Ph.D. THESIS MATHEMATICS

MEASURABLE DOMINATING FUNCTIONS

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CERTIFICATE

I hereby certify that the thesis entitled "**Measurable Dominating Functions**" is a bonafide work carried out by **Smt. Jisna P.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**Measurable Dominating Functions**" is based on the original work done by me under the supervision of **Dr. Raji Pilakkat**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

List of Symbols

G	Graph
$V(G)$	Vertex set of G
$E(G)$	Edge set of G
$n(G)$	Order of G
$m(G)$	Size of G
$\overline{G}$	Complement of G
$d_G(v)$	Degree of the vertex v in G
$G_1 \vee G_2$	Join of G_1 and G_2
$G_1 \circ G_2$	Corona of G_1 and G_2
$G_1[G_2]$	Lexicographic product of two graphs
$G_1 \otimes G_2$	Tensor product of two graphs
$G_1 \times G_2$	Cartesian product of two graphs
$G_1 \boxtimes G_2$	Normal product of two graphs
$G_1 * G_2$	Co-normal product of two graphs

$G_1 \ltimes G_2$	Homomorphic product of two graphs
P_n	Path on n vertices
C_n	Cycle on n vertices
K_n	Complete graph
$K_{m,n}$	Complete bipartite graph
$L(G)$	Line graph of G
$M(G)$	Middle graph of G
$T(G)$	Total graph of G
$Q_1(G)$	1-quasi-total graph of G
$Q_2(G)$	2-quasi-total graph of G
$G[S]$	Subgraph of G induced by S
$\Delta(G)$	Maximum degree of G
$\delta(G)$	Minimum degree of G
$\cong$	Isomorphic
$\mathcal{P}(X)$	Power set of a set X
$\mathcal{A}_G$	Neighborhood sigma algebra of G
μ	Measure
f_x	x-section of the function f
f^y	y-section of the function f
D_G	$\{v \in V(G) : d_G(v) = n(G) - 1\}$

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Chapter 0

Introduction

Graph theory is one of the most important and interesting areas in mathematics. It has experienced a rapid growth in last five decades. The main reason for the growth of graph theory is its wide range of applications in the areas such as chemistry, physics, genetics, psychology and computer science. Many practical problems can be visualized using graph theory.

Domination is one of the fastest growing areas in graph theory. The study of domination was started by C. Berge and O. Ore. The word dominating set was used first time by O. Ore in his book *Theory of Graphs* [21].

This thesis discusses both finite and infinite graphs. In this work we made an attempt to define dominating functions on infinite graphs and we call this as measurable dominating functions.

To extend the concept of dominating functions to infinite graphs we introduced a sigma algebra called neighborhood sigma algebra on the vertex sets of graphs. Using this sigma algebra measurable dominating functions of graphs (both finite and infinite) are defined. Minimal measurable dominating functions are defined and characterized. Apart from these a new type of graph polynomial called common neighborhood polynomial is introduced and discussed some of its properties. Neighborhood unique graphs are defined and common neighborhood polynomial of some classes of graphs are also found out.

0.1 Outline of the Thesis

Apart from this introductory chapter we presented our work in seven chapters. Chapters from two to six discuss only finite graphs and the seventh chapter includes discussion of infinite graphs.

In the **first chapter**, we provide some basic ideas and preliminary definitions which are essential for the development of the thesis. This chapter discusses some basic concepts of graph theory, measure theory and domination in graphs.

In the **second chapter**, neighborhood sigma algebra $\mathcal{A}_G$ of a graph G is introduced and studied its properties. We define the neighborhood sigma algebra of a graph G as the sigma algebra generated by the collection $\{N[v] : v \in V(G)\}$. Here a subset of the vertex set of a graph is measurable means it is measurable with respect to the neighborhood sigma algebra. We obtain that in the neighborhood sigma algebra of a graph G, the smallest measurable set containing a vertex v is the collection $\{u \in V(G) : N[u] = N[v]\}$ and we denote this set by E_v^G or E_v. It is proved that for a graph G with the neighborhood sigma algebra $\mathcal{P}(V(G))$, if P_3 is not a component of G and for $n > 2$, $K_{1,n}$ is not an induced subgraph of G, then the neighborhood sigma algebra of $L(G)$ is $\mathcal{P}(V(L(G)))$. It is also proved that for any graph G the neighborhood sigma algebra of its middle graph $M(G)$ is the power set of the vertex of $M(G)$. If G is a graph such that every component of G is different from P_2, then the neighborhood sigma algebra of its total graph $T(G)$ is the power set of the vertex of $T(G)$. We also determined the neighborhood sigma algebra of 2-quasi-total graph $Q_2(G)$ of a given graph G. If G is a graph without end vertices, then the neighborhood sigma algebra of $Q_2(G)$ is $\mathcal{P}(V(Q_2(G)))$.

In the **third chapter**, we determine the neighborhood sigma algebra of join two graphs and that of different graph products. We prove that if G and H are two vertex disjoint graphs with J as their join, then for each $v \in V(G)$ with $d_G(v) = n(G) - 1$, $E_v^J = E_v^G \bigcup \{u \in V(H) : d_H(u) = n(H) - 1\}$ and if $d_G(v) \neq n(G) - 1$, we obtain $E_v^J = E_v^G$. In the case of lexicographic product, tensor product, Cartesian product, normal product and co-normal product of two graphs G_1 and G_2, we prove that the product sigma algebra $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ is contained in the neighborhood sigma algebra of the product graph. In normal product these two sigma algebras coincide and whereas in homomorphic product there does not exist any such relationship.

In the **fourth chapter**, we introduce a new type of graph polynomial called common neighborhood polynomial. The common neighborhood polynomial of a graph G, denoted by $P(G,x)$, is the polynomial defined by

$$P(G,x) = \sum_{i=1}^{n(G)} a_i x^i,$$

where a_i is the number of E_v's of cardinality i in $\mathcal{A}_G$. We use the abbreviation CNP for the common neighborhood polynomial. Neighborhood unique graphs are also defined. A graph G is called a neighborhood unique graph if $P(G,x) = P(H,x)$ for any graph H implies that G is isomorphic to H. A characterization of such graphs are also given as follows. A graph G is neighborhood unique if and only if G is a complete graph or disjoint union of two complete graphs. The common neighborhood polynomials of line graph, middle graph, total graph, 1-quasi-total graph and 2-quasi-total graph of a given graph are also obtained in this chapter.

Fifth chapter deals with the CNP of join, corona and different graph products such as lexicographic product, tensor product, Cartesian product, normal product and co-normal product of two graphs..

Sixth chapter is a continuation of the work carried out in second and third chapters. The main new concept of this thesis, measurable dominating function of a finite graph is introduced in this chapter.

Let G be a graph. A function $f : V(G) \to [0,1]$ is called a measurable dominating function of G if the following conditions hold:

(i) f is measurable

(ii) $\displaystyle\int_{N[v]} f \, d\mu \geq 1$ for all $v \in V(G)$.

A necessary and sufficient condition for a measurable dominating function to be minimal is also obtained. Measurable k-dominating functions and measurable signed dominating functions are also defined. Characterizations of minimal measurable k-dominating function and minimal measurable signed dominating function are also derived. In the third chapter we proved that in the case of lexicographic product, tensor product, Cartesian product, normal product and co-normal product of two graphs G_1 and G_2, the product sigma algebra $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ is contained in the neighborhood sigma algebra of the product graph. Fortunately we could succeed to extend the product measure on $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ to the neighborhood sigma algebra of the vertex sets of the graph products.

If f_1 and f_2 are measurable dominating functions of two graphs G_1 and G_2 respectively, we check whether the function f defined on the vertex sets of graph products by $f((u, v)) = f_1(u)f_2(v)$ is a measurable dominating function of product graphs or not. We check the minimality of f also. The last section of this chapter deals with the measurability of x-section f_x and y-section f^y of a measurable function f defined on the vertex sets of different graph products. We prove that f_x is measurable but f^y is not always measurable in the case of lexicographic product, but whereas in tensor product, Cartesian product, co-normal product and homomorphic product f_x and f^y are in general, not measurable and in the case of normal product both f_x and f^y are measurable.

Seventh chapter deals with the measurable dominating functions of infinite graphs. In the case of an infinite graph G, if we define its neighborhood sigma algebra as the direct generalization of that of finite graphs, the smallest measurable set containing a vertex cannot be defined as the intersection of all measurable sets containing that vertex, because such a collection need not be countable. This realization blocks our work for a while, but by interpreting it in a slight different way we could overcome this situation. The neighborhood sigma algebra of an infinite graph is defined as the sigma algebra generated by $\mathcal{B} = \{N[v] : v \in V(G)\} \bigcup \{E_v : v \in V(G)\}$, where $E_v = \{u \in V(G) : N[u] = N[v]\}$. We stick on the notation E_v for the set $\{u \in V(G) : N[u] = N[v]\}$ because we prove that this set is the smallest measurable set containing v in parity with the finite graphs. Measurable dominating function of an infinite graph is defined and a characterization of minimal measurable dominating function is obtained. We concluded the thesis by introducing the concept of measurable signed dominating function of an infinite graph and characterized minimal measurable signed dominating functions.

The conclusion is given at the end and a bibliography is also given.

Chapter 1

Preliminaries

1.1 Introduction

Graph theory is a branch of mathematics which deals with the study of graphs. Many areas of mathematics such as group theory, operation research, topology and probability, have connections with graph theory. Also many real life problems can analize successfully using graphs.

The purpose of this chapter is to provide basic definitions and terminologies that we shall use in this work. It includes the basics of graph theory and measure theory and also discusses the concept of domination in graphs. For the notations and terminologies not given here, refer [3] and [6]

1.2 Basics of Graph Theory

Let us begin with the definition of a graph.

A (undirected) *graph* [6] G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to *join* u and v, and the vertices u and v are called the *ends* of e. In this case we also denote the edge by uv. Each vertex is indicated by a point, and each edge by a line joining the points representing its ends [6].

The number of vertices of the graph G is called the *order* [3] of G, denoted by $n(G)$ and the number of edges is called the *size* [3] of G, denoted by $m(G)$. A graph is called *finite* [28] if both its vertex set and edge set are finite. Otherwise it is called an infinite graph. That is if the vertex set or the edge set of a graph is infinite it is called an *infinite graph* [6].

A set of two or more edges of a graph G is called a set of *multiple edges* [3] if they have the same ends . An edge with identical ends is called a *loop* [3] . A graph is *simple* [3] if it has no loops and no multiple edges.

Every graph mentioned in this thesis is simple and undirected.

If u and v are distinct vertices and if $e = uv$ is an edge of the graph G, then u and v are said to *adjacent vertices*, the edge e is said to *incident with* u and v [7] and the vertices u and v are called the *end vertices* of the edge e [3]. If two distinct edges e and f are incident with a common vertex, they are called adjacent edges. Two adjacent vertices are referred to as neighbors of each other. In a graph G, the set of neighbors of a vertex v is called the *open neighborhood* [7] of v and it is denoted by $N_G(v)$. The set $N_G(v) \bigcup \{v\}$ is called the *closed neighborhood* [7] of v and it is denoted by $N_G[v]$ (or simply $N[v]$ if there is no confusion).

The *degree* [6] of a vertex v in a graph G, denoted by $d_G(v)$ (or $d(v)$), is the number of edges of G incident with v, each loop counting as two edges. In particular, if G is a simple graph, $d(v)$ is the number of neighbors of v in G. A vertex of degree zero is called an *isolated vertex* [22] . A vertex of degree one is called a *pendant vertex* or an *end vertex* [3] . A vertex adjacent to a pendant vertex is called a *support vertex* [22]. A *pendant edge* [3] is the edge incident with a pendant vertex. The minimum (respectively, maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ (respectively, $\Delta(G)$) [3].

The *complement* [6] of a simple graph G is the simple graph $\bar{G}$ whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. A *complete graph* [8] is a simple graph in which each pair of distinct vertices is joined by an edge. A complete graph on n vertices is denoted by K_n. A graph is said to be *bipartite* [3] if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a *bipartition* [3] of the bipartite graph. The bipartite graph with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is *complete* [3] if each vertex of X is adjacent to all the vertices of Y. A complete bipartite graph $G(X, Y)$ with $|X| = r$ and $|Y| = s$, is denoted by $K_{r,s}$.

Two graphs G and H are said to be *disjoint* [8] if they have no vertex in common. Two graphs G and H are *isomorphic* [6], written $G \cong H$, if there are bijections $\theta : V(G) \longrightarrow V(H)$ and $\phi : E(G) \longrightarrow E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; such a pair of mappings is called an *isomorphism* between G and H. Here the bijection θ satisfies the condition that u and v are end vertices of an edge e of G if and only if $\theta(u)$ and $\theta(v)$ are end vertices of the edge $\phi(e)$ in H [3].

A *walk* [3] in a graph G is an alternating sequence $W : v_0e_1v_1e_2v_2 \dots e_nv_n$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the *origin* and v_n is the *terminus* of W. The walk W is said to join v_0 and v_n. A walk is called a *trial* [3] if all the edges appearing in the walk are distinct. It is called a *path* [3] if all the vertices are distinct. Thus a path in G is automatically a trial in G. When writing a path, we usually omit the edges. A *cycle* [3] is a closed trial in which the vertices are all distinct. The number of edges in a walk is called its *length* [3]. A cycle of length n is denoted by C_n and P_n denotes a path on n vertices [3].

A graph H is called a *subgraph* [6] of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. If H is a subgraph of G, then G is said to be a *supergraph* [3] of H. A subgraph H of a graph G is said to be an *induced subgraph* [3] of G if each edge of G having its ends in $V(H)$ is also an edge of H. The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the *subgraph of* G *induced by* S and is denoted by $G[S]$ [3] . A subgraph H of a graph G is a *spanning subgraph* [3] of G, if $V(H) = V(G)$.

Let G be a graph and S a proper subset of the vertex set $V(G)$. The subgraph $G[V(G) \setminus S]$ is said to obtained from G by the *deletion* [3] of S. This subgraph is denoted by $G \setminus S$. If $S = \{v\}$, $G \setminus S$ is simply denoted by $G \setminus v$ [3].

A graph G is called *connected* [9] if any two of its vertices are linked by a path in G. A graph that is not connected is called *disconnected* [8]. *Components* [3] of a graph G are the maximal connected subgraphs of G. A connected graph without cycles is called a *tree* [3]. A subset V' of the vertex set $V(G)$ of a connected graph G is a *vertex cut* [3] of G, if $G \setminus V'$ is disconnected. A vertex v of G is a *cut vertex* [3] of G, if $\{v\}$ is a vertex cut of G. A vertex cut V' of G is *minimal* if no proper subset of V' is a vertex cut of G [1].

1.3 Operations on Graphs

We can construct new graphs from given graphs. This section deals with some methods of construction of new graphs from the given graphs.

The *union* [28] of graphs G_1 and G_2, written $G_1 \bigcup G_2$, has vertex set $V(G_1) \bigcup V(G_2)$ and edge set $E(G_1) \bigcup E(G_2)$. To specify the *vertex disjoint union* [3] with $V(G_1) \bigcap V(G_2) = \emptyset$, we write $G_1 + G_2$. The *join* [10] $G_1 \vee G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph with vertex set $V(G_1) \bigcup V(G_2)$ and edge set $E(G_1) \bigcup E(G_2) \bigcup \{uv : u \in V(G_1), v \in V(G_2)\}$. The *corona* [27] $G_1 \circ G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1 and $n(G_1)$ copies of G_2; and by joining each vertex of the i^{th} copy of G_2 to the i^{th} vertex of G_1, where $1 \leq i \leq n(G_1)$.

The *line graph* [3] $L(G)$ of a graph G is the graph with vertex set $E(G)$ in which two vertices are adjacent if they are adjacent edges in G.

The *middle graph* [26] $M(G)$ of a graph G is the graph with vertex set $V(G) \bigcup E(G)$ where two vertices are adjacent if they are either adjacent edges in G or one is a vertex and the other is an edge incident with it.

The *total graph* [15] $T(G)$ of a graph G is the graph with $V(G) \bigcup E(G)$ and two vertices x, y are adjacent in $T(G)$ if one of the following conditions holds:

(i) $x, y \in V(G)$ and x is adjacent to y in G

(ii) $x, y \in E(G)$ and x is adjacent to y in G

(iii) x is in $V(G)$ and y is in $E(G)$ and x, y are incident in G.

1.4 Domination in Graph Theory

The study of domination is the fastest growing area in graph theory. This section discusses the concept of dominating set and dominating function in a graph.

Let $G = (V(G), E(G)$ be the given graph.

A set $S \subseteq V(G)$ of vertices is called a *dominating set* [13] of G if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S.

A function $f : V(G) \to \{0, 1\}$ is called a *dominating function* [14] of G if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$.

A function $f : V(G) \longrightarrow [0,1]$ is called a *fractional dominating function* [14] of G if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$.

A function $f : V(G) \longrightarrow \{-1,1\}$ is called a *signed dominating function* [14] of G if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$.

A function $f : V(G) \longrightarrow \{0,1,2,\cdots,k\}$ is called a *k-dominating function* [14] of G if $\sum\limits_{u \in N[v]} f(u) \geq k$ for all $v \in V(G)$.

1.5 Measure Theory

This section focuses on some basic concepts of measure theory. For further details refer [23] and [11].

A distinguished collection $\mathcal{R}$ of subsets of a set X is called an *algebra* [11] if the following axioms are satisfied.

(i) If $E \in \mathcal{R}$ and $F \in \mathcal{R}$, then $E \bigcup F \in \mathcal{R}$

(ii) If $E \in \mathcal{R}$, then $E^c \in \mathcal{R}$, where $E^c := X \setminus E$ is the complement of E in X.

An algebra $\mathcal{R}$, of subsets of a set X is called a *sigma algebra* [23] if $\bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{R}$, whenever $E_1, E_2, \ldots \in \mathcal{R}$.

Proposition 1.5.1. *[23] If $\mathcal{F}$ is any family of subsets of a set X, there exists a smallest sigma algebra containing $\mathcal{F}$, called the sigma-algebra generated by $\mathcal{F}$.*

A set X together with a sigma algebra $\mathcal{R}$ of subsets of X is called a *measurable space* [23], and the members of $\mathcal{R}$ are called the *measurable sets* [23] in X.

Let X be a measurable space and Y be a topological space [20]. A mapping f from X into Y is said to be *measurable* [23] if $f^{-1}(S)$ is a measurable set in X for every open set S in Y. If f and g are measurable functions then $\alpha f + \beta g$ is measurable for any real numbers α and β [23]. Let $(X, \mathcal{R})$ be a measurable space. A *measure* [23] is a function μ, defined on the sigma algebra $\mathcal{R}$, whose range is in $[0, \infty]$ and which is countably additive. This means that if $\{E_i\}$ is a disjoint countable collection of members of $\mathcal{R}$ then $\mu(\bigcup\limits_{i=1}^{\infty} E_i) = \sum\limits_{i=1}^{\infty} \mu(E_i)$. In this thesis we consider only those measures which assume only finite values. A *measure space* [23] is a measurable space which has a measure defined on the sigma algebra of its measurable sets. Let P be a property concerning the points of a measure space $(X, \mathcal{R}, \mu)$ and let $E \in \mathcal{R}$. The statement "P holds almost everywhere on E "(abbreviated to "P holds a.e on E ") means that there exists $N \in \mathcal{R}$ such that $\mu(N) = 0$, $N \subset E$, and P holds at every point of $E \setminus N$.

A function s on a measure space X whose range consists of only finitely many points is called a *simple function* [23]. If $\alpha_1, \alpha_2, ..., \alpha_n$ are the distinct values of a simple function s, and if we set $A_i = \{x : s(x) = \alpha_i\}$ then $s = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i}$, where χ_{A_i} is the characteristic function of A_i. It can be proved that s is measurable if and only if each of the sets A_i is measurable [23]. Suppose $\mathcal{R}$ is a sigma algebra on the set X and μ is a measure on $\mathcal{R}$. If $s = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i}$ is a measurable simple function from X into $[0, \infty)$, where $\alpha_1, \alpha_2, ..., \alpha_n$ are the distinct values assumed by s and if $E \in \mathcal{R}$, then $\int\limits_E s \; d\mu$ is defined by $\sum\limits_{i=1}^{n} \alpha_i \mu(A_i \bigcap E)$ [23]. If $f : X \longrightarrow [0, \infty]$ is measurable and $E \in \mathcal{R}$, then $\int\limits_E f \; d\mu = \sup \int\limits_E s \; d\mu$, the supremum is taken over all simple measurable functions s such that $0 \leq s \leq f$. If

$0 \leq f \leq g$ then, $\int\limits_E f\ d\mu \leq \int\limits_E g\ d\mu$. If $A \subset B$ and $f \geq 0$, then $\int\limits_A f\ d\mu \leq \int\limits_B f\ d\mu$. If X and Y are two sets, their *Cartesian product* [23] $X \times Y$ is the set of all ordered pairs (x, y), with $x \in X$ and $y \in Y$. With each function f on $X \times Y$ and with each $x \in X$ we associate a function f_x defined on Y by $f_x(y) = f(x, y)$. Similarly, if $y \in Y$, f^y is the function defined on X by $f^y(x) = f(x, y)$ [23]. Call f_x and f^y, the x-section and y-section respectively, of f [23].

Suppose (X, S) and (Y, T) are two measurable spaces. A *measurable rectangle* [23] is any set of the form $A \times B$, where $A \in S$ and $B \in T$. The *product sigma algebra* $S \times T$ is defined to be the smallest sigma algebra in $X \times Y$ which contains every measurable rectangles [23]. If E is any subset of $X \times Y$, then for $x \in X$, we call the set $E_x = \{y : (x, y) \in E\}$ as a section of E determined by x and for $y \in Y$ we call the set $E^y = \{x : (x, y) \in E\}$ as a section of E determined by y [11]. Every section of a measurable set is a measurable set [11].

If (X, S, μ) and (Y, T, ν) are sigma finite measure spaces, then the set function λ, defined for every set E in $S \times T$ by $\lambda(E) = \int \nu(E_x)\ d\mu(x) = \int \mu(E^y)\ d\nu(y)$, is a sigma finite measure with the property that, for every measurable rectangle $A \times B$, $\lambda(A \times B) = \mu(A).\nu(B)$ [11]. The latter condition determines λ uniquely. The measure λ is called the *product* of the measures μ and ν and denote it by $\mu \times \nu$ [11].

Chapter 2

Neighborhood Sigma Algebra

In this chapter we study neighborhood sigma algebra of graphs. While studying dominating function of finite graphs we realized that the direct generalization of it into infinite graphs is not possible if we stick on the definition of dominating function of finite graphs [14]. So we tried to interpret this concept of dominating function with the help of theory of measures.

As we know the platform for working in measures is the algebra/sigma algebra of sets, we need such a structure on vertex sets of the graphs. But the problem now arising is that how to construct such a structure. As the power set of any set forms a sigma algebra, one way of escaping this trouble some situation is to use the power set as the sigma algebra. But this case is a least interesting one. In order to strengthen the theory we construct a sigma algebra which is most suitable to our study and a little bit fascinating which we call as neighborhood sigma algebra.

2.1 Neighborhood Sigma Algebra

We define the neighborhood sigma algebra of a graph as follows.

Definition 2.1.1. Let $G = (V(G), E(G))$ be a graph. The sigma algebra generated by $\mathscr{G} = \{N[v] : v \in V(G)\}$ on $V(G)$ is called the neighborhood sigma algebra of G and it is denoted by $\mathcal{A}_G$ (or simply $\mathcal{A}$ if there is no confusion) and $\mathscr{G}$ is called the generating set of $\mathcal{A}$.

Such a sigma algebra exists by Proposition 1.5.1. We build up our theory with this sigma algebra.

As the graphs considered here are finite their vertex sets are finite. So $\mathcal{A}$ is just an algebra. But this is not the case when the graph is infinite. As the thesis also discusses infinite graphs, we would like to use the terminology sigma algebra in both the cases, finite and infinite graphs.

Throughout this thesis, by a graph G, we mean the graph with its neighborhood sigma algebra $\mathcal{A}$ on the vertex set $V(G)$. Here a subset of $V(G)$ is measurable means it is measurable with respect to the neighborhood sigma algebra.

Definition 2.1.2. Let G be a graph. For $v \in V(G)$, we define E_v^G (or simply E_v if there is no confusion) to be the intersection of all measurable sets containing v. Hence it is the smallest measurable set containing v.

Example 2.1.3. For the graph G_1, in Figure 2.1: the neighborhood sigma algebra $\mathcal{A}$ is given by $\{\emptyset, \{u, v, w, x\}, \{u, v, x\}, \{v, w, x\}, \{u\}, \{w\}, \{v, x\}, \{u, w\}\}$. $E_u = \{u\}, E_v = \{v, x\}, E_w = \{w\}$ and $E_x = \{v, x\}$.

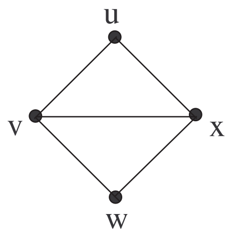

Figure 2.1: Graph G_1

Proposition 2.1.4. *Let G be a graph and $u, v \in V(G)$. Then $u \in E_v$ if and only if $v \in E_u$.*

Proof. Let $u \in E_v$. If $v \notin E_u$, $E_v \setminus E_u$ is a measurable set containing v and properly contained in E_v, which contradicts the fact that E_v is the smallest measurable set containing v. Hence $v \in E_u$. Also by interchanging the roles of u and v we get $u \in E_v$ whenever $v \in E_u$. □

Lemma 2.1.5. *Let G be a graph and u, v be two vertices of G such that $u \in E_v$. Then $E_u = E_v$.*

Proof. Since $u \in E_v$, by Proposition 2.1.4, $v \in E_u$. The sets E_u and E_v, being the smallest measurable sets containing u and v respectively, $u \in E_v$ and $v \in E_u$ imply that $E_u \subset E_v$ and $E_v \subset E_u$. Hence $E_u = E_v$. □

Lemma 2.1.6. *Let G be a graph and $u, v \in V(G)$ be such that $E_u \bigcap E_v \neq \emptyset$. Then $E_u = E_v$.*

Proof. Suppose that $E_u \bigcap E_v \neq \emptyset$. Let $w \in E_u \bigcap E_v$. Then by Lemma 2.1.5, $E_w = E_u = E_v$. □

Theorem 2.1.7 is an immediate consequence of Lemma 2.1.6.

Theorem 2.1.7. *Let G be a graph. Then $\{E_u : u \in V(G)\}$ forms a partition of $V(G)$.*

Remark 2.1.8. Each measurable set can be written as disjoint union of $E_v's$.

Definition 2.1.9. A vertex $v \in V(G)$ of the graph G is called a common neighborhood free vertex if $E_v = \{v\}$.

Proposition 2.1.10. *Let G be a connected graph with $n(G) \neq 2$ and let $v \in V(G)$ be such that it is either an end vertex or a support vertex. Then v is a common neighborhood free vertex.*

Proof. If $n(G) = 1$, then the result is trivially true. So assume that $n(G) > 2$. Let v be an end vertex of G with support vertex u. Since G is a connected graph of order greater than two, there exists a vertex $w \in N[u] \setminus \{v\}$. Therefore, $N[v] \bigcap N[w] = \{u\}$ is measurable. Hence $E_u = \{u\}$. Since $\{u\}$ is measurable, $N[v] \setminus \{u\}$ is measurable. That is $\{v\}$ is measurable. Hence $E_v = \{v\}$.

□

Remark 2.1.11. The converse of Proposition 2.1.10 is not true. That is $E_v = \{v\}$ does not imply, v is an end vertex or a support vertex.

Consider the path P_5 in Figure 2.2.

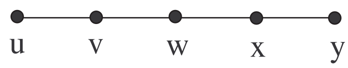

Figure 2.2: The path P_5

For the vertex w, $E_w = \{w\}$. But w is neither an end vertex nor a support vertex.

Next we observe the neighborhood sigma algebra of a complete graph. For a vertex v of a graph G, $E_v = V(G)$ if and only if $u \in E_v$ for all $u \in V(G)$. That is if and only if $E_u = E_v = V(G)$ for all $u \in V(G)$, by Lemma 2.1.5. That is if and only if $N[u] = N[v]$ for all $u \in V(G)$. Hence we have:

Proposition 2.1.12. *A graph G is complete if and only if $E_v = V(G)$, for some $v \in V(G)$.*

Note that for the graph G_1, in Figure 2.1, $E_v = E_x$. Note also that these two vertices v and x have the same closed neighborhoods, that is $N[v] = N[x]$. This result in fact has a general feature.

That is, for any two vertices v_1 and v_2 of a graph G, $E_{v_1} = E_{v_2}$ if and only if $N[v_1] = N[v_2]$. The proof of this result depends mainly on the neighborhood sigma algebra of the graph. Before proving this result, we characterize the neighborhood sigma algebras of graphs.

Proposition 2.1.13. *Let G be a graph with neighborhood sigma algebra $\mathcal{A}$. Then every member of $\mathcal{A}$ can be expressed as the union of sets, each of which can be expressed as the intersection of members of $\mathscr{F}$, where $\mathscr{F} = \{N[v] : v \in V(G)\} \bigcup \{N[v]^c : v \in V(G)\}$.*

Proof. Let $\mathscr{H}$ consists of all subsets of $V(G)$ which can be expressed as unions of members of $\mathscr{G}$, where $\mathscr{G}$ is the family of all intersections of members of $\mathscr{F}$. Then $\mathscr{H}$ contains $\{N[v] : v \in V(G)\}$ and it is contained in $\mathcal{A}$. Also $\mathscr{H}$ itself is a sigma algebra. As $\mathcal{A}$ is generated by $\{N[v] : v \in V(G)\}$, $\mathscr{H} = \mathscr{A}$. Hence the proposition. □

The following theorem helps to determine E_v's in a graph.

Theorem 2.1.14. *Let G be a graph. Then for $v_1, v_2 \in V(G)$, $E_{v_1} = E_{v_2}$ if and only if $N[v_1] = N[v_2]$.*

Proof. Assume that $E_{v_1} = E_{v_2}$ for some $v_1, v_2 \in V(G)$. Suppose $N[v_1] \neq N[v_2]$. Without loss of generality, assume that there exists $u \in V(G)$ such that $u \in N[v_1]$ but $u \notin N[v_2]$. Therefore, $N[u] \bigcap N[v_1]$ is a measurable set containing v_1 but not v_2. This implies that $v_2 \notin E_{v_1}$. This will contradict the fact that $E_{v_1} = E_{v_2}$.

Conversely, assume that $N[v_1] = N[v_2]$. This implies for any $v \in V(G)$ either $v_1, v_2 \in N[v]$ or $v_1, v_2 \in N[v]^c$. Therefore, by Proposition 2.1.13, if B is any measurable set, then either $v_1, v_2 \in B$ or $v_1, v_2 \in B^c$. This implies that $E_{v_1} = E_{v_2}$. □

If u and v are two vertices of a graph G then $u \in E_v$ if and only if $E_u = E_v$, by Lemma 2.1.5. That is if and only if $N[u] = N[v]$. Thus we have:

Corollary 2.1.15. *Let G be a graph and $v \in V(G)$. Then $E_v = \{u \in V(G) : N[u] = N[v]\}$.*

Remark 2.1.16. Let G be a graph and $v \in V(G)$. In general $E_v \neq \bigcap\{N[u] : u \in N[v]\}$.

Consider the path P_3 in Figure 2.3.

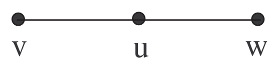

Figure 2.3: The path P_3

For the vertex v of P_3, $E_v = \{v\}$. But $\bigcap\{N[u] : u \in N[v]\} = \{v, u\}$.

Proposition 2.1.17. *Let G be a graph. If there exists only one vertex $v \in V(G)$ such that $u \in N[v]$ for all $u \in V(G)$, then $E_v = \{v\}$ and hence $\{v\}$ is measurable.*

Proof. In this case $\{v\} = \bigcap_{u \in V(G)} N[u]$. Hence $E_v = \{v\}$. □

Note 2.1.18. There are graphs in which $E_v \neq \{v\}$, for any vertex v. One such graph is given below.

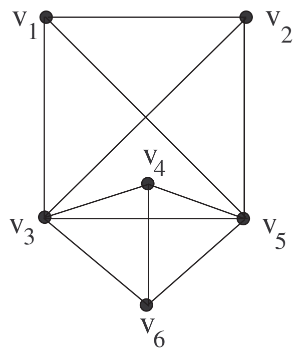

Figure 2.4: Graph G

In the graph G, $E_{v_1} = E_{v_2} = \{v_1, v_2\}$, $E_{v_3} = E_{v_5} = \{v_3, v_5\}$, $E_{v_4} = E_{v_6} = \{v_4, v_6\}$.

The following theorem says that in any graph a set of vertices of a particular degree is measurable.

Theorem 2.1.19. *Let G be a graph with vertex set $V(G)$. For each $k \in \mathbb{N}$ with $1 \leq k \leq \Delta(G)$, the collection $S_k := \{v \in V(G) : d(v) = k\}$ is a measurable set.*

Proof. Let $k \in \mathbb{N}$ be such that $1 \leq k \leq \Delta(G)$. If $S_k = \emptyset$, then it is measurable. So suppose that $S_k \neq \emptyset$. Let $v \in S_k$. Since $E_v = \{u \in V(G) : N[u] = N[v]\}$, $d(v) = d(u)$ for all $u \in E_v$. This implies that $E_v \subseteq S_k$ for all $v \in S_k$. Hence $S_k = \bigcup_{v \in S_k} E_v$. Therefore S_k is measurable. $\square$

Since the complement of a measurable set is measurable, we have:

Corollary 2.1.20. *Let G be a graph with vertex set $V(G)$. For each $k \in \mathbb{N}$ with $1 \leq k \leq \Delta(G)$, the collection $\{v \in V(G) : d(v) \neq k\}$ is measurable.*

Theorem 2.1.21. *Let G be a connected graph and C be a minimal vertex cut of G. Then C is measurable.*

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of $G \setminus C$ with vertex sets $V_1, V_2, \ldots, V_k$ respectively, where $k \geq 2$. Let $v \in C$. Since C is a minimal vertex cut, v is adjacent to vertices of at least two components, say G_1 and G_2. Suppose that $N(v) \bigcap V_1 = \{u_1, u_2, ..., u_{k_1}\}$ and $N(v) \bigcap V_2 = \{v_1, v_2, ..., v_{k_2}\}$. Then $v \in \bigcap_{i=1}^{k_1} N(u_i) \subseteq C \bigcup V_1$ and $v \in \bigcap_{i=1}^{k_2} N(v_i) \subseteq C \bigcup V_2$. Therefore $v \in (\bigcap_{i=1}^{k_1} N(u_i)) \bigcap (\bigcap_{i=1}^{k_2} N(v_i)) \subseteq C \bigcup (V_1 \bigcap V_2)) = C$. Therefore $(\bigcap_{i=1}^{k_1} N(u_i)) \bigcap (\bigcap_{i=1}^{k_2} N(v_i))$ is a measurable set containing v and contained in C. Thus C is a union of a collection of measurable sets. Hence C is measurable. □

Corollary 2.1.22. *If v is a cut vertex of a connected graph G, then $\{v\}$ is measurable.*

Corollary 2.1.23. *In a tree, if $\{v\}$ is a vertex of degree greater than one, then $\{v\}$ is measurable.*

Proof. If G is a tree and $v \in V(G)$ is such that $d(v) \geq 2$, then v is a cut vertex of G. Therefore $\{v\}$ is measurable. □

Note 2.1.24. In Proposition 2.1.10, it is proved that if v is an end vertex of a connected graph G of order not equal to two, then $\{v\}$ is measurable. Hence if G is a tree of order not equal to two, then $\{v\}$ is measurable for all $v \in V(G)$.

Hereafter a function defined on the vertex set of a graph is measurable means which is measurable with respect to the neighborhood sigma algebra of that graph.

Theorem 2.1.25. *Let G be a graph and $f : V(G) \longrightarrow [0,1]$ be a function. Then f is measurable if and only if f is constant on E_v for all $v \in V(G)$.*

Proof. Let $v \in V(G)$ and $f(v) = c$. Suppose $f(u) = d$ for some $u \in E_v$. Let, if possible, $c < d$. Then $f^{-1}(-\infty, d)$ is measurable and $v \in f^{-1}(-\infty, d)$. Therefore v belongs to the measurable set $f^{-1}(-\infty, d) \bigcap E_v$, which is a proper subset of E_v. This contradicts the fact that E_v is the smallest measurable set containing v. A similar kind of contradiction arises when $d < c$.

Conversely assume that f is constant on E_v for all $v \in V(G)$. Let U be an open subset of [0,1]. Suppose that $f(V(G)) \bigcap U = \{k_1, k_2, ..., k_m\}$. Then $f^{-1}(U) = f^{-1}(\{k_1\}) \bigcup f^{-1}(\{k_2\}) \bigcup ... \bigcup f^{-1}(\{k_m\})$. Let $1 \leq i \leq m$. As f is constant on each E_v, $f^{-1}(\{k_i\}) = \bigcup\limits_{f(v_j)=k_i} E_{v_j}$. Hence $f^{-1}(k_i)$ is measurable for all $1 \leq i \leq m$. Therefore $f^{-1}(U)$ is measurable. Hence f is measurable. □

As a consequence of Corollary 2.1.15 and Theorem 2.1.25, we have:

Corollary 2.1.26. *Let G be a graph with $u_1, u_2 \in V(G)$. If $f : V(G) \longrightarrow [0,1]$ is measurable and $N[u_1] = N[u_2]$ then $f(u_1) = f(u_2)$.*

Theorem 2.1.27. *Let G be a graph and $v \in V(G)$ be such that $d(v) = n(G)-1$. Then $E_v = \{u \in V(G) : d(u) = n(G) - 1\}$.*

Proof. Let $u \in E_v$. Then $N[u] = N[v]$. Hence $d(u) = n(G) - 1$. Therefore $E_v \subseteq \{u \in V(G) : d(u) = n(G)-1\}$. Let $u \in V(G)$ be such that $d(u) = n(G)-1$. Then $N[u] = V(G) = N[v]$. Hence $u \in E_v$. Thus, $E_v = \{u \in V(G) : d(u) = n(G) - 1\}$. $\square$

Corollary 2.1.28. *Let G be a graph with $\Delta(G) = n(G) - 1$ and $f : V(G) \longrightarrow [0, 1]$ be measurable. Then f is constant on the set,*
$\{v \in V(G) : d(v) = n(G) - 1\}$.

Note 2.1.29. The conclusion of Theorem 2.1.27 need not be true for the vertices of degree $< n(G) - 1$.

For example consider the cycle C_4 and the path P_3.

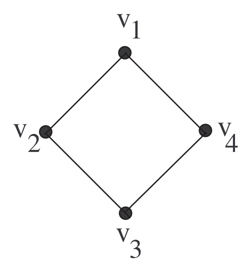

Figure 2.5: The cycle C_4

For the cycle C_4 given in Figure 2.5, $d(v_1)$=$d(v_2)$=$d(v_3)$=$d(v_4) = 2$.

But $E_{v_1} = \{v_1\}, E_{v_2} = \{v_2\}, E_{v_3} = \{v_3\}, E_{v_4} = \{v_4\}$.

For the path P_3 given in Figure 2.3, $d(v_1) = d(w) = 1$.

But $E_v = \{v\}, E_w = \{w\}$.

2.2 Vertex Deleted Graph

In this section we examine how the deletion of a vertex from a graph affect E_v's.

Let G be a graph and $v \in V(G)$. Consider the graph $G_v := G \setminus v$. It is clear that $N_{G_v}[u] = N_G[u] \setminus \{v\}$ for all $u \in V(G_v)$. For $u \in V(G_v)$, we expect that $E_u^{G_v} = E_u^G \setminus \{v\}$ but it is not true.

Example 2.2.1. Consider the graphs given in Figure 2.6.

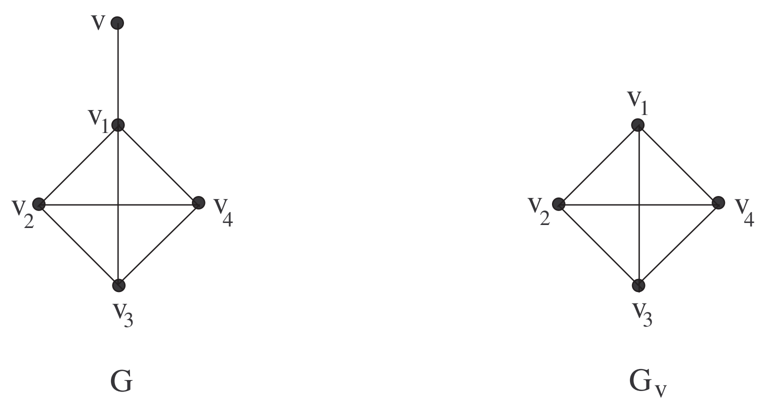

Figure 2.6: Graph G and its vertex deleted graph G_v

$E^G_{v_2} = \{v_2, v_3, v_4\}$ and $E^{G_v}_{v_2} = \{v_1, v_2, v_3, v_4\}$.

So, $E^G_{v_2} \setminus \{v\} \subseteq E^{G_v}_{v_2}$.

The following theorem says that this is true in general.

Theorem 2.2.2. *Let G be a graph and $v \in V(G)$. Then for $u \in V(G_v)$, $E_u^G \setminus \{v\} \subseteq E_u^{G_v}$.*

Proof. Let $u \in V(G_v)$ and $x \in E_u^G \setminus \{v\}$. Since $x \in E_u^G$, we have $N_G[x] = N_G[u]$. Therefore $N_G[x] \setminus \{v\} = N_G[u] \setminus \{v\}$. That is $N_{G_v}[x] = N_{G_v}[u]$. Therefore $E_x^{G_v} = E_u^{G_v}$. Hence $x \in E_u^{G_v}$. Therefore $E_u^G \setminus \{v\} \subseteq E_u^{G_v}$ for all $u \in V(G_v)$. □

Theorem 2.2.3. *Let G be a graph and $v \in V(G)$. Then $E_u^G \setminus \{v\} = E_u^{G_v}$ for all $u \in V(G_v)$ with $v \in E_u^G$.*

Proof. Let $u \in V(G_v)$ be such that $v \in E_u^G$. By theorem 2.2.2, $E_u^G \setminus \{v\} \subseteq E_u^{G_v}$. To obtain the reverse inclusion, let $x \in E_u^{G_v}$. Then $N_{G_v}[x] = N_{G_v}[u]$.

That means,

$$N_G[x] \setminus \{v\} = N_G[u] \setminus \{v\} \tag{2.1}$$

This implies, $x \in N_G[u]$. Since $N_G[u] = N_G[v]$, $x \in N_G[v]$. Hence $v \in N_G[x]$. Since $v \in E_u^G$, $v \in N_G[u]$. Therefore, equation (2.1), implies that $N_G[x] = N_G[u]$. Therefore, $E_u^{G_v} \subseteq E_u^G \setminus \{v\}$. Hence the theorem. □

2.3 Line Graph

This section deals with the neighborhood sigma algebra of the line graph of a graph.

Theorem 2.3.1. *Let G be a graph with neighborhood sigma algebra $\mathcal{P}(V(G))$. If P_3 is not a component of G and for $n > 2$, $K_{1,n}$ is not an induced subgraph of G, then the neighborhood sigma algebra of $L(G)$ is $\mathcal{P}(V(L(G)))$.*

Proof. For $x \in V(L(G))$, let $N_L[x]$ denote $\{x\} \bigcup \{u \in V(L(G)) : u$ is adjacent to x in $L(G)\}$. Let e and f be two distinct vertices of $L(G)$. Then e and f are two distinct edges of G. Suppose $N_L[e] = N_L[f]$. This implies e and f are two adjacent vertices of $L(G)$ and hence e and f are two adjacent edges of G. With out loss of generality assume that $e = uv$ and $f = vw$, where $u, v, w \in V(G)$. Suppose u and w are adjacent in G. Since $N_G[u] \neq N_G[w]$, there exists $x \in V(G)$ such that x belongs to $N_G[u]$ or $N_G[w]$ but not both. If $x \in N_G[u]$, $ux \in N_L[e]$ and $ux \notin N_L[f]$. If $x \in N_G[w]$, $wx \in N_L[f]$ and $wx \notin N_L[e]$. This will imply $N_L[e] \neq N_L[f]$. Therefore u and w are not adjacent in G. It is given that P_3 is not a component of G. Therefore, in G, u, v or w is adjacent to a vertex in $V(G) \setminus \{u, v, w\}$. Since $K_{1,n}$ is not an induced subgraph of G, we can say that, in G, u or w is adjacent to a vertex $x \in V(G) \setminus \{u, v, w\}$. If $ux \in E(G)$, then $ux \in N_L[e]$ but $ux \notin N_L[f]$, which is a contradiction. A similar contradiction arises when $wx \in E(G)$. Hence the theorem. □

Corollary 2.3.2. *(1). The neighborhood sigma algebra of $L(C_n)$ is $\mathcal{P}(V(L(C_n)))$, for all $n > 3$.*

(2). The neighborhood sigma algebra of $L(P_n)$ is $\mathcal{P}(V(L(P_n)))$, for all $n > 3$.

2.4 Middle Graph

In this section we determine the neighborhood sigma algebra of the middle graph of a graph.

Theorem 2.4.1. *Let G be a graph and $M(G)$ be its middle graph. Then the neighborhood sigma algebra of $M(G)$ is $\mathcal{P}(V(M(G)))$. In particular every function on $V(M(G))$ is measurable.*

Proof. For $x \in V(M(G))$, let $N_M[x]$ denote $\{x\} \bigcup \{u \in V(M(G)) : u$ is adjacent to x in $M(G)\}$. Let u and v be two distinct vertices of $M(G)$. We consider three cases.

Case 1. $u \in V(G)$ and $v \in E(G)$.

Let $v = xy$ with $x, y \in V(G)$. Then $N_M[u] = \{u\} \bigcup \{f \in E(G) : f$ is incident with u in $G\}$ and $N_M[v] = \{v, x, y\} \bigcup \{f \in E(G) : f$ is adjacent to v in $G\}$. Therefore $N_M[u] \neq N_M[v]$.

Case 2. $u, v \in V(G)$.

Then $N_M[u] \neq N_M[v]$, because no two vertices of G are adjacent in $M(G)$.

Case 3. $u, v \in E(G)$.

Let $u = u_1v_1$ and $v = u_2v_2$ with $u_1, v_1, u_2, v_2 \in V(G)$. Then $N_M[u]$ contains both u_1 and v_1. But, as $u \neq v$, not both u_1 and v_1 are in $N_M[v]$. Therefore $N_M[u] \neq N_M[v]$.

Hence $N_M[x]$ and $N_M[y]$ are distinct for any two distinct vertices x, y of $M(G)$. Therefore the neighborhood sigma algebra of $M(G)$ is $\mathcal{P}(V(M(G)))$. $\square$

Note 2.4.2. Let G be a graph with two vertices u and v such that $N[u] = N[v]$. Then G is not middle graph of any graph.

2.5 Total Graph

This section is devoted to determine the neighborhood sigma algebra of the total graph of a graph.

Theorem 2.5.1. *Let G be a graph such that every component of G is different from P_2 and $T(G)$ be its total graph. Then the neighborhood sigma algebra of $T(G)$ is $\mathcal{P}(V(T(G)))$.*

Proof. If $G = \bar{K}_n$, $n = 1, 2, ...$ then the result is obvious. Suppose that $G \neq \bar{K}_n$ for every n. Then $n(G) \geq 3$. For $x \in V(T(G))$, let $N_T[x]$ denote $\{x\} \bigcup \{u \in V(T(G)) : u$ is adjacent to x in $T(G)\}$. Let u and v be two distinct vertices of $T(G)$. We consider the following cases.

Case 1. $u, v \in E(G)$.

Let $u = u_1v_1$ and $v = u_2v_2$ with $u_1, v_1, u_2, v_2 \in V(G)$. Then $u_1, v_1 \in N_T[u]$. But not both of them belongs to $N_T[v]$. Therefore $N_T[u] \neq N_T[v]$.

Case 2. $u \in V(G)$ and $v \in E(G)$.

If possible assume that $N_T[u] = N_T[v]$. Then v is incident with u in G. Let $v = uw$, $w \in V(G)$. If $w' \neq w$ is adjacent to u in G then $w' \in N_T[u]$. But $w' \notin N_T[v]$. Hence in G, u is adjacent to w only. That means u is an end vertex. Suppose v is adjacent to an edge v' in G. Then u is not incident on v' since u is an end vertex. Hence $v' \in N_T[v]$ but $v' \notin N_T[u]$. So there does not exist $v' \in E(G)$, adjacent to v in G. This will imply P_2 is a component of G, a contradiction.

Case 3. $u, v \in V(G)$.

If possible assume that $N_T[u] = N_T[v]$. This implies u and v are adjacent in G. Suppose there exists $w(\neq v)$ in $V(G)$ which is adjacent to u in G. Then $e = uw$ will be an edge in G such that $e \in N_T[u]$ but $e \notin N_T[v]$. Therefore in G, u is adjacent to v only. Hence u is an end vertex of G. Similarly we can prove that v is an end vertex of G. Hence P_2 is a component of G, a contradiction.

Thus if P_2 is not a component of G, then $N_T[x] \neq N_T[y]$ for $x \neq y \in V(T(G))$. □

Corollaries 2.5.2 and 2.5.3 are immediate consequences of Theorem 2.5.1.

Corollary 2.5.2. *Since $T(P_2)$ is K_3 we have: Let G be a graph such that no component of G is K_3. If there exists two vertices u and v in G such that $N[u] = N[v]$. Then G is not total graph of any graph.*

Corollary 2.5.3. *Let G be the total graph of a graph such that no component of G is K_3. Then all functions defined from $V(G)$ are measurable.*

2.6 1-quasi-total Graph and 2-quasi-total Graph

This section deals with the neighborhood sigma algebras of 1-quasi-total graph and 2-quasi-total graph of a graph.

Definition 2.6.1. [25] Let G be a graph. The 1-quasi-total graph, $Q_1(G)$, of G is the graph with vertex set $V(G) \bigcup E(G)$ and in which two vertices u and v are adjacent if they satisfy one of the following conditions:

(1). u, v are in $V(G)$ and u, v are adjacent in G.

(2). u, v are in $E(G)$ and u, v are adjacent in G.

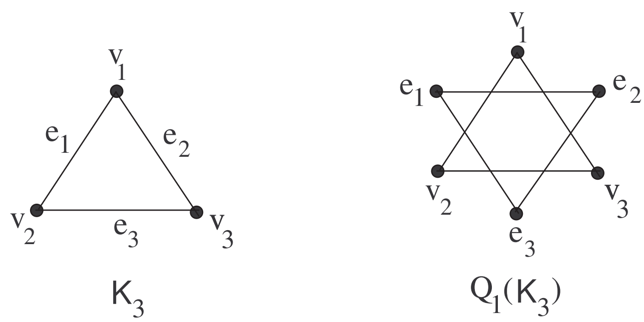

Figure 2.7: Complete graph K_3 and its 1-quasi-total graph $Q_1(K_3)$

Theorem 2.6.2. *Let G be a graph with neighborhood sigma algebra $\mathcal{P}(V(G))$. If P_3 is not a component of G and $K_{1,n}$, where $n > 2$, is not an induced subgraph of G, then the neighborhood sigma algebra of $Q_1(G)$ is $\mathcal{P}(V(Q_1(G)))$.*

Proof. 1-quasi-total graph $Q_1(G)$ of the graph G is the disjoint union of G and its line graph $L(G)$. By Theorem 2.3.1, neighborhood sigma algebra of $L(G)$ is $\mathcal{P}(V(L(G)))$. Therefore neighborhood sigma algebra of $Q_1(G)$ is $\mathcal{P}(V(Q_1(G)))$. $\square$

Corollary 2.6.3. *(1). The neighborhood sigma algebra of $Q_1(C_n)$ is $\mathcal{P}(V(Q_1(C_n)))$, for all $n > 3$.*

(2). The neighborhood sigma algebra of $Q_1(P_n)$ is $\mathcal{P}(V(Q_1(P_n)))$, for all $n > 3$.

Definition 2.6.4. [5] Let G be a graph. The 2-quasi-total graph, $Q_2(G)$, of G is the graph with vertex set $V(G) \bigcup E(G)$ and in which two vertices u and v are adjacent if they satisfy one of the following conditions:

(1) u and v are in $V(G)$ and u and v are adjacent in G.

(2) u is in $V(G)$, v is in $E(G)$ and v is incident u in G.

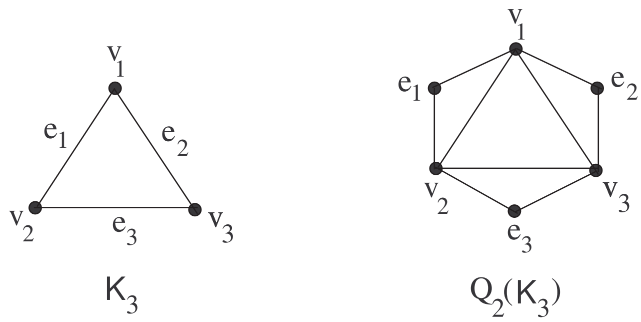

Figure 2.8: Complete graph K_3 and its 2-quasi-total graph $Q_2(K_3)$

Theorem 2.6.5. *Let G be a graph without end vertices, then the neighborhood sigma algebra of $Q_2(G)$ is $\mathcal{P}(V(Q_2(G)))$.*

Proof. For $u \in V(Q_2(G))$, let $N_{Q_2}[u] = \{u\} \bigcup \{v \in V(Q_2(G)) : v$ is adjacent to u in $Q_2(G)\}$. Let $v,\ e \in V(Q_2(G))$ be such that $v \in V(G)$ and $e \in E(G)$. Suppose $N_{Q_2}[v] = N_{Q_2}[e]$. Then e and v are adjacent vertices in $Q_2(G)$. Then from the definition of $Q_2(G)$, it is clear that v is incident on e in G. Let $e = uv$, $u \in V(G)$. Since G does not have end vertices, in G, v is adjacent to a vertex

$x \in V(G) \setminus \{u\}$. Then $x \in N_{Q_2}[v]$, but $x \notin N_{Q_2}[e]$, which is a contradiction to the fact that $N_{Q_2}[v] = N_{Q_2}[e]$. Hence $N_{Q_2}[v] \notin N_{Q_2}[e]$.

Suppose v_1 and v_2 be two distinct vertices of $Q_2(G)$ such that $v_1, v_2 \in V(G)$. Assume that $N_{Q_2}[v_1] = N_{Q_2}[v_2]$. This implies v_1 and v_2 are adjacent in G. Since G does not have end vertices there exists a vertex, $w \in V(G) \setminus \{v_2\}$ such that w is adjacent to v_1 in G. Then $e = v_1 w$ is an edge in G and hence it is a member of $N_{Q_2}[v_1]$. But then v_2 is not incident on e. So $e \notin N_{Q_2}[v_2]$. Therefore $N_{Q_2}[v_1] \neq N_{Q_2}[v_2]$.

Suppose e_1 and e_2 be two distinct vertices of $Q_2(G)$ such that $e_1, e_2 \in E(G)$. From the definition of $Q_2(G)$ it is clear that e_1 and e_2 are not adjacent in $Q_2(G)$. Therefore $N_{Q_2}[e_1] \neq N_{Q_2}[e_2]$. This completes the proof. □

Corollary 2.6.6. *(1). The neighborhood sigma algebra of $Q_2(C_n)$ is $\mathcal{P}(V(Q_2(C_n)))$, for all $n \geq 3$.*

(2). The neighborhood sigma algebra of $Q_2(K_n)$ is $\mathcal{P}(V(Q_2(K_n)))$, for all $n \neq 2$.

Chapter 3

Neighborhood Sigma Algebras of Join and Products of Two Graphs

In this chapter we discuss the neighborhood sigma algebra of join of two graphs and that of different products of two graphs.

3.1 Join of Two Graphs

This section deals with the neighborhood sigma algebra of join of two graphs

Notation 3.1.1. For any graph G, D_G denotes the set $\{v \in V(G) : d_G(v) = n(G) - 1\}$.

Theorem 3.1.2. *Let G_1 and G_2 be two vertex disjoint graphs and J be their join. Then for $i = 1, 2$ and $v \in V(G_i)$, $E_v^J = E_v^{G_i}$ if $v \notin D_{G_i}$.*

Proof. Let $v \in V(G_1)$ be such that $d_{G_1}(v) \neq n(G_1) - 1$.

First of all note that no vertex of G_2 belong to E_v^J. If possible let the vertex w of G_2 belong to E_v^J. Then $V(G_1) \subset N_J[w] = N_J[v]$, a contradiction. Thus $E_v^J \subset V(G_1)$.

Let u be a vertex G_1. Then,

$$\begin{aligned} u \in E_v^{G_1} &\Leftrightarrow E_u^{G_1} = E_v^{G_1} \\ &\Leftrightarrow N_{G_1}[u] = N_{G_1}[v], \text{ by Theorem 2.1.14.} \\ &\Leftrightarrow N_{G_1}[u] \cup V(G_2) = N_{G_1}[v] \cup V(G_2) \\ &\Leftrightarrow N_J[u] = N_J[v] \\ &\Leftrightarrow u \in E_v^J. \end{aligned}$$

Thus $E_v^J = E_v^{G_1}$.

A similar argument shows that if $v \in V(G_2)$ and $d_{G_2}(v) \neq n(G_2) - 1$, then $E_v^J = E_v^{G_2}$. Hence the theorem. □

Note 3.1.3. If $D_{G_i} = \emptyset$ for $i = 1, 2$ and $v \in G_i$, then $E_v^J = E_v^{G_i}$.

Theorem 3.1.4. *Let G and H be two vertex disjoint graphs and J be their join. Let $v \in V(G)$ be such that $v \in D_G$. Then $E_v^J = E_v^G \bigcup D_H$.*

Proof. Let $v \in V(G)$ be such that $d_G(v) = n(G) - 1$ and $u \in E_v^G$. Then $N_G[u] = N_G[v]$. Since $N_J[u] = N_G[u] \bigcup V(H)$ and $N_J[v] = N_G[v] \bigcup V(H)$, $N_J[u] = N_J[v]$. Hence $u \in E_v^J$. Therefore $E_v^G \subseteq E_v^J$. Suppose $u \in V(H)$ and $d_H(u) = n(H) - 1$.

Then

$$\begin{aligned} N_J[u] &= N_H[u] \bigcup V(G) \\ &= V(H) \bigcup V(G). \end{aligned}$$

Also

$$\begin{aligned} N_J[v] &= N_G[v] \bigcup V(H) \\ &= V(G) \bigcup V(H) \end{aligned}$$

Thus $N_J[u] = N_J[v]$. Hence $u \in E_v^J$. Therefore $D_H \subseteq E_v^J$.

To prove the reverse inclusion, let $u \in E_v^J$. Then $N_J[u] = N_J[v]$ and $u \in V(G)$ or $V(H)$. Let $u \in V(G)$. Since $N_G[u]$ and $N_G[v]$ are disjoint from $V(H)$ and since $N_G[u] \cup V(H) = N_J[u] = N_J[v] = N_G[v] \cup V(H)$, $N_G[u] = N_G[v]$. Hence $u \in E_v^G$. Suppose $u \in V(H)$. Hence $N_H[u] \bigcup V(G) = N_J[u] = N_J[v] = N_G[v] \bigcup V(H) = V(G) \bigcup V(H)$. Hence $N_H[u] = V(H)$. That is $d_H(u) = n(H) - 1$. Therefore $E_v^J = E_v^G \bigcup D_H$. □

Theorem 3.1.5. *Let G_1 and G_2 be two vertex disjoint graphs. Also let f_1 be a measurable function defined from $V(G_1)$ into $[0,1]$ and f_2 be a measurable function defined from $V(G_2)$ into $[0,1]$.*

(i) If $D_{G_1} = \emptyset$ or $D_{G_2} = \emptyset$, then the function $g : V(G_1 \vee G_2) \longrightarrow [0,1]$ defined by,

$$g(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1) \\ f_2(v) & \text{if } v \in V(G_2) \end{cases}$$

is a measurable function.

(ii) If $D_{G_1} \neq \emptyset$ and $D_{G_2} \neq \emptyset$, then the function $h : V(G_1 \vee G_2) \longrightarrow [0,1]$ defined by,

$$h(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1) \setminus D_{G_1} \\ f_2(v) & \text{if } v \in V(G_2) \setminus D_{G_2} \\ rs & \text{if } v \in D_{G_1} \bigcup D_{G_2} \end{cases}$$

where r is the value of f_1 on D_{G_1} and s is the value of f_2 on D_{G_2}, is a measurable function.

Proof. (i) Suppose $D_{G_1} = \emptyset$ or $D_{G_2} = \emptyset$.

To prove g is measurable it is enough to prove that g is constant on E_v^J for all $v \in V(G_1 \vee G_2)$ by theorem 2.1.25. Assume that $D_{G_1} = \emptyset$. Let $v \in V(G_1)$. Then $E_v^J = E_v^{G_1}$, by Theorem 3.1.2. Also $g \equiv f_1$ on $V(G_1)$. Since f_1 is measurable, f_1 is constant on $E_v^{G_1}$. This implies g is constant on E_v^J. Let $v \in V(G_2)$. If $v \notin D_{G_2}$, $E_v^J = E_v^{G_2}$, again by Theorem 3.1.2. Also $g \equiv f_2$ on $V(G_2)$. Since f_2 is measurable, f_2 is constant on $E_v^{G_2}$. This implies g is constant on E_v^J. If $v \in D_{G_2}$,

$$\begin{aligned} E_v^J &= E_v^{G_2} \bigcup D_{G_1} \\ &= E_v^{G_2}, \text{ since } D_{G_1} = \emptyset. \end{aligned}$$

Also $g \equiv f_2$ on G_2. Since f_2 is measurable, f_2 is constant on $E_v^{G_2}$. This implies g is constant on $E_v^{G_2}$. Therefore g is constant on E_v^J for all $v \in V(G_1 \vee G_2)$.

Similarly, if $D_{G_2} = \emptyset$ we can also prove that g is constant on E_v^J for all $v \in V(G_1 \vee G_2)$.

(ii) Suppose $D_{G_1} \neq \emptyset$ and $D_{G_2} \neq \emptyset$. Let $v \in V(G_1 \vee G_2)$. Without loss of generality suppose that $v \in V(G_1)$. If $v \notin D_{G_1}$, $E_v^J = E_v^{G_1}$, by Theorem 3.1.2. Let $u \in E_v^{G_1}$. Then $N_{G_1}[u] = N_{G_1}[v]$. Therefore $d_{G_1}(u) = d_{G_1}(v)$. Hence $u \notin D_{G_1}$. This implies $E_v^{G_1} \subseteq V(G_1) \setminus D_{G_1}$. Therefore $h \equiv f_1$ on $E_v^{G_1}$. Since f_1 is measurable, f_1 is a constant on $E_v^{G_1}$. Hence h is a constant on $E_v^{G_1}$. If $v \in D_{G_1}$, then $E_v^J = E_v^{G_1} \bigcup D_{G_2}$, by Theorem 3.1.4. Since $v \in D_{G_1}$, $E_v^{G_1} = D_{G_1}$ by Theorem 2.1.27. Therefore $E_v^J = D_{G_1} \bigcup D_{G_2}$. Since $h(u) = rs$ for all $u \in E_v^J$, h is a constant on E_v^J. Hence h is a constant on E_v^J for all $v \in V(G_1)$. Similarly we can prove that h is a constant on E_v^J for all $v \in V(G_2)$ also. Hence h is measurable.

□

3.2 Graph Products

A graph product of two graphs G and H is a new graph whose vertex set is $V(G) \times V(H)$ and for any two vertices (g, h) and (g', h') in the product, the adjacency is determined entirely by the adjacency of g and g' in G and that of h and h' in H. The commonly used graph products are lexicographic product, tensor product, Cartesian product, normal product, co-normal product and homomorphic product. In this section the neighborhood sigma algebras of these graph products are determined.

3.2.1 Lexicographic Product

The *lexicographic product* [17] $G_1[G_2]$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1[G_2]$ if either $u_1u_2 \in E(G_1)$ or $u_1 = u_2$ and $v_1v_2 \in E(G_2)$.

From the definition of the lexicographic product $G_1[G_2]$ of the graphs G_1 and G_2, it is clear that $N[(u, v)] = (N(u) \times V(G_2)) \bigcup(\{u\} \times N[v])$, for $(u, v) \in V(G_1[G_2])$.

Theorem 3.2.1. *Let G_1 and G_2 be two graphs and $(u, v), (x, y) \in V(G_1[G_2])$. Then $N[(u, v)] = N[(x, y)]$ if and only if one of the following conditions holds:*
$(i) u = x$ and $N[v] = N[y]$
$(ii) N[u] = N[x]$ and $N[v] = N[y] = V(G_2)$

Proof. Suppose that $N[(u, v)] = N[(x, y)]$. Then (u, v) and (x, y) are adjacent in $G_1[G_2]$. Therefore $u = x$ or u and x are adjacent. Suppose $u = x$. Then let, if possible, $w \in N[v] \setminus N[y]$ (or $w \in N[y] \setminus N[v]$). Then $(u, w) \in N[(u, v)] \setminus N[(x, y)]$ (or $(u, w) \in N[(x, y)] \setminus N[(u, v)]$), a contradiction. Therefore $N[v] = N[y]$.

Suppose $u \neq x$. Then u and x are adjacent. Let, if possible, $z \in N[u] \setminus N[x]$ (or $z \in N[x] \setminus N[u]$). Then $(z, v) \in N[(u, v)] \setminus N[(x, y)]$ (or $(z, y) \in N[(x, y)] \setminus N[(u, v)]$), a contradiction. Therefore $N[u] = N[x]$. Let, if possible, $w \in V(G_2) \setminus N[v]$. Then $(u, w) \in N[(x, y)] \setminus N[(u, v)]$, a contradiction. Therefore

$N[v] = V(G_2)$. Similarly we can prove that $N[y] = V(G_2)$.

Conversely assume that either (i) or (ii) holds.

If (i) holds then,

$$\begin{aligned} N[(u,v)] &= (N(u) \times V(G_2)) \bigcup (\{u\} \times N[v]) \\ &= (N(x) \times V(G_2)) \bigcup (\{x\} \times N[y]) \\ &= N[(x,y)] \end{aligned}$$

If (ii) holds then,

$$\begin{aligned} N[(u,v)] &= (N(u) \times V(G_2)) \bigcup (\{u\} \times N[v]) \\ &= (N(u) \times V(G_2)) \bigcup (\{u\} \times V(G_2)) \\ &= N[u] \times V(G_2) \\ &= N[x] \times V(G_2) \\ &= (N(x) \times V(G_2)) \bigcup (\{x\} \times V(G_2)) \\ &= (N(x) \times V(G_2)) \bigcup (\{x\} \times N[y]) \\ &= N[(x,y)] \end{aligned}$$

Hence the theorem. □

Lemma 3.2.2. *Let G_1 and G_2 be two graphs with $(u,v) \in V(G_1[G_2])$. Then $E_{(u,v)} \subseteq E_u \times E_v$.*

Proof. Let $(u',v') \in E_{(u,v)}$. Then $N[(u',v')] = N[(u,v)]$. Therefore by Theorem 3.2.1, either $u' = u$ and $N[v'] = N[v]$ or $N[u'] = N[u]$ and $N[v'] = N[v] = V(G_2)$.

In both the cases it is clear that $u' \in E_u$ and $v' \in E_v$. Hence the lemma. $\square$

Remark 3.2.3. *There are graphs in which the inclusion in the Lemma 3.2.2 is strict.*

To see this consider the graphs in Figure 3.1.

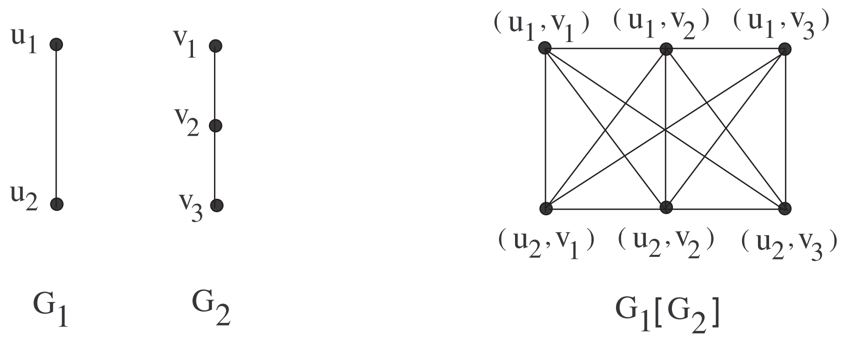

Figure 3.1: Lexicographic product of the graphs G_1 and G_2

Since $u_2 \in E_{u_1}$ *and* $v_1 \in E_{v_1}$, $(u_2, v_1) \in E_{u_1} \times E_{v_1}$. *But* $(u_2, v_1) \notin E_{(u_1,v_1)}$, *because* $(u_1, v_3) \in N[(u_2, v_1)] \setminus N[(u_1, v_1)]$.

Definition 3.2.4. [23] Suppose (X, S) and (Y, T) are two measurable spaces. A measurable rectangle is any set of the form $A \times B$, where $A \in S$ and $B \in T$.

Definition 3.2.5. [23] Suppose that (X, S) and (Y, T) are two measurable spaces. The product sigma algebra $S \times T$ is defined to be the smallest sigma algebra in $X \times Y$ which contains every measurable rectangles.

Proposition 3.2.6. *Let* G_1 *and* G_2 *be two graphs. Then* $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1[G_2]}$.

Proof. For $i = 1, 2$, any element of $\mathcal{A}_{G_i}$ can be written as the disjoint union of elements of the collection $\{E_u : u \in V(G_i)\}$. Therefore the generating sets of $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ can be written as the disjoint union of elements of the collection $\{E_u \times E_v : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Also $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ contains $\{E_u \times E_v : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Therefore to prove the proposition it is enough to prove that $E_u \times E_v \in \mathcal{A}_{G_1[G_2]}$ for all $(u, v) \in V(G_1) \times V(G_2)$.

Let $(u, v) \in V(G_1) \times V(G_2)$. Also let $(x, y) \in E_u \times E_v$. Then $x \in E_u$ and $y \in E_v$. This implies $E_x = E_u$ and $E_y = E_v$. Therefore, by Lemma 3.2.2, $E_{(x,y)} \subseteq E_x \times E_y = E_u \times E_v$. Hence for $(u, v) \in V(G_1) \times V(G_2)$, $E_u \times E_v$ can be written as countable disjoint union of the collection $\{E_{(x,y)} : (x, y) \in E_u \times E_v\}$. Therefore $E_u \times E_v \in \mathcal{A}_{G_1[G_2]}$ for all $(u, v) \in V(G_1) \times V(G_2)$. Hence the proposition. $\square$

Remark 3.2.7. *The reverse inclusion in Proposition 3.2.6 is not true in general.*

To see this consider the graphs in Figure 3.1.
$\{(u_1, v_1)\} \in \mathcal{A}_{G_1[G_2]}$. *As every measurable set in* G_1 *containing* u_1 *also contains* u_2*, every element in* $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ *containing* (u_1, v_1) *also contains* (u_2, v_1)*. Thus* $\{(u_1, v_1)\} \notin \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

Proposition 3.2.8. *Let* G_1 *and* G_2 *be two graphs. Also let* $f_1 : V(G_1) \longrightarrow [0, 1]$ *and* $f_2 : V(G_2) \longrightarrow [0, 1]$ *be two measurable functions. Then the function* $f : V(G_1[G_2]) \longrightarrow [0, 1]$ *defined by* $f((u, v)) = f_1(u)f_2(v)$ *is measurable.*

Proof. Let $(u, v) \in V(G_1[G_2])$ and $(u', v') \in E_{(u,v)}$. Then $(u', v') \in E_u \times E_v$ by Lemma 3.2.2. This implies $u' \in E_u$ and $v' \in E_v$. Hence $f_1(u') = f_1(u)$ and

$f_2(v') = f_2(v)$. Therefore $f((u', v')) = f((u, v))$. Hence by Proposition 2.1.25, f is measurable. $\square$

3.2.2 Tensor Product

The *tensor product* [19] $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \otimes G_2$ if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

It is clear that, $N[(u, v)] = \{(u, v)\} \bigcup (N(u) \times N(v))$, for $(u, v) \in V(G_1 \otimes G_2)$. Hence, if u is an isolated vertex of G_1 or v is an isolated vertex G_2, then (u, v) is an isolated vertex of $G_1 \otimes G_2$.

Theorem 3.2.9. *Let G_1 and G_2 be two graphs. Suppose u and v are distinct non isolated vertices of G_1 and x and y are distinct non isolated vertices of G_2. Then $N[(u, v)] = N[(x, y)]$ if and only if $N[u] = N[x] = \{u, x\}$ and $N[v] = N[y] = \{v, y\}$.*

Proof. Let $N[(u, v)] = N[(x, y)]$. Then u is adjacent to x and v is adjacent to y. Let, if possible $w(\neq u) \in N[u] \setminus N[x]$. Then for any $z \in N(v)$, $(w, z) \in N[(u, v)] \setminus N[(x, y)]$, which contradicts the assumption $N[(u, v)] = N[(x, y)]$. Therefore $N[u] = N[x]$.

If possible, let $p(\neq x) \in N(u)$. Then $(p, y) \in N[(u, v)]$. But $(p, y) \notin N[(x, y)]$. Hence $N[u] = N[x] = \{u, x\}$. Similarly we can prove that $N[v] = N[y] = \{v, y\}$.

Conversely assume that $N[u] = N[x] = \{u, x\}$ and $N[v] = N[y] = \{v, y\}$.

Then,

$$\begin{aligned} N[(u,v)] &= \{(u,v)\} \bigcup (N(u) \times N(v)) \\ &= \{(u,v)\} \bigcup \{(x,y)\} \\ &= \{(u,v),(x,y)\} \end{aligned}$$

and

$$\begin{aligned} N[(x,y)] &= \{(x,y)\} \bigcup (N(x) \times N(y)) \\ &= \{(x,y)\} \bigcup \{(u,v)\} \\ &= \{(x,y),(u,v)\} \end{aligned}$$

This completes the proof. □

Corollary 3.2.10. *Let G_1 and G_2 be two graphs. If G_1 or G_2 does not have P_2 as a component then the neighborhood sigma algebra of $G_1 \otimes G_2$ is $\mathcal{P}(V(G_1 \otimes G_2))$.*

Note 3.2.11. Let G_1 and G_2 be two graphs. For two distinct vertices u and x of G_1 and for two distinct vertices v and y of G_2, the conditions $N[u] = N[x]$ in G_1 and $N[v] = N[y]$ in G_2 are not sufficient to guarantee that $N[(u,v)] = N[(x,y)]$ in $G_1 \otimes G_2$.

For example consider $K_2 \otimes K_3$.

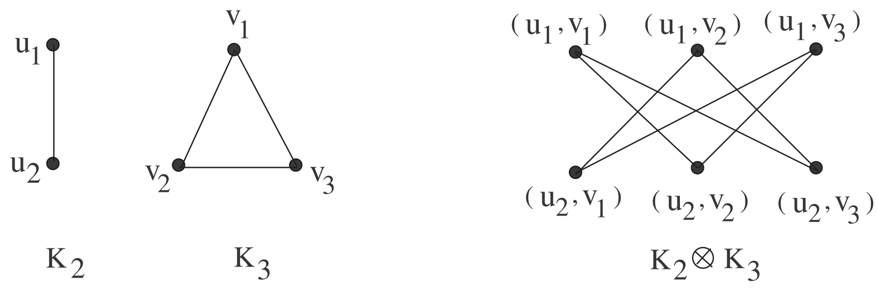

Figure 3.2: Tensor product of K_2 and K_3

Here $N[u_1] = N[u_2]$ in K_2 and $N[v_1] = N[v_2]$ in K_3. But $N[(u_1, v_1)] \neq N[(u_2, v_2)]$ in $K_2 \otimes K_3$.

Lemma 3.2.12. *Let G_1 and G_2 be two graphs with $(u, v) \in V(G_1 \otimes G_2)$. Then $E_{(u,v)} \subseteq E_u \times E_v$.*

Proof. If u is an isolated vertex of G_1 or v is an isolated vertex of G_2 then (u, v) is an isolated vertex of $G_1 \otimes G_2$. In this case $E_{(u,v)} = \{(u, v)\}$. Suppose u and v are two non isolated vertices of G_1 and G_2 respectively. If $(u', v') \in E_{(u,v)}$, then $N[(u', v')] = N[(u, v)]$. Therefore by Theorem 3.2.9, $N[u] = N[u'] = \{u, u'\}$ and $N[v] = N[v'] = \{v, v'\}$. So, $u' \in E_u$ and $v' \in E_v$. Hence the lemma. $\square$

Remark 3.2.13. *The reverse inclusion in Lemma 3.2.12 is not true in general.*

For example consider the graphs in Figure 3.3.

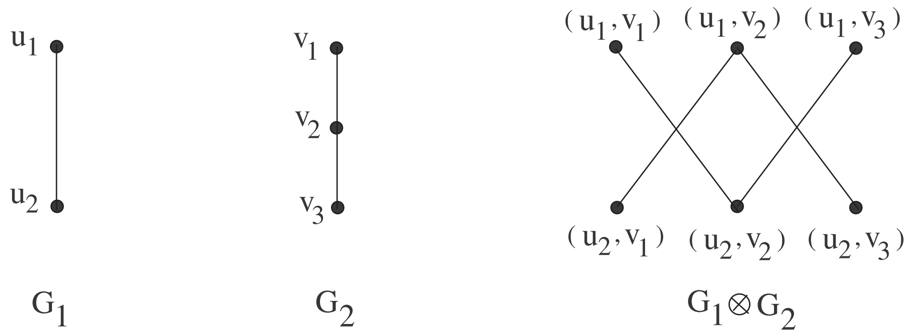

Figure 3.3: Tensor product of the graphs G_1 and G_2

Since $u_2 \in E_{u_1}$ *and* $v_1 \in E_{v_1}$, $(u_2, v_1) \in E_{u_1} \times E_{v_1}$. *But* $(u_1, v_2) \in N[(u_2, v_1)]$ *and* $(u_1, v_2) \notin N[(u_1, v_1)]$. *Therefore* $(u_2, v_1) \notin E_{(u_1,v_1)}$.

Proposition 3.2.14. *Let* G_1 *and* G_2 *be two graphs. Then* $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1 \otimes G_2}$.

Proof. Similar to the proof of Proposition 3.2.6 □

Remark 3.2.15. *The reverse inclusion in Proposition 3.2.14 is not true in general.*

Consider the graphs in Figure 3.3. By the same arguments in Remark 3.2.7, we get $\mathcal{A}_{G_1 \otimes G_2} \nsubseteq \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

Proposition 3.2.16. *Let* G_1 *and* G_2 *be two graphs. Also let* $f_1 : V(G_1) \longrightarrow [0, 1]$ *and* $f_2 : V(G_2) \longrightarrow [0, 1]$ *be two measurable functions. Then the function* $f : V(G_1 \otimes G_2) \longrightarrow [0, 1]$ *defined by* $f((u, v)) = f_1(u)f_2(v)$ *is measurable.*

Proof. For $(u, v) \in V(G_1 \otimes G_2)$, $E_{(u,v)} \subseteq E_u \times E_v$. Hence the proof is similar to the proof of Proposition 3.2.8. $\square$

3.2.3 Cartesian Product

The *Cartesian product* [16] $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ whenever $u_1 = u_2$ and v_1 adjacent to v_2 in G_2 or u_1 adjacent to u_2 in G_1 and $v_1 = v_2$.

For $(u, v) \in V(G_1 \times G_2)$, $N[(u, v)] = (\{u\} \times N[v]) \bigcup (N[u] \times \{v\})$.

Theorem 3.2.17. *Let G_1 and G_2 be two graphs. Suppose (u, v) and (x, y) are two distinct vertices of $G_1 \times G_2$. Then $N[(u, v)] = N[(x, y)]$ if and only if one of the following conditions holds:*
(i) $u = x$, u is an isolated vertex of G_1 and $N[v] = N[y]$
(ii) $v = y$, v is an isolated vertex of G_2 and $N[u] = N[x]$.

Proof. If $N[(u, v)] = N[(x, y)]$, then (u, v) is adjacent to (x, y). Therefore either $u = x$ or $v = y$.

Suppose that $u = x$. Then $v \neq y$ since $(u, v) \neq (x, y)$. If there exists a vertex $w \in N(u)$, then $(w, v) \in N[(u, v)]$. But $(w, v) \notin N[(x, y)]$. Hence u is an isolated vertex of G. If $N[v] \neq N[y]$, then without loss of generality we can assume that there exists a vertex $z \in N[v] \setminus N[y]$. Then $(u, z) \in N[(u, v)] \setminus N[(x, y)]$. Thus $N[v] = N[y]$.

If $v = y$, then $u \neq x$ since $(u, v) \neq (x, y)$. Then proceeding as above we can prove that v is an isolated vertex of G_2 and $N[u] = N[x]$.

Conversely if condition (i) holds, then

$$\begin{aligned}
N[(u,v)] &= (\{u\} \times N[v]) \bigcup (N[u] \times \{v\}) \\
&= (\{u\} \times N[v]) \bigcup (\{u\} \times \{v\}) \\
&= \{u\} \times N[v] \\
&= \{x\} \times N[y] \\
&= (\{x\} \times N[y]) \bigcup (\{x\} \times \{y\}) \\
&= (\{x\} \times N[y]) \bigcup (N[x] \times \{y\}) \\
&= N[(x,y)]
\end{aligned}$$

Similarly we can show that condition(ii) also implies $N[(u, v)] = N[(x, y)]$. Hence the theorem. □

Corollary 3.2.18. *If two graphs G_1 and G_2 have no isolated vertices then the neighborhood sigma algebra of $G_1 \times G_2$ is $\mathcal{P}(V(G_1 \times G_2))$.*

Corollary 3.2.19. *If G_1 is a graph with neighborhood sigma algebra $\mathcal{P}(V(G_1))$ and G_2 is a graph with neighborhood sigma algebra $\mathcal{P}(V(G_2))$, then neighborhood sigma algebra of $G_1 \times G_2$ is $\mathcal{P}(V(G_1 \times G_2))$.*

Lemma 3.2.20. *Let G_1 and G_2 be two graphs with $(u, v) \in V(G_1 \times G_2)$. Then $E_{(u,v)} \subseteq E_u \times E_v$.*

Proof. Let $(u', v') \in E_{(u,v)}$, then $N[(u', v')] = N[(u, v)]$. Therefore by Theorem

3.2.17, $N[u] = N[u']$ and $N[v] = N[v']$. So, $u' \in E_u$ and $v' \in E_v$. Hence the lemma. □

Remark 3.2.21. *The reverse inclusion in Lemma 3.2.20 is not true in general.*

Consider the graphs given in Figure 3.4.

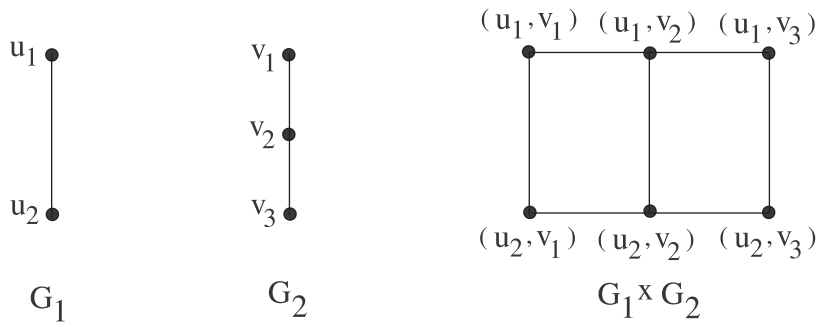

Figure 3.4: Cartesian product of the graphs G_1 and G_2

$E_{(u_1,v_1)} = \{(u_1, v_1)\}$ *and* $E_{u_1} \times E_{v_1} = \{(u_1, v_1), (u_2, v_1)\}$.

The following two propositions can be proved as in the proofs of Propositions 3.2.6 and 3.2.8.

Proposition 3.2.22. *Let G_1 and G_2 be two graphs. Then $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1 \times G_2}$.*

Proposition 3.2.23. *Let G_1 and G_2 be two graphs. Also let $f_1 : V(G_1) \longrightarrow [0,1]$ and $f_2 : V(G_2) \longrightarrow [0,1]$ be two measurable functions. Then the function $f : V(G_1 \times G_2) \longrightarrow [0,1]$ defined by $f((u,v)) = f_1(u)f_2(v)$ is measurable.*

Remark 3.2.24. *The reverse inclusion in Proposition 3.2.22 is not true in general. For example consider the graphs in Figure 3.4. By the same arguments in Remark 3.2.7, we get* $\mathcal{A}_{G_1 \times G_2} \nsubseteq \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

3.2.4 Normal Product

The *normal product* [24] $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \boxtimes G_2$ if one of the following conditions holds:

(i) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$

(ii) $u_1u_2 \in E(G_1)$ and $v_1 = v_2$

(iii) $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

For $(u, v) \in V(G_1 \boxtimes G_2)$, $N[(u, v)] = N[u] \times N[v]$. Hence we have the following theorem.

Theorem 3.2.25. *For the graphs G_1 and G_2, let (u, v) and (x, y) be two distinct vertices of $V(G_1 \boxtimes G_2)$. Then $N[(u, v)] = N[(x, y)]$ if and only if $N[u] = N[x]$ and $N[v] = N[y]$.*

Proof.

$$\begin{aligned} N[(u,v)] = N[(x,y)] &\Leftrightarrow N[u] \times N[v] = N[x] \times N[y] \\ &\Leftrightarrow N[u] = N[x] \text{ and } N[v] = N[y] \end{aligned}$$

$\square$

Lemma 3.2.26. *Let G_1 and G_2 be two graphs with $(u,v) \in V(G_1 \boxtimes G_2)$. Then $E_{(u,v)} = E_u \times E_v$.*

Proof.

$$\begin{aligned}
(u',v') \in E_u \times E_v &\Leftrightarrow u' \in E_u \text{ and } v' \in E_v \\
&\Leftrightarrow E_{u'} = E_u \text{ and } E_{v'} = E_v \\
&\Leftrightarrow N[u'] = N[u] \text{ and } N[v'] = N[v] \\
&\Leftrightarrow N[(u',v')] = N[(u,v)], \text{by Theorem 3.2.25.} \\
&\Leftrightarrow E_{(u',v')} = E_{(u,v)} \\
&\Leftrightarrow (u',v') \in E_{(u,v)}
\end{aligned}$$

Hence the lemma. $\square$

Proposition 3.2.27. *Let G_1 and G_2 be two graphs. Then $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} = \mathcal{A}_{G_1 \boxtimes G_2}$.*

Proof. Any element of $\mathcal{A}_{G_1}$ can be written as the disjoint union of elements of the collection $\{E_v : v \in V(G_1)\}$ and any element of $\mathcal{A}_{G_2}$ can be written as the disjoint union of elements of the collection $\{E_v : v \in V(G_2)\}$. Therefore the generating sets of $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ can be written as the disjoint union of elements of the collection $\{E_u \times E_v : u \in V(G_1) \text{ and } v \in V(G_2)\}$. But we have $E_{(u,v)} = E_u \times E_v$ by Lemma 3.2.26. Therefore all generating sets of $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ belong to $\mathcal{A}_{G_1 \boxtimes G_2}$. Hence $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1 \boxtimes G_2}$.

Any element of $\mathcal{A}_{G_1 \boxtimes G_2}$ can be written as the disjoint union of elements of the collection $\{E_{(u,v)} : (u,v) \in V(G_1 \boxtimes G_2)\}$. Since $E_{(u,v)} = E_u \times E_v$ for all $(u,v) \in V(G_1 \boxtimes G_2)$, it is clear that $\mathcal{A}_{G_1 \boxtimes G_2} \subseteq \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$. $\square$

Proposition 3.2.28. *Let G_1 and G_2 be two graphs. Also let $f_1 : V(G_1) \longrightarrow [0,1]$ and $f_2 : V(G_2) \longrightarrow [0,1]$ be two measurable functions. Then the function $f : V(G_1 \boxtimes G_2) \longrightarrow [0,1]$ defined by $f((u,v)) = f_1(u)f_2(v)$ is measurable.*

Proof. To prove f is measurable it is enough to prove that f is constant on $E_{(u,v)}$ for each $(u,v) \in V(G_1 \boxtimes G_2)$. Since f_1 and f_2 are measurable functions, f_1 is a constant on E_u for each $u \in V(G_1)$ and f_2 is a constant on E_v for each $v \in V(G_2)$. Therefore f is a constant on $E_u \times E_v = E_{(u,v)}$, for each $(u,v) \in V(G_1 \boxtimes G_2)$. $\square$

3.2.5 Co-normal Product

The *co-normal product* [2] $G_1 * G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 * G_2$ if either $u_1u_2 \in E(G_1)$ or $v_1v_2 \in E(G_2)$.

This section deals with the neighborhood sigma algebra of co-normal product of two graphs. It is clear that for $(u,v) \in V(G_1 * G_2)$, $N[(u,v)] = \{(u,v)\} \bigcup (N(u) \times V(G_2)) \bigcup (V(G_1) \times N(v))$.

Theorem 3.2.29. *Let (u_1, v_1) and (u_2, v_2) be two distinct vertices of $G_1 * G_2$. Then $N[(u_1, v_1)] = N[(u_2, v_2)]$ if and only if one of the following conditions holds:*

(i) $u_1 = u_2,\ N[u_1] = V(G_1)$ *and* $N[v_1] = N[v_2]$

(ii) $N[u_1] = N[u_2],\ v_1 = v_2$ *and* $N[v_1] = V(G_2)$

(iii) $N[u_1] = N[u_2] = V(G_1)$ *and* $N[v_1] = N[v_2] = V(G_2)$.

Proof. Suppose that $N[(u_1, v_1)] = N[(u_2, v_2)]$.

Let $u_1 = u_2 = u$. Since $N[(u_1, v_1)] = N[(u_2, v_2)]$, v_1 adjacent to v_2. Suppose $N[u] \neq V(G_1)$. Then there exists $u^{'} \in V(G_1) \setminus N[u]$ such that $(u^{'}, v_2) \in N[(u, v_1)] \setminus N[(u, v_2)]$, a contradiction. Therefore $N[u] = V(G_1)$. Suppose $N[v_1] \neq N[v_2]$. Without loss of generality, assume that there exists $v^{'} \in V(G_2)$ such that $v^{'} \in N[v_1] \setminus N[v_2]$. Then $(u, v^{'}) \in N[(u, v_1)] \setminus N[(u, v_2)]$, a contradiction. Hence $N[v_1] = N[v_2]$.
Similarly, if $v_1 = v_2$, we can prove that $N[v_1] = V(G_2)$ and $N[u_1] = N[u_2]$.

Suppose $u_1 \neq u_2$ and $v_1 \neq v_2$. Since $N[(u_1, v_1)] = N[(u_2, v_2)]$, u_1 is adjacent to u_2 or v_1 is adjacent to v_2. Suppose u_1 is adjacent to u_2. Assume that $N[v_1] \neq N[v_2]$. Without loss of generality, assume that there exists a $v^{'} \in N[v_1] \setminus N[v_2]$. Then $(u_2, v^{'}) \in N[(u_1, v_1)] \setminus N[(u_2, v_2)]$, a contradiction. Therefore $N[v_1] = N[v_2]$. Suppose $N[v_1] \neq V(G_2)$. Suppose $v^{''} \in V(G_2) \setminus N[v_1]$. Then $(u_2, v^{''}) \in N[(u_1, v_1)] \setminus N[(u_2, v_2)]$, a contradiction. Hence $N[v_1] = N[v_2] = V(G_2)$. This implies v_1 adjacent to v_2 also. Proceeding in the similar manner we get $N[u_1] = N[u_2] = V(G_1)$.

Conversely assume that condition (i) holds. Suppose $u_1 = u_2 = u$. Since (u_1, v_1) and (u_2, v_2) are two distinct vertices of $G_1 * G_2$, $v_1 \neq v_2$ and v_1 adjacent to v_2. Therefore, for $i = 1, 2$

$$\begin{aligned} N[(u, v_i)] &= \{(u, v_i)\} \bigcup (N(u) \times V(G_2)) \bigcup (V(G_1) \times N(v_i)) \\ &= \{(u, v_i)\} \bigcup ((V(G_1) \setminus \{u\}) \times V(G_2)) \bigcup (V(G_1) \times N(v_i)) \\ &= (\{u\} \times N[v_i]) \bigcup ((V(G_1) \setminus \{u\}) \times V(G_2)) \end{aligned}$$

By condition (i), $N[v_1] = N[v_2]$. Thus $N[(u, v_1)] = N[(u, v_2)]$. Similarly we can show that condition(ii) also implies $N[(u_1, v_1)] = N[(u_2, v_2)]$. Suppose condition (iii) holds. Then for $i = 1, 2$

$$\begin{aligned} N[(u_i, v_i)] &= \{(u_i, v_i)\} \bigcup (N(u_i) \times V(G_2)) \bigcup (V(G_1) \times N(v_i)) \\ &= \{(u_i, v_i)\} \bigcup ((V(G_1) \setminus \{u_i\}) \times V(G_2)) \bigcup (V(G_1) \times (V(G_2) \setminus \{v_i\})) \\ &= (V(G_1) \times V(G_2)) \end{aligned}$$

Hence the theorem. $\square$

The following corollaries are immediate consequences of the Theorem 3.2.29.

Corollary 3.2.30. *Let G_1 and G_2 be two graphs and $(u, v) \in V(G_1 * G_2)$. Then,*

$$E_{(u,v)} = \begin{cases} D_{G_1} \times D_{G_2} & \text{if } u \in D_{G_1} \text{ and } v \in D_{G_2} \\ \{u\} \times E_v & \text{if } u \in D_{G_1} \text{ and } v \notin D_{G_2} \\ E_u \times \{v\} & \text{if } u \notin D_{G_1} \text{ and } v \in D_{G_2} \\ \{(u, v)\} & \text{if } u \notin D_{G_1} \text{ and } v \notin D_{G_2} \end{cases}$$

Corollary 3.2.31. *Let G_1 and G_2 be two graphs with $D_{G_1} = D_{G_2} = \emptyset$. Then the neighborhood sigma algebra of $G_1 * G_2$ is $\mathcal{P}(V(G_1 * G_2))$.*

Corollary 3.2.32. *Let G_1 be a graph with neighborhood sigma algebra $\mathcal{P}(V(G_1))$ and G_2 be a graph with neighborhood sigma algebra $\mathcal{P}(V(G_2))$, then the neighborhood sigma algebra of $G_1 * G_2$ is $\mathcal{P}(V(G_1 * G_2))$.*

Proposition 3.2.33. *Let G_1 and G_2 be two graphs with $(u, v) \in V(G_1 * G_2)$. Then $E_{(u,v)} \subseteq E_u \times E_v$.*

Proof. Let $(u', v') \in E_{(u,v)}$. Therefore $N[(u', v')] = N[(u, v)]$. Hence by Theorem 3.2.29, $N[u'] = N[u]$ and $N[v'] = N[v]$. Therefore $u' \in E_u$ and $v' \in E_v$. Hence the proposition. □

Remark 3.2.34. *The reverse inclusion in Proposition 3.2.33 is not true in general. Consider the graphs in Figure 3.5.*

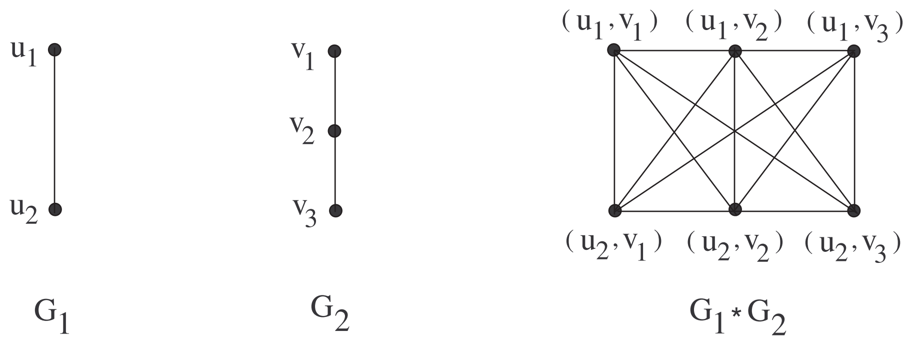

Figure 3.5: Co-normal product of the graphs G_1 and G_2

Since $u_2 \in E_{u_1}$ and $v_1 \in E_{v_1}$, $(u_2, v_1) \in E_{u_1} \times E_{v_1}$. But $(u_1, v_3) \in N[(u_2, v_1)] \setminus N[(u_1, v_1)]$. Therefore $(u_2, v_1) \notin E_{(u_1,v_1)}$.

The following proposition can be proved using the techniques of the proof of Proposition 3.2.6.

Proposition 3.2.35. *Let G_1 and G_2 be two graphs. Then $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1 * G_2}$.*

Remark 3.2.36. *The reverse inclusion in Proposition 3.2.35 is not true in general.*
To see this consider the graphs in Figure 3.5. By the same arguments in Remark 3.2.7, we get $\mathcal{A}_{G_1 * G_2} \nsubseteq \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

Proposition 3.2.37. *Let G_1 and G_2 be two graphs. Also let $f_1 : V(G_1) \longrightarrow [0,1]$ and $f_2 : V(G_2) \longrightarrow [0,1]$ are two measurable functions. Then the function $f : V(G_1 * G_2) \longrightarrow [0,1]$ defined by $f((u,v)) = f_1(u)f_2(v)$ is measurable.*

Proof. For $(u,v) \in V(G_1 * G_2)$, $E_{(u,v)} \subseteq E_u \times E_v$. Therefore the proof is similar to the proof of Proposition 3.2.8. $\square$

3.2.6 Homomorphic Product

The *homomorphic product* [18] $G_1 \ltimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \ltimes G_2$ if either $u_1 = u_2$ or u_1 is adjacent to u_2 and v_1 is not adjacent to v_2.

It is clear that for $(u,v) \in V(G_1 \ltimes G_2)$, $N[(u,v)] = (\{u\} \times V(G_2)) \bigcup (N(u) \times N(v)^c)$, where $N(v)^c$ denotes the complement of $N(v)$ in $V(G_2)$.

Theorem 3.2.38. *Let G_1 and G_2 be two graphs. If (u_1, v_1) and (u_2, v_2) are two distinct vertices of $G_1 \ltimes G_2$, then $N[(u_1, v_1)] = N[(u_2, v_2)]$ if and only if one of* *the following conditions holds:*

(*i*) $u_1 = u_2$ *and* u_1 *is an isolated vertex of* G_1

(*ii*) $u_1 = u_2,\ u_1$ *is a non isolated vertex of* G_1 *and* $N(v_1) = N(v_2)$

(*iii*) u_1 *is adjacent to* $u_2,\ N[u_1] = N[u_2]$ *and* v_1 *and* v_2 *are isolated vertices of* G_2.

Proof. Suppose that $N[(u_1, v_1)] = N[(u_2, v_2)]$. Then either $u_1 = u_2$ or u_1 is adjacent to u_2.

Suppose $u_1 = u_2 = u$ and u is a non isolated veretx of G_1. Then there exits $u' \in N(u)$. Assume that $N(v_1) \neq N(v_2)$. Without loss of generality assume that there exists a vertex $v' \in N(v_1) \setminus N(v_2)$. Then $(u', v') \notin N[(u, v_1)]$ and $(u', v') \in N[(u, v_2)]$, a contradiction. Therefore in this case $N(v_1) = N(v_2)$.

Suppose u_1 is adjacent to u_2. If $N[u_1] \neq N[u_2]$, without loss of generality assume that there exists a vertex u' which belongs to $N[u_1] \setminus N[u_2]$. Then $(u', v_1) \in N[(u_1, v_1)] \setminus N[(u_2, v_2)]$, a contradiction. Therefore $N[u_1] = N[u_2]$. If v_1 is not an isolated vertex of G_2, then $N(v_1) \neq \emptyset$. Let $v' \in N(v_1)$. This implies $(u_2, v') \in N[(u_2, v_2)] \setminus N[(u_1, v_1)]$, a contradiction. Hence v_1 is an isolated vertex of G_2. Similarly, we can prove that v_2 is also an isolated vertex of G_2.

Conversely assume that condition (i) holds. Suppose $u_1 = u_2 = u$. Since u is an isolated vertex of G_1, $N(u) = \emptyset$. Hence $N[(u, v_1)] = \{u\} \times V(G_2) = N[(u, v_2)]$.

Suppose condition (ii) holds. Then by the definition of $G_1 \ltimes G_2$, $N[(u_1, v_1)] = N[(u_2, v_2)]$.

Suppose condition (iii) holds. Since v_1 and v_2 are isolated vertices of G_2, $N(v_1)^c = N(v_2)^c = V(G_2)$. Then,

$$\begin{aligned} N[(u_1, v_1)] &= (\{u_1\} \times V(G_2)) \bigcup (N(u_1) \times V(G_2)) \\ &= N[u_1] \times V(G_2)) \end{aligned}$$

By condition (iii) $N[u_1] = N[u_2]$. This implies $N[(u_1, v_1)] = N[(u_2, v_2)]$. Hence the theorem. □

Corollary 3.2.39. *If G_1 and G_2 are two graphs without isolated vertices and $N(v_1) \neq N(v_2)$, for all $v_1 \neq v_2 \in V(G_2)$. Then the neighborhood sigma algebra of $G_1 \ltimes G_2$ is $\mathcal{P}(V(G_1 \ltimes G_2))$.*

Notation 3.2.40. Let G be a graph and $u \in V(G)$. Then E'_u denotes the collection $\{u' \in V(G) : N(u) = N(u')\}$ and I_G denotes the collection of all isolated vertices of G.

Corollary 3.2.41. *Let G_1 and G_2 be two graphs and $(u, v) \in V(G_1 \ltimes G_2)$. Then,*

(i) $E_{(u,v)} = \{u\} \times V(G_2)$*, if u is an isolated vertex of G_1.*

(ii) $E_{(u,v)} = \{u\} \times E'_v$*, if u is not an isolated vertex of G_1 and v is not an isolated vertex of G_2.*

(iii) $E_{(u,v)} = E_u \times I_{G_2}$*, if u is not an isolated vertex of G_1 and v is an isolated vertex of G_2.*

Note 3.2.42. Consider the graphs G_1, G_2 and $G_1 \ltimes G_2$ in Figure 3.6.

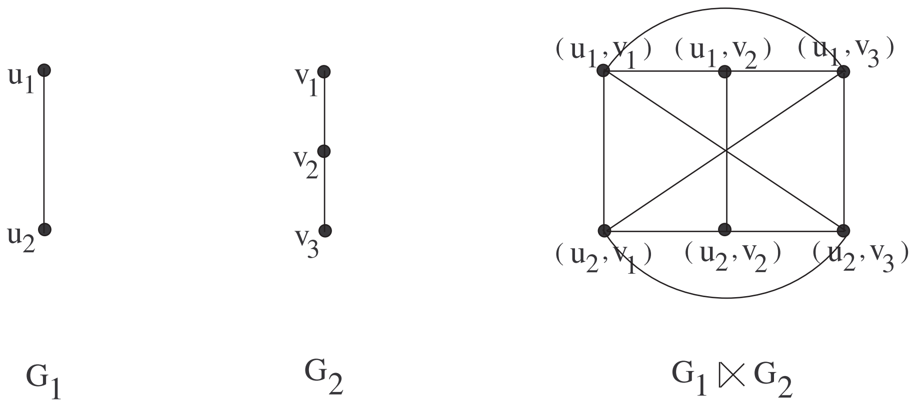

Figure 3.6: Homomorphic product of the graphs G_1 and G_2

It is clear that $E_{(u_1,v_1)} = \{(u_1, v_1), (u_1, v_3)\}$ and $E_{u_1} \times E_{v_1} = \{(u_1, v_1), (u_2, v_1)\}$. Therefore, $E_{(u_1,v_1)} \nsubseteq E_{u_1} \times E_{v_1}$ and $E_{u_1} \times E_{v_1} \nsubseteq E_{(u_1,v_1)}$.
In fact $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \nsubseteq \mathcal{A}_{G_1 \ltimes G_2}$ and $\mathcal{A}_{G_1 \ltimes G_2} \nsubseteq \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

$E_{u_1} \times E_{v_1} \in \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$. If $E_{u_1} \times E_{v_1} \in \mathcal{A}_{G_1 \ltimes G_2}$, then $E_{u_1} \times E_{v_1} \cap E_{(u_1,v_1)} = \{(u_1, v_1)\}$ will be a member of $\mathcal{A}_{G_1 \ltimes G_2}$. This will contradicts the fact that $E_{(u_1,v_1)} = \{(u_1, v_1), (u_1, v_3)\}$. Therefore $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \nsubseteq \mathcal{A}_{G_1 \ltimes G_2}$.

$\{(u_1, v_2)\} \in \mathcal{A}_{G_1 \ltimes G_2}$. As every measurable set in G_1 containing u_1 also contains u_2, every element in $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ containing (u_1, v_2) also contains (u_2, v_2). Thus $\{(u_1, v_2)\} \notin \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$. Therefore $\mathcal{A}_{G_1 \ltimes G_2} \nsubseteq \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

Chapter 4

Common Neighborhood Polynomial of a Graph

In this chapter we introduce a new type of graph polynomial called common neighborhood polynomial and discuss some of its properties. Neighborhood unique graphs are also defined and a characterization of these types of graphs is given. We also find the common neighborhood polynomial of middle graph, total graph, 1-quasi-total graph and 2-quasi-total graph of a given graph.

4.1 Common Neighborhood Polynomial

Definition 4.1.1. The common neighborhood polynomial of a graph G, denoted by $P(G, x)$, is the polynomial defined by

$$P(G, x) = \sum_{i=1}^{n(G)} a_i x^i,$$

where a_i is the number of E_v's of cardinality i in $\mathcal{A}_G$.

We use the abbreviation CNP for the common neighborhood polynomial.

Example 4.1.2. For the path P_n,

$$P(P_n, x) = \begin{cases} x^n, & n = 1, 2 \\ nx, & n > 2 \end{cases}$$

For the complete graph K_n,

$$P(K_n, x) = x^n, \text{for all } n \geq 1.$$

For the cycle C_n of order $n \geq 3$,

$$P(C_n, x) = \begin{cases} x^3, & n = 3 \\ nx, & n > 3 \end{cases}$$

The CNP of the graph G in Figure 4.1 is $x^2 + 2x$ and that of the graph H given in Figure 4.2 is $2x^2 + 4x$.

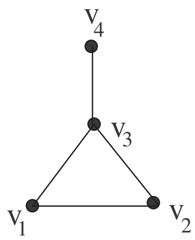

Figure 4.1: Graph G

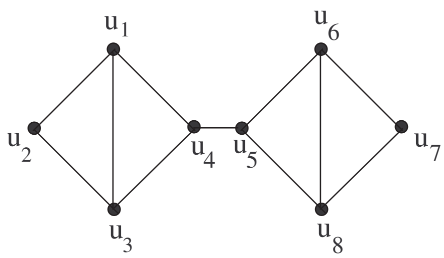

Figure 4.2: Graph H

Proposition 4.1.3 is a direct implication of the definition of CNP.

Proposition 4.1.3. *If a graph G has k components G_1, G_2, $\ldots$, G_k, then*

$$P(G,x) = P(G_1,x) + P(G_2,x) + \ldots + P(G_k,x).$$

Proposition 4.1.4. *Let G be a graph with $P(G,x) = \sum_{i=1}^{n(G)} a_i x^i$.*

Then $n(G) = \sum_{i=1}^{n(G)} i a_i$ and $m(G) \geq \sum_{i=1}^{n(G)} a_i \frac{i(i-1)}{2}$.

Proof. For $1 \leq i \leq n(G)$, a_i is the number of E_v's of cardinality i in $\mathcal{A}_G$. By

Theorem 2.1.7, the family of all E_v's forms a partition of $V(G)$. Hence we have $n(G) = \sum_{i=1}^{n(G)} ia_i$.

The subgraph induced by E_v is a complete subgraph and the number of edges in a complete subgraph with n vertices is $\frac{n(n-1)}{2}$. Therefore, $m(G) \geq \sum_{i=1}^{n(G)} a_i \frac{i(i-1)}{2}$. □

Proposition 4.1.5. *If* $P(G,x) = \sum_{i=1}^{n(G)} a_i x^i$ *for some graph* G*, then for* $1 \leq i \leq n(G)$*,* G *has* a_i *vertex disjoint subgraphs, each is isomorphic to* K_i *.*

Proof. For $1 \leq i \leq n(G)$, a_i is the number of E_v's of cardinality i in $\mathcal{A}_G$. The proposition now follows from the fact that E_v's are either identical or disjoint and the subgraph induced by each E_v is a complete subgraph. □

Note that for any graph G, $P(G,x)$ is a non zero polynomial over the set of non negative integers $\mathbb{N}\bigcup\{0\}$ without constant term, where $\mathbb{N}$ is the set of all natural numbers. Thus if a polynomial over $\mathbb{N}\bigcup\{0\}$ has a constant term it cannot be the CNP of any graph. On the other hand corresponding to every non zero polynomial P over $\mathbb{N}\bigcup\{0\}$ without constant term, there exists a graph G whose CNP is P.

For example if $P(x) = \sum_{i=1}^{n} a_i x^i$, with each a_i is a non negative integer for $1 \leq i \leq n$, then the graph G with exactly a_1 copies of K_1, a_2 copies of K_2,..., a_n copies of K_n has the given polynomial as its CNP.

We summarize these facts as follows.

Proposition 4.1.6. *Suppose* $P(x)$ *is a non zero polynomial in* x *over* $\mathbb{N}\bigcup\{0\}$

with $P(0) = 0$. Then there exists a graph G such that $P(G, x) = P(x)$.

The following proposition is an immediate consequence of the definition of CNP.

Proposition 4.1.7. *Two isomorphic graphs have the same CNP.*

The converse of proposition 4.1.7 is not true. For example the path P_n and the cycle C_n, for $n > 3$ have the same CNP but they are not isomorphic.

Some other non-isomorphic graphs having the same CNP are given in Figures 4.3 and 4.4.

Figure 4.3: Non-isomorphic graphs with same CNP

The CNP of the graphs G_1 and G_2 in Figure 4.3 is $x^2 + 2x$. But G_1 and G_2 are non-isomorphic.

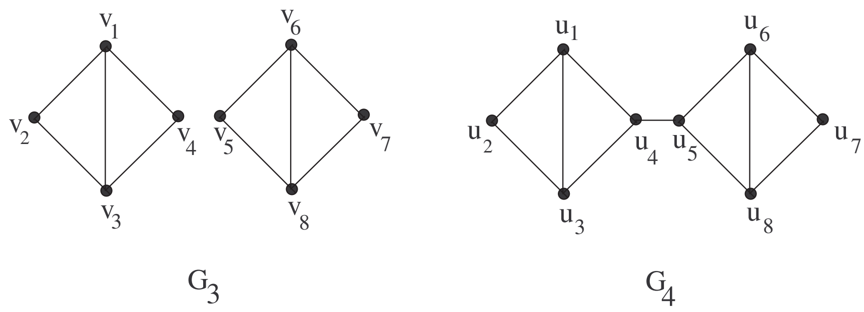

Figure 4.4: Non-isomorphic graphs with same CNP

$P(G_3, x) = P(G_4, x) = 2x^2 + 4x$. But G_3 and G_4 are non-isomorphic.

If the converse of Proposition 4.1.7 holds for a graph G in the sense that if H is any graph such that $P(H, x) = P(G, x)$ then G is isomorphic to H, such a graph G is called a neighborhood unique graph.

Definition 4.1.8. A graph G is called a neighborhood unique graph if $P(G, x) = P(H, x)$ for any graph H implies that G is isomorphic to H.

The graphs in Figure 4.3 and 4.4 are not neighborhood unique.
In the case of complete graph K_n on n vertices $P(K_n, x) = x^n$ and every graph having x^n as CNP is isomorphic to K_n. Hence K_n is a neighborhood unique graph for any $n \geq 1$.

Lemma 4.1.9. *Let G be the disjoint union of K_n and K_m, where $m, n \geq 1$. Then G is neighborhood unique.*

Proof. It is clear that $P(G, x) = x^n + x^m$. Suppose H is a graph with $P(H, x) =$

$x^n + x^m$. Then H has two vertex disjoint subgraphs H_1 and H_2 isomorphic to K_n and K_m respectively such that $E_v = V(H_i)$, for $v \in V(H_i)$, $i = 1,\ 2$. Since all vertices of an E_v have same neighbors, if one vertex of H_1 is adjacent to a vertex of H_2 then all other vertices of H_1 are adjacent to that vertex of H_2. Also we know that for each $u \in V(H_2)$, $E_u = V(H_2)$. Hence all the vertices of H_2 are adjacent to all the vertices of H_1. This will imply that for each $u \in V(H)$, $E_u = V(H)$, a contradiction. Hence no vertex of H_1 is adjacent to a vertex of H_2. Therefore $H \cong G$. Hence G is neighborhood unique. $\square$

Lemma 4.1.10. *Let n be a fixed positive integer. If a graph G is the disjoint union of more than two copies of the complete graphs K_n, then G is not neighborhood unique.*

Proof. Let k be an integer greater than 2. Let G be the disjoint union of k copies of K_n. Then $P(G_1, x) = kx^n$.

Construct a graph H in the following way. Draw k disjoint copies of K_n. Join all vertices of the first copy to all vertices of the second copy. Then join all vertices of the second copy to all vertices of the third copy. Continue this until all vertices of the $(k-1)^{th}$ copy are joined to all vertices of the k^{th} copy of K_n. Then common neighborhood polynomial of H is also kx^n. But $G \ncong H$. Hence G is not neighborhood unique. $\square$

Remark 4.1.11. For any positive integer $k > 2$ and $n \in \mathbb{N}$, there are non-isomorphic graphs with kx^n as their common neighborhood polynomial.

Lemma 4.1.12. *Let* $P(x) = \sum_{j=1}^{n} a_{i_j} x^{i_j}$ *be a non zero polynomial over* $\mathbb{N} \bigcup \{0\}$ *with* $P(0) = 0$. *If* $P(x)$ *has more than two terms or* $P(x)$ *is of the form* $a_{i_r} x^{i_r} + a_{i_s} x^{i_s}$, *with* $a_{i_r}, a_{i_s} > 0$ *and* a_{i_r} *or* $a_{i_s} > 1$, *then there exist non isomorphic graphs with* $P(x)$ *as common neighborhood polynomial.*

Proof. Without loss of generality assume that $a_{i_j} \neq 0$ for all $1 \leq j \leq n$. Let G_1 be the graph with exactly a_{i_1} disjoint copies of K_{i_1}, a_{i_2} disjoint copies of K_{i_2} ,..., a_{i_n} disjoint copies of K_{i_n}. Then $P(G_1, x) = \sum_{j=1}^{n} a_{i_j} x^{i_j}$.

Let G_2 be the graph obtained from G_1 as follows. For $j = 1, 2, ..., n$, join all vertices of the m^{th} copy of K_{i_j} to all vertices of the $(m+1)^{th}$ copy of K_{i_j}, $m = 1, 2, ..., a_{i_j} - 1$. Then join all vertices of the $a_{i_j}^{th}$ copy of K_{i_j} to all the vertices of first copy $K_{i_{j+1}}$, for $1 \leq j \leq n-1$. Then G_1 and G_2 are non isomorphic and $P(G_1, x) = P(G_2, x)$. □

Lemmas 4.1.9, 4.1.10 and 4.1.12, characterize the neighborhood unique graphs.

Theorem 4.1.13. *A graph* G *is neighborhood unique if and only if* G *is a complete graph or disjoint union of two complete graphs.*

The above theorem can be restated as:

"Let G be a graph with $P(G, x) = \sum_{i=1}^{n} a_i x^i$. Then G is neighborhood unique if and only if $P(G, x)$ is of the form $a_r x^r + a_s x^s$ with $1 \leq a_r + a_s \leq 2$ ".

4.2 CNP of Line Graph of a Graph

This section deals with the CNP of the line graph of a graph.

Theorem 4.2.1. *Let G be a graph with $m(G) = m$ and the neighborhood sigma algebra of G be $\mathcal{P}(V(G))$. If P_3 is not a component of G and for $n > 2$, $K_{1,n}$ is not an induced subgraph of G, then $P(L(G), x) = mx$.*

Proof. By the definition of $L(G)$, $V(L(G)) = E(G)$. Hence $n(G) = m$. It is proved in Theorem 2.3.1, that if the hypothesis holds, the neighborhood sigma algebra of $L(G)$ is $\mathcal{P}(V(L(G)))$. Therefore all E_v's in $\mathcal{A}_{L(G)}$ are of cardinality one. Hence the theorem. □

Corollary 4.2.2. *(1). $P(L(C_n), x) = nx$, for $n > 3$.*

(2). $P(L(P_n), x) = (n-1)x$, for $n > 3$.

4.3 CNP of Middle Graph of a Graph

In this section we determine the CNP of the middle graph of a graph.

Theorem 4.3.1. *Let G be a graph with $n(G) = n$ and $m(G) = m$. Then $P(M(G), x) = (m + n)x$.*

Proof. By the definition of $M(G)$, $V(M(G)) = V(G) \bigcup E(G)$. Hence $n(M(G)) = n(G) + m(G)$. By Theorem 2.4.1, $N[u] \neq N[v]$, for any two vertices u and v

of $M(G)$. Therefore for all $u \in V(M(G))$, $E_u = \{u\}$. Hence $P(M(G), x) = (m+n)x$. $\square$

Corollary 4.3.2. *(1) $P(M(P_n), x) = (2n-1)x$, for all $n \geq 1$.*

(2) $P(M(K_n), x) = \frac{n(n+1)}{2}x$, for all $n \geq 1$.

(3) $P(M(C_n), x) = 2nx$, for all $n \geq 3$.

4.4 CNP of Total Graph of a Graph

Here we determine the CNP of the total graph of a graph.

Lemma 4.4.1. *Let G be a graph with $n(G) = n$ and $m(G) = m$. If every component of G is different from P_2, then $P(T(G)) = (m+n)x$.*

Proof. By Theorem 2.5.1, if G is a graph such that every component of G is different from P_2, then for any two vertices $u, v \in V(T(G))$, $N[u] \neq N[v]$. Therefore for all $u \in V(T(G))$, $E_u = \{u\}$. Hence the theorem. $\square$

Corollary 4.4.2. *(1) $P(T(P_n), x) = (2n-1)x$, for all $n \neq 2$.*

(2) $P(T(K_n), x) = \frac{n(n+1)}{2}x$, for all $n \neq 2$.

(3) $P(T(C_n), x) = 2nx$, for all $n \geq 3$.

Note 4.4.3. Since $T(P_2) = K_3$, $P(T(P_2),\ x) = x^3$

Thus we have the following theorem.

Theorem 4.4.4. *Let G be any graph with order n and size m. If p components of G are P_2, where $p \geq 0$, then $P(T(G), x) = (m + n - 3p)x + px^3$.*

4.5 CNP of 1-quasi-total Graph and 2-quasi-total Graph

From the definition of 1-quasi-total graph $Q_1(G)$ of a graph G, it is clear that $Q_1(G)$ is the disjoint union of G and its line graph $L(G)$. Therefore CNP of $Q_1(G)$ is the sum of CNP of G and CNP of $L(G)$. Hence by Theorem 4.2.1, we have Theorem 4.5.1.

Theorem 4.5.1. *Let G be a graph with $n(G) = n_1$ and $m(G) = n_2$ and neighborhood sigma algebra of G be $\mathcal{P}(V(G))$. If P_3 is not a component of G and for $n > 2$, $K_{1,n}$ is not an induced subgraph of G, then $P(Q_1(G), x) = (n_1 + n_2)x$.*

Corollary 4.5.2. *(1). $P(Q_1(C_n), x) = 2nx$, for $n > 3$.*

(2). $P(Q_1(P_n), x) = (2n - 1)x$, for $n > 3$.

We explore Theorem 2.6.5, to get the common neighborhood polynomial of 2-quasi-total graphs of graphs having no end vertices.

Theorem 4.5.3. *Let G be a graph without end vertices. If $n(G) = n$ and $m(G) = m$, then $P(Q_2(G), x) = (m + n)x$.*

Proof. Since $V(Q_2(G)) = V(G) \bigcup E(G)$, $n(Q_2(G)) = m + n$. If G does not have end vertices, the neighborhood sigma algebra of $Q_2(G)$ is $\mathcal{P}(V(Q_2(G)))$, by Theorem 2.6.5. Therefore all E_v's in $\mathcal{A}_{Q_2(G)}$ are of cardinality one. Hence the theorem. □

Corollary 4.5.4. *(1). $P(Q_2(C_n), x) = 2nx$, for $n \geq 3$.*

(2). $P(Q_2(K_n), x) = (n + nC_2)x$, for $n > 3$.

Theorem 4.5.5. *Let G be a graph with $n(G) = n$ and $m(G) = m$. Suppose that G has s components. If exactly r of them are isomorphic to P_2, $1 \leq r \leq s$, and the total number of end vertices in the remaining components (if any) is k, then $P(Q_2(G), x) = rx^3 + kx^2 + (m + n - 3r - 2k)x$.*

Proof. Let G_1 be a component of G which is isomorphic to P_2. Let $V(G_1) = \{u, v\}$ and e be the edge joining u and v in G_1. Then $N_{Q_2}[u] = N_{Q_2}[v] = N_{Q_2}[e] \neq N_{Q_2}[w]$, for any $w \in V(Q_2(G)) \setminus \{u, v, e\}$. Hence $E_u^{Q_2} = E_v^{Q_2} = E_e^{Q_2} = \{u, v, e\}$, which is true for any component of G isomorphic to P_2. There are exactly r components in G which are isomorphic to P_2. They induce exactly r, E_v's of cardinality 3.

Let w be an end vertex of a component G_2 of G which is not isomorphic to P_2 and e' be the pendant edge of G incident to w. Let $e' = ww'$, $w' \in V(G)$. Then $N_{Q_2}[w] = N_{Q_2}[e] = \{w, e, w'\}$. Since $G_2 \ncong P_2$, w' is adjacent to a vertex $x(\neq w)$ in G. Therefore $N_{Q_2}[w'] \neq N_{Q_2}[w]$. Hence $E_w^{Q_2} = E_e^{Q_2} = \{w, e\}$.

Let $z \in V(G) \bigcup E(G)$ which is not an end vertex or a pendant edge of G. By imitating the proof of Theorem 2.6.5, we can prove that $E_z^{Q_2} = \{z\}$.

Hence the theorem. □

Corollary 4.5.6. *$P(Q_2(P_2), x) = x^3$ and $P(Q_2(P_n), x) = 2x^2 + (2n-5)x$, for $n \geq 3$.*

Example 4.5.7. Consider the graph G give below.

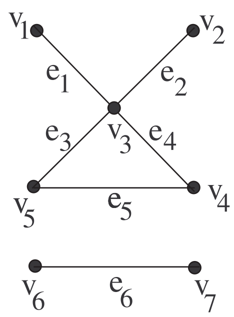

Figure 4.5: Graph G

For the graph G given in Figure 4.5.7,

$$P(Q_2(G), x) = x^3 + 2x^2 + 6x.$$

Chapter 5

CNP of Join, Corona and Product of Two Graphs

In this chapter we find the CNP of join, corona and different products of two graphs such as lexicographic product, tensor product, Cartesian product, normal product and co-normal product.

5.1 CNP of Join of Two Graphs

Here we discuss the CNP of join of two graphs. For this we are making use of the results proved in section 3.1.

Theorem 5.1.1. *Let G_1 and G_2 be two graphs. If*

1. *$D_{G_i} = \emptyset$ for $i = 1$ or 2, then $P(G_1 \vee G_2,\ x) = P(G_1,\ x) + P(G_2,\ x)$.*

2. for $i = 1, 2$, $D_{G_i} \neq \emptyset$ *and* $\mid D_{G_i} \mid = k_i$, *then* $P(G_1 \vee G_2,\ x) = P(G_1,\ x) + P(G_2,\ x) - x^{k_1} - x^{k_2} + x^{k_1+k_2}$.

Proof. Theorems 3.1.2 and 3.1.4 imply that, for $i = 1, 2$, if $v \in V(G_i) \setminus D_{G_i}$, $E_v^{G_1 \vee G_2} = E_v^{G_i}$ and if $v \in D_{G_i}$, $E_v^{G_1 \vee G_2} = D_{G_1} \bigcup D_{G_2}$.

1. By Theorem 2.1.27, for $i = 1, 2$ if $v \in D_{G_i}$, $E_v^{G_i} = D_{G_i}$. Thus if D_{G_1} or $D_{G_2} = \emptyset$, then for $i = 1, 2$ and $v \in V(G_i)$, $E_v^{G_1 \vee G_2} = E_v^{G_i}$. Hence in this case the number of E_v's of a particular cardinality in $\mathcal{A}_{G_1 \vee G_2}$ is equal to the sum of number of E_v's of that cardinality in $\mathcal{A}_{G_1}$ and $\mathcal{A}_{G_2}$. Therefore $P(G_1 \vee G_2,\ x) = P(G_1,\ x) + P(G_2,\ x)$.

2. For $i = 1, 2$, if $v \in D_{G_i}$, then $E_v^{G_1 \vee G_2} = D_{G_1} \bigcup D_{G_2}$. Therefore in $\mathcal{A}_{G_1 \vee G_2}$, for $v \in D_{G_1} \bigcup D_{G_2}$, $\mid E_v^{G_1 \vee G_2} \mid = \mid D_{G_1} \bigcup D_{G_2} \mid = k_1 + k_2$. For all other $v \in V(G_1) \bigcup V(G_2)$, E_v's does not change. Also by Theorem 2.1.27, for $i = 1, 2$, if $v \in D_{G_i}$, $E_v^{G_i} = D_{G_i}$. Thus we get $P(G_1 \vee G_2,\ x) = P(G_1,\ x) + P(G_2,\ x) - x^{k_1} - x^{k_2} + x^{k_1+k_2}$ if $D_{G_i} \neq \emptyset$ for $i = 1, 2$,where $\mid D_{G_1} \mid = k_1$ and $\mid D_{G_2} \mid = k_2$.

□

Corollary 5.1.2.

(1). $P(K_m \vee K_n,\ x) = x^{m+n}$, $m, n \geq 1$.

(2).
$$\begin{aligned} P(P_2 \vee P_3,\ x) &= x^2 + 3x - x^2 - x + x^3 \\ &= x^3 + 2x. \end{aligned}$$

(3). $P(P_m \vee P_n,\ x) = mx + nx$

$$= (m+n)x,\ m, n > 3.$$

(4). $P(K_{1,n} \vee K_n,\ x) = (n+1)x + x^n - x - x^n + x^{n+1}$

$$= x^{n+1} + nx,\ n > 1.$$

5.2 CNP of Corona of Two Graphs

This section is devoted to determine the CNP of corona of two graphs. Consider the graphs K_1 and K_n. $K_1 \circ K_n = K_{n+1}$. Hence $P(K_1 \circ K_n, x) = x^{n+1}$.

Let G be a graph such that $D_G \neq \emptyset$. For all $v \in V(G)$, $N_{K_1 \circ G}[v] = \{u\} \bigcup N_G[v]$. Therefore, for $v_1,\ v_2 \in V(G)$, $E_{v_1}^{K_1 \circ G} = E_{v_2}^{K_1 \circ G}$ if and only if $N_G[v_1] = N_G[v_2]$. As $N_{K_1 \circ G}[u] = \{u\} \bigcup V(G)$, for a vertex $v \in V(G)$, $N_{K_1 \circ G}[v] = N_{K_1 \circ G}[u]$ if and only if $v \in D_G$. Thus $E_v^{K_1 \circ G} = D_G \bigcup \{u\} = E_u^{K_1 \circ G}$ if $v \in D(G)$ and $E_v^{K_1 \circ G} = E_v^G$ otherwise.

If $D_G = \emptyset$, $E_v^{K_1 \circ G} = E_v^G$ for all $v \in V(G)$ and $E_u^{K_1 \circ G} = \{u\}$.

The preceding discussion may be summarized as follows.

Theorem 5.2.1. *For a graph G,*

$$P(K_1 \circ G, x) = \begin{cases} x^{k+1} + P(G, x) - x^k & \text{if } D_G \neq \emptyset,\ k = \mid D_G \mid \\ P(G, x) + x & \text{if } D_G = \emptyset \end{cases}$$

Proposition 5.2.2. *Suppose G_1 is a graph which does not have K_1 as a component and G_2 is any graph then $P(G_1 \circ G_2, x) = n(G_1)P(G_2, x) + n(G_1)x$.*

Proof. Let u and v be two distinct vertices of G_1. Then it is clear that $N_{G_1 \circ G_2}[u] \neq N_{G_1 \circ G_2}[v]$. Let $u \in V(G_1)$ be the i^{th} vertex of G_1 and v be a vertex of the i^{th} copy of G_2. Since G_1 does not have K_1 as a component, in $G_1 \circ G_2$, u is adjacent to at least one vertex w of G_1 and w is not adjacent to any vertex of the i^{th} copy of G_2. Therefore $N_{G_1 \circ G_2}[u] \neq N_{G_1 \circ G_2}[v]$ for any vertex v of the i^{th} copy of G_2. Hence for $u \in V(G_1)$ and for all $v(\neq u) \in V(G_1 \circ G_2)$, $N_{G_1 \circ G_2}[u] \neq N_{G_1 \circ G_2}[v]$. Therefore $E_u^{G_1 \circ G_2} = \{u\}$ for all $u \in V(G_1)$. That is there are $n(G_1)$, E_v's of cardinality one in $\mathcal{A}_{G_1 \circ G_2}$. Let w be a vertex of the i^{th} copy of G_2, $1 \leq i \leq n(G_1)$. Then w is not adjacent to vertices of any other copy of G_2. Also, if $v \in V(G_1)$, $N_{G_1 \circ G_2}[w] \neq N_{G_1 \circ G_2}[v]$. But for any vertices $w_1,\ w_2$ of the i^{th} copy of G_2, $N_{G_2}[w_1] = N_{G_2}[w_2]$ if and only if $N_{G_1 \circ G_2}[w_1] = N_{G_1 \circ G_2}[w_2]$, $1 \leq i \leq n(G_1)$. Hence $P(G_1 \circ G_2, x) = n(G_1)P(G_2, x) + n(G_1)x$. $\square$

Corollary 5.2.3.

1. $P(K_m \circ K_n, x) = mx^n + mx$, *for* $m, n \geq 1$.

2. $$\begin{aligned} P(P_m \circ P_n, x) &= mnx + mx \\ &= m(n+1)x,\ \ for\ m, n \geq 3. \end{aligned}$$

3. $P(C_m \circ K_n, x) = mx^n + mx$ *for* $m \geq 3,\ n \geq 1$.

5.3 CNP of Graph Products

This section deals with the common neighborhood polynomial of different graph products. If $P(x)$ is a polynomial in x, we use the notation $deg(P(x))$ to denote the degree of the polynomial $P(x)$.

5.3.1 Lexicographic Product

Theorem 5.3.1 leads to the CNP of lexicographic product of two graphs.

Theorem 5.3.1. *Let G_1 be a graph of order n_1 and G_2 be a graph of order n_2 with $\mid D_{G_2} \mid = k,\ k \neq 0$. If $P(G_1, x) = \sum_{i=1}^{n_1} a_i x^i$ and $P(G_2, x) = \sum_{j=1}^{n_2} b_j x^j$, then $P(G_1[G_2],\ x) = n_1 \sum_{\substack{i=1 \\ i \neq k}}^{n_2} b_i x^i + n_1(b_k - 1)x^k + \sum_{i=1}^{n_1} a_i x^{ik}$.*

Proof. From Theorem 3.2.1, it is clear that $E_{(u,v)} = (\{u\} \times E_v) \bigcup (E_u \times (E_v \bigcap D_{G_2}))$ for all $(u, v) \in V(G_1[G_2])$. Also, if $v \in D_{G_2}$ then $E_v = D_{G_2}$ and if $v \in V(G_2) \backslash D_{G_2}$ then $E_v \bigcap D_{G_2} = \emptyset$. Therefore for all $(u, v) \in V(G_1[G_2])$, if $v \notin D_{G_2}$, $E_{(u,v)} = \{u\} \times E_v$ and hence $\mid E_{(u,v)} \mid = \mid E_v \mid$. For $1 \leq j \leq n_2,\ j \neq k$ there are b_j, E_v's of cardinality j in $\mathcal{A}_{G_2}$ and hence $n_1 b_j$, $E_{(u,v)}$'s of cardinality j in $\mathcal{A}_{G_1[G_2]}$.

For all $(u, v) \in V(G_1[G_2])$, if $v \in D_{G_2}$, $E_{(u,v)} = E_u \times E_v$, and hence $\mid E_{(u,v)} \mid = \mid E_u \mid . \mid E_v \mid$. There are b_k, E_v's of cardinality k in $\mathcal{A}_{G_2}$ and D_{G_2} is an E_v of cardinality k in $\mathcal{A}_{G_2}$. Therefore in $\mathcal{A}_{G_1[G_2]}$, there are $n_1(b_k - 1)$, $E_{(u,v)}$'s of cardinality k and a_i, $E_{(u,v)}$'s of cardinality ik, $1 \leq i \leq n_1$. Hence the theorem. $\square$

Corollary 5.3.2. *Suppose G_1 is a graph of order n_1 and G_2 is a graph with $D_{G_2} = \emptyset$, If $P(G_2, x) = \sum_{i=1}^{n_2} b_i x^i$, then $P(G_1[G_2],\ x) = n_1 \sum_{i=1}^{n_2} b_i x^i$.*

Proof. Since $D_{G_2} = \emptyset$, for all $(u, v) \in V(G_1[G_2])$, $E_{(u,v)} = \{u\} \times E_v$ and hence $\mid E_{(u,v)} \mid = \mid E_v \mid$. For $1 \leq i \leq n_2$, there are b_i, E_v's of cardinality i in $\mathcal{A}_{G_2}$, hence there are $n_1 b_i$, $E_{(u,v)}$'s of cardinality i in $\mathcal{A}_{G_1[G_2]}$. Therefore $P(G_1[G_2],\ x) = n_1 \sum_{i=1}^{n_2} b_i x^i$. □

Corollary 5.3.3.

(1). $P(K_m[K_n],\ x) = x^{mn}$, for $m,\ n \geq 1$.

(2). $P(P_m[K_n],\ x) = mx^n$, for $m \geq 1,\ m \neq 2,\ n \geq 1$.

(3). $P(C_m[K_n],\ x) = mx^n$, for $m \geq 4,\ n \geq 1$.

5.3.2 Tensor Product

In this section, we determine the CNP of tensor product of two graphs.

Proposition 5.3.4. *Let G_1 and G_2 be two graphs with $n(G_1) = n_1$ and $n(G_2) = n_2$. If G_1 or G_2 does not have P_2 as a component, then $P(G_1 \otimes G_2,\ x) = n_1 n_2 x$.*

Proof. By Corollary 3.2.10, if the hypothesis of the proposition holds then the neighborhood sigma algebra of $G_1 \otimes G_2$ is $\mathcal{P}(V(G_1 \otimes G_2))$. Therefore all $E_{(u,v)}$'s in $\mathcal{A}_{G_1 \otimes G_2}$ are of cardinality one. Hence the proposition. □

Corollary 5.3.5. *1. $P(K_m \otimes K_n,\ x) = mnx$ for $m \neq 2$, $n \neq 2$.*

2. $P(P_m \otimes P_n,\ x) = mnx$ *for* $m \neq 2$, $n \neq 2$.

3. $P(C_m \otimes K_n,\ x) = mnx$ *for* $m \geq 3$, $n \neq 2$.

Proposition 5.3.6. *For any graphs G_1 and G_2, $deg(P(G_1 \otimes G_2,\ x))$ is either one or two.*

Proof. Let $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$. First of all suppose that u_1, u_2 are distinct and non isolated in G_1 and v_1, v_2 are distinct and non isolated in G_2. Then by Theorem 3.2.9, $N[(u_1, v_1)] = N[(u_2, v_2)]$ in $G_1 \otimes G_2$ if and only if $N[u_1] = N[u_2] = \{u_1, u_2\}$ in G_1 and $N[v_1] = N[v_2] = \{v_1, v_2\}$ in G_2. Hence in this case $N[(u_1, v_1)] = N[(u_2, v_2)] = \{(u_1, v_1), (u_2, v_2)\}$.

If $u_1 = u_2$ or $v_1 = v_2$ then (u_1, v_1) and (u_2, v_2) are not adjacent in $G_1 \otimes G_2$. Hence $N[(u_1, v_1)] \neq N[(u_2, v_2)]$. Also if u is an isolated vertex of G_1 or v is an isolated vertex of G_2, then (u, v) is an isolated vertex of $G_1 \otimes G_2$. Therefore all E_v's are of cardinality one or two. Hence the proposition. □

Proposition 5.3.7. *Let G_1 and G_2 be two graphs. If P_2 is a component of both G_1 and G_2, then $deg((P(G_1 \otimes G_2,\ x)) = 2$.*

Proof. Since P_2 is a component of both G_1 and G_2, there exist vertices $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$ such that $N[u_1] = N[u_2] = \{u_1, u_2\}$ in G_1 and $N[v_1] = N[v_2] = \{v_1, v_2\}$ in G_2. Hence $N[(u_1, v_1)] = N[(u_2, v_2)]$ in $G_1 \otimes G_2$. Thus there is an E_v of cardinality more than one in $\mathcal{A}_{G_1 \otimes G_2}$. Therefore $deg(P(G_1 \otimes G_2,\ x)) \geq 2$. Hence by Proposition 5.3.6, $deg(P(G_1 \otimes G_2,\ x)) = 2$. □

Theorem 5.3.8. *Let G be a graph of order n_1 and H be a graph of order n_2. If exactly r components of G are P_2 and exactly s components of H are P_2, then $P(G \otimes H,\ x) = 2rsx^2 + (n_1n_2 - 4rs)x$.*

Proof. As tensor product is commutative and distributive over disjoint union of graphs, if $G_1, G_2, ..., G_{k_1}$ are components of G and $H_1, H_2, ..., H_{k_2}$ are components of H, then

$$G \otimes H = \bigcup_{\substack{1\leq i\leq k_1\\1\leq j\leq k_2}} G_i \otimes H_j.$$

Also $\{G_i \otimes H_j :\ 1 \leq i \leq k_1 \text{ and } 1 \leq j \leq k_2\}$ is a family of disjoint subgraphs of $G \otimes H$. Therefore,

$$\begin{aligned} P(G \otimes H, x) &= P(\bigcup_{\substack{1\leq i\leq k_1\\1\leq j\leq k_2}} G_i \otimes H_j,\ x) \\ &= \sum_{\substack{1\leq i\leq k_1\\1\leq j\leq k_2}} P(G_i \otimes H_j,\ x). \end{aligned}$$

Without loss of generality assume that the first r components $G_1, G_2, ..., G_r$ of G and the first s components $H_1, H_2, ..., H_s$ of H are P_2. It is clear that $P(P_2 \otimes P_2,\ x) = 2x^2$. Hence

$$\begin{aligned} P(\bigcup_{\substack{1\leq i\leq r\\1\leq j\leq s}} G_i \otimes H_j,\ x) &= \sum_{\substack{1\leq i\leq r\\1\leq j\leq s}} P(G_i \otimes H_j,\ x) \\ &= 2rsx^2. \end{aligned}$$

None of the components $G_i,\ r+1 \leq i \leq k_1$ of G and $H_j,\ s+1 \leq j \leq k_2$ of H

are P_2, if such i and j exist. Hence

$$\begin{aligned} P(G \otimes H, x) &= \sum_{\substack{1 \le i \le k_1 \\ 1 \le j \le k_2}} P(G_i \otimes H_j,\ x) \\ &= 2rsx^2 + (mn - 4rs)x. \end{aligned}$$

□

Example 5.3.9. Consider the graphs G and H in Figure 5.1 and 5.2.

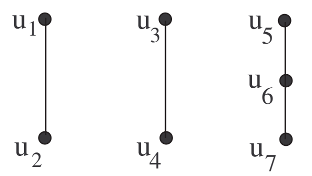

Figure 5.1: Graph G

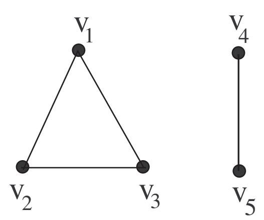

Figure 5.2: Graph H

$P(G \otimes H,\ x) = 4x^2 + 27x.$

5.3.3 Cartesian Product

Here we determine the CNP of Cartesian product of two graphs.

Proposition 5.3.10. *Let G_1 be a graph of order n_1 and G_2 be a graph of order n_2. If G_1 and G_2 have no isolated vertices then $P(G_1 \times G_2,\ x) = n_1 n_2 x$.*

Proof. By Corollary 3.2.18, if G_1 and G_2 have no isolated vertices then the neighborhood sigma algebra of $G_1 \times G_2$ is $\mathcal{P}(V(G_1 \times G_2))$. Therefore all E_v's in $\mathcal{A}_{G_1 \times G_2}$ are of cardinality one. Hence $P(G_1 \times G_2,\ x) = n_1 n_2 x$. □

Corollary 5.3.11.

(1). $P(K_m \times K_n,\ x) = mnx$, for $m,\ n \geq 2$.

(2). $P(K_m \times P_n,\ x) = mnx$, for $m,\ n \geq 2$.

(3). $P(C_m \times K_n,\ x) = mnx$, for $m \geq 3,\ n \geq 2$.

Proposition 5.3.12. *Let G_1 be a graph of order n_1 and G_2 be a graph of order n_2. If the neighborhood sigma algebra of G_1 is $\mathcal{P}(V(G_1))$ and that of G_2 is $\mathcal{P}(V(G_2))$, then $P(G_1 \times G_2,\ x) = n_1 n_2 x$.*

Proof. By Corollary 3.2.19, if the hypothesis of the Proposition holds then the neighborhood sigma algebra of $G_1 \times G_2$ is $\mathcal{P}(V(G_1 \times G_2))$. Therefore all E_v's in $\mathcal{A}_{G_1 \times G_2}$ are of cardinality one. Hence the proposition. □

Corollary 5.3.13.

(1). $P(P_m \times P_n,\ x) = mnx$, *for* $m,\ n \geq 3$.

(2). $P(C_m \times C_n,\ x) = mnx$, *for* $m,\ n \geq 4$.

Note 5.3.14. From the adjacency relation in $G_1 \times G_2$ and from Theorem 3.2.17, we could conclude that for a vertex $(u, v) \in V(G_1 \times G_2)$, if

(i) u is an isolated vertex of G_1 and $v \in V(G_2)$, then $E_{(u,v)} = \{u\} \times E_v$.

(ii) $u \in V(G_1)$ and v is an isolated vertex of G_2, then $E_{(u,v)} = E_u \times \{v\}$.

(iii) u and v are not isolated vertices, then $E_{(u,v)} = \{(u, v)\}$.

Theorem 5.3.15. *Let* G_1 *be a graph of order* n_1 *with* k_1 *isolated vertices and* G_2 *be a graph of order* n_2 *with* k_2 *isolated vertices. If* $P(G_1, x) = \sum_{i=1}^{n_1} a_i x^i$ *and* $P(G_2, x) = \sum_{j=1}^{n_2} b_j x^j$ *with* $n_1 < n_2$, *then* $P(G_1 \times G_2,\ x) = ((m - k_1)(n - k_2) + (a_1 - k_1)k_2 + b_1 k_1)x + \sum_{i=2}^{n_1}(a_i k_2 + b_i k_1)x^i + \sum_{i=n_1+1}^{n_2} b_i k_1 x^i$.

Proof. Let $(u, v) \in V(G_1 \times G_2)$. By Note 5.3.14, $\mid E_{(u,v)} \mid = 1$ if and only if u is an isolated vertex of G_1 and $v \in V(G_2)$ is such that $\mid E_v \mid = 1$ or $u \in V(G_1)$ is such that $\mid E_u \mid = 1$ and v is an isolated vertex of G_2 or both u and v are non isolated vertices of G_1 and G_2 respectively.

Hence number of E_v's of cardinality one in $\mathcal{A}_{G_1 \times G_2}$ is given by $(m - k_1)(n - k_2) + (a_1 - k_1)k_2 + b_1 k_1$.

If u is an isolated vertex of G_1 and $v \in V(G_2)$ is such that $\mid E_v \mid = j$ for $1 \leq j \leq n_2$, then $\mid E_{(u,v)} \mid = j$. Also if $u \in V(G_1)$ is such that $\mid E_u \mid = i$ for $1 \leq i \leq n_1$ and v is an isolated vertex of G_2, then $\mid E_{(u,v)} \mid = \mid E_u \mid = i$ for $1 \leq i \leq n_1$. Thus in $\mathcal{A}_{G_1 \times G_2}$, there are $k_1 b_j$, E_v's of cardinality j for $1 \leq j \leq n_2$ and $k_2 a_i$, E_v's of cardinality i for $1 \leq i \leq n_1$. Hence the theorem. $\square$

Corollary 5.3.16. *Let G_1 be a graph of order n with k_1 isolated vertices and G_2 be a graph of order n with k_2 isolated vertices. If $P(G_1, x) = \sum_{i=1}^{n} a_i x^i$ and $P(G_2, x) = \sum_{j=1}^{n} b_j x^j$, then $P(G_1 \times G_2,\ x) = ((n - k_1)(n - k_2) + (a_1 - k_1)k_2 + b_1 k_1)x + \sum_{i=2}^{n} (a_i k_2 + b_i k_1)x^i$.*

5.3.4 Normal Product

This section deals with the CNP of normal product of two graphs.

Let G_1 and G_2 be two graphs and (u_1, v_1), $(u_2, v_2) \in V(G_1 \boxtimes G_2)$. Then $N[(u_1, v_1)] = N[(u_2, v_2)]$ if and only if $N[u_1] = N[u_2]$ and $N[v_1] = N[v_2]$. Also for $(u, v) \in V(G_1 \boxtimes G_2)$, $E_{(u,v)} = E_u \times E_v$, by Lemma 3.2.26.

Theorem 5.3.17. *Let G_1 be a graph of order n_1 and G_2 be a graph of order n_2. If $P(G_1, x) = \sum_{i=1}^{n_1} a_i x^i$ and $P(G_2, x) = \sum_{j=1}^{n_2} b_j x^j$, then $P(G_1 \boxtimes G_2,\ x) = \sum_{\substack{1 \leq i \leq n_1 \\ 1 \leq j \leq n_2}} a_i b_j x^{ij}$.*

Proof. For $(u, v) \in V(G_1 \boxtimes G_2)$, $E_{(u,v)} = E_u \times E_v$. Hence $\mid E_{(u,v)} \mid =$ $\mid E_u \times E_v \mid = \mid E_u \mid \times \mid E_v \mid$. Therefore, if there are a_i, E_v's of cardinality i in

$\mathcal{A}_{G_1}$ and b_j, E_v's of cardinality j in $\mathcal{A}_{G_2}$, then in $\mathcal{A}_{G_1 \boxtimes G_2}$ there are $a_i b_j$, $E_{(u,v)}$'s of cardinality ij. Hence the theorem. □

Corollary 5.3.18.

(1). $P(K_m \boxtimes K_n,\ x) = x^{mn}$, *for* $m,\ n \geq 1$.

(2). $P(K_m \boxtimes P_n,\ x) = nx^m$, *for* $m \geq 1,\ n \geq 3$.

(3). $P(P_m \boxtimes P_n,\ x) = mnx$, *for* $m,\ n \geq 3$.

(4). $P(C_m \boxtimes K_n,\ x) = mx^n$, *for* $m > 3,\ n \geq 1$.

Corollary 5.3.19. *Let G be a graph. Then $P(G \boxtimes K_1,\ x) = P(G, x)$.*

Corollary 5.3.20. *Let G_1 and G_2 be two graphs. Then $deg(P(G_1 \boxtimes G_2),\ x) = deg(P(G_1,\ x)).deg(P(G_2,\ x))$.*

5.3.5 Co-normal Product

Here we determine the CNP of co-normal product of two graphs.

Proposition 5.3.21. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. If $D_{G_1} = D_{G_2} = \emptyset$ or if G_1 is a graph with neighborhood sigma algebra $\mathcal{P}(V(G_1))$ and G_2 is a graph with neighborhood sigma algebra $\mathcal{P}(V(G_2))$ then $P(G_1 * G_2,\ x) = n_1 n_2 x$.*

Proof. By Corollary 3.2.31, if G_1 and G_2 are two graphs with $D_{G_1} = D_{G_2} = \emptyset$, then the neighborhood sigma algebra of $G_1 * G_2$ is $\mathcal{P}(V(G_1 * G_2))$. By Corollary

3.2.32, if G_1 is a graph with neighborhood sigma algebra $\mathcal{P}(V(G_1))$ and G_2 is a graph with neighborhood sigma algebra $\mathcal{P}(V(G_2))$, then the neighborhood sigma algebra of $G_1 * G_2$ is $\mathcal{P}(V(G_1 * G_2))$. Therefore, in both the cases all $E_{(u,v)}$'s are of cardinality one. Hence the proposition. $\square$

Theorem 5.3.22. *Let G_1 be a graph of order n_1 and G_2 be a graph of order n_2 with $\mid D_{G_1} \mid= k_1$ and $\mid D_{G_2} \mid= k_2$ $(k_1,\ k_2 \neq 0)$. If $P(G_1, x) = \sum_{i=1}^{n_1} a_i x^i$ and $P(G_2, x) = \sum_{j=1}^{n_2} b_j x^j$, then $P(G_1 * G_2,\ x) = (n_1 - k_1)(n_2 - k_2)x + x^{k_1 k_2} + k_1 \sum_{\substack{j=1 \\ j\neq k_2}}^{n_2} b_j x^j + k_2 \sum_{\substack{i=1 \\ i\neq k_1}}^{n_1} a_i x^i + k_1(b_{k_2} - 1)x^{k_2} + k_2(a_{k_1} - 1)x^{k_1}$.*

Proof. Let $(u, v) \in V(G_1 * G_2)$. If $u \notin D_{G_1}$ and $v \notin D_{G_2}$, $E_{(u,v)} = \{(u, v)\}$, by Corollary 3.2.30. There are such $(n_1 - k_1)(n_2 - k_2)$ vertices (u, v) in $G_1 * G_2$. If $u \in D_{G_1}$ and $v \notin D_{G_2}$, $E_{(u,v)} = \{u\} \times E_v$ and hence $\mid E_{(u,v)} \mid=\mid E_v \mid$. For $1 \leq j \leq n_2$, $j \neq k_2$, there are $k_1 b_j$ $E_{(u,v)}$'s of cardinality j in $\mathcal{A}_{G_1 * G_2}$. As D_{G_2} is an E_v of cardinality k_2 in $\mathcal{A}_{G_2}$, there are $b_{k_2} - 1$, E_v's of cardinality k_2 in $\mathcal{A}_{G_2}$ other than D_{G_2}. Corresponding to these E_v's there are $k_1(b_{k_2} - 1)$, $E_{(u,v)}$'s of cardinality k_2 in $\mathcal{A}_{G_1 * G_2}$. If $u \notin D_{G_1}$ and $v \in D_{G_2}$, $E_{(u,v)} = E_u \times \{v\}$ and hence $\mid E_{(u,v)} \mid=\mid E_u \mid$. For $1 \leq i \leq n_1$, $i \neq k_1$, there are $k_2 a_i$ $E_{(u,v)}$'s of cardinality i in $\mathcal{A}_{G_1 * G_2}$. As D_{G_1} is an E_v of cardinality k_1 in $\mathcal{A}_{G_1}$, there are $a_{k_1} - 1$, E_v's of cardinality k_1 in $\mathcal{A}_{G_1}$ other than D_{G_1}. Corresponding to these E_v's there are $k_2(a_{k_1} - 1)$, $E_{(u,v)}$'s of cardinality k_1 in $\mathcal{A}_{G_1 * G_2}$. If $u \in D_{G_1}$ and $v \in D_{G_2}$, $E_{(u,v)} = D_{G_1} \times D_{G_2}$. This implies the existence of an $E_{(u,v)}$ of cardinality $k_1 k_2$ in $\mathcal{A}_{G_1 * G_2}$. Hence the theorem. $\square$

Example 5.3.23. Consider the graphs G_1 and G_2.

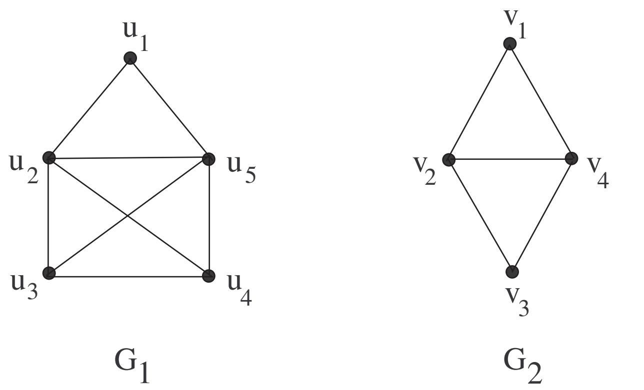

Figure 5.3: Graphs G_1 and G_2

For the graph G_1,

$$P(G_1,\ x) = 2x^2 + x,\ n(G_1) = 5,\quad |\, D_{G_1} \,| = k_1 = 2 \text{ and } a_{k_1} = 2.$$

For the graph G_2,

$$P(G_2,\ x) = x^2 + 2x,\ n(G_2) = 4,\quad |\, D_{G_2} \,| = k_2 = 2 \text{ and } b_{k_2} = 1.$$

By Theorem 5.3.22,

$$P(G_1 * G_2,\ x) = x^4 + 2x^2 + 12x.$$

Corollary 5.3.24.

(1). $P(K_m * K_n,\ x) = x^{mn}$, *for* $m,\ n \geq 1$.

(2). $P(P_3 * K_n,\ x) = x^n + 2nx$, *for* $n \geq 1$.

Theorem 5.3.25. *Let* G_1 *be a graph of order* n_1 *with* $P(G_1, x) = \sum_{i=1}^{n_1} a_i x^i$ *and* G_2 *be a graph of order* n_2 *with* $P(G_2, x) = \sum_{j=1}^{n_2} b_j x^j$. *If* $\mid D_{G_1} \mid = k_1 (\neq 0)$ *and* $D_{G_2} = \emptyset$, *then* $P(G_1 * G_2,\ x) = (n_1 - k_1)n_2 x + k_1 \sum_{j=1}^{n_2} b_j x^j$. *Also if* $D_{G_1} = \emptyset$ *and* $\mid D_{G_2} \mid = k_2 (\neq 0)$, *then* $P(G_1 * G_2,\ x) = n_1(n_2 - k_2)x + k_2 \sum_{i=1}^{n_1} a_i x^i$.

Proof. Let $(u, v) \in V(G_1 * G_2)$. Suppose $\mid D_{G_1} \mid = k_1 (\neq 0)$ and $D_{G_2} = \emptyset$. Hence, if $u \notin D_{G_1}$, $E_{(u,v)} = \{(u, v)\}$. There are such $(n_1 - k_1)n_2$ vertices in $G_1 * G_2$. If $u \in D_{G_1}$, then $E_{(u,v)} = \{u\} \times E_v$ and hence $\mid E_{(u,v)} \mid = \mid E_v \mid$. Corresponding to these there are $k_1 b_j$, $E_{(u,v)}$'s of cardinality j in $\mathcal{A}_{G_1 * G_2}$, for $1 \leq j \leq n_2$. Therefore $P(G_1 * G_2,\ x) = (n_1 - k_1)n_2 x + k_1 \sum_{j=1}^{n_2} b_j x^j$. Suppose $D_{G_1} = \emptyset$ and $\mid D_{G_2} \mid = k_2 (\neq 0)$. By interchanging the roles of G_1 and G_2, $P(G_2 * G_1,\ x) = n_1(n_2 - k_2)x + k_2 \sum_{i=1}^{n_1} a_i x^i$. Since $G_1 * G_2 \cong G_2 * G_1$ and since isomorphic graphs have same CNP $P(G_1 * G_2,\ x) = n_1(n_2 - k_2)x + k_2 \sum_{i=1}^{n_1} a_i x^i$. □

Corollary 5.3.26.

(1). $P(K_m * C_n),\ x) = mnx$, *for* $m \geq 1,\ n > 3$.

(2). $P(P_3 * C_n,\ x) = 3nx$, *for* $n > 3$.

Chapter 6

Measurable Dominating Functions of Finite Graphs

6.1 Measurable Dominating Functions

The mathematical study of dominating sets in graphs began around 1960. The concept of domination was first studied by O. Ore and C. Berge. C. Berge in his book "Theory of Graphs and its Applications" [4], has introduced the concept of dominating sets and he called it as the "externally stable sets ".

Let $G = (V(G), E(G))$ be a graph. A function $f : V(G) \to \{0, 1\}$ is called a *dominating function* of G if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$ [14]. As we know every nonempty set X can be made into a measure space by taking the power set $\mathcal{P}(X)$ of X as the sigma algebra and the counting measure as the measure, the vertex set $V(G)$ of G can also be made into a measure space in a similar manner. Also as V(G) is finite, every function $f : V(G) \to \{0,1\}$ is simple [23]. With these notions we can redefine the dominating function as the function $f : V(G) \to \{0,1\}$ such that $\int\limits_{N[v]} f\ d\mu \geq 1$, for all $v \in V(G)$, where $\int\limits_{N[v]} f\ d\mu = \sum\limits_{u \in N[v]} f(u)$. It is further extended for functions from $V(G)$ to $[0,1]$.

Now at this stage we can think about the extension of this notion to an arbitrary measure space, $(V(G), \mathcal{R}, \mu)$. There arise two questions. The first question is that 'is for every $v \in V(G)$, $N[v] \in \mathcal{R}$ or not 'and the second is that 'though $\mathcal{R}$ contains all $N[v]$, is $\int\limits_{N[v]} f\ d\mu$ meaningful'. From the theory of measures, the integral of a real valued function is defined only if the function is measurable. Taking all these into consideration, we consider only those sigma algebra $\mathcal{R}$ which contains all $N[v]$, $v \in V(G)$ and only those functions defined on $V(G)$ which are measurable. To make this theory more effective we take the neighborhood sigma algebra, that is the sigma algebra generated by the collection $\{N[v] : v \in V(G)\}$ and functions $f : V(G) \to [0,1]$ which are measurable with respect to this sigma algebra.

In this chapter by a graph G, we mean a graph with the neighborhood sigma algebra $\mathcal{A}$ on the vertex set $V(G)$ and a measure μ on $\mathcal{A}$.
We define a measurable dominating function of a graph G as follows.

Definition 6.1.1. Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to [0,1]$ is called a measurable dominating function of G if the following conditions hold:

(i) f is measurable

(ii) $\displaystyle\int_{N[v]} f \, d\mu \geq 1$ for all $v \in V(G)$.

Remark 6.1.2. Let f be a measurable dominating function of a graph G. Then for all $v \in V(G)$, $f(N[v]) > 0$ and $\mu(N[v]) > 0$, where $f(N[v]) = \displaystyle\sum_{u \in N[v]} f(u)$.

Example 6.1.3. Consider the graph G in Figure 6.1.

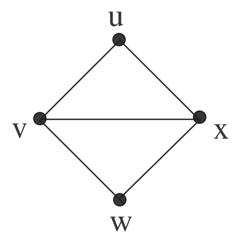

Figure 6.1: Graph G

For the graph G, $E_u = \{u\}$, $E_v = E_x = \{v, x\}$ and $E_w = \{w\}$.

Let μ be the measure defined on $V(G)$ by, $\mu(\{u\}) = 1/2$, $\mu(\{v, x\}) = 2$ and $\mu(\{w\}) = 1/3$.

Consider the function $f : V(G) \longrightarrow [0, 1]$, defined by, $f(u) = 1$ and $f(v) = f(w) = f(x) = 1/2$.

Then f is measurable, since f is constant on each E_v's.

$$\begin{aligned}\int\limits_{N[u]} f\ d\mu &= f(u)\mu(E_u) + f(v)\mu(E_v)\\ &= \tfrac{3}{2}\\ &> 1\end{aligned}$$

$$\begin{aligned}\int\limits_{N[v]} f\ d\mu &= f(u)\mu(E_u) + f(v)\mu(E_v) + f(w)\mu(E_w)\\ &= \tfrac{5}{3}\\ &> 1\end{aligned}$$

$$\begin{aligned}\int\limits_{N[w]} f\ d\mu &= f(w)\mu(E_w) + f(v)\mu(E_v)\\ &= \tfrac{7}{6}\\ &> 1\end{aligned}$$

Therefore, f is a measurable dominating function of G.

Theorem 6.1.4. *Let f and g be two measurable dominating functions of a graph G. Then all convex linear combinations of f and g are measurable dominating functions of G.*

Proof. Let $\alpha \in \mathbb{R}$ be such that $0 \leq \alpha \leq 1$ and let $h = \alpha f + (1-\alpha)g$. Since f and g are measurable functions, h is also measurable. Then for $v \in V(G)$,

$$\begin{aligned}
\int_{N[v]} h \, d\mu &= \int_{N[v]} [\alpha f + (1-\alpha) g] \, d\mu \\
&= \int_{N[v]} \alpha f \, d\mu + \int_{N[v]} (1-\alpha) g \, d\mu \\
&= \alpha \int_{N[v]} f \, d\mu + (1-\alpha) \int_{N[v]} g \, d\mu \\
&\geq \alpha + (1-\alpha) \\
&= 1
\end{aligned}$$

Therefore, h is a measurable dominating function of G. Hence the theorem. □

Definition 6.1.5. Let G be a graph with vertex set $V(G)$. A measurable dominating function f of G is said to be minimal if there does not exist a measurable dominating function g of G such that $g \leq f$ a.e and $g < f$ on some set of positive measure.

Next we derive a necessary and sufficient condition for a measurable dominating function to be minimal.

Theorem 6.1.6. *Let G be a graph with vertex set $V(G)$. A measurable dominating function f of G is minimal if and only if for every vertex $v \in V(G)$ with $\mu(E_v) > 0$ and $f > 0$ on E_v there exists a vertex $u \in N[v]$ with $\int_{N[u]} f \, d\mu = 1$.*

Proof. Let f be a minimal measurable dominating function of G. Suppose there exists a vertex $v \in V(G)$ with $\mu(E_v) > 0$ and $f > 0$ on E_v such that $\int_{N[u]} f \, d\mu > 1$ for all $u \in N[v]$.

Let $m = \min \{ \int\limits_{N[u]\setminus E_v} f \, d\mu : u \in N[v]\}$.

We consider the cases $m \geq 1$ and $m < 1$ separately.

Case 1. $m \geq 1$

Let $g = f - f\chi_{E_v}$, where χ_{E_v} denotes the characteristic function of E_v. That is for $w \in V(G)$,

$$g(w) = \begin{cases} 0 & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

Since the product and difference of measurable functions are measurable, the function g is measurable. Also $g(w) \leq f(w)$ for every $w \in V(G)$ and $g < f$ on E_v.

For $u \in V(G)$ with $u \in N[v]$,

$$\begin{aligned} \int\limits_{N[u]} g \, d\mu &= \int\limits_{E_v} g \, d\mu + \int\limits_{N[u]\setminus E_v} g \, d\mu \\ &= \int\limits_{N[u]\setminus E_v} g \, d\mu \\ &= \int\limits_{N[u]\setminus E_v} f \, d\mu \\ &\geq m \\ &\geq 1. \end{aligned}$$

Also, for $u \in V(G)$ with $u \notin N[v]$,

$$\begin{aligned} \int\limits_{N[u]} g \, d\mu &= \int\limits_{N[u]} f \, d\mu \\ &\geq 1. \end{aligned}$$

Therefore, g is also a measurable dominating function, a contradiction.

Case 2. $m < 1$

For $u \in N[v]$, $\int\limits_{N[u]} f \, d\mu > 1$ by the assumption.

Suppose $f = c$ on E_v. Then,

$$\begin{aligned}\int\limits_{N[u]} f \, d\mu &= \int\limits_{E_v} f \, d\mu + \int\limits_{N[u]\setminus E_v} f \, d\mu \\ &= c\mu(E_v) + \int\limits_{N[u]\setminus E_v} f \, d\mu\end{aligned}$$

Since $m < 1$, for at least one vertex $u \in N[v]$, $\int\limits_{N[u]\setminus E_v} f \, d\mu < 1$.

For such a u,

$$\begin{aligned}c\mu(E_v) &> 1 - \int\limits_{N[u]\setminus E_v} f \, d\mu \\ &> 0\end{aligned}$$

This implies,

$$c > \frac{1 - \int\limits_{N[u]\setminus E_v} f \, d\mu}{\mu(E_v)} = R_u, \text{ say}$$

Let $U = \{u \in N[v] : \int\limits_{N[u]\setminus E_v} f \, d\mu < 1\}$. Since $V(G)$ is finite, U is also finite.

Now choose d so that $c > d > R_u$ for all $u \in U$.

Let $h = f - (f - d)\chi_{E_v}$.

That is for $w \in V(G)$,

$$h(w) = \begin{cases} d & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

The function h is measurable, since it is the difference of the measurable functions f and $(f-d)\chi_{E_v}$. Also $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on E_v.

Let $u \in U$,

$$\begin{aligned} \int\limits_{N[u]} h \, d\mu &= \int\limits_{N[u]\setminus E_v} h \, d\mu + \int\limits_{E_v} h \, d\mu \\ &= \int\limits_{N[u]\setminus E_v} f \, d\mu + d\mu(E_v) \\ &> \int\limits_{N[u]\setminus E_v} f d\mu + R_u \mu(E_v) \\ &= \int\limits_{N[u]\setminus E_v} f \, d\mu \ + \left(1 - \int\limits_{N[u]\setminus E_v} f \, d\mu \right) \\ &= 1 \end{aligned}$$

Let $u \notin U$.

If $u \notin N[v]$,

$$\begin{aligned} \int\limits_{N[u]} h \, d\mu &= \int\limits_{N[u]} f \, d\mu \\ &\geq 1 \end{aligned}$$

If $u \in N[v]$, $\int\limits_{N[u]\setminus E_v} f \, d\mu \geq 1$

Therefore,

$$\begin{aligned}\int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]\setminus E_v} h\ d\mu + \int\limits_{E_v} h\ d\mu \\ &= \int\limits_{N[u]\setminus E_v} f\ d\mu + d\mu(E_v) \\ &> 1.\end{aligned}$$

Therefore, h is a measurable dominating function with $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on E_v, a contradiction.

Conversely, let f be a measurable dominating function of G such that for every vertex v with $\mu(E_v) > 0$ and $f > 0$ on E_v, there exists a vertex $u \in N[v]$ such that $\int\limits_{N[u]} f\ d\mu = 1$. Suppose f is not minimal. Then there exists a measurable dominating function l such that $l \leq f$ a.e and $l < f$ on a set of positive measure. So there exists a $v \in V(G)$ with $\mu(E_v) > 0$ and $l < f$ on E_v. This implies $f(v) > 0$. Now by assumption, there exists a $u \in V(G)$ with $u \in N[v]$ and $\int\limits_{N[u]} f\ d\mu = 1$.

Therefore,

$$\begin{aligned}1 &\leq \int\limits_{N[u]} l\ d\mu \\ &= \int\limits_{N[u]\setminus E_v} l\ d\mu + \int\limits_{E_v} l\ d\mu \\ &< \int\limits_{N[u]\setminus E_v} f\ d\mu + \int\limits_{E_v} f\ d\mu \\ &= 1, \text{ a contradiction.}\end{aligned}$$

Therefore, f is a minimal measurable dominating function. Hence the theorem.

□

6.2 Measurable k-Dominating Functions

The co-domain of the measurable dominating function is usually taken as [0,1]. In fact we can take any interval $[0, k]$ with k as a positive integer instead of [0,1]. In this section we prove that all the results proved in the case of [0,1] in section 6.1 are also true in the case of $[0, k]$. We call such dominating function as measurable k-dominating function.

Definition 6.2.1. Let G be a graph with vertex set $V(G)$. If k is a positive integer, a function $f : V(G) \longrightarrow [0, k]$ is called a measurable k-dominating function of G if the following conditions hold:

(i) f is measurable

(ii) $\displaystyle\int_{N[v]} f \, d\mu \geq k$ for all $v \in V(G)$.

Measurable 1-dominating functions are the measurable dominating functions.

Definition 6.2.2. Let G be a graph with vertex set $V(G)$. A measurable k-dominating function f of G is said to be minimal if there does not exist a measurable k- dominating function g of G such that $g \leq f$ a.e and $g < f$ on some set of positive measure.

Theorem 6.2.3. *Let G be a graph with vertex set $V(G)$. A measurable k-dominating function f of G is minimal if and only if for every vertex $v \in V(G)$ with $\mu(E_v) > 0$ and $f > 0$ on E_v there exists a vertex $u \in N[v]$ with* $\displaystyle\int_{N[u]} f \, d\mu = k.$

Proof. Let f be a minimal measurable k-dominating function of G. Suppose there exists a vertex v with $\mu(E_v) > 0$ and $f > 0$ on E_v such that $\int\limits_{N[u]} f \, d\mu > k$ for all $u \in N[v]$.

Let $m = \min \left\{ \int\limits_{N[u]\setminus E_v} f \, d\mu : u \in N[v] \right\}$.

We consider the cases $m \geq k$ and $m < k$ separately.

Case 1. $m \geq k$

Define $g : V(G) \longrightarrow [0, k]$ by

$$g(w) = \begin{cases} 0 & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

As the product and difference of measurable functions are measurable, the function $f - f\chi_{E_v} = g$ is measurable. Also $g(w) \leq f(w)$ for every $w \in V(G)$ and $g < f$ on E_v.

For $u \in V(G)$ with $u \in N[v]$,

$$\begin{aligned} \int\limits_{N[u]} g \, d\mu &= \int\limits_{E_v} g \, d\mu + \int\limits_{N[u]\setminus E_v} g \, d\mu \\ &= \int\limits_{N[u]\setminus E_v} g \, d\mu \\ &= \int\limits_{N[u]\setminus E_v} f \, d\mu \\ &\geq m \\ &\geq k. \end{aligned}$$

On the other hand for $u \in V(G)$ with $u \notin N[v]$,

$$\begin{aligned}\int_{N[u]} g \; d\mu &= \int_{N[u]} f \; d\mu \\ &\geq k\end{aligned}$$

Therefore g is also a measurable k- dominating function of G, a contradiction.

Case 2. $m < k$

For $u \in N[v]$, $\int\limits_{N[u]} f \; d\mu > k$ by the assumption.

Suppose $f = c$ on E_v. Then,

$$\begin{aligned}\int_{N[u]} f \; d\mu &= \int_{E_v} f \; d\mu + \int_{N[u]\setminus E_v} f \; d\mu \\ &= c\mu(E_v) + \int_{N[u]\setminus E_v} f \; d\mu \\ &> k.\end{aligned}$$

Since $m < k$, for at least one $u \in N[v]$, $\int\limits_{N[u]\setminus E_v} f \; d\mu < k$.

For such a u,

$$\begin{aligned}c\mu(E_v) &> k - \int_{N[u]\setminus E_v} f \; d\mu \\ &> 0.\end{aligned}$$

This implies $c > \dfrac{k - \int\limits_{N[u]\setminus E_v} f \; d\mu}{\mu(E_v)} = R_u(\text{say})$

Let $U = \{u \in N[v] : \int\limits_{N[u]\setminus E_v} f \; d\mu \; < k\}$. Since $V(G)$ is finite U is also finite. Now choose d so that $c > d > R_u$ for all $u \in U$.

Define $h : V(G) \longrightarrow [0, k]$ as,

$$h(w) = \begin{cases} d & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

The function h is measurable since it is the difference of the measurable functions f and $(f - d)\chi_{E_v}$. Also $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on E_v.

Let $u \in U$,

$$\begin{aligned} \int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]\setminus E_v} h\ d\mu + \int\limits_{E_v} h\ d\mu \\ &= \int\limits_{N[u]\setminus E_v} f\ d\mu + d\mu(E_v) \\ &> \int\limits_{N[u]\setminus E_v} f\ d\mu + R_u\mu(E_v) \\ &= \int\limits_{N[u]\setminus E_v} f\ d\mu + (k - \int\limits_{N[u]\setminus E_v} f\ d\mu) \\ &= k. \end{aligned}$$

Let $u \notin U$.

In this case we proceed as follows.

If $u \notin N[v]$,

$$\begin{aligned} \int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]} f\ d\mu \\ &\geq k \end{aligned}$$

If $u \in N[v]$ then $\displaystyle\int\limits_{N[u]\setminus E_v} f\ d\mu \geq k$.

Therefore,

$$\begin{aligned}
\int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]\setminus E_v} h\ d\mu + \int\limits_{E_v} h\ d\mu \\
&= \int\limits_{N[u]\setminus E_v} f\ d\mu + d\mu(E_v) \\
&> k.
\end{aligned}$$

Therefore h is a measurable k-dominating function with $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on E_v, a contradiction.

Conversely let f be a measurable k-dominating function of G such that for every vertex v with $\mu(E_v) > 0$ and $f > 0$ on E_v there exists a vertex $u \in N[v]$ such that $\int\limits_{N[u]} f\ d\mu = k$. Suppose f is not minimal. Then there exists a measurable k-dominating function l such that $l \leq f$ a.e and $l < f$ on a set of positive measure. So there exists a $v \in V(G)$ with $\mu(E_v) > 0$ and $l < f$ on E_v. This implies $f(v) > 0$. Now by our assumption there exists a $u \in V(G)$ with $u \in N[v]$ and $\int\limits_{N[u]} f\ d\mu = k$.
This implies

$$\begin{aligned}
k &\leq \int\limits_{N[u]} l\ d\mu \\
&= \int\limits_{N[u]\setminus E_v} l\ d\mu + \int\limits_{E_v} l\ d\mu \\
&< \int\limits_{N[u]\setminus E_v} f\ d\mu + \int\limits_{E_v} f\ d\mu \\
&= k, \text{ a contradiction.}
\end{aligned}$$

Therefore f is a minimal measurable k-dominating function of G.

Hence the theorem. □

6.3 Measurable Signed Dominating Functions

This section guarantees that the theory of measurable dominating functions will also work if the function takes negative values.

Definition 6.3.1. Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to [-1, 1]$ is called a measurable signed dominating function of G, if the following conditions hold:

(i) f is measurable

(ii) $\int\limits_{N[v]} f \, d\mu \geq 1$ for all $v \in V(G)$.

Example 6.3.2. Consider the graph G_1 in Figure 6.2.

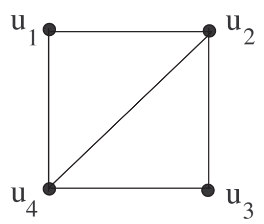

Figure 6.2: Graph G_1

In G_1, $E_{u_1} = \{u_1\}$, $E_{u_2} = E_{u_4} = \{u_2, u_4\}$, $E_{u_3} = \{u_3\}$.

Let μ be the measure defined on $V(G_1)$ by, $\mu(\{u_1\}) = \frac{1}{2}$, $\mu(\{u_2, u_4\}) = 2$, $\mu(\{u_3\}) = \frac{1}{2}$.

Consider the function $f : V(G_1) \longrightarrow [-1, 1]$ defined by,

$f(u_1) = -\frac{1}{4}$, $f(u_2) = f(u_4) = \frac{3}{4}$, $f(u_3) = -\frac{1}{2}$. Then f is measurable by Theorem

2.1.25.

$$\int\limits_{N[u_1]} f\ d\mu = -\tfrac{1}{4}\mu(E_{u_1}) + \tfrac{3}{4}\mu(E_{u_2}) = \tfrac{11}{8} > 1$$

$$\int\limits_{N[u_2]} f\ d\mu = -\tfrac{1}{4}\mu(E_{u_1}) + \tfrac{3}{4}\mu(E_{u_2}) + -\tfrac{1}{2}\mu(E_{u_3}) = \tfrac{9}{8} > 1$$

$$\int\limits_{N[u_3]} f\ d\mu = -\tfrac{1}{2}\mu(E_{u_3}) + \tfrac{3}{4}\mu(E_{u_2}) = \tfrac{5}{4} > 1$$

Hence f is a measurable signed dominating function of G_1 relative to μ.

But f is not a signed dominating function [23] of G_1 . Because $f(N[u_2]) = \sum\limits_{u_i \in N[u_2]} f(u_i) = -\dfrac{1}{4} + \dfrac{3}{4} - \dfrac{1}{2} + \dfrac{3}{4} = \dfrac{3}{4} < 1$.

Example 6.3.3. Consider the graph G_2 in Figure 6.3.

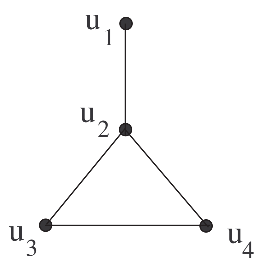

Figure 6.3: Graph G_2

Consider the function $f : V(G_2) \longrightarrow [-1, 1]$ defined by, $f(u_1) = 1$, $f(u_2) = \frac{3}{4}$, $f(u_3) = -\frac{1}{4}$, $f(u_4) = \frac{1}{2}$.

For the function f, $f(N[u_1]) = \frac{7}{4}$, $f(N[u_2]) = 2$, $f(N[u_3]) = 1$ and $f(N[u_4]) = 1$.

Hence f is a signed dominating function of G_2.

Since $N[u_3] = N[u_4]$ and $f(u_3) \neq f(u_4)$, f is not measurable by Corollary 2.1.26.

This is an example of a non measurable signed dominating function.

If f is signed dominating and measurable then f is measurable signed dominating relative to the counting measure restricted to $\mathcal{A}$.

Proposition 6.3.4. *Let G be a graph and μ be the counting measure restricted to $\mathcal{A}$. Then a function $f : V(G) \longrightarrow [-1, 1]$ is signed dominating and measurable if and only if it is a measurable signed dominating function of G relative to μ.*

Definition 6.3.5. Let G be a graph with vertex set $V(G)$. A measurable signed dominating function f of G is said to be minimal if there does not exist a measurable signed dominating function g of G such that $g \leq f$ a.e and $g < f$ on some set of positive measure.

Theorem 6.3.6. *Let G be a graph with vertex set $V(G)$. A measurable signed dominating function f of G is minimal if and only if for every vertex $v \in V(G)$ with $\mu(E_v) > 0$ and $f > -1$ on E_v, there exists a vertex $u \in N[v]$ such that* $\int\limits_{N[u]} f \, d\mu = 1$.

Proof. Let f be a minimal measurable signed dominating function of G. Suppose there exists a vertex v with $\mu(E_v) > 0$ and $f > -1$ on E_v such that $\int\limits_{N[u]} f \, d\mu > 1$ for all $u \in N[v]$.

Then, for all $u \in N[v]$, $\int\limits_{N[u]} f \, d\mu = 1 + d_u,$ where $d_u > 0$.

Since, for $u \in N[v]$

$$\int\limits_{N[u]} f \, d\mu = \int\limits_{E_v} f \, d\mu + \int\limits_{N[u]\setminus E_v} f \, d\mu$$

we have,

$$c\mu(E_v) + \int\limits_{N[u]\setminus E_v} f \, d\mu = 1 + d_u$$

where c is the value of f on E_v.

Note that the $min\{d_u; u \in N[v]\} > 0$. Therefore $k = \frac{1}{\mu(E_v)} min\{d_u; u \in N[v]\} > 0$. Then $c - k < c$. Choose a real number $d \in [-1, 1]$ such that $c - k \leq d < c$ and define, $g : V(G) \to [-1, 1]$ as

$$g(w) = \begin{cases} d & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

Then $g = f - (f - d)\chi_{E_v}$ is measurable. Also $g(w) \leq f(w)$ for all $w \in V(G)$ and $g < f$ on E_v. We assert that g is measurable signed dominating.

Let $u \in N[v]$.

$$\begin{aligned}
\int_{N[u]} g \; d\mu &= \int_{E_v} g \; d\mu + \int_{N[u]\setminus E_v} g \; d\mu \\
&= d\mu(E_v) + \int_{N[u]\setminus E_v} f \; d\mu \\
&= d\mu(E_v) + 1 + d_u - c\mu(E_v) \\
&\geq 1 + (d + k - c)\mu(E_v) \\
&= 1 + [d - (c - k)]\mu(E_v) \\
&\geq 1.
\end{aligned}$$

Let $u \in V(G) \setminus N[v]$. Then,

$$\int_{N[u]} g \; d\mu = \int_{N[u]} f \; d\mu \geq 1.$$

Therefore g is a measurable signed dominating function of G, a contradiction.

Conversely, let f be a measurable signed dominating function of G such that for every vertex v with $\mu(E_v) > 0$ and $f > -1$ on E_v there exists a vertex $u \in N[v]$ such that $\int_{N[u]} f \; d\mu = 1$. Suppose f is not minimal. Then there exists a measurable signed dominating function g of G such that $g \leq f$ a.e and $g < f$ on a set of positive measure. So there exists $v \in V(G)$ with $\mu(E_v) > 0$ and $g < f$ on

E_v. This implies $f(v) > -1$. So $f > -1$ on E_v, by Theorem 2.1.25. Therefore there exists a $u \in N[v]$ such that $\int\limits_{N[u]} f \, d\mu = 1$.

This implies,

$$\begin{aligned} 1 &\leq \int\limits_{N[u]} g \, d\mu \\ &= \int\limits_{E_v} g \, d\mu + \int\limits_{N[u]\setminus E_v} g \, d\mu \\ &< \int\limits_{E_v} f d\mu + \int\limits_{N[u]\setminus E_v} f \, d\mu \\ &= \int\limits_{N[u]} f \, d\mu \\ &= 1, \text{ a contradiction.} \end{aligned}$$

Therefore f is a minimal measurable signed dominating function of G. Hence the theorem. □

6.4 Measure on Graph Products

In this section we define a measure on vertex sets of lexicographic product, tensor product, Cartesian product, normal product and co-normal product. Also we check whether the function f defined from the vertex sets of these products to $[0, 1]$, by $f((u, v)) = f_1(u)f_2(v)$, where f_1 and f_2 are measurable dominating functions of component graphs, is a measurable dominating function with respect to this measure or not.

Let $G_1 \triangle G_2$ be an arbitrary graph product such that $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1 \triangle G_2}$

and $E^{G_1 \triangle G_2}_{(u,v)} \subseteq E^{G_1}_u \times E^{G_2}_v$, for $(u,v) \in V(G_1 \triangle G_2)$.

Let $(u,v) \in V(G_1 \triangle G_2)$. If $(x,y) \in E^{G_1}_u \times E^{G_2}_v$, $x \in E^{G_1}_u$ and $y \in E^{G_2}_v$. This implies $E^{G_1 \triangle G_2}_{(x,y)} \subseteq E^{G_1}_x \times E^{G_2}_y = E^{G_1}_u \times E^{G_2}_v$. Hence $E^{G_1}_u \times E^{G_2}_v$ can be written as countable disjoint union of the collection $\{E^{G_1 \triangle G_2}_{(x,y)} : (x,y) \in E^{G_1}_u \times E^{G_2}_v\}$.

In this case, if μ_1 is a measure on $\mathcal{A}_{G_1}$ and μ_2 is a measure on $\mathcal{A}_{G_2}$, we can extend the product measure $\mu_1 \times \mu_2$ on $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ to $\mathcal{A}_{G_1 \triangle G_2}$ in a natural way. More explicitly, if

$\mathscr{F}_{(u,v)} := \{E_{(x,y)} : (x,y) \in E_u \times E_v\},$ for $(u,v) \in V(G_1 \triangle G_2)$,

then for any $(x,y) \in V(G_1 \triangle G_2)$, define μ as

$$\begin{aligned} \mu(E_x \times E_y) &= \mu_1(E_x)\mu_2(E_y) \\ \mu(E_{(x,y)}) &= \frac{1}{|\mathscr{F}_{(x,y)}|}\mu(E_x \times E_y) \end{aligned} \tag{6.1}$$

and extend μ to be a measure on $\mathcal{A}_{G_1 \triangle G_2}$.

Then for $A \in \mathcal{A}_{G_1}$ and $B \in \mathcal{A}_{G_2}$,

$$\begin{aligned} \mu(A \times B) &= \sum_{\substack{u \in A \\ v \in B}} \mu(E_u \times E_v) \\ &= \sum_{u \in A} \mu_1(E_u) \sum_{v \in B} \mu_2(E_v) \\ &= \mu_1(A)\mu_2(B) \end{aligned}$$

where the sums are taken over distinct $E_u \times E_v$'s, distinct E_u's and distinct E_v's. Therefore the measure μ defined on $\mathcal{A}_{G_1 \triangle G_2}$ agrees with the product measure $\mu_1 \times \mu_2$ on the collection $\{A \times B : A \in \mathcal{A}_{G_1},\ B \in \mathcal{A}_{G_2}\}$ of generators of $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$

Hereafter, whenever $G_1 \triangle G_2$ is a graph product such that $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2} \subseteq \mathcal{A}_{G_1 \triangle G_2}$ and $E_{(u,v)}^{G_1 \triangle G_2} \subseteq E_u^{G_1} \times E_v^{G_2}$, for $(u,v) \in V(G_1 \triangle G_2)$, we use the measure defined in Equation 6.1 as the measure of the product. For example lexicographic product, tensor product, Cartesian product, normal product and co-normal product have these properties.

There are graph products which do not have these properties, for example homomorphic product. In this case the natural extension of product measure of $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ to $\mathcal{A}_{G_1 \ltimes G_2}$ is not possible.

Let G_1 and G_2 be two graphs and f_1 and f_2 be measurable dominating functions of G_1 and G_2 respectively. In the following subsections, we define a function f on $V(G_1 \triangle G_2)$, by $f((u,v)) = f_1(u)f_2(v)$ and check whether f is a measurable dominating function of $G_1 \triangle G_2$ or not. We also examine whether the minimality of f_1 and f_2 implies that of f.

6.4.1 Lexicographic Product

Theorem 6.4.1. *Let G_1 and G_2 be two graphs and f_1 and f_2 be measurable dominating functions of G_1 and G_2 respectively. Then the function f defined on $V(G_1[G_2])$, by $f((u,v)) = f_1(u)f_2(v)$ is a measurable dominating function of $G_1[G_2]$.*

Proof. By Proposition 3.2.8, f is a measurable function defined from $V(G_1[G_2])$ to $[0,1]$. Let $(u,v) \in V(G_1[G_2])$. Then $N[(u,v)] = (N(u) \times V(G_2)) \bigcup (\{u\} \times$

$N[v]$). Hence $N[u] \times N[v] \subseteq N[(u,v)]$. Let $N[u] = \bigcup_{i=1}^{m} E_{u_i}$ and $N[v] = \bigcup_{j=1}^{n} E_{v_i}$, where $u_i \in N[u]$ and each E_{u_i}'s are distinct for $1 \leq i \leq m$ and $v_i \in N[v]$ and each E_{v_i}'s are distinct for $1 \leq j \leq n$. It is clear that f is constant on $E_{u_i} \times E_{v_j}$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Therefore.

$$\begin{aligned}
\int\limits_{N[(u,v)]} f \, d\mu &\geq \int\limits_{N[u] \times N[v]} f \, d\mu \\
&= \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} f(u_i, v_j) \mu(E_{u_i} \times E_{v_j}) \\
&= \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} f_1(u_i) f_2(v_j) \mu_1(E_{u_i}) \mu_2(E_{v_j}) \\
&= \sum_{1 \leq i \leq m} f_1(u_i) \mu_1(E_{u_i}) \sum_{1 \leq j \leq n} f_2(v_j) \mu_2(E_{v_j}) \\
&\geq 1
\end{aligned}$$

Therefore f is a measurable dominating function of $G_1[G_2]$. □

The functions f_1 and f_2 in Example 6.4.2 are minimal measurable dominating functions, but their product $f = f_1 f_2$ is not minimal.

Example 6.4.2. Consider the graphs G_1, G_2 and $G_1[G_2]$ given in Figure 3.1.

Let $f_1 : V(G_1) \longrightarrow [0,1]$ be defined by $f_1(u_1) = f_1(u_2) = \frac{1}{2}$. Let $f_2 : V(G_2) \longrightarrow [0,1]$ be defined by $f_2(v_1) = f_2(v_2) = f_2(v_3) = \frac{1}{2}$.

Let the measures μ_1 on $\mathcal{A}_{G_1}$ and μ_2 on $\mathcal{A}_{G_2}$ be defined as follows. $\mu_1(E_{u_1}) = 2$

and $\mu_2(E_{v_1}) = \mu_2(E_{v_2}) = \mu_2(E_{v_3}) = 1$.

$$\begin{aligned}\int\limits_{N[u_1]} f_1 \ d\mu_1 = \int\limits_{N[u_2]} f_1 \ d\mu_1 &= f_1 \ (u_1)\mu_1(E_{u_1}) \\ &= 1.\end{aligned}$$

$$\begin{aligned}\int\limits_{N[v_1]} f_2 \ d\mu_2 &= f_2(v_1)\mu_2(E_{v_1}) + f_2(v_2)\mu_2(E_{v_2}) \\ &= 1.\end{aligned}$$

$$\begin{aligned}\int\limits_{N[v_2]} f_2 \ d\mu_2 &= f_2(v_1)\mu_2(E_{v_1}) + f_2(v_2)\mu_2(E_{v_2}) + f_2(v_3)\mu_2(E_{v_3}) \\ &= \tfrac{3}{2}\end{aligned}$$

$$\begin{aligned}\int\limits_{N[v_3]} f_2 \ d\mu_2 &= f_2(v_2)\mu_2(E_{v_2}) + f_2(v_3)\mu_2(E_{v_3}) \\ &= 1.\end{aligned}$$

By Theorem 6.1.6, f_1 and f_2 are minimal measurable dominating functions of G_1 and G_2 respectively.

$E_{(u_1,v_1)} = \{(u_1, v_1)\}$, $E_{(u_2,v_1)} = \{(u_2, v_1)\}$ and $\mathscr{F}_{(u_1,v_1)} = \mathscr{F}_{(u_2,v_1)} = \{E_{(u_1,v_1)}, E_{(u_2,v_1)}\}$.

Therefore,

$$\begin{aligned}\mu(E_{(u_1,v_1)}) &= \tfrac{1}{2}\mu(E_{u_1} \times E_{v_1}) \\ &= 1\end{aligned}$$

$$\begin{aligned}\mu(E_{(u_2,v_1)}) &= \tfrac{1}{2}\mu(E_{u_2} \times E_{v_1}) \\ &= 1\end{aligned}$$

$E_{(u_1,v_2)} = E_{(u_2,v_2)} = \{(u_1, v_2), (u_2, v_2)\}$.

Therefore,

$$\begin{aligned}\mu(E_{(u_1,v_2)}) &= \mu(E_{(u_2,v_2)}) \\ &= \mu(E_{u_1} \times E_{v_2}) \\ &= 2\end{aligned}$$

Similarly we get,

$$\begin{aligned}\mu(E_{(u_1,v_3)}) &= \tfrac{1}{2}\mu(E_{u_1} \times E_{v_3}) \\ &= 1\end{aligned}$$

$$\begin{aligned}\mu(E_{(u_2,v_3)}) &= \tfrac{1}{2}\mu(E_{u_2} \times E_{v_3}) \\ &= 1\end{aligned}$$

Let $f := f_1 f_2$.

Then, $f((u_i, v_j)) = \frac{1}{4}$, $i = 1, 2$ and $j = 1, 2, 3$.

$$\begin{aligned}\int\limits_{N[(u_1,v_1)]} f\, d\mu &= f((u_1, v_1))\mu(E_{u_1,v_1}) + f((u_2, v_1))\mu(E_{(u_2,v_1)}) \\ &\quad + f((u_1, v_2))\mu(E_{u_1,v_2}) + f((u_2, v_3))\mu(E_{u_2,v_3}) \\ &= \tfrac{5}{4}\end{aligned}$$

Similarly we get,

$$\int\limits_{N[(u_2,v_1)]} f\ d\mu = \tfrac{5}{4}$$
$$\int\limits_{N[(u_1,v_3)]} f\ d\mu = \tfrac{5}{4}$$
$$\int\limits_{N[(u_2,v_3)]} f\ d\mu = \tfrac{5}{4}$$

$$\begin{aligned}\int\limits_{N[(u_1,v_2)]} f\ d\mu &= f((u_1,v_2))\mu(E_{u_1,v_2}) + f((u_1,v_1))\mu(E_{(u_1,v_1)}) \\ &\quad + f((u_2,v_1))\mu(E_{u_2,v_1}) + f((u_1,v_3))\mu(E_{u_1,v_3}) \\ &\quad + f((u_2,v_3))\mu(E_{u_2,v_3}) \\ &= \tfrac{3}{2}\end{aligned}$$

Since $N[(u_1,v_2)] = N[(u_2,v_2)]$, $\displaystyle\int\limits_{N[(u_2,v_2)]} f\ d\mu = \frac{3}{2}$.

Therefore f is measurable dominating function of $G_1[G_2]$. But f is not a minimal measurable dominating function, by Theorem 6.1.6.

6.4.2 Tensor Product

Let f_1 be a measurable dominating function of the graph G_1 and f_2 be a measurable dominating function of the graph G_2, then the function f defined by $f((x,y)) = f_1(x)f_2(y)$, $(x,y) \in G_1 \otimes G_2$, need not be a measurable dominating function of $G_1 \otimes G_2$.

Example 6.4.3. Consider the graphs G_1, G_2 and $G_1 \otimes G_2$ given in Figure 3.3.

Let $f_1 : V(G_1) \longrightarrow [0,1]$ be defined by $f_1(u_1) = f_1(u_2) = \frac{1}{2}$ and

$f_2 : V(G_2) \longrightarrow [0,1]$ be defined by $f_2(v_1) = f_2(v_2) = f_2(v_3) = \frac{1}{2}$.

Let the measures μ_1 on $\mathcal{A}_{G_1}$ and μ_2 on $\mathcal{A}_{G_2}$ be defined by $\mu_1(E_{u_1}) = 2$ and $\mu_2(E_{v_1}) = \mu_2(E_{v_2}) = \mu_2(E_{v_3}) = 1$.

In Example 6.4.2, it is proved that f_1 is a measurable dominating function of G_1 and f_2 is a measurable dominating function of G_2.

$E_{(u_1,v_1)} = \{(u_1, v_1)\}$, $E_{(u_2,v_1)} = \{(u_2, v_1)\}$, $\mu(E_{u_1} \times E_{v_1}) = 2$ and $|\mathscr{F}_{(u_1,v_1)}| = 2$.

Therefore,

$$\begin{aligned} \mu(E_{(u_1,v_1)}) &= \tfrac{1}{2}\mu(E_{u_1} \times E_{v_1}) \\ &= 1 \end{aligned}$$

$E_{(u_1,v_2)} = \{(u_1, v_2)\}$, $E_{(u_2,v_2)} = \{(u_2, v_2)\}$, $\mu(E_{u_2} \times E_{v_2}) = 2$, and $|\mathscr{F}_{(u_2,v_2)}| = 2$.

Therefore,

$$\begin{aligned} \mu(E_{(u_2,v_2)}) &= \tfrac{1}{2}\mu(E_{u_1} \times E_{v_1}) \\ &= 1 \end{aligned}$$

$f((u_i, v_j)) = \frac{1}{4}$, $i = 1, 2$ and $j = 1, 2, 3$.

$$\begin{aligned} \int\limits_{N[(u_1,v_1)]} f \, d\mu &= f((u_1, v_1))\mu(E_{u_1,v_1}) + f((u_2, v_2))\mu(E_{(u_2,v_2)}) \\ &= \tfrac{1}{2} < 1 \end{aligned}$$

Hence f is not a measurable dominating function of $G_1 \otimes G_2$.

6.4.3 Cartesian Product

In the case of Cartesian product, measurability of functions does not behave smoothly in the following sense. Let f_1 be a measurable dominating function of G_1 and f_2 be a measurable dominating function of G_2. Then $f = f_1 f_2$ need not be a measurable dominating function of $G_1 \times G_2$.

Example 6.4.4. Consider the graphs G_1, G_2 and $G_1 \times G_2$ given in Figure 3.4.

Let μ_1, μ_2, f_1 and f_2 be as in Example 6.4.2. Then f_1 is a measurable dominating function of G_1 and f_2 is a measurable dominating function of G_2. $E_{(u_1,v_1)} = \{(u_1, v_1)\}$, $E_{(u_2,v_1)} = \{(u_2, v_1)\}$, $\mu(E_{u_1} \times E_{v_1}) = 2$ and $|\mathscr{F}_{(u_1,v_1)}| = 2$. Therefore,

$$\begin{aligned} \mu(E_{(u_1,v_1)}) &= \tfrac{1}{2}\mu(E_{u_1} \times E_{v_1}) \\ &= 1 \end{aligned}$$

Similarly, we get $\mu(E_{(u_2,v_1)}) = \mu(E_{(u_1,v_2)}) = 1$.

Let $f := f_1 f_2$.

Then, $f((u_i, v_j)) = \frac{1}{4}$, $i = 1, 2$ and $j = 1, 2, 3$.

$$\begin{aligned} \int\limits_{N[(u_1,v_1)]} f \; d\mu &= f((u_1, v_1))\mu(E_{(u_1,v_1)}) + f((u_2, v_1))\mu(E_{(u_2,v_1)}) + \\ & \quad f((u_1, v_2))\mu(E_{(u_1,v_2)}) \\ &= \tfrac{3}{4} < 1 \end{aligned}$$

Hence f is not a measurable dominating function of $G_1 \times G_2$.

6.4.4 Normal Product

Let G_1 and G_2 be two graphs. Let μ_1 be the measure on $\mathcal{A}_{G_1}$ and μ_2 be the measure on $\mathcal{A}_{G_2}$. We have $\mathcal{A}_{G_1 \boxtimes G_2} = \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ and $E_{(x,y)} = E_x \times E_y$, for $(x, y) \in V(G_1 \boxtimes G_2)$. Therefore the measure defined in Equation 6.1 on $\mathcal{A}_{G_1 \boxtimes G_2}$ coincides with the product measure of $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$.

Theorem 6.4.5. *Let G_1 and G_2 be two graphs and f_1 and f_2 be measurable dominating functions of G_1 and G_2 respectively. Then the function f defined on $V(G_1 \boxtimes G_2)$ by $f((u, v)) = f_1(u)f_2(v)$ is a measurable dominating function of $G_1 \boxtimes G_2$.*

Proof. By Proposition 3.2.28, f is a measurable function defined from $V(G_1 \boxtimes G_2)$ to $[0, 1]$. Let $(u, v) \in V(G_1 \boxtimes G_2)$. Let $N[u] = \bigcup_{i=1}^{m} E_{u_i}$ and $N[v] = \bigcup_{j=1}^{n} E_{v_j}$, where $u_i \in N[u]$ and E_{u_i}'s are distinct for $1 \leq i \leq m$ and $v_i \in N[v]$ and E_{v_i}'s are distinct for $1 \leq j \leq n$. It is clear that f is constant on $E_{u_i} \times E_{v_j}$, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Therefore,

$$
\begin{aligned}
\int\limits_{N[(u,v)]} f \, d\mu &= \int\limits_{N[u]\times N[v]} f \, d\mu \\
&= \sum_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} f(u_i, v_j)\mu(E_{u_i} \times E_{v_j}) \\
&= \sum_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} f_1(u_i)f_2(v_j)\mu_1(E_{u_i})\mu_2(E_{v_j})
\end{aligned}
$$

$$\begin{aligned} &= \sum_{1\le i\le m} f_1(u_i)\mu_1(E_{u_i}) \sum_{1\le j\le n} f_2(v_j)\mu_2(E_{v_j}) \\ &\geq 1 \end{aligned}$$

Hence f is a measurable dominating function of $G_1 \boxtimes G_2$. □

Let G_1 and G_2 be two graphs and f_1 and f_2 be minimal measurable dominating functions of G_1 and G_2 respectively. Next we check whether f_1f_2 is a minimal measurable dominating function of $G_1 \boxtimes G_2$.

Theorem 6.4.6. *Let G_1 and G_2 be two graphs and f_1 and f_2 are minimal measurable dominating functions of G_1 and G_2 respectively. Then $f := f_1f_2$ is a minimal measurable dominating function of $G_1 \boxtimes G_2$.*

Proof. By Theorem 6.4.5, f is a measurable dominating function of $G_1 \boxtimes G_2$. Let $(u,v) \in V(G_1 \boxtimes G_2)$ be such that $f(u,v) > 0$. Then $f_1(u) > 0$ and $f_2(v) > 0$. Since f_1 and f_2 are minimal measurable dominating functions of G_1 and G_2 respectively, by Theorem 6.1.6, there exit $u' \in N[u]$ and $v' \in N[v]$ such that $\int\limits_{N[u']} f_1 \, d\mu_1 = 1$ and $\int\limits_{N[v']} f_2 \, d\mu_2 = 1$. Then $(u',v') \in N[u] \times N[v] = N[(u,v)]$. By imitating the procedure used in Theorem 6.4.5, we get

$$\begin{aligned} \int\limits_{N[(u',v')]} f \, d\mu &= \int\limits_{N[u']\times N[v']} f \, d\mu \\ &= \int\limits_{N[u']} f_1 \, d\mu_1 \int\limits_{N[v']} f_2 \, d\mu_2 \\ &= 1 \end{aligned}$$

Hence by Theorem 6.1.6, f is a minimal measurable dominating function of $G_1 \boxtimes G_2$. $\square$

6.4.5 Co-normal Product

Theorem 6.4.7. *Let G_1 and G_2 be two graphs and f_1 and f_2 be measurable dominating functions of G_1 and G_2 respectively. Then the function f defined on $V(G_1 * G_2)$ by $f((u,v)) = f_1(u)f_2(v)$ is a measurable dominating function of $G_1 * G_2$.*

Proof. Let $(u,v) \in V(G_1[G_2])$. Then $N[(u,v)] = \{(u,v)\} \bigcup (N(u) \times V(G_2)) \bigcup (V(G_1) \times N(v))$. Hence $N[u] \times N[v] \subseteq N[(u,v)]$. Thus the theorem can be proved in a way similar to that of Theorem 6.4.1. $\square$

Consider the graphs G_1, G_2 and $G_1 * G_2$ given in the Figure 3.5. Let μ_1, μ_2, f_1 and f_2 be as in Example 6.4.2. Then f_1 is a minimal measurable dominating function of G_1 and f_2 is a minimal measurable dominating function of G_2. Also for these graphs G_1 and G_2, $G_1 * G_2 \cong G_1[G_2]$. In Example 6.4.2, it is proved that $f_1 f_2$ is not a minimal measurable dominating function of $G_1[G_2]$. Therefore $f_1 f_2$ is not a minimal measurable dominating function of $G_1 * G_2$. Thus in the case of co-normal product the minimality of f_1 and f_2 does not imply the minimality of $f_1 f_2$.

6.5 x-section and y-section of Measurable Functions Defined on Graph Products

In this section we are considering the measurability of x-sections and y-sections of measurable functions defined on the vertex set of different types of graph products.

6.5.1 Lexicographic Product

Theorem 6.5.1. *Let G_1 and G_2 be two graphs. If a function f defined on $V(G_1[G_2])$ is measurable, then for each $x \in V(G_1)$, f_x is measurable.*

Proof. Let $x \in V(G_1)$ and $y_1, y_2 \in V(G_2)$ be such that $N[y_1] = N[y_2]$. To prove f_x is measurable, we have to prove that $f_x(y_1) = f_x(y_2)$, by Theorem 2.1.25. That is $f((x, y_1)) = f((x, y_2))$. For this it is enough to prove that $N[(x, y_1)] = N[(x, y_2)]$, by Theorem 2.1.25.

$$\begin{aligned} N[(x, y_1)] &= (N(x) \times V(G_2)) \bigcup (\{x\} \times N[y_1]) \\ &= (N(x) \times V(G_2)) \bigcup (\{x\} \times N[y_2]) \\ &= N[(x, y_2)] \end{aligned}$$

Hence f_x is measurable. $\square$

Remark 6.5.2. In the case of lexicographic product $G_1[G_2]$ of two graphs G_1 and G_2, though the x-sections f_x of a measurable function f defined on

$V(G_1) \times V(G_2)$ are measurable for all $x \in V(G_1)$, the y-section f^y need not be measurable for $y \in V(G_2)$.

For example consider the graphs G_1 and G_2 given below.

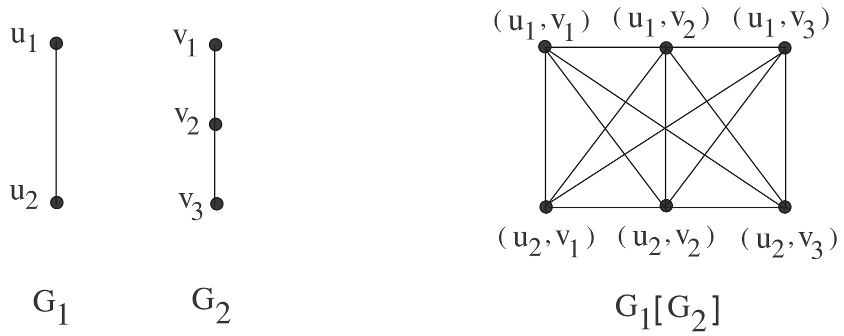

Figure 6.4: The lexicographic product $G_1[G_2]$ of two graphs G_1 and G_2.

Define $f : V(G_1[G_2]) \to [0,1]$ as,

$f((u_1,v_1)) = \frac{1}{2}$, $f((u_2,v_1)) = 1$, $f((u_1,v_2)) = f((u_2,v_2)) = \frac{1}{2}$, $f((u_1,v_3)) = \frac{1}{4}$, $f((u_2,v_3)) = 1$. Then f is measurable.

Consider,

$$f^{v_1} : V(G_1) \to [0,1]$$

In G_1,

$$N[u_1] = N[u_2]$$

But,

$$\begin{aligned} f^{v_1}(u_1) &= f((u_1,v_1)) = \tfrac{1}{2} \\ \text{and} \quad f^{v_1}(u_2) &= f((u_2,v_1)) = 1. \end{aligned}$$

Hence by Corollary 2.1.26, f^{v_1} is not measurable.

6.5.2 Cartesian Product

In the case of Cartesian product the measurability condition is weaker than that of the lexicographic product in the sense that if G_1 and G_2 are two graphs and f is a measurable function defined on $V(G_1 \times G_2)$, then the functions f_x and f^y need not be measurable for $x \in V(G_1)$ and for $y \in V(G_2)$.
For example consider the graphs G_1 and G_2 given below.

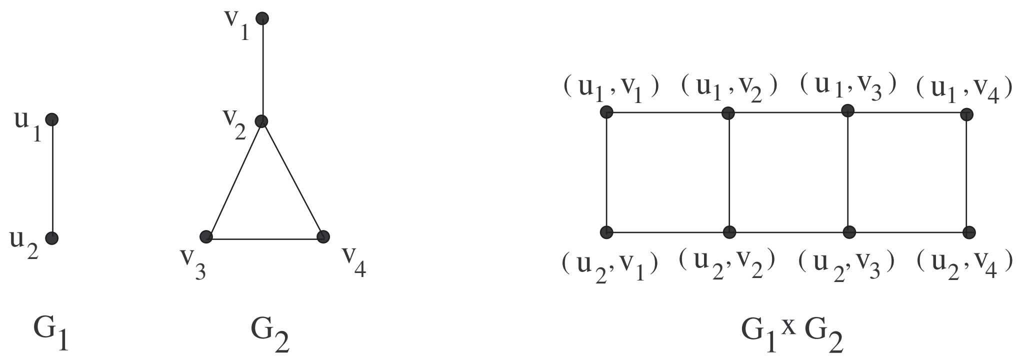

Figure 6.5: The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2.

Consider the function $f : V(G_1 \times G_2) \to [0, 1]$ defined by,
$f((u_1, v_1)) = f((u_2, v_2)) = f((u_2, v_3)) = 1$, $f((u_1, v_2)) = f((u_1, v_3)) = f((u_2, v_4)) = \frac{1}{2}$, $f((u_1, v_4)) = \frac{1}{4}$, $f((u_2, v_1)) = \frac{1}{8}$.
The function f is measurable.

In G_2, $N[v_3] = N[v_4]$ but $f_{u_1}(v_3) \neq f_{u_1}(v_4)$. Hence by Corollary 2.1.26, f_{u_1} is not measurable. In G_1, $N[u_1] = N[u_2]$ but $f^{v_1}(u_1) \neq f^{v_1}(u_2)$. Again by Corollary 2.1.26, f^{v_1} is not measurable.

Thus in the case of Cartesian product, measurability of a function f defined on the vertex set of $V(G_1 \times G_2)$ does not imply measurability of its x-sections and y-sections.

6.5.3 Tensor Product

Likewise in Cartesian product the situation of measurability is not fair in the case of tensor product. Let G_1 and G_2 be two graphs. The measurability of a function f defined on $V(G_1 \otimes G_2)$ will not imply that of f_x and f^y for $x \in V(G_1)$ and $y \in V(G_2)$.

For example consider the graphs given below.

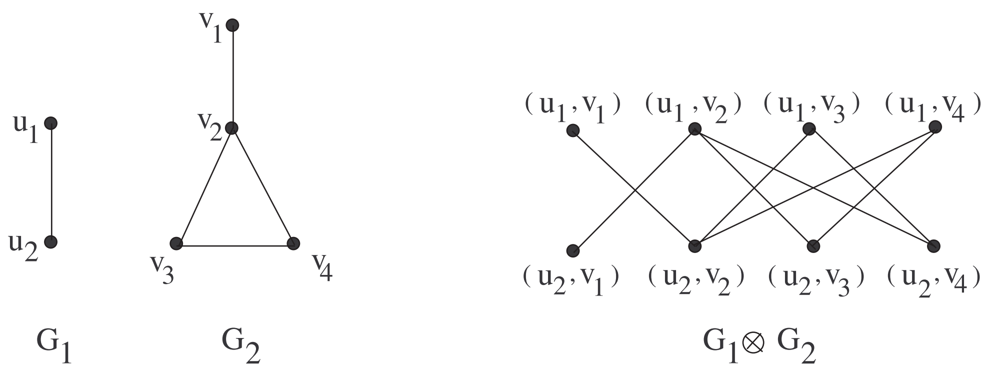

Figure 6.6: The tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2.

Consider the function $f : V(G_1 \otimes G_2) \to [0, 1]$ defined by,

$f((u_1, v_1)) = f((u_1, v_2)) = f((u_1, v_4)) = f((u_2, v_3)) = \frac{1}{2}$, $f((u_1, v_3)) =$ $f((u_2, v_4)) = 1$, $f((u_2, v_1)) = \frac{1}{4}$, $f((u_2, v_2)) = \frac{1}{8}$.

The function f is measurable by Theorem 2.1.25. But f_{u_1} and f^{v_1} are not measurable.

In G_2, $N[v_3] = N[v_4]$, but $f_{u_1}(v_3) \neq f_{u_1}(v_4)$. Therefore f_{u_1} is not measurable. In G_1, $N[u_1] = N[u_2]$, but $f^{v_1}(u_1) \neq f^{v_1}(u_2)$. Therefore f^{v_1} is not measurable.

6.5.4 Co-normal Product

The measurability of a function defined on the vertex set of the co-normal product of two graphs does not imply the measurability of its x-sections and y-sections. For example consider the graphs given below.

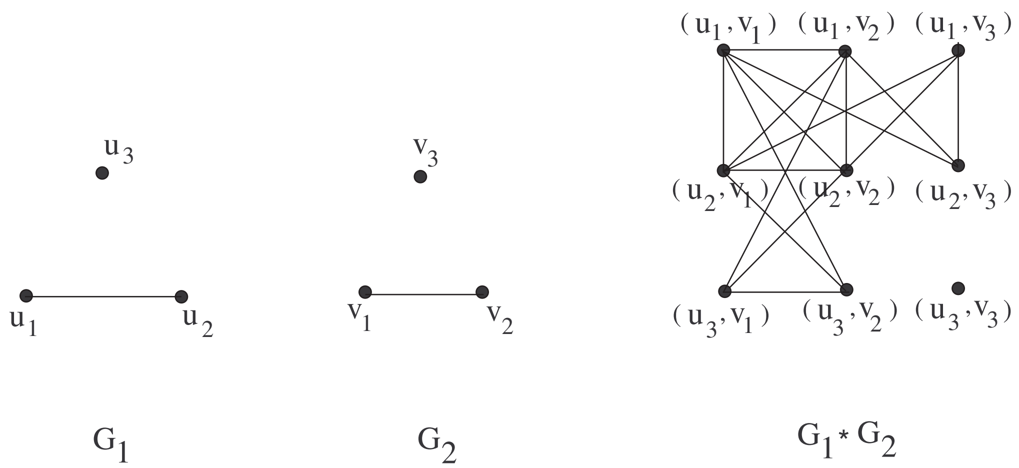

Figure 6.7: The co-normal product $G_1 * G_2$ of two graphs G_1 and G_2.

Let $f : V(G_1 * G_2) \to [0, 1]$ be defined by,

$f((u_1, v_1)) = f((u_1, v_3)) = f((u_2, v_2)) = f((u_2, v_3)) = f((u_3, v_1)) = f((u_3, v_2)) = f((u_3, v_3)) = 1$, $f((u_1, v_2)) = f((u_2, v_1)) = 2$.

The neighborhood sigma algebra of $G_1 * G_2$ is $\mathcal{P}(V(G_1 * G_2))$. Hence all functions from $V(G_1 * G_2)$ are measurable. In particular f is measurable.

In G_2, $N[v_1] = N[v_2]$, but $f_{u_1}(v_1) \neq f_{u_1}(v_2)$. Therefore f_{u_1} is not measurable. Because of a similar reason f^{v_1} is not measurable.

6.5.5 Homomorphic Product

The measurability of a function defined on the vertex set of the homomorphic product of two graphs does not imply the measurability of its x-sections and y-sections. For example consider the graphs given below.

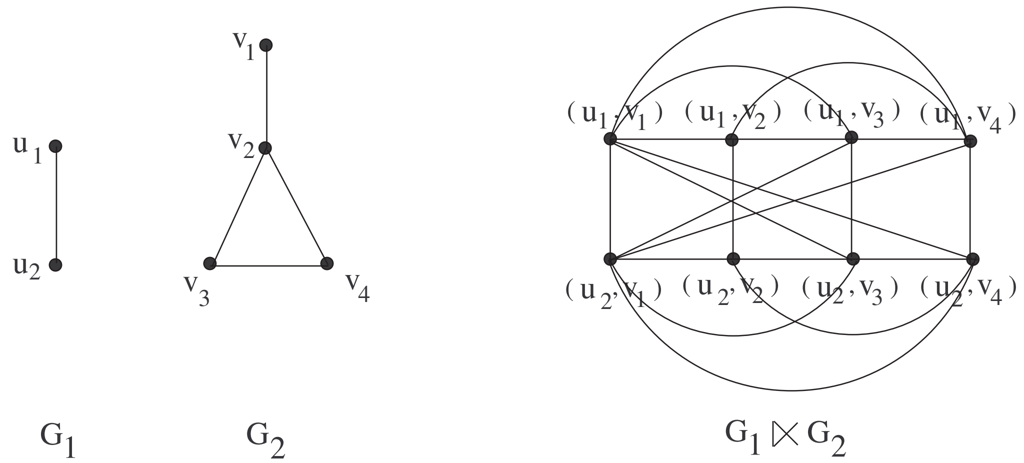

Figure 6.8: The homomorphic product $G_1 \ltimes G_2$ of two graphs G_1 and G_2.

Consider the function $f : V(G_1 \ltimes G_2) \to [0,1]$ defined by,

$f((u_1, v_1)) = f((u_1, v_2)) = f((u_1, v_3)) = f((u_2, v_2)) = f((u_2, v_3)) =$

$f((u_2, v_4)) = 1$, $f((u_1, v_4)) = f((u_2, v_1)) = 2$.

The neighborhood sigma algebra of $G_1 \ltimes G_2$ is $\mathcal{P}(V(G_1 \ltimes G_2))$. Hence all functions from $V(G_1 \ltimes G_2)$ are measurable. In particular f is measurable.

In G_2, $N[v_3] = N[v_4]$, but $f_{u_1}(v_3) \neq f_{u_1}(v_4)$. Therefore f_{u_1} is not measurable. A similar reason implies that f^{v_1} is not measurable.

6.5.6 Normal Product

Apart from all other products, in the case of the normal product, the measurability of functions defined on $V(G_1 \boxtimes G_2)$ and that of their x-sections and y-sections behave very nicely.

Theorem 6.5.3. *Let G_1 and G_2 be two graphs. If a function f defined on $V(G_1 \boxtimes G_2)$ is measurable, then f_x and f^y are measurable for each $x \in V(G_1)$ and $y \in V(G_2)$.*

Proof. Let $x \in V(G_1)$ and $y_1, y_2 \in V(G_2)$ be such that $N[y_1] = N[y_2]$. To prove f_x is measurable, we have to prove that $f_x(y_1) = f_x(y_2)$. That is $f((x, y_1)) = f((x, y_2))$. For this it is enough to prove that $N[(x, y_1)] = N[(x, y_2)]$.

$$\begin{aligned} N[(x, y_1)] &= N[x] \times N[y_1] \\ &= N[x] \times N[y_2] \\ &= N[(x, y_2)] \end{aligned}$$

Hence f_x is measurable.

Similarly, we can prove that f^y is measurable for each $y \in V(G_2)$. $\square$

Theorem 6.5.3 is a natural phenomenon because $\mathcal{A}_{G_1 \boxtimes G_2} = \mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ (Theorem 3.2.27). The crux of it is in fact a consequence of the following theorem of the general measure theory.

Theorem 6.5.4. *[23] Suppose (X, S) and (Y, T) are measurable spaces. Let f be a $(S \times T)$ - measurable function on $X \times Y$. Then*

(i) For each $x \in X$, f_x is a T - measurable function.

(ii) For each $y \in Y$, f^y is a S - measurable function.

In this section we examined the measurability of x-sections, f_x and y-sections, f^y of a measurable function f defined on the vertex set of different types graph products. It is proved that in the case of lexicographic product f_x is measurable but f^y need not be measurable, in the case of tensor product, Cartesian product, co-normal product and homomorphic product f_x and f^y need not be measurable and in the case of normal product both f_x and f^y are measurable.

Chapter 7

Measurable Dominating Functions of Infinite Graphs

7.1 Neighborhood Sigma Algebra of Infinite Graphs

In the case of a finite graph $G = (V(G), E(G))$, we have proved that the smallest measurable set in the neighborhood sigma algebra $\mathcal{A}$ of G, containing a vertex v is the set $\{u \in V(G) : N[u] = N[v]\}$. As in this case the neighborhood sigma algebra contains only a finite number of elements, intersection of any collection of elements of $\mathcal{A}$ is again in $\mathcal{A}$. Hence smallest measurable set containing a vertex is meaningful. But this is not quite obvious in the case of infinite graphs and the existence of such a set is even doubtful.

Consider the following example.

Let G be a graph with vertex set $V(G) = [0,1] \bigcup (2,3)$. Let f be a bijection from $(0,1)$ to $(2,3)$. The adjacency of G is as follows.

$N[0] = N[1] = [0,1]$, $N[x] = [0,1] \bigcup f(x)$ for each $x \in (0,1)$ and $N[y] = \{y, f^{-1}(y)\}$ for each $y \in (2,3)$. Then $\{u \in V(G) : N[u] = N[0]\} = \{0,1\}$. But we cannot assure that $\{0,1\}$ is measurable.

In order to overcome this difficulty we define the neighborhood sigma algebra of any graph by a different way but will not contradict the definition of neighborhood sigma algebra of finite graphs. Motivated by the notations of finite graphs we denote the set $\{u \in V(G) : N[u] = N[v]\}$ by E_v in all the cases whether G is finite or not and define the neighborhood sigma algebra as the sigma algebra generated by the collection $\mathcal{B} = \{N[v] : v \in V(G)\} \bigcup \{E_v : v \in V(G)\}$. Later we prove that E_v is the smallest measurable set containing v in parity with the finite graphs.

Example 7.1.1. Let G be a graph with $V(G) = \mathbb{Z}$. The adjacency relation in G is as follows. Fix $r \in \mathbb{N}$. Two vertices a, b of G are adjacent if and only if $a \neq b$ and $a \equiv b (mod\ r)$.

For $n \in \mathbb{Z}$, let $< n >= \{kr + n : k \in \mathbb{Z}\}$. Then for $m = kr + n$ with $k \in \mathbb{Z}$ and $n = 0, 1, 2, \ldots, r-1$, $E_m =< n >$. In this case order of G is countably infinite but there are only a finite number of E_v's. Note that each E_v is infinite.

Example 7.1.2. Let G be a graph with $V(G) = \mathbb{R}$. Two vertices $a, b \in \mathbb{R}$ are adjacent in G if and only if and $b = -a$. Then for $v \in V(G)$, $E_v = \{v, -v\}$.

Here order of G is uncountable and there are uncountable number of E_v's.

Example 7.1.3. Let G be a graph with vertex set $V(G) = \mathbb{R}^2$ with the following as the closed neighborhoods.

$N[(0,0)] = \mathbb{R}^2$. Let x be an irrational number and q be a rational number. Then $N[(x,q)] = \{(x,r) : r \text{ rational}\} \bigcup \{(0,0)\}$. For all (u,v) not in any of $N[(x,q)]$, $N[(u,v)] = W$, where $W = (\bigcup_{x\in\mathbb{R}\setminus\mathbb{Q}} N[x,0])^c \bigcup \{(0,0)\}$. Then $E_{(0,0)} = \{(0,0)\}$, $E_{(x,q)} = \{(x,r) : r \text{ rational}\}$ and for $w \in W \setminus \{(0,0)\}$, $E_w = W \setminus \{(0,0)\}$

For a graph G, let $\mathscr{N} = \{N[v] : v \in V(G)\}$ and $\mathscr{M} = \{E_v : v \in V(G)\}$.

Definition 7.1.4. Let G be any graph (whether finite or infinite). The sigma algebra generated by the collection $\mathscr{N} \bigcup \mathscr{M}$ is called the neighborhood sigma algebra of G and it is denoted by $\mathcal{A}_G$ or simply by $\mathcal{A}$ if there is no confusion.

Hereafter by a graph G, we mean an infinite graph with the neighborhood sigma algebra $\mathcal{A}$ on the vertex set $V(G)$ and a measure μ on $\mathcal{A}$.

Theorem 7.1.5. *Let G be a graph. Then for $v \in V(G)$ and $A \in \mathcal{A}$, either $E_v \subset A$ or $E_v \subset A^c$.*

Proof. Consider the collection, $\mathscr{A} = \{A \subset V(G) : \text{ for any } E_v \in \mathscr{M}, \text{ either } E_v \subset A \text{ or } E_v \subset A^c\}$. It is clear that $\mathscr{N} \subset \mathscr{A}$. We can also show that $\mathscr{A}$ is a sigma algebra. Clearly if $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$. Now let $\{A_n\}$ be a sequence in $\mathscr{A}$. For $E_v \in \mathscr{M}$, if $E_v \subset A_m$ for some m, then $E_v \subset \bigcup A_n$. Otherwise $E_v \subset A_n^c$ for all n and $E_v \subset \bigcap A_n^c = (\bigcup A_n)^c$. This implies that $\bigcup A_n \in \mathscr{A}$. Hence $\mathscr{A}$ is a sigma algebra containing $\mathscr{N} \bigcup \mathscr{M}$ and hence $\mathcal{A} \subset \mathscr{A}$. Proving the assertion. $\square$

Corollary 7.1.6. *If $A \in \mathcal{A}$ and if $A \subset E_v$ for some $v \in V(G)$, then either A is empty or $A = E_v$.*

Proof. Suppose $A \in \mathcal{A}$ and $A \subset E_v$ for some $v \in V(G)$. By Theorem 7.1.5, either $E_v \subset A$ or $E_v \subset A^c$. In the first case $A = E_v$. In the second case $A = \emptyset$. $\square$

Corollary 7.1.6 shows that no proper non empty subset of E_v is measurable. Thus the smallest measurable set containing each vertex v of G exists and it is E_v. As a consequence of this result, if some real function defined on the vertex set of a graph is measurable with respect to $\mathcal{A}$ then it cannot assume different values on E_v, for each $v \in V(G)$. In other words any measurable real function takes constant values on each E_v, $v \in V(G)$.

Proposition 7.1.7. *Let G be a graph and let f be a measurable real valued function defined on $V(G)$. Then for each $v \in V(G)$, f is constant on E_v.*

Proof. Let $v \in V(G)$ and $f(v) = c$. Suppose $f(u) = d$ for some $u \in E_v$. Let, if possible, $d \neq c$, suppose that $c < d$. Then $f^{-1}(-\infty, d)$ is measurable and $v \in f^{-1}(-\infty, d)$. Therefore $v \in f^{-1}(-\infty, d) \bigcap E_v$ and $f^{-1}(-\infty, d) \bigcap E_v$ is a measurable set properly contained in E_v, which contradicts the fact that E_v is the smallest measurable set containing v. A similar kind of contradiction arises when $d < c$. $\square$

Remark 7.1.8. *The converse of this proposition is not true in the case of infinite* *graphs. That is even though a function defined on the vertex set of an infinite graph is constant on all E_v's, it need not be a measurable function.*

Example 7.1.9. Let G be a graph with vertex set $V(G) = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Suppose that no two vertices in G are adjacent.
Then $N[v] = E_v = \{v\}$ for all $v \in V(G)$.
Hence the neighborhood sigma algebra $\mathcal{A}$ of G is the sigma algebra generated by the collection $\{\{x\} : x \in [0, 1]\}$.
The identity function f on $V(G)$ is constant on each E_v, but it is not measurable.
This follows from the fact that, $\mathcal{A} = \{A \subseteq [0, 1] : A \textit{ or } A^c \textit{ is countable}\}$.

This example also suggests that, in the case of infinite graphs the measurability of all singleton subsets of $V(G)$ does not imply the measurability of functions on $V(G)$.

7.2 Measurable Dominating Functions

In this section we define measurable dominating function of an infinite graph and characterize minimal measurable dominating function.

Definition 7.2.1. Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to [0, 1]$ is called a measurable dominating function of G if the following conditions are satisfied:

(i) f is measurable

(ii) $\int\limits_{N[v]} f\ d\mu \geq 1$ for all $v \in V(G)$.

Definition 7.2.2. Let G be a graph with vertex set $V(G)$. A measurable dominating function f of G is said to be minimal if there does not exist a measurable dominating function g of G such that $g \leq f$ a.e and $g < f$ on some set of positive measure.

Now we derive a necessary and sufficient condition for a measurable dominating function to be minimal.

Theorem 7.2.3. *Let G be a graph with vertex set $V(G)$. A measurable dominating function f of G is minimal if and only if $inf\{\int\limits_{N[u]} f\ d\mu : u \in N[v]\} = 1$, for every vertex $v \in V(G)$ with $\mu(E_v) > 0$ and $f > 0$ on E_v.*

Proof. Let f be a minimal measurable dominating function of G and $v \in V(G)$ be such that $\mu(E_v) > 0$ and $f(v) > 0$.

Suppose $inf\{\int\limits_{N[u]} f\ d\mu : u \in N[v]\} \neq 1$. Since f is a measurable dominating function of G, $\int\limits_{N[u]} f\ d\mu \geq 1$ for all $u \in N[v]$. This implies $\int\limits_{N[u]} f\ d\mu > 1$ for all $u \in N[v]$.

Let $S = \{\int\limits_{N[u] \setminus E_v} f\ d\mu : u \in N[v]\}$.

Suppose $inf S \geq 1$.

Define a function $g : V(G) \longrightarrow [0,1]$ as

$$g(w) = \begin{cases} f(w) & \text{if } w \notin E_v \\ 0 & \text{if } w \in E_v \end{cases}$$

Then $g = f - f\chi_{E_v}$, where χ_{E_v} denotes the characteristic function of E_v. Hence g is measurable.

Let $u \in N[v]$, then

$$\begin{aligned}
\int\limits_{N[u]} g \; d\mu &= \int\limits_{E_v} g \; d\mu + \int\limits_{N[u]\setminus E_v} g \; d\mu \\
&= \int\limits_{N[u]\setminus E_v} g \; d\mu \\
&= \int\limits_{N[u]\setminus E_v} f \; d\mu \\
&\geq infS \\
&\geq 1.
\end{aligned}$$

Also, for $u \notin N[v]$,

$$\begin{aligned}
\int\limits_{N[u]} g \; d\mu &= \int\limits_{N[u]} f \; d\mu \\
&\geq 1
\end{aligned}$$

Therefore g is a measurable dominating function such that $g(w) \leq f(w)$ for every $w \in V(G)$ and $g < f$ on E_v, which is a contradiction.

Suppose $infS < 1$.

Let $f(v) = c$. Then $f \equiv c$ on E_v.

For $u \in N[v]$,

$$\begin{aligned}\int\limits_{N[u]} f\ d\mu &= \int\limits_{N[u]\setminus E_v} f\ d\mu + \int\limits_{E_v} f d\mu \\ &= \int\limits_{N[u]\setminus E_v} f\ d\mu + c\mu(E_v) \\ &> 1 \end{aligned} \tag{7.1}$$

This implies

$$c > \frac{1 - \int\limits_{N[u]\setminus E_v} f\ d\mu}{\mu(E_v)},\ \text{for all } u \in N[v].$$

Also for $u \in N[v]$, $\int\limits_{N[u]\setminus E_v} f\ d\mu \geq infS$.

This implies $1 - infS \geq 1 - \int\limits_{N[u]\setminus E_v} f\ d\mu$, for all $u \in N[v]$.

Hence for all $u \in N[v]$,

$$\frac{1 - infS}{\mu(E_v)} \geq \frac{1 - \int\limits_{N[u]\setminus E_v} f\ d\mu}{\mu(E_v)}.$$

Also by (7.1), for all $u \in N[v]$, $\int\limits_{N[u]\setminus E_v} f\ d\mu > 1 - c\mu(E_v)$. This implies, $infS \geq 1 - c\mu(E_v)$.

But $inf\{\int\limits_{N[u]} f\ \ d\mu : u \in N[v]\} \neq 1$. This implies that $infS \neq 1 - c\mu(E_v)$.

Hence $infS > 1 - c\mu(E_v)$.

That is,

$$c > \frac{1 - infS}{\mu(E_v)}$$

Since $inf S < 1$, there exists at least one $u \in N[v]$, such that $\int\limits_{N[u]\setminus E_v} f\ d\mu < 1$.

Let $U = \{u \in N[v] : \int\limits_{N[u]\setminus E_v} f\ d\mu < 1\}$.

If $u \in U$,

$$\frac{1 - \int\limits_{N[u]\setminus E_v} f\ d\mu}{\mu(E_v)} > 0.$$

Therefore there exits $d \in [0,1]$, such that $\frac{1-\int\limits_{N[u]\setminus E_v} f\ d\mu}{\mu(E_v)} < d < c$, for all $u \in U$.

Define a function $h : V(G) \longrightarrow [0,1]$ as,

$$h(w) = \begin{cases} f(w) & \text{if } w \notin E_v \\ d & \text{if } w \in E_v \end{cases}$$

Then $h = f - (f-d)\chi_{E_v}$, hence measurable.

Let $u \in U$.

$$\begin{aligned} \int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]\setminus E_v} h\ d\mu + \int\limits_{E_v} h\ d\mu \\ &= \int\limits_{N[u]\setminus E_v} f\ d\mu + d\mu(E_v) \\ &> \int\limits_{N[u]\setminus E_v} f\ d\mu + \left(\frac{1 - \int\limits_{N[u]\setminus E_v} f\ d\mu}{\mu(E_v)} \right) \mu(E_v) \\ &= 1. \end{aligned}$$

Let $u \notin U$.

If $u \notin N[v]$,

$$\begin{aligned}\int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]} f\ d\mu \\ &\geq 1\end{aligned}$$

If $u \in N[v]$, $\int\limits_{N[u]\setminus E_v} f\ d\mu \geq 1$

Therefore,

$$\begin{aligned}\int\limits_{N[u]} h\ d\mu &= \int\limits_{N[u]\setminus E_v} h\ d\mu + \int\limits_{E_v} h\ d\mu \\ &= \int\limits_{N[u]\setminus m_v} f\ d\mu + d\mu(E_v) \\ &> 1\end{aligned}$$

Therefore h is a measurable dominating function such that $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on E_v, which is a contradiction.

Conversely, let f be a measurable dominating function of G such that $inf\{\int\limits_{N[u]} f\ d\mu : u \in N[v]\} = 1$, for every vertex v with $\mu(E_v) > 0$ and $f > 0$ on E_v.

Suppose f is not minimal. Then there exists a measurable dominating function l of G such that $l \leq f$ a.e and $l < f$ on a set of positive measure. So, there exists $v \in V(G)$ such that $\mu(E_v) > 0$ and $l < f$ on E_v. This implies $f(v) > 0$. Also $l(v)\mu(E_v) < f(v)\mu(E_v)$, since $\mu(E_v) > 0$. Hence $1 - f(v)\mu(E_v) < 1 - l(v)\mu(E_v)$. Since $inf\{\int\limits_{N[u]} f\ d\mu : u \in N[v]\} = 1$, $infS = 1 - \mu(E_v)f(v)$. Hence for each $r > 1 - \mu(E_v)f(v)$, there exists $u \in N[v]$ such that $r > \int\limits_{N[u]\setminus E_v} f\ d\mu \geq 1 - \mu(E_v)f(v)$. Therefore by taking $r = 1 - l(v)f(v)$, we get a $u \in N[v]$ such that $1 - \mu(E_v)f(v) \leq$

$\int\limits_{N[u]\setminus E_v} f\ d\mu < 1 - l(v)\mu(E_v)$. But $\int\limits_{N[u]\setminus E_v} l\ d\mu \leq \int\limits_{N[u]\setminus E_v} f\ d\mu < 1 - l(v)\mu(E_v)$. This implies $\int\limits_{N[u]} l\ d\mu < 1$, which contradicts the fact that l is a measurable dominating function of G. Hence the theorem. $\square$

Example 7.2.4. Consider the graph G with $V(G) = \{u_1, u_2, ..., v_1, v_2, ...\}$ and $N[u_i] = \{v_i, u_1, u_2, ...\},\ i \in \mathbb{N}$ and $N[v_i] = \{u_i, v_1, v_2, ...\},\ i \in \mathbb{N}$.

Then $E_w = \{w\}$ for all $w \in V(G)$. Let $\{x_i\}$ and $\{y_i\}$ be two sequences in $\mathbb{R}^+$ such that $\sum\limits_{i\in\mathbb{N}} x_i = \sum\limits_{i\in\mathbb{N}} y_i = 1$.

Let $\mu(\{u_i\}) = x_i$ and $\mu(\{v_i\}) = y_i$ for all $i \in \mathbb{N}$.

Since $\sum\limits_{i\in\mathbb{N}} x_i$ and $\sum\limits_{i\in\mathbb{N}} y_i$ are convergent, $lim\ x_n = lim\ y_n = 0$.

Define $f : V(G) \longrightarrow [0, 1]$ as $f(w) = 1$ for all $w \in V(G)$.

Being a constant function f measurable.

For $j \in \mathbb{N}$,

$$\begin{aligned}\int\limits_{N[u_j]} f\ d\mu &= \sum_{i\in\mathbb{N}} x_i + y_j \\ &= 1 + y_j\end{aligned}$$

and

$$\begin{aligned}\int\limits_{N[v_j]} f\ d\mu &= \sum_{i\in\mathbb{N}} y_i + x_j \\ &= 1 + x_j\end{aligned}$$

Thus for each $w \in V(G)$, $inf\{\int\limits_{N[w']} f\ d\mu : w' \in N[w]\} = 1$.

Hence by Theorem 7.2.3, f is a minimal measurable dominating function of G.

Remark 7.2.5. *In Example 7.2.4,* $\int\limits_{N[w]} f\ d\mu \neq 1$ *for any* $w \in V(G)$. *So as* *in the case of finite graphs for a minimal measurable dominating function, for $v \in V(G)$ with $f(v) > 0$ and $\mu(E_v) > 0$, there need not exists a vertex $u \in N[v]$ such that* $\displaystyle\int_{N[u]} f \, d\mu = 1$.

We conclude our work with the following section, which is a generalization of measurable signed dominating functions to infinite graphs.

7.3 Measurable Signed Dominating Functions

Definition 7.3.1. Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to [-1, 1]$ is called a measurable signed dominating function of G if the following conditions are satisfied:

(i) f is measurable

(ii) $\displaystyle\int_{N[v]} f \, d\mu \geq 1$ for all $v \in V(G)$.

Definition 7.3.2. Let G be a graph with vertex set $V(G)$. A measurable signed dominating function f of G is said to be minimal if there does not exist a measurable signed dominating function g of G such that $g \leq f$ a.e on $V(G)$ and $g < f$ on some set of positive measure.

Theorem 7.3.3 characterizes minimal measurable signed dominating functions.

Theorem 7.3.3. *Let G be a graph and μ be a measure on $V(G)$. A measurable signed dominating function f of G is minimal if and only if* $inf\{\int_{N[u]} f\ d\mu : u \in N[v]\} = 1$, *for every vertex* $v \in V(G)$ *with* $\mu(E_v) > 0$ *and* $f > -1$ *on* E_v.

Proof. Let f be a minimal measurable signed dominating function of G relative to μ and $v \in V(G)$ be such that $\mu(E_v) > 0$ and $f(v) > -1$.

Suppose $inf\{\int_{N[u]} f\ d\mu : u \in N[v]\} \neq 1$. Then $inf\{\int_{N[u]} f\ d\mu : u \in N[v]\} > 1$. This implies $\int\limits_{N[u]} f\ d\mu > 1$, for all $u \in N[v]$.

Then for all $u \in N[v]$,

$$\int\limits_{N[u]} f\ d\mu = 1 + d_u,\ d_u > 0. \tag{7.2}$$

Also,

$$\begin{aligned} \int\limits_{N[u]} f\ d\mu &= \int\limits_{E_v} f\ d\mu + \int\limits_{N[u]\setminus E_v} f\ d\mu \\ &= c\mu(E_v) + \int\limits_{N[u]\setminus E_v} f\ d\mu \end{aligned} \tag{7.3}$$

where c is the value of f on E_v.

By (7.2) and (7.3), for all $u \in N[v]$,

$$\int\limits_{N[u]\setminus E_v} f\ d\mu = 1 + d_u - c\mu(E_v). \tag{7.4}$$

Since $inf\{\int_{N[u]} f\ d\mu : u \in N[v]\} > 1$, $inf\{d_u : u \in N[v]\} > 0$.

Therefore $k = \dfrac{1}{\mu(E_v)} inf\{d_u; u \in N[v]\} > 0$. Then $c - k < c$. Choose a real number $d \in [-1, 1]$ such that $c - k \leq d < c$ and define, $g : V(G) \to [-1, 1]$ as

$$g(w) = \begin{cases} d & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

Then $g = f - (f - d)\chi_{E_v}$ is measurable, $g(w) \leq f(w)$ for all $w \in V(G)$ and $g < f$ on E_v.

Also g is measurable signed dominating.

For let $u \in N[v]$.

$$\begin{aligned} \int\limits_{N[u]} g \; d\mu &= \int\limits_{E_v} g \; d\mu + \int\limits_{N[u] \setminus E_v} g \; d\mu \\ &= d\mu(E_v) + \int\limits_{N[u] \setminus E_v} f \; d\mu \\ &= d\mu(E_v) + 1 + d_u - c\mu(E_v), \text{ from (7.4)} \\ &\geq 1 + (d + k - c)\mu(E_v) \\ &= 1 + [d - (c - k)]\mu(E_v) \\ &\geq 1. \end{aligned}$$

Now let $u \in V(G) \setminus N[v]$. Then,

$$\int\limits_{N[u]} g \; d\mu = \int\limits_{N[u]} f \; d\mu \geq 1.$$

Therefore g is a measurable signed dominating function of G, a contradiction.

Conversely, let f be a measurable signed dominating function of G such that for every vertex v with $\mu(E_v) > 0$ and $f > -1$ on E_v, $inf\{\int\limits_{N[u]} f\ d\mu : u \in N[v]\} = 1$. Suppose f is not minimal. Then there exists a measurable signed dominating function l of G such that $l \leq f$ a.e and $l < f$ on some set of positive measure. So there exists $v \in V(G)$ with $\mu(E_v) > 0$ and $l < f$ on E_v. This implies $f(v) > -1$. So $f > -1$ on E_v. Therefore $inf\{\int\limits_{N[u]} f\ d\mu : u \in N[v]\} = 1$.
Let $S = \{\int\limits_{N[u]\setminus E_v} f\ d\mu : u \in N[v]\}$.
Then $infS = 1 - \mu(E_v)f(v)$. Also $1 - \mu(E_v)f(v) < 1 - \mu(E_v)l(v)$. Therefore, there exists $u \in N[v]$ such that $1 - \mu(E_v)l(v) > \displaystyle\int\limits_{N[u]\setminus E_v} f\ d\mu$.
But $\displaystyle\int\limits_{N[u]\setminus E_v} l\ d\mu \leq \int\limits_{N[u]\setminus E_v} f\ d\mu$. This implies $\displaystyle\int\limits_{N[u]} l\ d\mu < 1$, a contradiction. Hence the theorem. □

Conclusion

In this thesis we extended the concept of dominating functions to infinite graphs. We introduced a new type of sigma algebra called neighborhood sigma algebra on the vertex sets of graphs and discussed its properties. We determined neighborhood sigma algebras of some graphs that are derived from given graphs. We defined common neighborhood polynomial of a graph and neighborhood unique graphs. We also found out common neighborhood polynomials of join, corona and different types of graph products of two graphs. We checked the measurability of x-section and y-section of a measurable function defined on the vertex sets of different graph products.

In the case of lexicographic product, tensor product, Cartesian product, normal product and co-normal product of two graphs G_1 and G_2, we proved that the product sigma algebra $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$ is contained in the neighborhood sigma algebra of the product graph. We made an attempt to define a measure on the neighborhood sigma algebras of graph products as an extension of the product measure on $\mathcal{A}_{G_1} \times \mathcal{A}_{G_2}$. We defined measurable dominating functions and measurable signed dominating functions of both finite and infinite graphs. Characterizations of minimal measurable dominating functions and minimal measurable signed dominating functions are also obtained.

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List of Publications

Papers Published/Accepted

1. P. Jisna, Raji Pilakkat, *Measurable dominating functions*, International Electronic Journal of Pure and Applied Mathematics, 10(2), 167-176,(2016).

2. P. Jisna, Raji Pilakkat, *Neighborhood sigma algebras of middle graph, total graph and join of two graphs*, Global Journal of Pure and Applied Mathematics , 13(9), 6673-6680,(2017).

3. P. Jisna, Raji Pilakkat, *Measurable signed dominating functions*, Bulletin of Kerala Mathematical Association, 15(2), 165-171, (2017).

4. P. Jisna, Raji Pilakkat, *Neighborhood sigma algebra of graph products*, International Electronic Journal of Pure and Applied Mathematics, Accepted.

5. P. Jisna, Raji Pilakkat, *Sections of measurable functions*, Far East Journal of Mathematical Sciences, Accepted.

Papers Presented

1. Presented a paper on "Neighborhood sigma algebra and compatible graph products" in the MESMAC international conference organized by MES

Mambad college in association with FLAIR on 14, 15 and 16 February 2017.

2. Presented a paper on "Neighborhood sigma algebra of a graph " in the three day UGC sponsored national seminar on 'Topology and its Applications ' organized by department of mathematics, University of Calicut on 23, 24 and 25 March 2017.

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