\wp(F) \cup \mathcal{U}$ where F is the union of all sets in the topology τ' not in the topology τ .

Proof

Let τ' be a topology finer than τ .

Now we prove that $\tau' = \wp(F) \cup \mathcal{U}$.

Let $A \in \tau'$. If A is empty then $A \in \wp(F)$. If A is non-empty and $A \in \tau$ then $A \in \mathcal{U}$.

Otherwise $A \in \tau' \backslash \tau$. then by lemma 2.2.2 every singleton subset of A is open in the topology τ' and hence the set A and its subsets are in $\wp(F)$.

Hence $A \in \wp(F)$.

In other words, every element A in the topology τ' is also a member of the topology $\wp(F) \cup \mathcal{U}$.

Hence $\tau' \subset \wp(F) \cup \mathcal{U}$ --------(i)

if A is a member of the topology $\wp(F) \cup \mathcal{U}$, then, either $A \in \mathcal{U}$ or $A \in \wp(F)$.

If $A \in \mathcal{U}$ then $A \in \tau$ and hence $A \in \tau'$.

Otherwise $A \in \wp(F)$ and A is not a member of $\mathcal{U}$. Since $\mathcal{U}$ is an ultra filter we have the complement $X \setminus A \in \mathcal{U}$.

If $a \in A$ then $(X \setminus A) \cup \{a\} \in \mathcal{U}$ and hence $\{a\} = A \cap ((X \setminus A) \cup \{a\})$ is a member of the topology $\mathcal{U} \cup \wp(F)$.

Also $\{a\}$ is not in the ultra filter $\mathcal{U}$. So $\{a\} \in \wp(F)$ for all $a \in A$.

So $a \in F$. So $\{a\} \in \tau'$ for all $a \in A$

Since τ' is a topology we have $A \in \tau'$.

So every member of the topology $\wp(F) \cup \mathcal{U}$ is also a member of the topology τ'.

Hence $\wp$ (F) $\cup$ $\mathcal{U} \subset \tau'$ -----------(ii)

From (i) & (ii) we have $\tau' = \wp$ (F) $\cup$ $\mathcal{U}$.

Thus every topology finer than τ is of the form $\wp$ (F) $\cup$ $\mathcal{U}$.

CHAPTER 3

LOWER NEIGHBOURS OF TOPOLOGIES

In this chapter we consider the dual problem in the previous chapter. Here we consider lower neighbours. Recall that a lower neighbour of topology τ is a topology $\tau`$ such that $\tau$ is an upper neighbour of $\tau`$.

A topology in which every non empty open set is dense is called a D-topology by N. Levine [18]. Levine proved that a topology is a D topology if and only if every pair of non empty open sets has a nonempty intersection if and only if every openset in the topology is connected.

P. Agashe and N. Levine [1] initiated the investigation of the existence of lower neighbours in the Lattice of Topologies. They proved that a topology which is not a D-topology in which singleton sets are G-closed has a lower neighbour. From this it follows that T_1 non D- topology and every Hausdorff topology have lower neighbours.

In the Lattice of T_1 Topologies, Larson and Thron [16] started investigating this problem and we generalise their results from the Lattice of T_1 Topologies to the Lattice of Topologies.

3.1 Lower neighbours in the lattice of topologies

3.1.1 Lemma

Let τ be a topology on X and $A \subseteq X$ such that $A \notin \tau$ and $\tau = \tau(A) \cap \Im(x, \mathcal{U})$ for some ultratopology $\Im(x, \mathcal{U})$. Then $x \in A \setminus \text{int}_{\tau} A$

Proof

We have always $A \in \tau(A)$ and also we have the relation $A \notin \tau = \tau(A) \cap \Im(x, \mathcal{U})$. So $A \notin \Im(x, \mathcal{U})$ and hence $x \in A$. Also if $x \in int_\tau A \in \tau$, then $int_\tau A \in \Im(x, \mathcal{U})$ and so $int_\tau A \in \mathcal{U}$. Therefore the set A, being a superset of $int_\tau A$ will be in $\mathcal{U}$ and hence it will be in the ultra topology $\Im(x, \mathcal{U})$. So A is an element of the topology τ, since it is in $\tau(A)$, which is not true. Hence

$x \notin int_\tau A$. Therefore $x \in A \setminus int_\tau A$.

3.1.2 Proposition

Let τ be a topology on X and $A \subseteq X$ such that $A \notin \tau$ and $\tau = \tau(A) \cap \Im(x, \mathcal{U})$ for some ultratopology $\Im(x, \mathcal{U})$. If $A \setminus int_\tau A$ is indiscrete then

$A \setminus int_\tau A = Cl_\tau(\{x\})$

Proof

We have by lemma 3.1.1, $x \in A \setminus int_\tau A$

Since $A \setminus int_\tau A$ is indiscrete, by theorem 2.1.2 it is contained in the closure of each of its points. So $A \setminus int_\tau A. \subseteq Cl_\tau(\{x\})$.

On the other hand if there exist $t \in A \setminus int_\tau A$ such that $t \notin Cl_\tau(\{x\})$, then there is an open set U containing t but not x. So U and hence $U \cap A$ are elements of the ultra topology $\mathfrak{I}(x, \mathcal{U})$. Also $U \cap A$ is an element of $\tau(A)$. Hence $U \cap A \in \tau(A) \cap \mathfrak{I}(x, \mathcal{U}) = \tau$.

We have $t \in A$ and $t \in U$. Therefore $U \cap A$ is a nonempty τ-open set containing t contained in A and hence $t \in int_\tau A$. This contradicts the choice of t as an element of $A \setminus int_\tau A$.

Hence there is no $t \in A \setminus \text{int}_\tau A$ not in $Cl_\tau(\{x\})$. This together with $A \setminus \text{int}_\tau A. \subseteq Cl_\tau(\{x\})$ means $A \setminus \text{int}_\tau A = Cl_\tau(\{x\})$.

Next we generalise the results of Larson and Thron from the lattice of T_1 topologies to the Lattice of Topologies.

3.1.3 Definition [16]

Let τ and τ^1 be topologies on X such that τ^1 is an upper neighbour of τ. We say that τ^1 is an upper neighbourof τ by x if there is an ultratopology such that $\tau = \tau^1 \cap \Im(b, \mathcal{U})$.

3.1.4 Theorem

If τ and τ^1 are two topologies on X such that τ^1 is an upper neighbour of τ by two points a and b.

Then $Cl_{\tau}(\{a\}) = Cl_{\tau}(\{b\})$.

Proof

Since τ^{1} is an upper neighbour of τ we have

$\tau^{1} = \tau(A)$ for some subset A of X, not open in τ. Also by assumption there are ultra topologies such that

$$\tau = \tau^{1} \cap \Im(a, \mathcal{U}) \text{ and}$$

$$\tau = \tau^{1} \cap \Im(b, \mathcal{U}_{1}).$$

Then by lemma 3.1.1 $a \in A \setminus int_{\tau} A$

and $b \in A \setminus int_{\tau} A$

We have to prove that $Cl_{\tau}(\{a\}) = Cl_{\tau}(\{b\})$. Equivalently it is enough to prove that $b \in Cl_{\tau}(\{a\})$ and $a \in Cl_{\tau}(\{b\})$.

Suppose $a \notin Cl_{\tau}(\{b\})$. Since $X \setminus Cl_{\tau}(\{b\}) \in \tau$ it is an element of $\tau(A)$ and hence $A \cap (X \setminus Cl_{\tau}(\{b\})) = A \setminus Cl_{\tau}(\{b\}) \in \tau(A)$ and hence

$$\tau \subseteq \tau(A \setminus Cl_{\tau}(\{b\})) \subseteq \tau(A).$$

If $\tau(A \setminus Cl_{\tau}(\{b\})) = \tau(A)$, then by theorem 2.1.9 we have

$$A \setminus int_{\tau} A \subseteq A \setminus Cl_{\tau}(\{b\}) \subseteq X \setminus Cl_{\tau}(\{b\}).$$

Here $b \in A \setminus int_{\tau} A$, but $b \notin X \setminus Cl_{\tau}(\{b\})$, a contradiction. Hence $\tau(A \setminus Cl_{\tau}(\{b\}))\quad \tau(A)$.

Also $(A \setminus Cl_{\tau}(\{b\}) \notin \tau$ as $a \in A \setminus int_{\tau} A$

If $\tau = \tau (A \setminus Cl_{\tau}(\{b\}))$ then $A \setminus Cl_{\tau}(\{b\} \in \tau$.

Then $A \setminus Cl_{\tau}(\{b\} = int_{\tau} (A \setminus Cl_{\tau}(\{b\})$

$$= int_{\tau} A \cap (X \setminus Cl_{\tau}(\{b\})$$

$$= int_{\tau} A \setminus Cl_{\tau}(\{b\}$$

But since $a \notin Cl_{\tau}(\{b\}$, $a \in A$, we have $a \in A \setminus Cl_{\tau}(\{b\}$. So $a \in int_{\tau} A$, a contradiction to the choice of a as an element of $A \setminus int_{\tau} A$.

Hence we have the strict inclusions

$\tau \subset \tau (A \setminus Cl_{\tau}(\{b\})) \subset \tau (A)$ which contradicts the fact that τ is lower neighbour of $\tau (A)$.

So $a \in Cl_{\tau}(\{b\})$. Similarly $b \in Cl_{\tau}(\{a\})$.

Therefore $Cl_{\tau}(\{a\}) = Cl_{\tau}(\{b\})$.

3.1.5 Corollary

If τ and τ^1 are two T_0 topologies such that τ^1 upper neighbours τ . Then there exist a unique element x of X such that τ^1 is an upper neighbourof τ by x.

Proof

Suppose that τ^1 is an upper neighbourof τ by x and y. Then by the previous result we have $Cl_{\tau}(\{x\}) = Cl_{\tau}(\{y\})$. So $x = y$, since the topology τ is T_0.

Thus there exist a unique element x of X such that τ^1 is an upper neighbourof τ by x.

3.1.6 Corollary [16]

If τ and τ^1 are two T_1 topologies such that τ^1 upper neighbours τ . Then there exist a unique element x of X such that τ^{1} is an upper neighbourof τ by x.

3.1.7 Definition

A subset B is said to be a neighbourhood of the set A in the topology τ if $A \subseteq int_{\tau} B$.

3.1.8 Definition

The collection $\wp(X \setminus \{x\}) \cup \mathcal{F}$, $x \in X$ where $\mathcal{F}$ is a filter on X such that $\{x\} \notin \mathcal{F}$ is a topology on X. We denote this topology by $\Im(x, \mathcal{F})$.

3.1.9 Lemma

If τ is a topology and $\Im(x, \mathcal{F})$ is a topology on X which is not finer than τ. Then if τ is an upper neighbour of $\tau' = \tau \cap \Im(x, \mathcal{F})$ then $Cl_{\tau'}(\{x\})$ is indiscrete

Proof

Since τ is an upper neighbour of $\tau` = \tau \cap \Im(x, \mathcal{J})$, there is an element $A \in \tau \setminus \tau`$. Hence by theorem 1.2.1 $\tau = \tau`(A)$. Hence $\tau` = \tau`(A) \cap \Im(x, \mathcal{U})$. Also by theorems 2.1.3 and 3.1.2 $A \setminus . \, int_{\tau`} A$ is indiscrete and by proposition 3.1.2 $A \setminus . \, int_{\tau`} A. = Cl_{\tau`}(\{x\})$. So $Cl_{\tau`}(\{x\})$ is indiscrete.

3.1.10 Lemma

Let τ be a topology and let $\Im(x, \mathcal{J})$ be a topology on X which is not finer than τ. If τ is an upper neighbour of $\tau \cap \Im(x, \mathcal{J})$, then for any two neighbourhoods G and H of $Cl_\tau \{x\}$ in τ such that $H \notin F$, there exist another neighbourhood N of $Cl_\tau \{x\}$ such that $N \in \mathcal{J}$, $N \cap H \subseteq G$.

Proof

Let τ' be the topology $\tau \cap \Im(x, \mathcal{J})$.

Let τ be an upper neighbour of $\tau \cap \Im(x, \mathcal{J})$ i.e. τ is an upper neighbour of τ' and let G and H be neighbourhoods of $Cl_{\tau}(\{x\})$ in τ such that $H \notin \mathcal{J}$. Then $H \notin \Im(x, \mathcal{J})$ and so $H \notin \tau'$. There fore $H \in \tau \setminus \tau'$. Since τ is an upper neighbour of τ' and $H \in \tau \setminus \tau'$ we have by theorem $\tau = \tau'(H)$.

Now $G \in \tau'(H)$ and so by lemma 2.1.1 open sets A and B in τ' such that $G = A \cap (H \cup B)$. So $G \subseteq A$.

Since G is a neighbourhood of $Cl_{\tau}(\{x\})$ and $G \subseteq A$, A is a neighbourhood of $Cl_{\tau}(\{x\})$. Also $G = A \cap (H \cup B) = (A \cap H) \cup (A \cap B)$. Hence $(A \cap H) \subseteq G$.

Also $A \in \tau' = \tau \cap \Im(x, \mathcal{F})$ and hence $A \in \mathcal{F}$ (since $x \in A$). So we may take $N = A$.

The neighbourhood N of $Cl_\tau\{x\}$ is such that $N \in \mathcal{F}$, $N \cap H \subseteq G$.

3.1.11 Theorem

Let τ be a topology and let $\Im(x, \mathcal{F})$ be a topology on X which is not finer than τ. Then τ is an upper neighbour of $\tau \cap \Im(x, \mathcal{F})$ if and only if $Cl_\tau(\{x\})$ is indiscrete and for any two neighbourhoods G and H of x in τ such that $H \notin \mathcal{F}$, there exist another neighbourhood N of x such that $N \in \mathcal{F}$, $N \cap H \subseteq G$.

Proof

If τ is an upper neighbour of $\tau \cap \Im(x, \mathcal{F})$

then the result follows from lemmas 3.1.9 & 3.1.10

For the converse , let the topology τ and the topology $\Im(x, \mathcal{J})$ be such that $Cl_{\tau}(\{x\})$ is indiscrete and for any two neighbourhoods G and H of x in τ such that $H \notin \mathcal{J}$, there exist another neighbourhood N of x such that $N \in \mathcal{J}$, $N \cap H \subseteq G$,

Since $Cl_{\tau}(\{x\})$ is indiscrete, any neighbourhood of x , is a neighbourhood of $Cl_{\tau}(\{x\})$.

Let H and G be any two elements of $\tau \setminus \tau`$. So $H \notin \Im(x, \mathcal{J})$. They are neighbourhoods of x in τ. Since

$Cl_{\tau}(\{x\})$ is indiscrete , they are neighbourhoods of $Cl_{\tau}(\{x\})$ in τ.

Also H $\notin$ $\mathcal{J}$. So by assumption there exist another neighbourhood N of x such that N $\in$ $\mathcal{J}$, $N \cap H \subseteq G$, N is also a neighbourhood of Cl_{τ} ({x}).

In this case we have $H \in \tau \setminus \tau'$ and since $G \setminus Cl_{\tau}(\{x\}) \in \mathfrak{I}(x, \mathcal{J})$, $G \setminus Cl_{\tau}(\{x\}) \in \tau$ we have $G \setminus Cl_{\tau}(\{x\}) \in \tau'$. Also $N \in \mathcal{J}$, $N \in \tau$ implies $N \in \tau'$ and since N and H are neighbourhoods of Cl_{τ} ({x}) $N \cap H$ is also a neighbourhood.

Since τ is a larger topology than τ', Cl_{τ} ({x} is subset of $Cl_{\tau'}$ ({x})). Since $Cl_{\tau'}$ ({x})is indiscrete (theorem 3.1.7) , Cl_{τ} ({x}) is also indiscrete.

Now we have $G = (N \cap H) \cup (G \setminus Cl_{\tau'}(\{x\}))$

Hence

$$G \subseteq (N \cap H) \cup (G \setminus Cl_{\tau'}(\{x\}))$$

$$\subseteq G \cup (G \setminus Cl_{\tau`}(\{x\})) = G$$

There fore $G = (N \cap H) \cup (G \setminus Cl_{\tau`}(\{x\})) \in \tau`(H)$. Hence $\tau`(G) \subseteq \tau`(H)$. Since G and H are arbitrary elements of $\tau \setminus \tau`$, it follows that $\tau`(G) = \tau`(H)$ That is $\tau`(G) = \tau`(H)$for all G and H in $\tau$ not in $\tau`$. Therefore by theorem 1.2.1 τ is an upper neighbour of $\tau`$. Hence the theorem.

The above theorem generalises the following them of Larson and Thron [16]. We note that in a T_1 topology $Cl_{\tau}(\{x\}) = \{x\}$ is always indiscrete

3.1.12 Theorem [16]

Let τ be a topology and let $\mathfrak{I}(x, \mathcal{F})$ be a topology on X which is not finer than τ. Then τ is an upper neighbour of $\tau \cap \mathfrak{I}(x, \mathcal{F})$ if and only if for any two neighbourhoods G and H of x in τ such that

$H \notin \mathcal{J}$, there exist another neighbourhood N of x such that $N \in \mathcal{J}$, $N \cap H \subseteq G$.

3.1.13 Corollary

Suppose τ is a topology and $\Im(x, \mathcal{J})$ is an ultra topology on X which is not finer than τ such that τ is an upper neighbour of $\tau \cap \Im(x, \mathcal{J})$ then τ is an upper neighbour of $\tau \cap \Im(y, \mathcal{J})$ for every topology

$\Im(y, \mathcal{J})$ such that $Cl(\{x\}) = Cl(\{y\})$.

Proof

Let $\tau`$ be the topology $\tau \cap \Im(x, \mathcal{J})$ where $Cl(\{x\}) = Cl(\{y\})$.

Let G and H be any two neighbourhoods of $Cl(\{y\})$ in τ such that $H \notin \mathcal{J}$. So G and H are any two neighbourhoods of Cl $(\{x\})$ in τ such that $H \notin \mathcal{I}$, hence by the previous theorem there exist a neighbourhood N of Cl $(\{x\})$ such that $N \in \mathcal{I}$, $N \cap H \subseteq G$. Since every neighbourhood of $Cl(\{x\})$ is a neighbourhood of $Cl(\{y\})$, it follows that there exist a neighbourhood N of Cl $(\{y\})$ such that $N \in \mathcal{I}$, $N \cap H \subseteq G$. Hence by the previous theorem τ is an upper neighbour of $\tau \cap \Im(y, \mathcal{I})$.

Since the element y such that $Cl(\{x\}) = Cl(\{y\})$ was arbitrary it follows that τ is an upper neighbour of $\tau \cap \Im(y, \mathcal{I})$ for every y such that $Cl(\{x\}) = Cl(\{y\})$.

3.1.14 Theorem

If τ is a topology and $\Im(x, \mathcal{I})$ is a topology on X which is not finer than τ. Then τ is an upper neighbour of $\tau \cap \Im(x, \mathcal{I})$ if and only if $Cl_{\tau}(\{x\})$ is indiscrete and for any two neighbourhoods

G and H of Cl_τ ({x}) in τ such that H $\notin \mathcal{J}$, we have

Int_τ [G ∪ (X \ H)] ∈ $\mathcal{J}$,

Proof

Let τ is an upper neighbour of τ ∩ $\Im$ (x, $\mathcal{J}$). Let G and H be any two neighbourhoods of Cl ({x}) in τ such that H $\notin \mathcal{J}$. Then by the previous theorem there exist some N ∈ $\mathcal{J}$, which is a neighbourhood of Cl ({x}) such that N ∩ H ⊆ G. Therefore N ⊆ G ∪ (X \ H). Since N is open in τ , N ⊆ Int [G ∪ (X \ H)] . Since N ∈ $\mathcal{J}$ we have Int [G ∪ (X \ H)] ∈ $\mathcal{J}$.

Conversely,

Let for any two neighbourhoods G and H of Cl ({x}) in τ such that H $\notin \mathcal{J}$, we have

Int [G ∪ (X \ H)] ∈ $\mathcal{J}$.

Let $N = Int [G \cup (X \setminus H)]$ Then we have

$$N \cap H = Int [G \cup (X \setminus H)] \cap H \subseteq G.$$

By the previous theorem τ is an upper neighbour of $\tau \cap \Im (x , \mathcal{F})$.

3.1.15 Corollary

Let τ be a topology such that super sets of non empty open sets are open in τ Then if $\Im (x , \mathcal{U}(y))$ is a principal ultra topology on X which is not finer than τ, then τ is an upper neighbour of $\tau \cap \Im (x , \mathcal{U}(y))$.

Proof

Let H be neighbourhood of $Cl(\{x\})$ in τ such that $H \notin \mathcal{U}(y)$, that is $y \notin H$.

If G is any neighbourhood of $Cl(\{x\})$ in τ, then

$[G \cup (X \setminus H)]$, being super set of a nonempty open set is open in τ.

Hence $Int[G \cup (X \setminus H)] = [G \cup (X \setminus H)]$

Also $y \notin H$ implies that $y \in X \setminus H$.

So $y \in [G \cup (X \setminus H)]$. There fore $[G \cup (X \setminus H)] \in \mathcal{U}(y)$. By the previous theorem τ is an upper neighbour of $\tau \cap \Im(x, \mathcal{U}(y))$.

CHAPTER 4

TOPOLOGIES WITHOUT UPPER NEIGHBOURS IN LATTICE OF TOPOLOGIES

P. Agashe and N. Levine first gave examples of topologies without upper neighbours. They proved that a first countable completely normal topology is without upper neighbours in the lattice of topologies[1]. Later R.E. Larson and R. Valent proved that a first countable Hausdorff topology is also without upper neighbour[34].

4.1 Topologies without upper neighbours

In this section the above result of Larson and Valent is extended to the case of countably accessible topologies.

4.1.1 Lemma

Let τ be a topology such that every non-closed subset has two disjoint subsets with a common limit point out side the set. Then τ does not have an upper neighbour.

Proof

Suppose on the other hand let $\tau`$ be an upper neighbour of $\tau$ then $\tau` = \tau(A)$ for some $A \subseteq X$ $A \notin \tau$. So by assumption $X \setminus A$ contains two disjoint subsets C_1 and C_2 with a limit point $x \in A$. So $x \in A \setminus int_\tau A$. Also this x is such that $x \in ClC_1$ and $x \in ClC_2$. The sets C_1 and C_2 are such that $C_1 \cap C_2 = \varnothing$ and $C_1 \cap C_2 \cap int_\tau A = \varnothing$

$$\text{Let } O = X \setminus Cl(C_1 \cup \{x\})$$

$$= X \setminus [ClC_1 \cup Cl(\{x\})]$$

$= X \setminus Cl(C_1)$,

(since $x \in Cl(C_1) \Rightarrow ClC_1 \cup Cl(\{x\}) = ClC_1$)

Also $C_1 \subseteq X \setminus A$ implies int $A \subseteq X \setminus Cl(C_1) = O$

Now $O \cup (A \setminus \text{int } A)$

$= (X \setminus Cl(C_1)) \cup [A \cap (X \setminus \text{int } A)]$

$= [A \cup (X \setminus Cl(C_1)] \cap [(X \setminus Cl(C_1)) \cup (X \setminus \text{int } A)]$

$= [A \cup (X \setminus Cl(C_1)] \cap [(X \setminus (Cl(C_1) \cap \text{int } A)]$

$= [A \cup (X \setminus Cl(C_1)] \cap X$

$= [A \cup (X \setminus Cl(C_1)]$

$= A \cup O$

If $O \cup (A \setminus \text{int } A) = O \cup A \in \tau$ then $O \cup A$ is an open set containing x and hence intersects C_1, which cannot happen.

Therefore $O \cup A \notin \tau$. This means $O \cup A \in \tau(A) \setminus \tau$ and hence $\tau(A) = \tau(O \cup A)$. So A is of the form $U \cup (V \cap (O \cup A)$ for U, V open in τ.

Therefore

$$A = U \cup (V \cap (O \cup ((A \setminus \text{int } A) \cup \text{int } A)))$$

$$= U \cup (V \cap \text{int } A) \cup (V \cap O) \cup (V \cap (A \setminus \text{int } A))$$

and therefore $V \cap (O \setminus \text{int } A) \subseteq V \cap O \subseteq \text{int } A$

Also $V \cap (O \setminus \text{int } A) \subseteq V \cap O \cap (X \setminus \text{int } A) \subseteq X \setminus \text{int } A$

Hence $V \cap (O \setminus \text{int } A) = \varnothing$

Now $A \notin \tau$, $V \cap (A \setminus \text{int } A) = \varnothing$

$$\Rightarrow A \setminus \text{int } A \subseteq V \subseteq X \setminus Cl((O \setminus \text{int } A)),$$

since

$(A \setminus \text{int } A)$ is indiscrete

$$(A \setminus \text{int } A) \cap Cl((O \setminus \text{int } A)) = \phi$$

But we have

$C_2 \subseteq X \setminus \text{int } A$ and $C_2 \subseteq O$ So $C_2 \subseteq (O \setminus \text{int } A)$

Hence $x \in Cl(C_2) \subseteq Cl(O \setminus \text{int } A)$ so $x \notin (A \setminus \text{int } A)$ which is a contradiction to the choice of x.

Hence $\tau(A)$ can not be a upper neighbour of τ.

Hence the theorem.

A topology is called countably accessible if the topology can be written as the intersection of ultra topologies whose associated ultrafilters contain countable sets but no finite sets([14] page 33). This type of topologies were introduced by R. E. Larson. Larson constructied these topologies by imitating A.K. Steiner's construction of principal topologies. Since a countably accessible topology is an intersection of non-principal ultra topologies, every countably accessible topology is T_1. R. E. Larson gave the charecterisation of a countably accessible topology as the topology such that every non-closed subset contains a countable subset with a limit point outside the set [14].

4.1.2 Corollary

No countably accessible topology which is not an ultra topology has an upper neighbour in the lattice of topologies

Proof

Let (X, τ) be a countably accessible topology and let A be a non-closed set in this topology. By the charecterisation of Larson mentioned above there is a countably infinite subset B of A with a limit point x lying out side A.

Let $f : N \rightarrow B$ be a bijection, where N is the set of natural numbers. Such a bijection exists since the set B is countably infinite.

Let

$C_1 = \{ x \in B$ such that $f(i) = x$ for an odd integer $i\}$ and

$C_2 = \{ x \in B$ such that $f(i) = x$ for an even integer $i\}$.

These sets C_1 and C_2 are disjoint. Also the limit point x is such that it is a common limit point of the sets C_1 and C_2.

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Hence every non closed set has two disjoint subsets having a common limit point outside the set. So by the previous theorem τ cannot have an upper neighbour.

4.1.3 Corollary

No first countable T_1 space has an upper neighbour.

Proof

A T_1 first countable space is always countably accessible.

4.1.4 Corollary [3]

No first countable Hausdorff space has an upper neighbour.

Proof

A first countable T_1 space is always countably accessible.

4.2 Lattice of Cech Closure operators

4.2.1 Definitions [23]

Let $\wp(X)$ denote the power set of set X. The topological closure operator associated with a topology τ is defined as a function on $\wp(X)$defined by $c(A) = Cl_\tau(A)$. A Cech closure operator is a similar operator on $\wp(X)$ and can be considered as a generalisation of the topological closure operator.

Formally a Čech closure operator V on a set X is a function V from $\wp(X)$ into $\wp(X)$ having the following properties.

1. V fixes the empty set. i.e. $V(\varphi) = \varphi$.

2. V is expansive. i.e. $V(A) = A$.

3. V preserves the union. i.e. for any two subsets A and B of X, we have $V(A \cup B) = V(A) \cup V(B)$.

For convenience we call a Cech closure operator on a set X as a closure operator on X. Also the pair (X , V) is called a closure space.

A subset A in a closure operator (X , V) is said to be closed when $V(A) = A$. A subset is said to be open when its complement $X \setminus A$ is closed. i.e. when $V(X \setminus A) = X \setminus A$. The set of all open sets of a closure operator forms a topology on X. This topology is called the topology associated with the closure operator. Also this topology is such that, different closure operators on the same set, can have the same associated topology.

A closure operator on set X is topological if and only if it is the closure operator associated with a topology on X.

Now have a partial order among the set of all closure operators on a set X.

Suppose V_1 and V_2 are closure operators on a set X. We define $V_1 \leq V_2$ if and only if $V_2(A) \subseteq V_1(A)$ for every subset A of X. We denote the set of all closure operators on a fixed set by C(X). With the partial order defined as above the set of all closure operators on X forms a complete lattice.

The smallest element of this lattice is the indiscrete closure operator. For a subset a of the set X the indiscrete closure operator I is defined by

$$I(A) = \phi \quad \text{if } A = \phi$$

$$= X \quad \text{if } A \neq \phi .$$

This closure operator is the closure operator associated with the indiscrete topology on X.

The largest element of the lattice of closure operators is the discrete closure operator. The discrete closure operator D on X is defined by D(A) = A for every subset A of X. It is the closure operator associated with the discrete topology on X. Also D is the unique closure operator whose associated topology is discrete.

An infra closure operator is an atom in lattice of closure operators. i.e. A closure operator is called an infra closure operator if it is different from the indiscrete closure operator I and the only closure operator strictly smaller than it is the indiscrete closure operator.

An ultra closure operator is an dual atom in the lattice of closure operators. i.e. A closure operator is an ultra closure operator if it is different from the discrete closure operator and the only closure operator strictly larger than it is the discrete closure operator.

The Lattice of Cech Closure Operators is dually atomic and distributive. So by theorem 1.1.5 every element of it has a lower neighbour.

4.3 Open Problems

Our present investigations have proved successful in solving many intricate problems confronted during the course of this project. Nevertheless a few problems still remain unanswered. This will evidently open up new avenues in this area of research. Here we list some of the open problems.

1. Charecterisation of topologies having upper neighbours in the lattice of topologies.

2, Finding topologies without lower neighbours in the Lattice of Topologies

3. The upper and lower neighbours in some of the important sublattices of the Lattice of Topologies like lattice of regular topologies, lattice of completely regular topologies etc., are to be studied.

4, The relation between covers in the lattice of topologies and lattice of Cech closure operators is to be studied.

5, In areas like digital topology we require strengthening or weakening of topologies. In these problems some times it is required to change the neighbourhoods of some points in certain special ways without affecting the neighbourhoods of other points. It is felt that the methods in this thesis can be used in such situations.Detailed investigations in this direction are to be made.

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