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CERTIFICATE

I hereby certify that the thesis entitled "A STUDY ON SOME GRAPH POLYNOMIALS AND ITS STABILITY" is a bonafide work carried out by **Smt. Shyama M. P.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Dr. Anil Kumar V.
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DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON SOME GRAPH POLYNOMIALS AND ITS STABILITY**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Associate Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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INTRODUCTION

Graph theory is one of the most flourishing branches of modern mathematics and computer science. The study of graph polynomials and their location of roots in the complex plane can be treated as one of the main areas in graph theory. The roots of various graph polynomials including the chromatic polynomial, the independence polynomial, the matching polynomial and the domination polynomial have been studied extensively.

The general definition of graph polynomial is : *"Let $\mathscr{G}$ be the class of graphs and let R be a ring and X be a (not necessarily finite) set of indeterminates. A graph polynomial is a function $\mathfrak{p} : \mathscr{G} \to R[X]$ such that for isomorphic graphs H and G we have $\mathfrak{p}(H) = \mathfrak{p}(G)$ [14]".* Intuitively, graph polynomials are polynomials assigned to graphs. Observe that applications of graph polynomials arise in many areas outside *graph theory* as well. For example, matching polynomial and Hossoya polynomial have many applications in Statistical Physics and Theoretical Chemistry. In the past few decades, many graph polynomials have been studied and plenty of theoretical and practical approaches have been developed. In this sequel we review the history of some well known graph polynomials :

Edge difference polynomial Historically first polynomial in graph theory was introduced by J.J. Sylvester in 1878 [17] and further studied by J. Petersen in [19]. It is a multivariate polynomial depending on the ordering of the vertices $V = \{v_1, v_2, \ldots, v_n\}$ of a graph $G = (V(G), E(G))$ and defined as

$$P_G(x_1, x_2, \ldots, x_n) = \sum_{i<j,\ (v_i,v_j)\in E} (x_i - x_j).$$

This polynomial is not a graph invariant, but it was used as a tool in studying regularity and colorability questions of graphs.

Domination polynomial The domination polynomial of a graph was first introduced by Saied Alikhani in 2009 in his Ph.D thesis [34]. He derived a recursive formula for domination polynomials of some specific graphs. Further more, proved some relationships between domination polynomial and the geometrical properties of graphs. He studied the roots of the domination polynomial of certain graphs and characterized graphs with one, two and three distinct domination roots and studied the $\mathcal{D}$-equivalence classes of some graphs.

Total domination polynomial S. Sanal Kumar introduced total domination polynomial of a graph, which is an analogue of domination polynomial, in his Ph.D thesis [42]. He derived the total domination polynomial for path P_n, cycle C_n and wheel W_n.

Hosoya polynomial The Hosoya polynomial of a connected graph is defined as :

$$\mathtt{H}(G, x) = \sum_{\{u,v\}\subset G} x^{\mathsf{d}(u,v)},$$

where $\mathsf{d}(u, v)$ denotes the distance between vertices u and v. This polynomial was introduced by Hosoya [12] in 1988. This polynomial has many Chemical applications. Especially, the two well-known topological indices, namely, Wiener index and hyper-Wiener index, can be directly obtained from the Hosoya polynomial. The value of the first derivative of Hosoya polynomial $\mathtt{H}(G, x)$ of a graph G at $x = 1$ equals the Wiener index of G. The hyper-Wiener index of G is equal to the half of the second derivative of the polynomial $x\,\mathtt{H}(G, x)$ at $x = 1$.

An overview of the thesis

The works of Saeid Alikhani [34–40] and Janson I. Brown and Julia [16] motivated me to select the present topic. The thesis is organized into eight chapters preceded by an introduction.

In **Chapter One**, basic definitions and terminologies are provided which are used in the subsequent chapters. For the graph theoretic terminologies we refer to [4, 6, 10, 13, 27, 46] and terminologies for polynomials and its nature of roots we refer to [24, 25, 28, 41, 45].

Chapter Two mainly deals with domination polynomial of graphs. In Section 2.1 we define domination polynomial of a graph and find domination polynomial of some graphs. In Subsection 2.1.1 we find domination polynomial of square of some graphs. In Section 2.2, we define domination root and introduce the concept, $\mathtt{d}$-number of a graph and also find $\mathtt{d}$-number of some graphs. We obtained bounds for domination roots of some graphs in Section 2.3. We introduce $\mathtt{d}$-stable graphs and $\mathtt{d}$-unstable graphs in Section 2.4. We include some examples of $\mathtt{d}$-stable graphs and $\mathtt{d}$-unstable graphs.

Chapter Three focus on distance-k domination polynomial of graphs. In Section 3.1 we define distance-k domination polynomial of graph and find distance-k domination polynomial of some graphs. In Section 3.2, we define distance-k domination root and introduce a new concept, $\mathtt{d}^k$-number of graphs and also find $\mathtt{d}^k$-number of some graphs. We obtained bounds for distance-k domination roots of some graphs in Section 3.3. We introduce $\mathtt{d}^k$-stable graphs and $\mathtt{d}^k$-unstable graphs in Section 3.4 and find some examples of $\mathtt{d}^k$-stable graphs and $\mathtt{d}^k$-unstable graphs.

Chapter Four mainly deals with the total domination polynomial of graphs. In Section 4.1 we define total domination polynomial of graph and find total domination polynomial of some graphs. In Subsection 4.1.1 we find total domination polynomial of square of some graphs. In Section 4.2, we define total domination root and introduce $\mathtt{d}_t$-number of graphs and also find $\mathtt{d}_t$-number of some graphs. We obtained bounds for total domination roots of some graphs in Section 4.3. We introduce $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs in Section 4.4 and provide some examples of $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs.

Chapter Five is devoted to distance-k total domination polynomial of graphs. In Section 5.1 we define distance-k total domination polynomial of graph and find distance-k total domination polynomial of some graphs. In Section 5.2, we define distance-k total domination root and introduce a new concept, $\mathtt{d}_t^k$-number of a graph and also find $\mathtt{d}_t^k$-number of some graphs. We obtained bounds for distance-k total domination roots of some graphs in Section 5.3. We introduce $\mathtt{d}_t^k$-stable and $\mathtt{d}^k$-unstable graphs in Section 5.4 and find some examples of $\mathtt{d}_t^k$-stable and $\mathtt{d}_t^k$-unstable graphs.

Chapter Six mainly deals with the Hosoya polynomial of a graph. In Section 6.1 we define Hosoya polynomial of graph and find Hosoya polynomial of some graphs. In Subsection 6.1.1 we find Hosoya polynomial of square of some graphs. In Section 6.2, we define Hosoya root and introduce new definition, $\mathtt{h}$-number of graph and also find $\mathtt{h}$-number of some graphs. We obtained bounds for Hosoya roots of some graphs in Section 6.3. We introduce $\mathtt{h}$-stable graphs and $\mathtt{h}$-unstable graphs in Section 6.4 and find some examples of $\mathtt{h}$-stable graphs and $\mathtt{h}$-unstable graphs.

In **Chapter Seven** we include some general properties of graph polynomials which are studied in the earlier chapters. In Section 7.1 we prove that for odd n, $-\tau^n$ can never be a domination (distance-k domination, total domination, distance-k total domination, Hosoya) root, where τ denotes the golden ratio. In Section 7.2 we prove that all the integer distance-k domination roots are even. Also we prove that there is no connected graphs G such that $\mathbb{Z}(D^k(G,x)) = \left\{0, \frac{-3\pm\sqrt{5}}{2}\right\}$.

Chapter Eight is the concluding chapter of the thesis. Some conjectures and open problems are proposed on this chapter.

A bibliography and Index are also provided.

The results of the thesis have been published/communicated in the form of research papers, a list of which is given below.

1. M. P. Shyama and V. Anil Kumar : *Total domination polynomials of complete partite graphs*, Advances and Applications in Discrete Mathematics, Vol. 13, No. 1, 23-28, 2014.

2. M. P. Shyama and V. Anil Kumar : *Total domination polynomials of square of some graphs*, Advances and Applications in Discrete Mathematics, Vol. 15, No. 2, 167-175, 2015.

3. M. P. Shyama and V. Anil Kumar : *On the roots of Hosoya polynomial*, Journal of Discrete Mathematical Science and Cryptography, Vol. 19, No.

1, 199-219, 2016.

4. M. P. Shyama and V. Anil Kumar : *Distance-k domination polynomial of some graphs*, Journal of Pure and Applied Mathematics, Vol. 16, No. 2, 71-86, 2016.

5. M. P. Shyama and V. Anil Kumar : *Distance-k total domination polynomial of some graphs*, Advances and Applications in Discrete Mathematics, (Accepted).

6. M. P. Shyama and V. Anil Kumar : *Domination stable graphs*, Advances and Applications in Discrete Mathematics, (Communicated).

7. M. P. Shyama and V. Anil Kumar : *Distance-k domination stable Graphs*, Journal of Pure and Applied Mathematics, (Communicated).

8. M. P. Shyama and V. Anil Kumar : *Total domination stable graphs*, Iranian Journal of Mathematical Science and Informatics, (Communicated).

9. M.P. Shyama and V. Anil Kumar : *Distance-k total domination stable graphs*, Journal of Discrete Mathematical Science and Cryptography, (Communicated).

10. M.P. Shyama and V. Anil Kumar : Hosoya polynomial of square of some graphs, Advances and Applications in Discrete Mathematics, (Communicated).

11. M.P. Shyama and V. Anil Kumar : *Hosoya stable graphs*, Far East Journal of Mathematics, (Communicated).

CHAPTER 1 PRELIMINARIES

In this chapter, we want to collect most of the terminology and notations used in thesis. For those not given here, they will be defined when needed.

1.1 Graphs

A *graph* $G = (V(G), E(G))$ is a finite nonempty set $V(G)$ of objects called *vertices* together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called *edges*. The number of elements in $V(G)$ and in $E(G)$ are called the *order* and the *size* of G respectively. Two vertices u and v in G are *adjacent* if there exists an edge between them, that is, if $\{u, v\} \in E(G)$. We often write uv instead of $\{u, v\}$. The vertices u and v are the ends of $e = uv$ and e is *incident* with both u and v; both u and v are incident with e. The *degree* of a vertex $v \in V(G)$, written $deg(v)$, is the number of edges in G which are incident with v. A *pendent vertex (or end vertex)* is any vertex of degree 1 (that is, a vertex adjacent to exactly one other vertex).

A graph G is said to be *isomorphic* to a graph H, if there is a bijection $\psi : V(G) \longrightarrow V(H)$ and a bijection $\phi : E(G) \longrightarrow E(H)$ such that u and v are adjacent in G if and only if $\psi(u)$ and $\psi(v)$ are adjacent in H.

A graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If G and H are two graphs such that $V(G) = V(H) = V$ and for all distinct u and v in V, $uv \in E(G)$ if and only if $uv \notin E(H)$, then H is the *complement* of G. We write $H = \overline{G}$. The *union* $G = G_1 \cup G_2$ of two graphs G_1 and G_2 is the graph with vertices $V(G) = V(G_1) \cup V(G_2)$ and edges $E(G) = E(G_1) \cup E(G_2)$. If in addition, $V_1 \cap V_2 = \emptyset$, then G is the *disjoint union* of G_1 and G_2, written $G = G_1 \dot{\cup} G_2$.

A graph G is said to be *complete* if every vertex is adjacent to all other vertices. A complete graph with n vertices is denoted by K_n and its complement is the *null graph* of order n, written as $\overline{K}_n$.

For vertices $u, v \in V(G)$, a $u - v$ *path* is an alternating sequence of vertices and edges that begins with the vertex u and ends with the vertex v in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. Moreover, no vertex is repeated in this sequence. A path on n vertices is denoted by P_n. The number of edges in the sequence is considered as the *length* of the path.The *distance* $\mathsf{d}(u, v)$ between two vertices u and v is the minimum of the lengths of paths between u and v. The *diameter* D of a graph G is defined as $D(G) := \max\limits_{u,v \in V(G)} \{\mathsf{d}(u, v)\}$.

A graph G is *connected* if for every pair of vertices in $V(G)$, there exists a path between them. A graph G is *disconnected* if it is not connected. If a graph G is disconnected then its diameter is defined to be infinity. A maximal connected subgraph of a graph G is called a *component* of G. A *bridge* of a connected graph G is an edge of G whose removal disconnects the graph G.

A *cycle* on n vertices, denoted C_n, is a path which originates and concludes at the same vertex. The length of a cycle is the number of edges in the cycle. A *wheel* W_n is a graph with n vertices, obtained from a cycle C_{n-1} by adding a new vertex and edges joining it to all vertices of the cycle.

A *tree* is a graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree.

For $m > 1$, a $m-$*partite graph* is a graph G whose vertex set $V(G)$ can be partitioned into m non-empty subsets $V_1, V_2, \ldots, V_m$ such that each edge of G joins a vertex in V_i to a vertex in V_j for some distinct i, j in $\{1, 2, \ldots, m\}$. We call $\{V_1, V_2, \ldots, V_m\}$ a $m-$partition of G. An $m-$partite graph is called a *complete* $m-$*partite graph* if every vertex in V_i is adjacent to every vertex in V_j for all distinct i, j in $\{1, 2, \ldots, m\}$. Such a complete $m-$partite graph is denoted by $K_{n_1, n_2, \ldots, n_m}$ if $|V_i| = n_i$ for each $i = 1, 2, \ldots, m$. If $n_i = n$ then we denote the complete m-partite graph by $K_{n[m]}$. A complete $2-$partite graph is called a *complete bipartite graph*, denoted by $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$. The complete bipartite graph $K_{1,n}$ is called a *star graph* denoted by S_n. The *spider graph* Sp_{2n+1} is the graph obtained by subdividing each edge once in the star graph S_n. The *bipartite cocktail party graph* B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

The *corona* $H \circ G$ of two graphs H and G is the graph formed from one copy of H and $|V(H)|$ copies of G, where the i^{th} vertex of H is adjacent to every vertex in the i^{th} copy of G. If H and G are any two graphs, then $H \vee G$ is the *join* of H and G obtained from $H \cup G$ by joining each vertex of H to every vertex of G. If H and G are any two graphs, then the *cartesian product* $H \Box G$ of H and G is a graph such that

- the vertex set of $H \Box G$ is the cartesian product $V(H) \times V(G)$ and
- any two vertices (h, g) and (h', g') are adjacent if and only if either $h = h'$ and g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H.

The *Dutch windmill graph* G_3^n is the graph obtained by selecting one vertex in each of n triangles and identifying them. The graph $Q(m, n)$ is obtained by identifying each vertex of K_m with a vertex of a unique K_n.
For a positive integer k, the k^{th} *power* of a graph G is the graph with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most k between them, and that graph is denoted by G^k. If $k = 2$, we call G^2 as the *square* of G.

The *generalized star-tree* graph $T^*_{l,m,n}$ is a graph obtained by connecting two star graphs S_l and S_m by a path P_n. A tree $S_{(n_1,n_2,\ldots,n_r)}$ with $n = n_1+n_2+\ldots+n_r$ edges is called *star-like tree graph* if it arises from the star graphs $S_{n_1}, S_{n_2}, \ldots, S_{n_r}$ by taking exactly one leaf of each S_{n_i}, $i = 1, 2, \ldots, r$ and identifying them with each other. A *bi-star graph* $B_{(m,n)}$ is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v. The *generalized barbell graph* $B_{l,m,n}$ is a graph obtained by connecting two complete graphs K_l and K_m by a path P_n. The *n-barbell graph* $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge. The *lollipop graph* $L_{m,n}$ is the graph obtained by joining a complete graph K_m to a path P_n, with a bridge.

Consider two copies of paths P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ respectively, join each pair of vertices v_i, u_i, $i = 1, 2, \ldots, n$ with a new edge. The resulting graph is called a *ladder* L_n. A *square grid graph* $G_{n,m}$ is the graph whose vertices correspond to the points in the plane with integer coordinates, x-coordinates being in the range $1, 2, \ldots, n$, y-coordinates being in the range

$1, 2, \ldots, m$, and two vertices are connected by an edge whenever the corresponding points are at distance 1.

The Petersen graph P is shown in Figure 1.1.

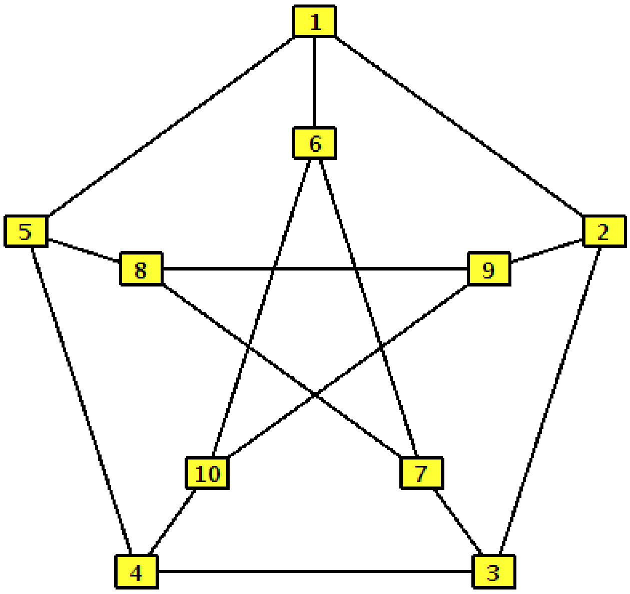

Figure 1.1: Petersen graph P.

1.2 Polynomials

In order to study the nature of the roots of graph polynomials, we need the following results [11, 45]. These results will be used in Sections 2.2, 3.2, 4.2, 5.2 and 6.2.

Theorem 1.2.1 (Descartes rule). *The number of positive roots of the polynomial* $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ *does not exceed the number of sign changes in the sequence* $a_0, a_1, \ldots, a_n$.

Theorem 1.2.2 (De Gua's theorem). *If the polynomial lacks* $2m$ *consecutive terms, that is, the coefficients of these terms vanish, then this polynomial has no less than* $2m$ *imaginary roots. If* $2m + 1$ *consecutive terms are missing, then if they are between terms of different signs, the polynomial has no less than* $2m$

imaginary roots, whereas if the missing terms are between terms of the same sign the polynomial has no less than $2m+2$ imaginary roots.

Theorem 1.2.3 (The intermediate value theorem). *Suppose $f(x)$ is continuous on an interval I and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.*

Here we need the following results which will be used in Sections 2.3, 3.3, 4.3, 5.3 and 6.3. These results are taken from [16, 45].

Theorem 1.2.4.

$$\lim_{n\to\infty} \ln n \left(\frac{\ln n - 1}{\ln n}\right)^n = 0.$$

Theorem 1.2.5. *Let $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, where $a_i \in \mathbb{C}$. Then, inside the circle $|z| = 1 + \max_i |a_i|$, there are exactly n roots of f, multiplicities counted.*

Theorem 1.2.6 (Enestrom-Kakeya theorem). *If $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ has positive real coefficients, then all roots of f lie in the annulus $r \le |z| \le R$, where*

$$r = \min\left\{\frac{a_i}{a_{i+1}} \ : 0 \le i \le n-1\right\} \ \textit{and}\ R = \max\left\{\frac{a_i}{a_{i+1}} \ : 0 \le i \le n-1\right\}.$$

In various problems on stability one has to investigate whether all the roots of a given polynomial belong to the left half-plane, that is, whether the real parts of the roots are negative. The polynomial with this property is said to be *stable*. The *Routh-Hurwitz problem* is : how to find out directly by looking at the coefficients of the polynomial whether it is stable or not. Several different solutions of the problem are known. Throughout this work we use the *Routh-Hurwitz criteria* [45] which is useful to locate the roots of some of the graph polynomials.

Theorem 1.2.7 (Routh-Hurwitz criteria). *Given the polynomial,*

$$P(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n,$$

where the coefficients a_i are real constants, $i = 1, 2, \ldots, n$, define the n Hurwitz *matrices using the coefficients a_i of the above polynomial :*

$$H_1 = \begin{bmatrix} a_1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} \qquad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix},$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_n \end{bmatrix},$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive :

$$\det\, H_j > 0, \quad j = 1, 2, \dots, n.$$

We use the following definitions and results to prove some graph polynomials which are not stable. These definitions and theorems are taken from [41].

Definition 1.2.1. *If $f_n(x)$ is a family of complex polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{Z}(f_n(x))$, $\mathbb{Z}(f_n(x))$ is the set of the roots of the family $f_n(x)$.*

Now, a family $f_n(x)$ of polynomials is a recursive family of polynomials if $f_n(x)$ satisfy a homogeneous linear recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x), \tag{1.1}$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x) \neq 0$. The number k is the order of the recurrence. We can form from equation (1.1) its associated characteristic equation

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2\lambda^{k-2} - \dots - a_k(x) = 0 \tag{1.2}$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity.

If these roots, say $\lambda_1(x), \lambda_2(x), \ldots, \lambda_k(x)$, are distinct, then the general solution to equation (1.1) is known to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^n \tag{1.3}$$

with the usual variant if some of the $\lambda_i(x)$ are repeated. The functions

$$\alpha_1(x), \alpha_2(x), \ldots, \alpha_k(x)$$

are determined from the initial conditions, that is, the k linear equations in the α_i obtained by letting $n = 0, 1, \ldots, k-1$ in equation (1.3) or its variant. The details are available in [41]. Beraha, Kahane and Weiss [41] proved the following results on recursive families of polynomials and their roots.

Theorem 1.2.8. *If $f_n(x)$ is a recursive family of polynomials, then a complex number z is a limit of roots of $f_n(x)$ if and only if there is a sequence (z_n) in $\mathbb{C}$ such that $f_n(z_n) = 0$ for all n and $z_n \to z$ as $n \to \infty$.*

Theorem 1.2.9. *Under the non-degeneracy requirements that in equation (1.3) no $\alpha_i(x)$ is identically zero and that for no pair $i \neq j$ is it true that $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either*

(1) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(2) for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.

Corollary 1.2.10 (see [15])**.** *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \ldots + \alpha_k(x)\lambda_k(x)^n \tag{1.4}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then the limits of roots of $f_n(x)$ are exactly those z satisfying (i) or (ii) of Theorem 1.2.9.

CHAPTER 2

DOMINATION STABLE GRAPHS

In this chapter we mainly deals with the domination polynomial of graphs. In Section 2.1 we define domination polynomial of a graph and find domination polynomial of some graphs. In Subsection 2.1.1 we find domination polynomial of the square of some graphs. In Section 2.2, we define domination roots and introduce a new concept, $\mathtt{d}$-number of a graph and also find $\mathtt{d}$-number of some graphs. Bounds for domination roots of some graphs are included in Section 2.3. We introduce $\mathtt{d}$-stable graphs and $\mathtt{d}$-unstable graphs in Section 2.4 and include some examples of $\mathtt{d}$-stable and $\mathtt{d}$-unstable graphs.

2.1 Domination polynomial of graphs

We begin this section by defining the domination polynomial of a graph.

Definition 2.1.1 ([34]). *Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V$ is a dominating set if every vertex $v \in V - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. Let $\mathcal{D}(G, i)$ be the family of dominating sets of G with cardinality i and let $\mathtt{d}(G, i) = |\mathcal{D}(G, i)|$. The polynomial*

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} \mathtt{d}(G, i) x^i$$

is defined as domination polynomial of G.

Example 2.1.1. *Consider the graph G in Figure 2.1. Clearly, $\gamma(G) = 2$, $\mathtt{d}(G, 2) = 1$, $\mathtt{d}(G, 3) = 6$, $\mathtt{d}(G, 4) = 11$, $\mathtt{d}(G, 5) = 6$ and $\mathtt{d}(G, 6) = 1$. Therefore the domination polynomial of G is $D(G, x) = x^6 + 6x^5 + 11x^4 + 6x^3 + x^2$.*

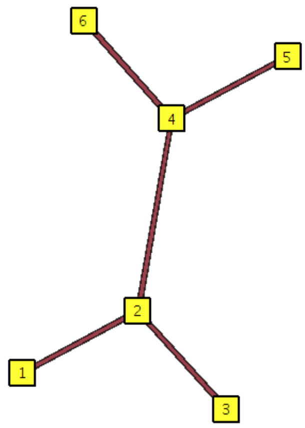

Figure 2.1: Graph G.

In this sequel we state the following results without proof due to Saeid Alikhani [34, 35], Janson I. Brown and Julia Tufts [16].

Results 2.1.2. *Some important results in [34, 35] are follows :*

(1) If G and H are isomorphic graphs then $D(G, x) = D(H, x)$.

(2) Let G_1 and G_2 be graphs of order n_1 and n_2 respectively. Then $D(G_1 \vee G_2, x) = ((1 + x)^{n_1} - 1)((1 + x)^{n_2} - 1) + D(G_1, x) + D(G_2, x)$.

(3) $D(K_n, x) = (1 + x)^n - 1$.

(4) $D(S_n, x) = x^n + x(1 + x)^n$.

(5) $D(K_{m,n}, x) = ((1 + x)^m - 1)((1 + x)^n - 1) + x^m + x^n$.

(6) $D(G \circ K_1, x) = x^n(x + 2)^n$, where n is the order of G.

(7) $D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}}(x^2 + 3x + 1)^{\frac{n}{3}}$, where n is the order of G.

(8) $D(G \circ H, x) = (x(1 + x)^n + D(H, x))^m$, where G and H are nonempty graphs of order m and n respectively.

(9) $D(G,x) = x^n(x+2)^n$ *if and only if* $G = H \circ K_1$, *for some graph* H *of order* n.

(10) *Let* G *be a connected graph with exactly two distinct domination roots. Then* $D(G,x) = x^n(x+2)^n$, *where* n *is a natural number.*

(11) *Let* G *be a connected graph of order* n. *Then* $\mathbb{Z}(D(G,x)) = \{0, \frac{-3\pm\sqrt{5}}{2}\}$ *if and only if* $G = H \circ \overline{K_2}$ *for some graph* H.

(12) *Let* G *be a graph without pendent vertices, and* $D(G,x)$ *be its domination polynomial. If* $D(G,x)$ *has exactly three distinct roots, then* $\mathbb{Z}(D(G,x)) \subseteq \{0, -2 \pm i\sqrt{2}, \frac{-3\pm i\sqrt{3}}{2}\}$.

(13) *Let* G *be a graph and* $D(G,x)$ *be its domination polynomial. If* $D(G,x)$ *has exactly three distinct roots, then* $\mathbb{Z}(D(G,x)) \subset \left\{-2, 0, \frac{-3\pm\sqrt{5}}{2}, -2 \pm i\sqrt{2}, \frac{-3\pm i\sqrt{3}}{2}\right\}$.

Results 2.1.3. *Some important results in [16] are follows :*

(1) $D(B_n,x) = ((1+x)^n - nx - 1)^2 + nx^2(2(1+x)^{n-1} - 1) + 2x^n$.

(2) *The bipartite cocktail party graphs* B_n *have domination roots in the right half-plane for* $n \geq 10$.

(3) *The closure of the domination roots is the whole complex plane.*

Now we compute the domination polynomials of some special types of graphs.

Theorem 2.1.4. *For* $n \geq 2$ *the domination polynomial of the lollipop graph* $L_{n,1}$ *is*

$$D(L_{n,1},x) = x\left((1+x)^n + (1+x)^{n-1} - 1\right).$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the complete graph K_n and v be the path P_1 and let v is adjacent to v_1. Clearly, $\gamma(L_{n,1}) = 1$ and $\mathtt{d}(L_{n,1}, 1) = 1$. For $2 \leq i \leq n-1$, the only non dominating sets of i vertices of $L_{n,1}$ are the subset of $\{v_2, v_3, \ldots, v_n\}$. Therefore $\mathtt{d}(L_{n,1}, i) = \binom{n+1}{i} - \binom{n-1}{i}$. Also $\mathtt{d}(L_{n,1}, n) = n+1$ and $\mathtt{d}(L_{n,1}, n+1) = 1$. Hence $D(L_{n,1},x) = x\left((1+x)^n + (1+x)^{n-1} - 1\right)$. □

Theorem 2.1.5. *For* $n \geq 2$ *the domination polynomial of the Dutch windmill graph* G_3^n *is*

$$D(G_3^n, x) = x(1+x)^{2n} + (2x+x^2)^n.$$

Proof. Let v be the central vertex of G_3^n. It is clear that $\{v\}$ is the only dominating set of cardinality 1. Therefore $\gamma(G_3^n) = 1$ and $\mathtt{d}(G_3^n, 1) = 1$. For $1 \leq i \leq 2n+1$, the number of ways of selecting dominating sets of cardinality i which containing the center is $\binom{2n}{i-1}$. Also there are 2^n dominating sets of cardinality n which does not contain the central vertex v. Similarly there are $\binom{n}{i}2^{n-i}$ ways to select a dominating set of cardinality $n+i$, $1 \leq i \leq n$, which does not contain the central vertex v. Therefore $D(G_3^n, x) = x(1+x)^{2n} + (2x+x^2)^n$. □

Theorem 2.1.6. *The domination polynomial of the generalized barbell graph $B_{m,n,1}$ is*

$$D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]\,.$$

Proof. Without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_m and u_n are adjacent. Observe that there is no one element dominating sets and $\{v_i, u_j\}$ is a dominating set of cardinality 2 of $B_{m,n,1}$. Therefore $\gamma(B_{m,n,1}) = 2$ and $\mathtt{d}(B_{m,n,1}, 2) = mn$. Also observe that for $2 \leq i \leq m+n$, a subset S of vertices of $B_{m,n,1}$ of cardinality i is not a dominating set if either $S \subset V$ or $S \subset U$. Therefore $\mathtt{d}(B_{m,n,1}, i) = \binom{m+n}{i} - \binom{n}{i} - \binom{m}{i}$; for $2 \leq i \leq m$, $\mathtt{d}(B_{m,n,1}, i) = \binom{m+n}{i} - \binom{n}{i}$; for $m+1 \leq i \leq n$ and $\mathtt{d}(B_{m,n,1}, i) = \binom{m+n}{i}$; for $n+1 \leq i \leq m+n$. This implies that $D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]\,.$ □

Corollary 2.1.7. *The domination polynomial of the n-barbell graph $B_{n,1}$ is*

$$D(B_{n,1}, x) = ((1+x)^n - 1)^2\,.$$

Proof. It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Theorem 2.1.8. *The domination polynomial of the bi-star graph $B_{(m,n)}$ is*

$$D(B_{(m,n)}, x) = x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m.$$

Proof. Without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$, $U = \{u_1, u_2, \ldots, u_n\}$ and $\{u, v\}$ be the vertices of $B_{(m,n)}$ such that u and v are adjacent, every vertices in U are adjacent to u and every vertices in V are adjacent to v. Clearly, there is no one element dominating set. The set $\{u, v\}$

is the only dominating set of cardinality 2 of $B_{(m,n)}$. Therefore $\gamma(B_{(m,n)}) = 2$ and $\mathtt{d}(B_{(m,n)}, 2) = 1$. For $3 \leq i \leq m$, the dominating sets of cardinality i of $B_{(m,n)}$ must contain $\{u, v\}$, and the remaining $i - 2$ elements can have $\binom{m+n}{i-2}$ choices. For $m + 1 \leq i \leq n$, there are $\binom{m+n}{i-2}$ dominating sets of cardinality i of $B_{(m,n)}$ containing $\{u, v\}$ and $\binom{n}{i-m-1}$ dominating sets of cardinality i of $B_{(m,n)}$ containing $V \cup \{u\}$. For $n+1 \leq i \leq m+n-1$, there are $\binom{m+n}{i-2}$ dominating sets of cardinality i of $B_{(m,n)}$ containing $\{u, v\}$, $\binom{n}{i-m-1}$ dominating sets of cardinality i of $B_{(m,n)}$ containing $V \cup \{u\}$ and $\binom{m}{i-n-1}$ dominating sets of cardinality i of $B_{(m,n)}$ containing $U \cup \{v\}$. Also there are $\binom{m+n}{i-2}$ dominating sets of cardinality $(m+n)$ of $B_{(m,n)}$ containing $\{u, v\}$, n dominating sets of cardinality $(m + n)$ of $B_{(m,n)}$ containing $V \cup \{u\}$, m dominating sets of cardinality $(m+n)$ of $B_{(m,n)}$ containing $U \cup \{v\}$ and one dominating set of cardinality $(m + n)$ of $B_{(m,n)}$ not containing $\{u, v\}$. Also $\mathtt{d}(B_{(m,n)}, m+n+1) = m+n+2$ and $\mathtt{d}(B_{(m,n)}, m+n+2) = 1$. That is,

$$\mathtt{d}(B_{(m,n)}, i) = \begin{cases} 1 & \text{if } i = 2, m+n+2, \\ \binom{m+n}{i-2} & \text{if } 3 \leq i \leq m, \\ \binom{m+n}{i-2} + \binom{n}{i-m-1} & \text{if } m+1 \leq i \leq n, \\ \binom{m+n}{i-2} + \binom{n}{i-m-1} + \binom{m}{i-n-1} & \text{if } n+1 \leq i \leq m+n-1, \\ \binom{m+n}{i-2} + n + m + 1 & \text{if } i = m+n, \\ m+n+2 & \text{if } i = m+n+1. \end{cases}$$

Hence $D(B_{(m,n)}) = x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m$. $\square$

Corollary 2.1.9. *The domination polynomial of the bi-star graph $B_{(n,n)}$ is*

$$D(B_{(n,n)}, x) = (x(1+x)^n + x^n)^2 .$$

Theorem 2.1.10. *Let K_m and K_n be the complete graphs. Then the domination polynomial of $K_m \circ K_n$ is*

$$D(K_m \circ K_n, x) = \left((1+x)^{n+1} - 1\right)^m .$$

Proof. The proof follows from (8) in Results 2.1.2. $\square$

Corollary 2.1.11. *For $m \geq 2$, the domination polynomial of $Q(m, n)$ is*

$$D(Q(m, n), x) = ((1+x)^n - 1)^m .$$

Proof. It follows from the fact that $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. $\square$

2.1.1 Domination polynomial of square of some graphs

In this section we obtain an explicit formula for the domination polynomial of the square of some specific graphs. Next results will give domination polynomial of the square of some graphs with some specified properties.

Theorem 2.1.12. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$D(G^2, x) = \prod_{i=1}^{m} D(G_i^2, x).$$

Proof. We have $G = G_1 \cup G_2 \cup \ldots \cup G_m$, then $G^2 = G_1^2 \cup G_2^2 \cup \ldots \cup G_m^2$. Therefore $D(G^2, x) = \prod_{i=1}^{m} D(G_i^2, x)$. $\square$

Theorem 2.1.13. *Let G be a graph of order n. Then the domination polynomial of G^2 is $(1+x)^n - 1$ if and only if $D(G) \leq 2$.*

Proof. It follows from the facts that the complete graph K_n is $\mathcal{D}-$unique [34] and the graphs G^2 and the complete graph K_n are isomorphic if and only if $D(G) \leq 2$. $\square$

Corollary 2.1.14. *For the complete graph K_n,*

$$D(K_n^2, x) = (1+x)^n - 1.$$

Corollary 2.1.15. *For the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$,*

$$D(K_{n_1,n_2,\ldots,n_m}^2, x) = (1+x)^N - 1,$$

where $N = n_1 + n_2 + \ldots + n_m$.

Corollary 2.1.16. *For the complete bipartite graph $K_{m,n}$,*

$$D(K_{m,n}^2, x) = (1+x)^{m+n} - 1.$$

Corollary 2.1.17. *For the star graph S_n,*

$$D(S_n^2, x) = (1+x)^{n+1} - 1.$$

Corollary 2.1.18. *For the wheel graph W_n,*

$$D(W_n^2, x) = (1+x)^n - 1.$$

Corollary 2.1.19. *Let H and G be two graphs of order m and n respectively. Then the domination polynomial of the square of $H \vee G$ is*

$$D((H \vee G)^2, x) = (1+x)^{m+n} - 1.$$

Corollary 2.1.20. *For the complete graphs K_m and K_n,*

$$D((K_m \Box K_n)^2, x) = (1+x)^{mn} - 1.$$

Corollary 2.1.21. *Let P be the Petersen graph, then*

$$D(P^2, x) = (1+x)^{10} - 1.$$

Corollary 2.1.22. *The domination polynomial of the square of the Dutch windmill graph G_3^n is*

$$D(G_3^{n^2}, x) = (1+x)^{2n+1} - 1.$$

Corollary 2.1.23. *The domination polynomial of the square of the lollipop graph $L_{n,1}$ is*

$$D(L_{n,1}^2, x) = (1+x)^{n+1} - 1.$$

Lemma 2.1.24. *Let H and G be two graphs. Then $(H \vee G)^2$ and $H^2 \vee G^2$ are isomorphic if and only if $D(H)$ and $D(G)$ are less than or equal to two.*

Proof. Observe that $(H \vee G)^2$ is complete. Therefore it is enough to show that $H^2 \vee G^2$ is complete if and only if $D(H)$ and $D(G)$ are less than or equal to two. Suppose $D(H)$ and $D(G)$ are less than or equal to two. Then H^2 and G^2 are complete. Therefore $H^2 \vee G^2$ is complete.
Conversely, suppose that $H^2 \vee G^2$ is complete. Suppose $D(H) > 2$, H^2 is not complete. Therefore there exist a vertex v of H such that v is not adjacent with all vertices of H^2. This implies that, the vertex v is not adjacent with all vertices of $H^2 \vee G^2$. That is, then $H^2 \vee G^2$ is not complete, which is a contradiction. Therefore $D(H) \leq 2$. □

Theorem 2.1.25. *Let H and G be two graphs. Then the domination polynomial of $(H \vee G)^2$ and $H^2 \vee G^2$ are equal if and only if $D(H)$ and $D(G)$ are less than or equal to two.*

Corollary 2.1.26. *Let G be a graph of order n such that G^2 is a complete graph. Then $D(K_m \vee G^2, x) = D((K_m \vee G)^2, x) = (1+x)^{m+n} - 1$.*

Theorem 2.1.27. *Let B_n be the bipartite cocktail party graph. Then for $n \geq 3$, $D(B_n^2, x) = (1+x)^{2n} - 2nx - 1$.*

Proof. It is clear that $\gamma(B_n^2) = 2$ and for $2 \leq i \leq n$, any subset of vertices of B_n^2 of cardinality i is a dominating set. Therefore $D(B_n^2, x) = (1+x)^{2n} - 2nx - 1$. □

Remark 2.1.28. *Note that $B_1 = 2K_1$ and $B_2 = 2K_2$, so $D(B_1^2, x) = x^2$ and $D(B_2^2, x) = x^2(x+2)^2$.*

Theorem 2.1.29. *The domination polynomial of the square of the generalized barbell graph $B_{m,n,1}$ is*

$$D(B_{m,n,1}^2, x) = (1+x)^{m+n} - (1+x)^{n-1} - (1+x)^{m-1} + 1.$$

Proof. Without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_m and u_n are adjacent. Then $\{v_m\}$ and $\{u_n\}$ are the only dominating sets of cardinality 1 of $B_{m,n,1}^2$. Therefore $\gamma(B_{m,n,1}^2) = 1$ and $\mathtt{d}(B_{m,n,1}^2, 1) = 2$. For $2 \leq i \leq m+n$, a subset S of vertices of $B_{m,n,1}^2$ of cardinality i is not a dominating set if either $S \subset V - \{v_m\}$ or $S \subset U - \{u_n\}$. Therefore $\mathtt{d}(B_{m,n,1}^2, i) = \binom{m+n}{i} - \binom{n-1}{i} - \binom{m-1}{i}$; for $2 \leq i \leq m-1$, $\mathtt{d}(B_{m,n,1}^2, i) = \binom{m+n}{i} - \binom{n-1}{i}$; for $m \leq i \leq n-1$ and $\mathtt{d}(B_{m,n,1}^2, i) = \binom{m+n}{i}$; for $n \leq i \leq m+n$. This implies that $D(B_{m,n,1}^2, x) = (1+x)^{m+n} - (1+x)^{n-1} - (1+x)^{m-1} + 1$. □

Corollary 2.1.30. *Let $B_{n,1}$ be the n-barbell graph. Then for all n,*

$$D(B_{n,1}^2, x) = (1+x)^{2n} - 2(1+x)^{n-1} + 1.$$

Proof. It follows from the fact that the square of the n-barbell graph $B_{n,1}$ and the square of the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Theorem 2.1.31. *The domination polynomial of the square of the bi-star graph $B_{(m,n)}$ is*

$$D(B_{(m,n)}^2, x) = (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1.$$

Proof. It follows from the fact that the square of the bi-star graph $B_{(m,n)}$ and the square of the generalized barbell graph $B_{m+1,n+1,1}$ are isomorphic. □

Theorem 2.1.32. *Let K_m and K_n be the complete graphs. Then for $m \geq 2$ the domination polynomial of the square of $K_m \circ K_n$ is*

$$D((K_m \circ K_n)^2, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1.$$

Proof. The proof is similar to the proof of the Theorem 2.1.6. □

Corollary 2.1.33. *For $m \geq 2$ the domination polynomial of the square of $Q(m,n)$ is*

$$D(Q^2(m,n), x) = (1+x)^{mn} - m(1+x)^{n-1} + m - 1.$$

Proof. It follows from the fact that $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

2.2 d-number of graphs

In this section we find the number of real roots of domination polynomial of some graphs. First we define domination root of a graph.

Definition 2.2.1. *Let G be a graph with domination polynomial $D(G,x)$. A root of $D(G,x)$ is called a domination root of G and the set of all the domination roots of G is denoted by $\mathbb{Z}(D(G,x))$.*

Remark 2.2.1. *Let G be a graph with domination polynomial $D(G,x)$. Since the coefficients of $D(G,x)$ are positive, $(0,\infty)$ is a zero-free interval for $D(G,x)$.*

We mainly focus on the number of real domination roots of some specific graphs. So we introduce a new definition as follows.

Definition 2.2.2. *Let G be a graph. The number of distinct real domination roots of the graph G is called d-number of G and is denoted by $\mathtt{d}(G)$.*

Example 2.2.2. *The domination polynomial of the graph G in Figure 2.1 is*

$$D(G,x) = x^6 + 6x^5 + 11x^4 + 6x^3 + x^2.$$

The domination roots of G are $-2.618033989, -0.3819660113, 0,$ all have multiplicity 2. Therefore $\mathtt{d}(G) = 3$.

Theorem 2.2.3. *For any graph G, $\mathtt{d}(G) \geq 1$.*

Proof. It follows from the fact that 0 is a domination root of any graph. $\square$

Theorem 2.2.4. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}(G) \leq \sum_{i=1}^{m} \mathtt{d}(G_i) - m + 1.$$

Proof. It follows from the fact that $D(G,x) = \prod_{i=1}^{m} D(G_i, x)$. $\square$

Theorem 2.2.5. *If G and H are isomorphic, then $\mathtt{d}(G,x) = \mathtt{d}(H,x)$.*

Proof. It follows from the fact that if G and H are isomorphic, then $D(G,x) = D(H,x)$. $\square$

Theorem 2.2.6. *If G has exactly two distinct domination roots, then $\mathtt{d}(G) = 2$.*

Proof. It follows from the fact that 0 is a domination root and complex roots occurs in conjugate pairs. $\square$

Theorem 2.2.7 (Ore's theorem [44])**.** *If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Remark 2.2.8. *Suppose $D(G,x)$ has exactly two distinct domination roots. Since 0 is a domination root with multiplicity $\gamma(G)$ for every graph G, we have*

$$D(G,x) = x^{\gamma(G)}(x+a)^{n-\gamma(G)}$$

where $-a$ is the remaining domination root of G. Then the coefficient of x^{n-1} is $a(n-\gamma(G))$. Therefore $a(n-\gamma(G))$ is a positive integer. Since n and $\gamma(G)$ are positive integers and $n-\gamma(G) > 0$, we have a is a positive integer. Since G is connected, the coefficient of x^{n-1} is n. Hence, $n = a(n-\gamma(G))$. Since $\gamma(G) \geq 1$, we must have $a \geq 2$. By Ore's theorem 2.2.7, $\gamma(G) \leq \frac{n}{2}$. Therefore $\frac{n(a-1)}{a} \leq \frac{n}{2}$. This implies that $a \leq 2$. Therefore $D(G,x) = x^{\gamma(G)}(x+2)^{n-\gamma(G)}$. This implies $\mathbb{Z}(D(G,x) = \{0,-2\}$, that is, $\mathtt{d}(G) = 2$.

Theorem 2.2.9. *Let G be a graph without pendent vertices. If G has exactly three distinct domination roots, then $\mathtt{d}(G) = 1$.*

Proof. It follows from (12) in Results 2.1.2. $\square$

Theorem 2.2.10. *For all n we have the following :*

$$\mathtt{d}(K_n) = \begin{cases} 1 & \text{; if } n \text{ is odd,} \\ 2 & \text{; if } n \text{ is even.} \end{cases}$$

Proof. We have the domination polynomial of K_n is

$$D(K_n, x) = (1+x)^n - 1. \tag{2.1}$$

The result follows from the transformation $y = 1 + x$ in equation (2.1). □

Remark 2.2.11. *For even n the nonzero real domination root of the complete graph K_n is -2 with multiplicity 1.*

Theorem 2.2.12. *For any graph G, $\mathtt{d}(G \circ K_1) = 2$.*

Proof. By (6) in Results 2.1.2, we have $D(G \circ K_1, x) = x^n(x+2)^n$, where n is the order of G. Therefore $\mathtt{d}(G \circ K_1) = 2$. □

Theorem 2.2.13. *For any graph G, $\mathtt{d}(G \circ \overline{K_2}) = 3$.*

Proof. By (7) in Results 2.1.2 we have $D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}}(x^2+3x+1)^{\frac{n}{3}}$, where n is the order of G. Therefore $\mathbb{Z}(D(G,x)) = \{0, \frac{-3\pm\sqrt{5}}{2}\}$. This implies that $\mathtt{d}(G \circ \overline{K_2}) = 3$. □

Theorem 2.2.14. *For all n the $\mathtt{d}$-number of the star graph S_n is*

$$\mathtt{d}(S_n) = \begin{cases} 2 & \text{; if } n \text{ is odd,} \\ 3 & \text{; if } n \text{ is even.} \end{cases}$$

Proof. We have the domination polynomial of S_n is

$$D(S_n, x) = x(1+x)^n + x^n. \tag{2.2}$$

Therefore it suffices to prove that $f(x) = (1+x)^n + x^{n-1}$ has exactly one real root if n is odd and two real roots if n is even. But the number of real roots of $f(x)$ is equal to the number of real roots of $g(x) = \left(1 + \frac{1}{x}\right)^n + \frac{1}{x}$. Again the number of real roots of $g(x)$ is equal to the number of real roots of $g(\frac{1}{x}) = (1+x)^n + x$. Consider $g(\frac{1}{y-1}) = y^n + y - 1$, we find the number of real roots of $h(y) = y^n + y - 1$. We have $h(0) = -1 < 0$ and $h(1) = 1 > 0$. Therefore by the intermediate value theorem 1.2.3, $h(y)$ has at least one real root in $(0, 1)$. Also by De Gua's rule 1.2.2 for imaginary roots, there are at least $n-1$ complex roots for odd n and there are at least $n-2$ complex roots for even n. Therefore we can conclude that $h(y)$ has exactly one real root for odd n and two real roots for even n. It remains to show that all the real roots of $f(x)$ are distinct. Suppose $a \in \mathbb{R}$ is a double root of $f(x)$. Therefore

$$(1+a)^n + a^{n-1} = 0 \tag{2.3}$$

$$n(1+a)^{n-1} + (n-1)a^{n-2} = 0. \tag{2.4}$$

From equation (2.3) we get

$$(1+a)^{n-1} = -\frac{a^{n-1}}{1+a} \quad (\text{ since } a \neq -1). \tag{2.5}$$

Putting the value of $(1+a)^{n-1}$ in 2.4 and simplify, we obtain $a = n-1$. Which is a contradiction since $a < 0$. □

Theorem 2.2.15. *For all n the* d*-number of $K_{2n,2n}$ is* 1.

Proof. We have the domination polynomial of $K_{2n,2n}$ is

$$D(K_{2n,2n}, x) = \left((1+x)^{2n} - 1\right)^2 + 2x^{2n}. \tag{2.6}$$

Suppose for $a \in \mathbb{R}$, $((1+a)^{2n} - 1)^2 + 2a^{2n} = 0$, then $((1+a)^{2n} - 1)^2 = -2a^{2n}$. But this is true only if $a = 0$, hence $\mathtt{d}(K_{2n,2n}) = 1$. □

The domination roots of the complete bipartite graph $K_{2n,2n}$ for $1 \leq n \leq 20$ are shown in Figure 2.2.

Theorem 2.2.16. *The* d*-number of $K_{2n+1,2n+1}$ is greater than or equal to* 3 *for all n.*

Proof. We have the domination polynomial of $K_{2n+1,2n+1}$ is

$$D(K_{2n+1,2n+1}, x) = \left((1+x)^{2n+1} - 1\right)^2 + 2x^{2n+1}.$$

It is easy to verify that

$$D\left(K_{2n+1,2n+1}, -\frac{1}{2}\right) = 1 + \frac{1}{2^{2n-1}}\left(\frac{1}{2^{2n+3}} - 1\right) > 0$$

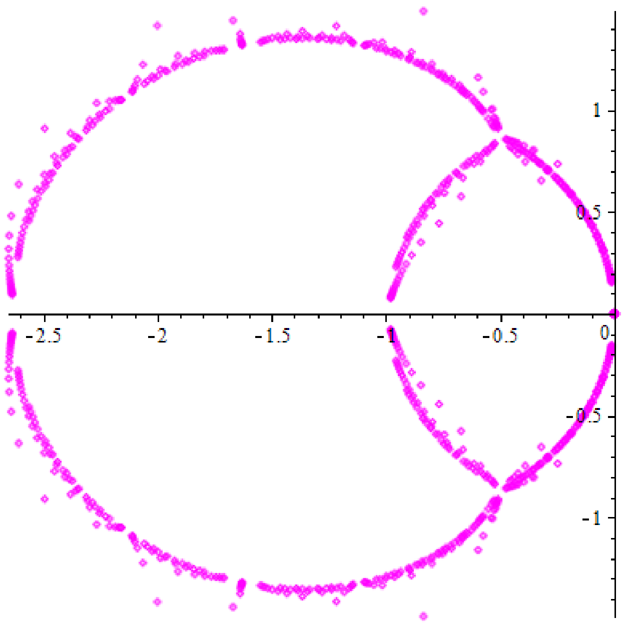

Figure 2.2: Domination roots of $K_{2n,2n}$ for $1 \leq n \leq 20$.

$$\begin{aligned} D\left(K_{2n+1,2n+1}, -1\right) &= -1 < 0 \\ D\left(K_{2n+1,2n+1}, -2\right) &= 2^2(1 - 2^{2n}) < 0 \\ D\left(K_{2n+1,2n+1}, -3\right) &= (2^{2n+1} + 1)^2 - 2 \times 3^{2n+1} > 0 \end{aligned}$$

Therefore by the intermediate value theorem, $K_{2n+1,2n+1}$ has at least one real domination root in $(-1, -\frac{1}{2})$ and at least one in $(-3, -2)$, hence $\mathtt{d}(K_{2n+1,2n+1}) \geq 3$. $\square$

Remark 2.2.17. *Using Maple, we observe that $D(K_{2n+1,2n+1}, x)$ has exactly three distinct real roots for $1 \leq n \leq 600$. So we conjectured that $\mathtt{d}(K_{2n+1,2n+1}) = 3$ for all n.*

The real domination roots of the complete bipartite graph $K_{2n+1,2n+1}$ for $1 \leq n \leq 600$ are shown in Figure 2.3.

Theorem 2.2.18. *For all n the $\mathtt{d}$-number of the Dutch windmill graph G^3_{2n+1} is 1.*

Proof. We have the domination polynomial of the Dutch windmill graph G^3_{2n+1} is

$$D(G^3_{2n+1}, x) = x(1 + x)^{4n+2} + (2x + x^2)^{2n+1}.$$

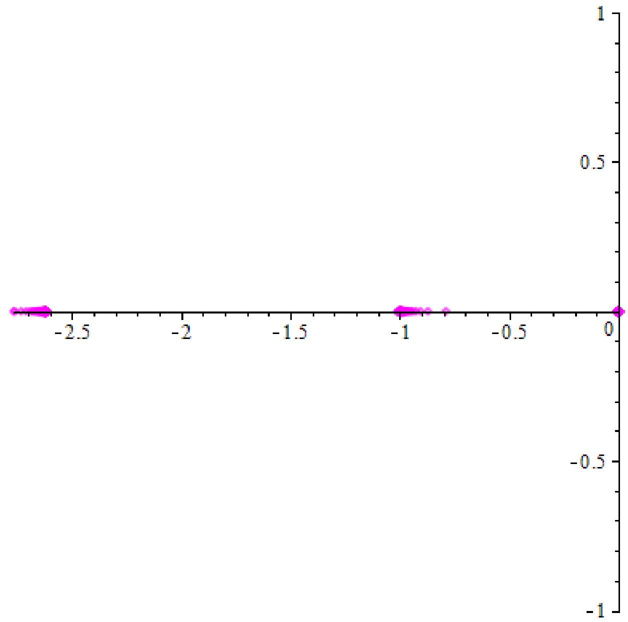

Figure 2.3: Real domination roots of $K_{2n+1,2n+1}$ for $1 \leq n \leq 600$.

Suppose there is a number $a \in \mathbb{R}$ with $a \neq 0$ such that $a(1+a)^{4n+2} + (2a + a^2)^{2n+1} = 0$. Then we have $a < 0$ and by a simple calculation we have

$$a = -\left(1 - \frac{1}{(1+a)^2}\right). \tag{2.7}$$

Suppose $-2 < a < 0$, then the left side of the equation (2.7) is negative but the right side is positive, a contradiction. Now suppose $a \leq -2$. Then the left side of the equation (2.7) is less than or equal to -2 but the right side is greater than -1, a contradiction. Therefore there is no nonzero real domination root for G^3_{2n+1} and hence $\mathtt{d}(G^3_{2n+1}) = 1$. □

Theorem 2.2.19. *The* d*-number of G^3_{2n} is greater than or equal to 3 for all n.*

Proof. We have the domination polynomial of the Dutch windmill graph G^3_{2n} is

$$D(G^3_{2n}, x) = x(1+x)^{4n} + (2x + x^2)^{2n}.$$

It is easy to verify that $D(G^3_{2n}, -1) > 0$ and $D(G^3_{2n}, -2) < 0$. Also if a is a negative real number near to 0, then $D(G^3_{2n}, a) < 0$. Therefore by the intermediate value theorem 1.2.3, we have G^3_{2n} has a real domination root in $(-2,-1)$ and a real domination root in $(-1,0)$ and hence $\mathtt{d}(G^3_{2n+1}) \geq 3$. $\square$

Remark 2.2.20. *Using Maple, we observe that G^3_{2n} has exactly three distinct real domination roots for $1 \leq n \leq 100$. So we conjectured that $\mathtt{d}(G^3_{2n}) = 3$ for all n.*

The real domination roots of the Dutch windmill G^3_{2n} for $1 \leq n \leq 100$ are shown in Figure 2.4.

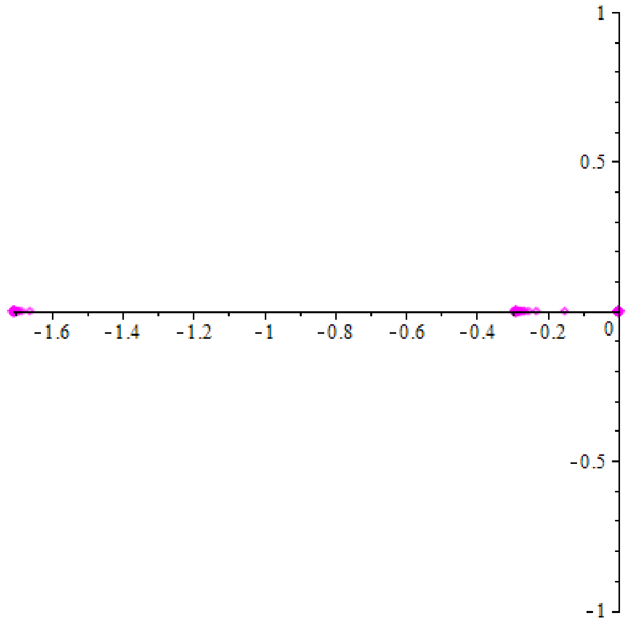

Figure 2.4: Real domination roots of G^3_{2n} for $1 \leq n \leq 100$.

Theorem 2.2.21. *For all $n \geq 2$ the* d*-number of the lollipop graph $L_{n,1}$ is*

$$\mathtt{d}(L_{n,1}) = \begin{cases} 2 & ; \textit{if } n \textit{ is odd,} \\ 3 & ; \textit{if } n \textit{ is even.} \end{cases}$$

Proof. By Theorem 2.1.4 it is enough to prove that $f(y) = y^n + y^{n-1} - 1$ has only one real root if n is odd and has exactly two real roots if n is even. By De Gua's rule 1.2.2 for imaginary roots, there are at least $n-1$ complex roots if n is odd and there are at least $n-2$ complex roots if n is even. Now, $f(0) = -1 < 0$ and $f(1) = 2 > 0$ for all n and $f(-1) = -1 < 0$ and $f(-2) = 2^{n-1} - 1 > 0$

for all even n. Therefore by the intermediate value theorem 1.2.3, we have the result. $\square$

Theorem 2.2.22. *For all m, n the* d*-number of the generalized barbell graph $B_{m,n,1}$ is*

$$\mathtt{d}(B_{m,n,1}) = \begin{cases} 1 & \text{; if both } m \text{ and } n \text{ are odd,} \\ 2 & \text{; otherwise.} \end{cases}$$

Proof. The result follows from the transformation $y = 1 + x$ in the domination polynomial of $B_{m,n,1}$. $\square$

Corollary 2.2.23. *For all n, the* d*-number of the n-barbell graph $B_{n,1}$ is*

$$\mathtt{d}(B_{n,1}) = \begin{cases} 1 & \text{; if } n \text{ is odd,} \\ 2 & \text{; if } n \text{ is even.} \end{cases}$$

Remark 2.2.24. *Note that -2 is the only nonzero real domination root of generalized barbell graph $B_{m,n,1}$ with multiplicity 1 or 2 according as if exactly one of m or n is even or both m and n are even. So -2 is the only nonzero real domination root of n-barbell graph $B_{n,1}$ with multiplicity 2 if n is even.*

Theorem 2.2.25. *For the bi-star graph $B_{(m,n)}$, $m \neq n$ we have the following :*

$$\mathtt{d}(B_{(m,n)}) = \begin{cases} 3 & \text{; if both } m \text{ and } n \text{ are odd,} \\ 5 & \text{; if both } m \text{ and } n \text{ are even,} \\ 4 & \text{; if } m \text{ and } n \text{ have opposite parity.} \end{cases}$$

Proof. By Theorem 2.1.8 we have,

$$\begin{aligned} D(B_{(m,n)}, x) &= x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m \\ &= x^2\left(x^{m+n-2} + (1+x)^{m+n} + x^{m-1}(1+x)^n + x^{n-1}(1+x)^m\right) \\ &= x^2\left(x^{m-1}\left((1+x)^n + x^{n-1}\right) + (1+x)^m\left((1+x)^n + x^{n-1}\right)\right) \\ &= x^2\left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right). \end{aligned}$$

We have there is no real number satisfying both the equations $(1+x)^m + x^{m-1} = 0$ and $(1+x)^n + x^{n-1} = 0$ simultaneously. Therefore it suffices to prove that $(1+x)^m + x^{m-1}$ has exactly one real root for odd m and two real roots for even m. The remaining proof is similar to the proof of Theorem 2.2.14. $\square$

Theorem 2.2.26. *For bi-star graph $B_{(n,n)}$, we have the following :*

$$\mathtt{d}(B_{(n,n)}) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is odd,} \\ 3 & ;\ \textit{if } n \textit{ is even.} \end{cases}$$

Proof. The proof similar to the proof of Theorem 2.2.14. □

Theorem 2.2.27. *For the corona $K_m \circ K_n$, we have the following :*

$$\mathtt{d}(K_m \circ K_n) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is odd,} \\ 1 & ;\ \textit{if } n \textit{ is even.} \end{cases}$$

Proof. It follows from the transformation $y = 1+x$ in the domination polynomial $D(K_m \circ K_n, x)$. □

Corollary 2.2.28. *For the graph $Q(m,n)$, we have the following :*

$$\mathtt{d}(Q(m,n)) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is odd,} \\ 2 & ;\ \textit{if } n \textit{ is even.} \end{cases}$$

Remark 2.2.29. *Note that -2 is the only nonzero real domination root with multiplicity m of the corona $K_m \circ K_n$ if and only if n is odd. So -2 is the only nonzero real domination root with multiplicity m of the graph $Q(m,n)$ if and only if n is even.*

Theorem 2.2.30. *Let G be a graph of order n and diameter D. If $D \leq 2$, then*

$$\mathtt{d}(G^2) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is odd,} \\ 2 & ;\ \textit{if } n \textit{ is even.} \end{cases}$$

Proof. The result follows from the transformation $y = 1 + x$ in the domination polynomial $D(G^2, x)$. □

Corollary 2.2.31. *For all n we have the following :*

$$\mathtt{d}(K_n^2) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is odd,} \\ 2 & ;\ \textit{if } n \textit{ is even.} \end{cases}$$

Corollary 2.2.32. *For all n we have the following :*

$$\mathtt{d}(S_n^2) = \begin{cases} 1 & ; \textit{if } n \textit{ is even,} \\ 2 & ; \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 2.2.33. *For all n we have the following :*

$$\mathtt{d}(W_n^2) = \begin{cases} 1 & ; \textit{if } n \textit{ is odd,} \\ 2 & ; \textit{if } n \textit{ is even.} \end{cases}$$

Corollary 2.2.34. *For all n we have the following :*

$$\mathtt{d}(L_{n,1}^2) = \begin{cases} 1 & ; \textit{if } n \textit{ is even,} \\ 2 & ; \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 2.2.35. *Let H and G be two graphs of order m and n respectively. Then the* d*-number of $(H \vee G)^2$ is*

$$\mathtt{d}((H \vee G)^2) = \begin{cases} 1 & ; \textit{if } m \textit{ and } n \textit{ have opposite parity,} \\ 2 & ; \textit{otherwise.} \end{cases}$$

Corollary 2.2.36. *For all m, n we have the following :*

$$\mathtt{d}((K_m \square K_n)^2) = \begin{cases} 1 & ; \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 2 & \textit{otherwise.} \end{cases}$$

Corollary 2.2.37. *The* d*-number of the square of the Petersen graph P is 2.*

Corollary 2.2.38. *For all m, n the* d*-number of the square of the complete bipartite graph $K_{m,n}$ is*

$$\mathtt{d}(K_{m,n}^2) = \begin{cases} 1 & ; \textit{if } m \textit{ and } n \textit{ have opposite parity,} \\ 2 & ; \textit{otherwise.} \end{cases}$$

Corollary 2.2.39. *For all n the* d*-number of the square of the complete bipartite graph $K_{n,n}$ is 2.*

Corollary 2.2.40. *For all n the* d*-number of the square of the Dutch windmill graph G_3^n is 1.*

In the next theorem we will prove that square of the bipartite cocktail party graph B_n has no nonzero real domination roots for $n \geq 3$.

Theorem 2.2.41. *For $n \geq 3$ the $\mathtt{d}$-number of the square of the bipartite cocktail party graph B_n is 1.*

Proof. We have domination polynomial of the square of the bipartite cocktail party graph B_n is

$$D(B_n^2, x) = (1+x)^{2n} - 2nx - 1.$$

Put $y = 1+x$, then $D(B_n^2, y-1) = f(y) = y^{2n} - 2ny + 2n - 1$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes rule 1.2.1, it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Since there is no variations of the signs of the coefficients of $f(-y)$, $f(y)$ has no negative real roots. This implies that the only real domination root of the square of the bipartite cocktail party graph B_n is zero, hence $\mathtt{d}(B_n^2) = 1$. □

Theorem 2.2.42. *For the generalized barbell graph $B_{m,n,1}$; $m, n \geq 2$, we have the following :*

$$\mathtt{d}(B_{m,n,1}^2) = \begin{cases} 4 & ;\textit{ if both } m \textit{ and } n \textit{ are odd,} \\ 2 & ;\textit{ if both } m \textit{ and } n \textit{ are even,} \\ 3 & ;\textit{ if } m \textit{ and } n \textit{ have opposite parity.} \end{cases}$$

Proof. We have $D(B_{m,n,1}^2, y-1) = f(y) = y^{m+n} - y^{n-1} - y^{m-1} + 1$. The proof of the existence of the positive real roots, the proof is similar to the proof of Theorem 2.2.41. Now consider $f(-y)$. If m and n have same parity, the proof is similar to the proof of Theorem 2.2.41. So we need to consider the remaining two cases :

Case 1 : If m is even and n is odd.

$f(-y) = -y^{m+n} - y^{n-1} + y^{m-1} + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 1, by Descartes rule 1.2.1, it has at most one negative real root. Clearly, $y = -1$ is a negative root of $f(y)$. Therefore $D(B_{m,n,1}^2, x)$ has exactly two nonzero real roots.

Case 2 : If m is odd and n is even.

$f(-y) = -y^{m+n} + y^{n-1} - y^{m-1} + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 3, by Descartes rule, it has at most three negative real roots. Clearly, $y = -1$ is a negative root of $f(y)$. Since the graphs in case

1 and case 2 are isomorphic, we can conclude that $D(B^2_{m,n,1}, x)$ has exactly two nonzero real roots. $\square$

Corollary 2.2.43. *For n-barbell graph $B_{n,1}$; $n \geq 2$, we have the following :*

$$\mathtt{d}(B^2_{n,1}) = \begin{cases} 4 & ;\ \textit{if } n \textit{ is odd,} \\ 2 & ;\ \textit{if } n \textit{ is even.} \end{cases}$$

Remark 2.2.44. *Note that -2 is a domination root of the square of the generalized barbell graph $B_{m,n,1}$ if either m and n are odd or m and n have opposite parity. Hence -2 is a domination root of the square of the n-barbell graph $B_{n,1}$ if n is odd.*

Theorem 2.2.45. *For the bi-star graph $B_{(m,n)}$ we have the following :*

$$\mathtt{d}(B^2_{(m,n)}) = \begin{cases} 2 & ;\ \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 4 & ;\ \textit{if both } m \textit{ and } n \textit{ are even,} \\ 3 & ;\ \textit{if } m \textit{ and } n \textit{ have opposite parity.} \end{cases}$$

Proof. Suppose $m \leq n$. We have $D(B^2_{(m,n)}, x) = (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1$. Put $y = 1+x$, then $D(B^2_{(m,n)}, y-1) = f(y) = y^{m+n+2} - y^n - y^m + 1$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes rule 1.2.1, it has at most two positive real roots. Clearly, $y = 1$ is a root of $f(y)$. Therefore $f(y)$ has exactly 2 positive real roots. Since $y = 1$ is a simple root, $D(B^2_{(m,n)}, x)$ has a nonzero real root. Now consider $f(-y)$.

Case 1 : If m and n are even.

$f(-y) = y^{m+n+2} - y^n - y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 2, by Descartes rule, it has at most two negative real roots. Clearly, $y = -1$ is a negative root of $f(y)$. Therefore $f(y)$ has exactly 2 negative real roots. Since $y = -1$ is a simple root, $D(B^2_{(m,n)}, x)$ has exactly three nonzero real roots.

Case 2 : If m and n are odd.

$f(-y) = y^{m+n+2} + y^n + y^m + 1$. There is no sign changes, $f(y)$ has no negative real roots. Therefore $D(B_{(m,n)}, x)$ has exactly one nonzero real root.

Case 3 : If m is odd and n is even.

$f(-y) = -y^{m+n+2} - y^n + y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 1, by Descartes rule, it has at most one negative real roots. Clearly, $y = -1$ is a negative root of $f(y)$. Therefore $D(B^2_{(m,n)}, x)$ has exactly two nonzero real roots.

Case 4 : If m is even and n is odd.

$f(-y) = -y^{m+n+2} + y^n - y^m + 1$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 3, by Descartes rule, it has at most three negative real roots. Clearly, $y = -1$ is a negative root of $f(y)$. Since the graphs in case 3 and case 4 are isomorphic, we can conclude that $D(B^2_{(m,n)}, x)$ has exactly two nonzero real roots. □

Remark 2.2.46. *Note that -2 is a domination root of the square of the bi-star graph $B^2_{(m,n)}$ if either m and n are even or m and n have opposite parity.*

Theorem 2.2.47. *For $m \geq 2$ and $n \geq 1$ we have the following :*

$$\mathtt{d}((K_m \circ K_n)^2) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is odd,} \\ 3 & ;\ \textit{if } m \textit{ is odd and } n \textit{ is even,} \\ 4 & ;\ \textit{if both } m \textit{ and } n \textit{ are even.} \end{cases}$$

Proof. The proof is similar to the proof of the Theorem 2.2.42. □

Corollary 2.2.48. *For $m \geq 2$ and $n \geq 1$ we have the following :*

$$\mathtt{d}(Q^2(m,n)) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is even,} \\ 3 & ;\ \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 4 & ;\ \textit{if } m \textit{ is even and } n \textit{ is odd.} \end{cases}$$

Remark 2.2.49. *Note that -2 is a domination root of the square of $K_m \circ K_n$ if both m and n are even. So -2 is a domination root of the square of $Q(m,n)$ if m is even and n is odd.*

2.3 Bounds for the domination roots of some graphs

In this section we estimate the bounds for the domination roots of some graphs.

Theorem 2.3.1. *All the domination roots of the complete graph K_n lie on the unit circle with center $(-1, 0)$.*

Proof. It follows from the fact that all the n^{th} roots of unity lie on the unit circle with center $(0, 0)$. $\square$

The domination roots of the complete graph K_n for $1 \leq n \leq 30$ are shown in Figure 2.5

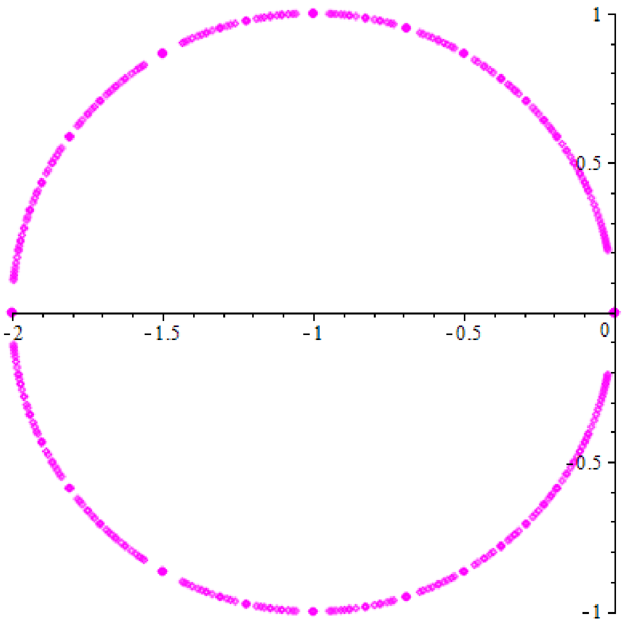

Figure 2.5: Domination roots of K_n for $1 \leq n \leq 30$.

Theorem 2.3.2. *The lollipop graph $L_{n,1}$ has no domination roots on the unit circle centered at $(-1, 0)$.*

Proof. By Theorem 2.1.4 it is enough to prove that $f(y) = y^n + y^{n-1} - 1$ has no roots on the unit circle centered at the origin. Suppose $f(y)$ has a root z such that $|z| = 1$. Let $z = \exp(i\theta)$, then $\exp(in\theta) + \exp(i(n-1)\theta) = 1$. These two complex numbers $\exp(in\theta)$ and $\exp(i(n-1)\theta)$ have modulus 1 and must be conjugates to sum to 1. The only possible pair is $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ which can be written as $\exp(\pm i\frac{\pi}{3})$. Focusing our attention on the angles $n\theta$ and $(n-1)\theta$, we see there exist integers a and b satisfying the pair of equations

$$\begin{aligned} n\theta \quad &= \pm \tfrac{\pi}{3+2\pi a} \\ (n-1)\theta \quad &= \mp \tfrac{\pi}{3+2\pi b}. \end{aligned}$$

Solving both equations for θ and equating the results gives

$$2n - 1 = \pm 6(n(b-a) + a) \tag{2.8}$$

The left side of the equation (2.8) is odd and but the right side is even, this is a contradiction, hence the result. $\square$

Theorem 2.3.3. *All the nonzero domination roots of the lollipop graph $L_{n,1}$ lie inside the circle with center $(-1, 0)$ and radius 2.*

Proof. We have $D(L_{n,1}, y-1) = y^n + y^{n-1} - 1$. Here $\max_i |a_i| = 1$, where a_i's are the coefficients of $D(L_{n,1}, y-1)$ for $i = 0, 1, \ldots, n$. Then by Theorem 1.2.5 we have the result. $\square$

The domination roots of the lollipop graph $L_{n,1}$ for $1 \leq n \leq 30$ are shown in Figure 2.6.

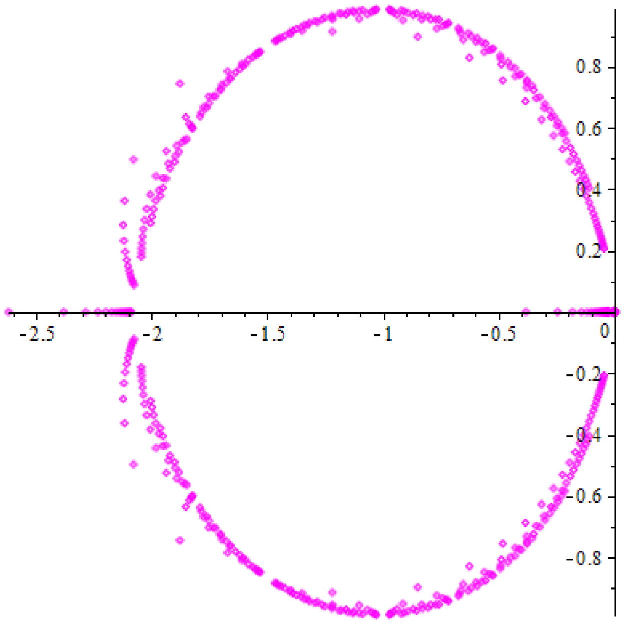

Figure 2.6: Domination roots of $L_{n,1}$ for $1 \leq n \leq 30$.

Theorem 2.3.4. *All the domination roots of the corona $K_m \circ K_n$ lie on the unit circle centered at $(-1, 0)$.*

Proof. It follows from the fact that all $(n+1)^{th}$ roots of unity lie on the unit circle centered at $(0,0)$. □

Corollary 2.3.5. *All the domination roots of the graph $Q(m,n)$ lie on the unit circle centered at* $(-1,0)$.

Theorem 2.3.6. *All the domination roots of the generalized barbell graph $B_{m,n,1}$ lie on the unit circle centered at* $(-1,0)$.

Proof. It follows from the fact that all n^{th} and m^{th} roots of unity lie on the unit circle centered at $(0,0)$. □

Corollary 2.3.7. *All the domination roots of the n-barbell graph $B_{n,1}$ lie on the unit circle centered at* $(-1,0)$.

Next, we prove that there are real domination roots of arbitrarily large modulus.

Theorem 2.3.8. *The domination polynomial of the bi-star graph, $B_{(m,n)}$ has a real root in the interval $(-2m,-\ln m)$ and a real root in the interval $(-2n,-\ln n)$ for m,n sufficiently large.*

Proof. We have $D(B_{(m,n)},x) = x^2\left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right)$. Therefore it suffices to prove that

$$f_n(x) = x(1+x)^n + x^n$$

has a real root in the interval $(-2n,-\ln n)$ for n sufficiently large. But

$$\begin{aligned} f_n(x) &= x(1+x)^n + x^n \\ &= x\left(1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n-1}x^{n-1}+x^n\right)+x^n \\ &= x+\binom{n}{1}x^2+\binom{n}{2}x^3+\ldots+(n+1)x^n+x^{n+1}. \end{aligned}$$

We claim that $f_n(-2n)$ has sign $(-1)^{n+1}$ and $f_n(-\ln n)$ has sign $(-1)^n$ for sufficiently large n. Therefore by the intermediate value theorem 1.2.3, for sufficiently large n, $f_n(x)$ has a real root in in the interval $(-2n,-\ln n)$. Now consider

$$\begin{aligned} f_n(-2n) &= -2n+\binom{n}{1}(-2n)^2+\binom{n}{2}(-2n)^3+\ldots+(n+1)(-2n)^n+(-2n)^{n+1} \\ &= (-2n)^{n+1}\left(\frac{(-1)^n}{(2n)^n}+\frac{(-1)^{n-1}\binom{n}{1}}{(2n)^{n-1}}+\frac{(-1)^{n-2}\binom{n}{2}}{(2n)^{n-2}}+\ldots+\frac{\binom{n}{2}}{(2n)^2}-\frac{n+1}{2n}+1\right). \end{aligned}$$

To prove $f_n(-2n)$ has sign $(-1)^{n+1}$ for sufficiently large n, it suffices to show that

$$\frac{(-1)^n}{(2n)^n}+\frac{(-1)^{n-1}\binom{n}{1}}{(2n)^{n-1}}+\frac{(-1)^{n-2}\binom{n}{2}}{(2n)^{n-2}}+\ldots+\frac{\binom{n}{2}}{(2n)^2}-\frac{n+1}{2n}<1.$$

We have

$$\frac{(-1)^n}{(2n)^n}+\frac{(-1)^{n-1}\binom{n}{1}}{(2n)^{n-1}}+\frac{(-1)^{n-2}\binom{n}{2}}{(2n)^{n-2}}+\ldots+\frac{\binom{n}{2}}{(2n)^2}-\frac{n+1}{2n}<\frac{1}{(2n)^n}+\frac{\binom{n}{1}}{(2n)^{n-1}}++\frac{\binom{n}{2}}{(2n)^{n-2}}+\ldots+\frac{\binom{n}{2}}{(2n)^2}+\frac{n+1}{2n}.$$

But

$$\begin{aligned}
\frac{1}{(2n)^n}+\frac{\binom{n}{1}}{(2n)^{n-1}}+\ldots+\frac{\binom{n}{2}}{(2n)^2}+\frac{n+1}{2n} &< \frac{1}{2^n n!}+\frac{1}{2^{n-1}(n-1)!}+\ldots+\frac{1}{2^2.2!}+\frac{1}{2}+\frac{1}{2n}\\
&= \frac{1}{2}\left(\frac{1}{2^n\frac{n!}{2}}+\frac{1}{2^{n-1}\frac{(n-1)!}{2}}+\ldots+\frac{1}{2^2}+1\right)+\frac{1}{2n}\\
&< \frac{1}{2}\left(\frac{1}{2^n}+\frac{1}{2^{n-1}}+\ldots+\frac{1}{2^3}+\frac{1}{2^2}\right)+\frac{1}{2}+\frac{1}{2n}\\
&= \frac{1}{8}\left(\frac{1}{2^{n-2}}+\frac{1}{2^{n-3}}+\ldots+\frac{1}{2}+1\right)+\frac{1}{2}+\frac{1}{2n}\\
&< \frac{1}{8}\left(\frac{1}{1-\frac{1}{2}}\right)+\frac{1}{2}+\frac{1}{2n}\\
&= \frac{3}{4}+\frac{1}{2n}\\
&< 1\,;\quad \text{if } n\geq 3.
\end{aligned}$$

Thus $f_n(-2n)$ has sign $(-1)^{n+1}$ for $n\geq 3$.
Finally, consider

$$\begin{aligned}
f_n(-\ln n) &= (-\ln n)(1-\ln n)^n+(-\ln n)^n\\
&= (-1)^n(-\ln n)(\ln n-1)^n+(-1)^n(\ln n)^n\\
&= (-1)^n(\ln n)^n\left(1-\ln n\left(\tfrac{\ln n-1}{\ln n}\right)^n\right).
\end{aligned}$$

By Theorem 1.2.4, we know that $\lim_{n\to\infty}\ln n\left(\frac{\ln n-1}{\ln n}\right)^n=0$, which implies that $f_n(-\ln n)$ has sign $(-1)^n$ for sufficiently large n, this completes the proof. □

Theorem 2.3.9. *Let G be a graph with diameter D. If $D\leq 2$, then all the domination roots of the graph G^2 lie on the unit circle with center $(-1,0)$.*

Proof. It follows from the fact that all n^{th} roots of unity lie on the unit circle centered at $(0,0)$. □

Corollary 2.3.10. *Let G be a graph of order n and diameter D. If $D\leq 2$, then*

(1) $D(G^2,x)$ has no nonzero integer root, if n is odd.

(2) -2 *is the only nonzero integer root of* $D(G^2, x)$*, if* n *is even.*

Corollary 2.3.11. *All the domination roots of the square of the complete graph* K_n *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.12. *All the domination roots of the square of the complete m-partite graph* $K_{n_1,n_2,\ldots,n_m}$ *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.13. *All the domination roots of the square of the complete bipartite graph* $K_{m,n}$ *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.14. *All the domination roots of the square of the star graph* S_n *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.15. *All the domination roots of the square of the wheel graph* W_n *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.16. *For any two graphs* H *and* G*, all the domination roots of the graph* $H \vee G$ *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.17. *For the complete graphs* K_m *and* K_n*, all the domination roots of the square of the graph* $K_m \Box K_n$ *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.18. *All the domination roots of the square of the Petersen graph* P *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.19. *All the domination roots of the square of the Dutch windmill graph* G_3^n *lie on the unit circle centered at* $(-1, 0)$.

Corollary 2.3.20. *All the domination roots of the square of the lollipop graph of* $L_{n,1}$ *lie on the unit circle centered at* $(-1, 0)$.

Theorem 2.3.21. *All the nonzero domination roots of the square of the bipartite cocktail party graph* B_n *lie in the annulus* $1 < |z + 1| \leq 2$.

Proof. We have the domination polynomial of the square of the bipartite cocktail party graph B_n is

$$D(B_n^2, x) = (1 + x)^{2n} - 2nx - 1.$$

Put $y = 1 + x$ and consider $f(y) = y^{2n} - 2ny + 2n - 1$. Then $y = 1$ is a double root of $f(y)$. Therefore $f(y) = (y - 1)^2 g(y)$, where

$$g(y) = y^{2n-2} + 2y^{2n-3} + 3y^{2n-4} + \ldots + (2n - 2)y + 2n - 1.$$

It suffices to show that all the roots of $g(y)$ lie in the annulus $1 < |z| \leq 2$. By Enestrom-Kakeya theorem 1.2.6, if $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ has positive real coefficients, then all roots of f lie in the annulus $r \leq |z| \leq R$, where

$$r = \min\left\{\frac{a_i}{a_{i+1}} \ : 0 \leq i \leq n-1\right\} \text{ and } R = \max\left\{\frac{a_i}{a_{i+1}} \ : 0 \leq i \leq n-1\right\}.$$

In this case

$$r = \min\left\{\frac{n}{n-1}, \frac{n-1}{n-2}, \ldots, 2\right\} \text{ and } R = \max\left\{\frac{n}{n-1}, \frac{n-1}{n-2}, \ldots, 2\right\}.$$

So we have the result. □

Corollary 2.3.22. *The square of the bipartite cocktail party graph B_n has no nonzero integer domination roots.*

The domination roots of the square of the bipartite cocktail party graph B_n for $1 \leq n \leq 30$ are shown in Figure 2.7.

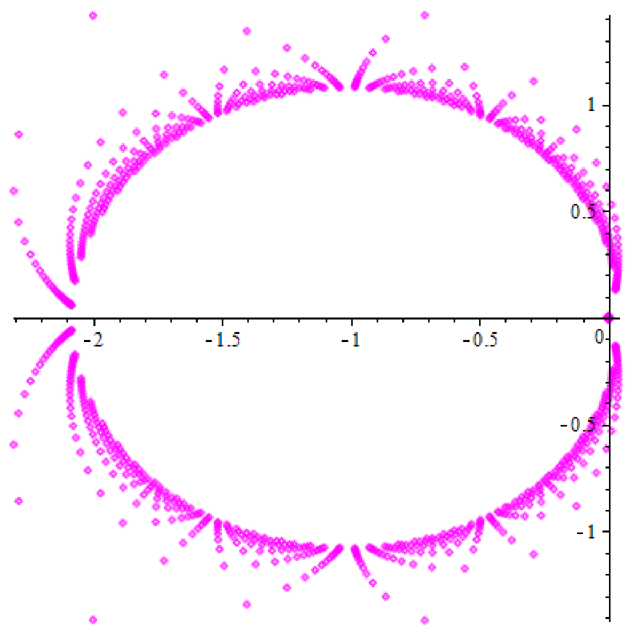

Figure 2.7: Domination roots of B_n^2 for $1 \leq n \leq 30$.

Theorem 2.3.23. *All the domination roots of the square of the generalized barbell graph $B_{m,n,1}$ lie inside the circle with center $(-1, 0)$ and radius 2.*

Proof. We have $D(B^2_{m,n,1}, y-1) = f(y) = y^{m+n} - y^{n-1} - y^{m-1} + 1$. In this case $\max_i |a_i| = 1$, where a_i's are the coefficients of $f(y)$ for $i = 0, 1, \ldots, m+n$. Then by Theorem 1.2.5 we have the result. □

The domination roots of the square of the generalized barbell graph $B_{m,n,1}$ for $2 \leq m \leq 10$ and $2 \leq n \leq 30$ are shown in Figure 2.8.

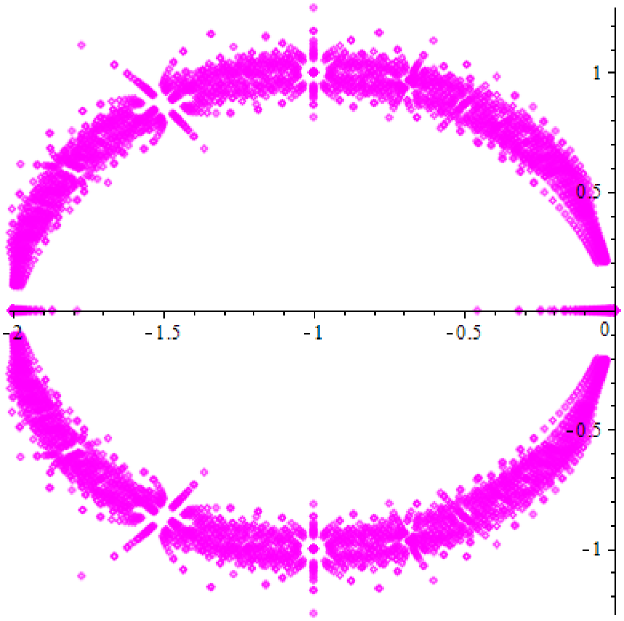

Figure 2.8: Domination roots of $B^2_{m,n,1}$ for $2 \leq m \leq 10$ and $2 \leq n \leq 30$.

Corollary 2.3.24. *For all m, n we have the following :*

1. *If m and n are odd, then -2 is the only nonzero integer root of $D(B^2_{m,n,1}, x)$.*
2. *If m and n are even, then $D(B^2_{m,n,1}, x)$ has no nonzero integer root.*
3. *If m and n have opposite parity, then -2 is the only nonzero integer root of $D(B^2_{m,n,1}, x)$.*

Corollary 2.3.25. *For $n \geq 2$ we have the following :*

1. *If n is even, then $D(B^2_{n,1}, x)$ has no nonzero integer root.*
2. *If n is odd, then -2 is the only nonzero integer root of $D(B^2_{n,1}, x)$.*

Theorem 2.3.26. *All the domination roots of the square of the bi-star graph $B_{(m,n)}$ lie inside the circle with center $(-1,0)$ and radius 2.*

Proof. The proof is similar to the proof of the Theorem 2.3.23. □

Corollary 2.3.27. *For all m, n we have the following :*

1. *If m and n are even, then -2 is the only nonzero integer root of $D(B^2_{(m,n)}, x)$.*
2. *If m and n are odd, then $D(B^2_{(m,n)}, x)$ has no nonzero integer root.*
3. *If m and n have opposite parity, then -2 is the only nonzero integer root of $D(B^2_{(m,n)}, x)$.*

The domination roots of the square of the bi-star graph $B_{(m,n)}$ for $1 \leq m \leq 15$ and $1 \leq n \leq 30$ are shown in Figure 2.9.

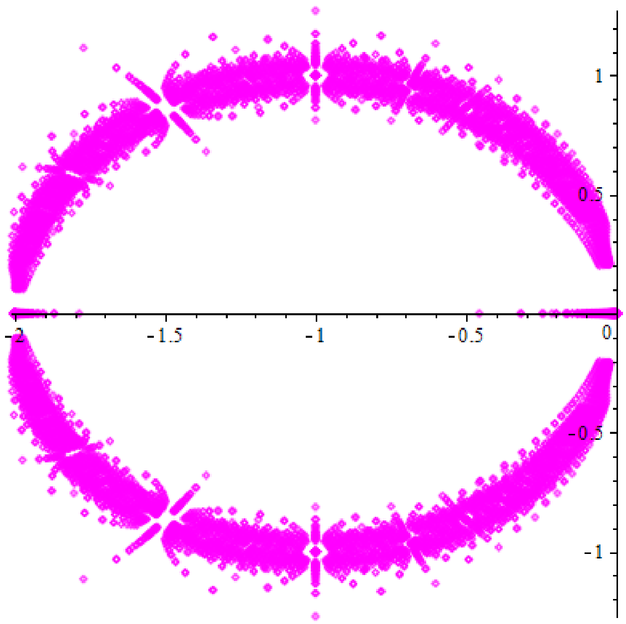

Figure 2.9: Domination roots of $B^2_{(m,n)}$ for $1 \leq m \leq 15$ and $1 \leq n \leq 30$.

Theorem 2.3.28. *All the domination roots of the square of the corona $K_m \circ K_n$ lie inside the circle with center $(-1,0)$ and radius m.*

Proof. We have the domination polynomial of the square of the corona $K_m \circ K_n$ is

$$D((K_m \circ K_n)^2, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1. \tag{2.9}$$

Replace $1+x$ by y in equation (2.9) we get

$$D((K_m \circ K_n)^2, y-1) = y^{m(n+1)} - my^n + m - 1.$$

But $y^{m(n+1)} - my^n + m - 1 = (y-1)f(y)$, where

$$f(y) = y^{m(n+1)-1} + y^{m(n+1)-2} + \ldots + y^n - (m-1)(y^{n-1} + y^{n-2} + \ldots + y + 1).$$

In this case $\max_i |a_i| = m-1$, where a_i's are the coefficients of $f(y)$ for $i = 0, 1, \ldots, m(n+1)-1$. Thus by Theorem 1.2.5 we have the result. $\square$

The domination roots of the square of the corona $K_m \circ K_n$ for $2 \le m \le 5$ and $1 \le n \le 15$ are shown in Figure 2.10.

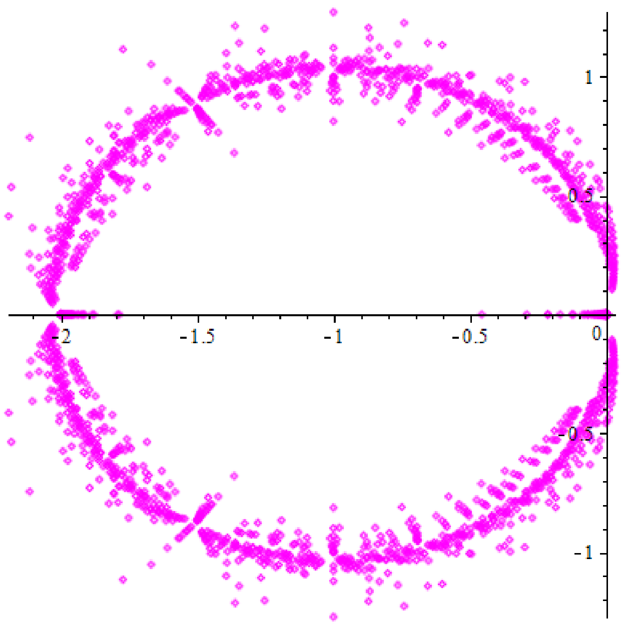

Figure 2.10: Domination roots of $(K_m \circ K_n)^2$ for $2 \le m \le 5$ and $1 \le n \le 15$.

Corollary 2.3.29. *All the domination roots of the square of the graph $Q(m,n)$ lie inside the circle with center $(-1,0)$ and radius m.*

The domination roots of the square of the graph $Q(m, n)$ for $2 \leq m \leq 5$ and $2 \leq n \leq 15$ are shown in Figure 2.11.

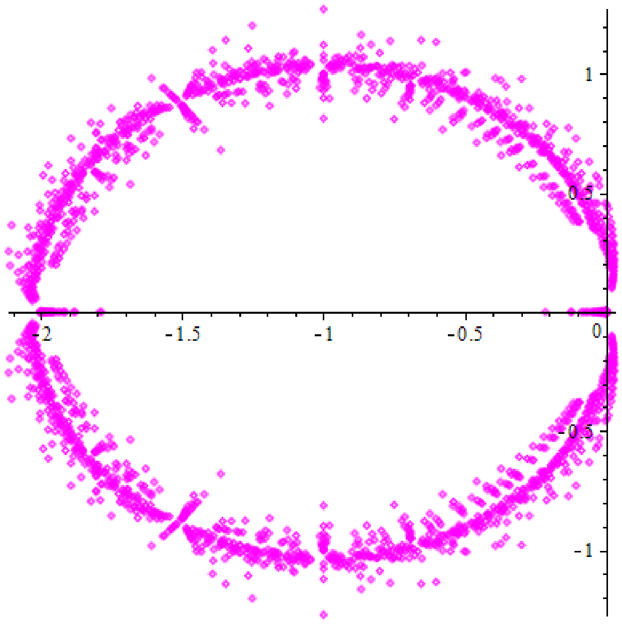

Figure 2.11: Domination roots of $Q^2(m, n)$ for $2 \leq m \leq 5$ and $2 \leq n \leq 15$.

Theorem 2.3.30. *All the domination roots of the square of the corona $K_n \circ K_n$ lie inside the circle with center $(-1, 0)$ and radius $n^{\frac{1}{n}}$.*

Proof. We have the domination polynomial of the square of the corona $K_n \circ K_n$ is

$$D((K_n \circ K_n)^2, x) = (1 + x)^{n(n+1)} - n(1 + x)^n + n - 1. \tag{2.10}$$

So it suffices to show that all the roots of

$$f(y) = y^{n+1} - ny + n - 1$$

lie in the circle center at the origin and having radius n. We have $y = 1$ is a root of $f(y)$. Therefore $f(y) = (y - 1)g(y)$, where

$$g(y) = y^{n(n+1)-1} + y^{n(n+1)-2} + \ldots + y^n - (n - 1)(y^{n-1} + y^{n-2} + \ldots + y + 1).$$

In this case $\max\limits_i |a_i| = n - 1$, where a_i's are the coefficients of $g(y)$ for $i =$

$0, 1, \ldots, n(n+1)-1$. Thus by Theorem 1.2.5, if $g(y) = 0$ then

$$\begin{aligned} |y| &\leq n \\ |(1+x)^n| &\leq n \\ |(1+x)| &\leq n^{\frac{1}{n}}, \end{aligned}$$

we have the result. □

Corollary 2.3.31. *All the domination roots of the square of the graph $Q(n, n)$ lie inside the circle with center $(-1, 0)$ and radius $n^{\frac{1}{n}}$.*

2.4 Stable graphs related to domination polynomial

In this section we introduce d-stable and d-unstable graphs. We obtained some examples of d-stable and d-unstable graphs.

Definition 2.4.1. *Let $G = (V(G), E(G))$ be a graph. The graph G is said to be a domination stable graph or simply* d*-stable graph if all the nonzero domination roots of G lie in the left open half-plane, that is, if real part of the nonzero domination roots is negative. If G is not* d*-stable graph, then G is said to be a domination unstable graph or simply* d*-unstable graph.*

Example 2.4.1. *Using Maple, we find that real part of all the domination roots of path graph P_n and the cycle graph C_n are negative for $n \leq 15$. Therefore the path graph P_n and the cycle graph C_n are* d*-stable for $n \leq 15$.*

Example 2.4.2. *The domination polynomial of Dutch windmill graph G_3^7 is*

$$D(G_3^7, x) = x(1+x)^{14} + (2x + x^2)^7.$$

With the aid of Maple, the domination roots of G_3^7 are :
$\mathbb{Z}(D(G_3^7, x)) = \{-1.8465 - .5747i, -1.8465 + .5747i, -1.8224 - 2.0627i, -1.8224 + 2.0627i,$
$-1.7335 - .2651i, -1.7335 + .2651i, -1.6976 - 0.8041i, -1.6976 + 0.8041i,$
$-0.2537 - 0.8373i, -0.2537 + 0.8373i, -0.1942 - 0.2815i, -0.1942 + 0.2815i,$
$0, 0.4785 - 0.6883i, 0.4785 + 0.6883i.\}$

The domination roots $0.4785 - 0.6883i$ *and* $0.4785 + 0.6883i$ *lie in the right half plane. Hence* G_3^7 *is* $\mathtt{d}$*-unstable graph.*

Theorem 2.4.3. *If* G *and* H *are isomorphic graphs, then* G *is* $\mathtt{d}$*-stable if and if* H *is* $\mathtt{d}$*-stable.*

Proof. It follows from the fact that if G and H are isomorphic graphs then $D(G,x) = D(H,x)$. □

Corollary 2.4.4. *If* G *and* H *are isomorphic graphs then* G *is* $\mathtt{d}$*-unstable if and if* H *is* $\mathtt{d}$*-unstable.*

Theorem 2.4.5. *If a graph* G *consists of* m *components* $G_1, G_2, \ldots, G_m$*, then* G *is* $\mathtt{d}$*-stable if and if each* G_i *is* $\mathtt{d}$*-stable.*

Proof. It follows from the fact that $D(G,x) = \prod_{i=1}^{m} D(G_i,x)$. □

Corollary 2.4.6. *If a graph* G *consists of* m *components* $G_1, G_2, \ldots, G_m$*, then* G *is* $\mathtt{d}$*-unstable if and if one of the* G_i *is* $\mathtt{d}$*-unstable.*

Remark 2.4.7. *Using Maple, we find that real part of all the domination roots of all the graphs of order upto* 6 *is negative. Therefore there is no* $\mathtt{d}$*-unstable graph of order upto* 6*.*

Theorem 2.4.8. *Let* G *be a connected graph of order* $n > 2$ *without pendent vertices. If* G *is* $\mathtt{d}$*-stable, then*

$$n < 1 + 2\,\mathtt{d}(G, n-3).$$

Proof. Suppose G is $\mathtt{d}$-stable. Then by Routh-Hurwitz criteria 1.2.7, we have Routh-Hurwitz matrix $H_2 > 0$. This implies that

$$\mathtt{d}(G, n-1)\mathtt{d}(G, n-3) - \mathtt{d}(G, n-2) > 0.$$

Since G is connected and without pendent vertices we have

$$\mathtt{d}(G, n-1) = n \text{ and } \mathtt{d}(G, n-2) = \frac{1}{2}n(n-1).$$

This completes the proof. □

Theorem 2.4.9. *The complete graph K_n is* d*-stable graph for all n.*

Proof. The domination polynomial of K_n is

$$D(K_n, x) = (1 + x)^n - 1.$$

Therefore

$$\mathbb{Z}(D(K_n, x)) = \left\{ \exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, 1, \ldots, n - 1 \right\}.$$

Clearly, real part of all the roots are non-positive. This implies that K_n is d-stable for all n. □

Theorem 2.4.10. *The complement of the complete graph K_n is* d*-stable graph for all n.*

Proof. It follows from the fact that the graph $\overline{K_n}$ has no nonzero domination roots. □

Remark 2.4.11. *We have the domination polynomial of S_n is*

$$\begin{aligned} D(S_n, x) &= x^n + x(1 + x)^n \\ &= 1(x)^n + x(1 + x)^n \\ &= \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n, \end{aligned}$$

where $\alpha_1 = 1$, $\lambda_1 = x$, $\alpha_2 = x$ and $\lambda_2 = 1+x$. Clearly 1 and x are not identically zero and $\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now, $|\lambda_1| = |\lambda_2|$ holds if and only if $|x - 0| = |x - (-1)|$, that is, if and only if x is equidistant from 0 and -1. This holds if and only if real part of x is $-\frac{1}{2}$. Also α_1 is never 0 and $\alpha_2 = 0$ if and only if $x = 0$ and in this case $|\lambda_2(0)| = 1 > 0 = |\lambda_1(0)|$. By these arguments we have 0 and the complex numbers z such that $\mathcal{R}(z) = -\frac{1}{2}$ are the limits of roots of $D(S_n, x)$. Therefore we think that that there is no complex number z with positive real part is a root of $D(S_n, x)$. We conjectured that the star graph S_n is d*-stable graph for all n.*

The domination roots of the star graph S_n for $1 \leq n \leq 60$ are shown in Figure 2.12.

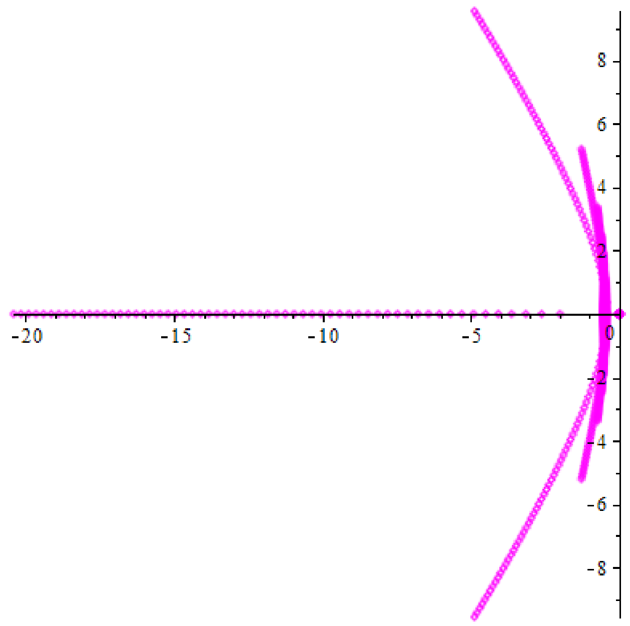

Figure 2.12: Domination roots of S_n for $1 \leq n \leq 60$.

Remark 2.4.12. *We have the domination polynomial of $K_{m,n}$ is*

$$D(K_{m,n}, x) = ((1+x)^m - 1)\,((1+x)^n - 1) + x^m + x^n.$$

Let m be fixed and rewrite $D(K_{m,n}, x)$ as :

$$\begin{aligned} D(K_{m,n}, x) &= ((1+x)^m - 1)\,(1+x)^n + ((1 + x^m - (1+x)^m))\,(1)^n + 1(x)^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \end{aligned}$$

where $\alpha_1 = (1+x)^m - 1$, $\lambda_1 = 1+x$, $\alpha_2 = 1 + x^m - (1+x)^m$, $\lambda_2 = 1$, $\alpha_3 = 1$ and $\lambda_3 = x$. Clearly α_1,α_2 and α_3 are not identically zero and $\lambda_i \neq \omega\lambda_j$ for $i \neq j$ and any complex number ω of modulus 1. Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now, applying part(i) of Theorem 1.2.9, we consider the following four different cases :

(i) $|\lambda_1| = |\lambda_2| = |\lambda_3|$

(ii) $|\lambda_1| = |\lambda_2| > |\lambda_3|$

(iii) $|\lambda_1| = |\lambda_3| > |\lambda_2|$

(iv) $|\lambda_2| = |\lambda_3| > |\lambda_1|$

Case (i) : *Assume that* $|1+x| = |1| = |x|$. *Then* $|x-(-1)| = |x-0|$ *implies that* x *lies on the vertical line* $z = -\frac{1}{2}$, $|x-(-1)| = 1$ *implies that* x *lies on the unit circle centered at* $(-1,0)$ *and* $1 = |x-0|$ *implies that* x *lies on the unit circle centered at the origin. Therefore the two points of intersection,* $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ *are limits of roots.*

Case (ii) : *Assume that* $|1+x| = |1| > |x|$. *Then* $|x-(-1)| = 1$ *implies that* x *lies on the unit circle centered at* $(-1,0)$, $|x-(-1)| > |x-0|$ *implies that* x *lies to the right of the vertical line* $z = -\frac{1}{2}$. *Therefore the complex numbers* x *that satisfy* $|x-(-1)| = 1$ *and* $\mathcal{R}(x) > -\frac{1}{2}$ *are limits of roots.*

Case (iii) : *Assume that* $|1+x| = |x| > |1|$. *Then* $|x-(-1)| = |x-0|$ *implies that* x *lies on the vertical line* $x = -\frac{1}{2}$ *and* $|x-0| > 1$ *implies that* x *lies outside the unit circle centered at the origin. Therefore the complex numbers* x *that satisfy* $|x| > 1$ *and* $\mathcal{R}(x) > -\frac{1}{2}$ *are limits of roots.*

Case (iv) : *Assume that* $|1| = |x| > |1+x|$. *Then* $1 = |x-0|$ *implies that* x *lies on the unit circle centered at the origin and* $|x-0| > |x-(-1)|$ *implies that* x *lies to the left of the vertical line* $x = -\frac{1}{2}$. *Therefore the complex numbers* x *that satisfy* $|x| = 1$ *and* $\mathcal{R}(x) < -\frac{1}{2}$ *are limits of roots.*

Also there may be some additional isolated limits of roots, being roots of α_2 *inside* $|1+x| = 1$ *and* $|x| = 1$. *The union of the curves and points above yield that for* m *fixed, the limits of roots of the domination polynomial of the complete bipartite graph* $K_{m,n}$ *consists of the part of the circle* $|z| = 1$ *with real part at most* $-\frac{1}{2}$, *the part of the circle* $|z+1| = 1$ *with real part at least* $-\frac{1}{2}$ *and the part of the line* $\mathcal{R}(z) = -\frac{1}{2}$ *with modulus at least 1. So we conjectured that the complete bipartite graph* $K_{m,n}$ *is* $\mathtt{d}$*-stable for all* m, n.

The domination roots of the complete bipartite graphs $K_{m,n}$ for $1 \leq m \leq 15$, $1 \leq n \leq 30$ and $K_{n,n}$ for $1 \leq n \leq 30$ are respectively shown in Figures 2.13 and 2.14.

Theorem 2.4.13. *The generalized barbell graph* $B_{m,n,1}$ *is* $\mathtt{d}$*-stable for all* m, n.

Proof. We have by Theorem 2.1.6, the domination polynomial of $B_{m,n,1}$ is

$$D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]\,.$$

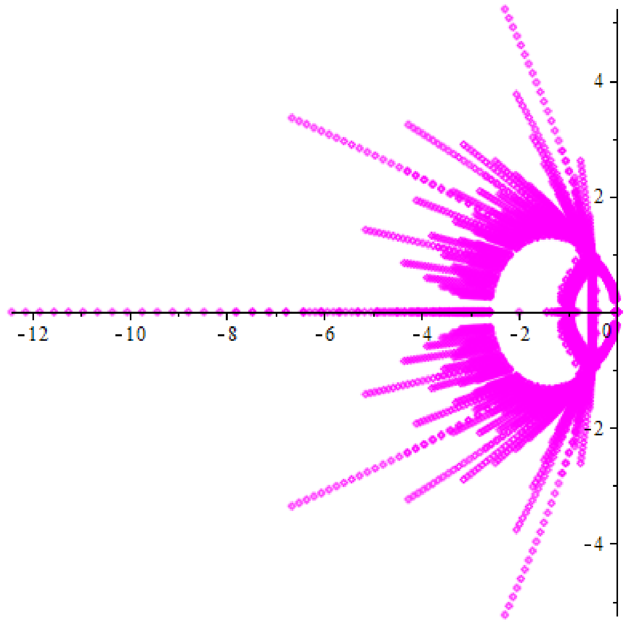

Figure 2.13: Domination roots of $K_{m,n}$ for $1 \leq m \leq 15$ and $1 \leq n \leq 30$.

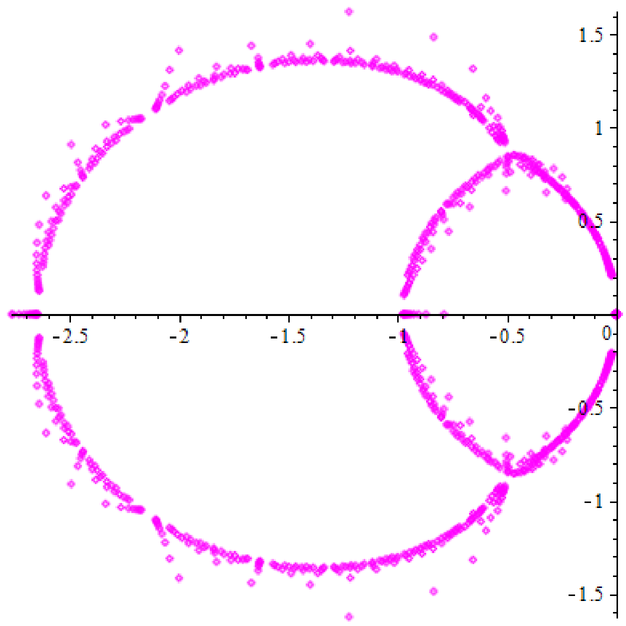

Figure 2.14: Domination roots of $K_{n,n}$ for $1 \leq n \leq 30$.

Therefore

$$\mathbb{Z}(D(B_{m,n,1},x)) = \left\{\exp\left(\frac{2k\pi i}{m}\right) - 1 | k = 0,\dots,m-1\right\} \cup \left\{\exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0,\dots,n-1\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the generalized barbell graph $B_{m,n,1}$ is $\mathtt{d}$-stable for all m, n. $\square$

The domination roots of the generalized barbell graph $B_{m,n,1}$ for $1 \leq m, n \leq 30$ are shown in Figure 2.15.

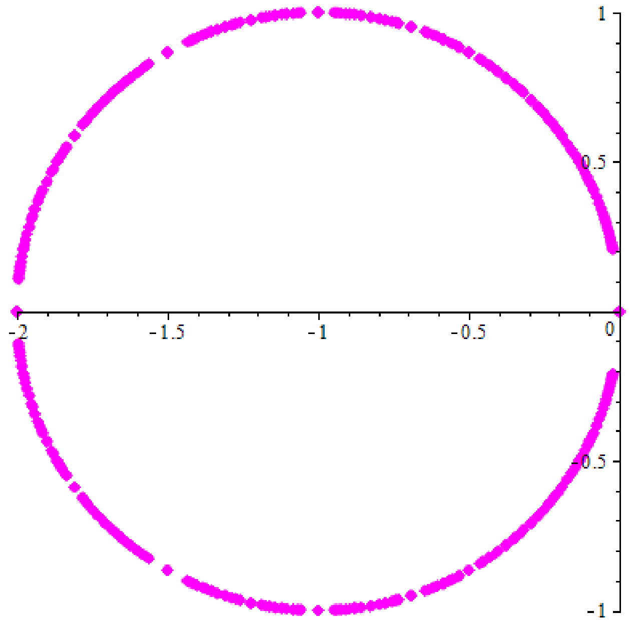

Figure 2.15: Domination roots of $B_{m,n,1}$ for $1 \leq m, n \leq 30$.

Corollary 2.4.14. *The n-barbell graph $B_{n,1}$ is $\mathtt{d}$-stable for all n.*

Proof. It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. $\square$

The domination roots of the n-barbell graph $B_{n,1}$ for $1 \leq n \leq 60$ are shown in Figure 2.16.

Theorem 2.4.15. *The corona $K_m \circ K_n$ is $\mathtt{d}$-stable for all m, n.*

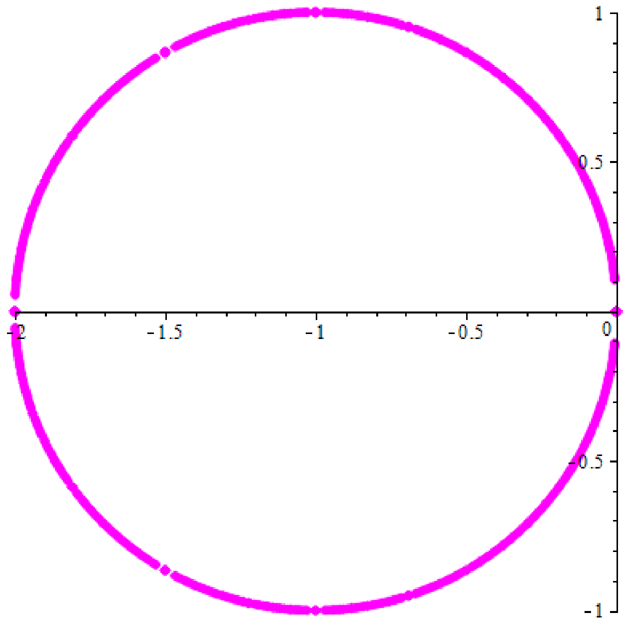

Figure 2.16: Domination roots of $B_{n,1}$ for $1 \leq n \leq 60$.

Proof. We have the domination polynomial of $K_m \circ K_n$ is

$$D(K_m \circ K_n, x) = \left((1+x)^{n+1} - 1\right)^m.$$

Therefore

$$\mathbb{Z}(D(K_m \circ K_n, x)) = \left\{\exp\left(\frac{2k\pi i}{n+1}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the corona $K_m \circ K_n$ is $\mathtt{d}$-stable for all m, n. □

Corollary 2.4.16. *The graph $Q(m,n)$ is $\mathtt{d}$-stable for all m, n.*

Proof. It follows from the fact that the graph $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Remark 2.4.17. *We have $D(B_{(m,n)}, x) = x^2\left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right)$. Let m be fixed, we rewrite $D(B_{(m,n)}, x)$ as $f_n(x)$:*

$$\begin{aligned} f_n(x) &= \left(x^{m+1} + x^2(1+x)^m\right)(1+x)^n + \left(x^m + x(1+x)^m\right)x^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \end{aligned}$$

where

$$\alpha_1 = \left(x^{m+1} + x^2(1+x)^m\right), \ \lambda_1 = 1+x, \ \alpha_2 = (x^m + x(1+x)^m) \ \text{and} \ \lambda_2 = x.$$

Clearly $(x^{m+1} + x^2(1+x)^m)$ *and* $(x^m + x(1+x)^m)$ *are not identically zero and* $\lambda_1 \neq \omega\lambda_2$ *for any complex number* ω *of modulus* 1. *Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now,* $|\lambda_1| = |\lambda_2|$ *holds if and only if* $|x-(-1)| = |x-0|$, *that is, if and only if* x *is equidistant from* -1 *and* 0. *The latter holds if and only if* $\mathcal{R}(x) = -\frac{1}{2}$. *Notice that* $\alpha_1(0) = 0$ *and* $\alpha_1(0) = 1+0 = 1$ *has modulus strictly greater than* $\lambda_2(0) = 0$. *Note that there may be some additional limits of roots, being roots of* α_1 *and* α_2. *But from the Remark 2.4.11, we can conclude that* α_1 *and* α_2 *have no roots in the right-half plane. By these arguments we have* 0 *and the complex numbers* z *that satisfy* $\mathcal{R}(z) = -\frac{1}{2}$ *are the limits of roots of* $D(B_{(m,n)}, x)$. *So we conjectured that the bi-star graph* $B_{(m,n)}$ *is* `d`*-stable for all* m, n.

The domination roots of the bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 50$ are shown in Figure 2.17.

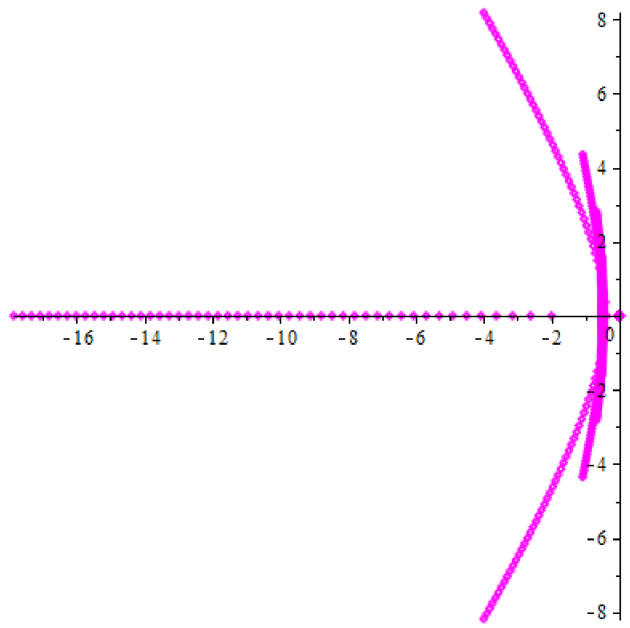

Figure 2.17: Domination roots of bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 50$.

Theorem 2.4.18. *Let G be a connected graph of order n and $D(G,x)$ be its domination polynomial. If $D(G,x)$ has exactly two distinct domination roots, then G is* $\mathtt{d}$*-stable for all n.*

Proof. It follows from the fact that the two distinct roots are real. □

Theorem 2.4.19. *Let G be a graph of order n, then the corona $G \circ K_1$ is* $\mathtt{d}$*-stable for all n.*

Proof. We have domination polynomial of $G \circ K_1$ is

$$D(G \circ K_1, x) = x^n(x+2)^n.$$

Therefore $\mathbb{Z}(D(G \circ K_1, x) = \{0, -2\}$, that is, $G \circ K_1$ is $\mathtt{d}$-stable for all n. □

Theorem 2.4.20. *Let G be a graph of order n, then the corona $G \circ \overline{K_2}$ is* $\mathtt{d}$*-stable for all n.*

Proof. We have domination polynomial of $G \circ \overline{K_2}$ is

$$D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}}(x^2+3x+1)^{\frac{n}{3}}.$$

Therefore $\mathbb{Z}(D(G \circ \overline{K_2}, x) = \left\{0, \frac{-3\pm\sqrt{5}}{2}\right\}$, That is, $G \circ \overline{K_2}$ is $\mathtt{d}$-stable for all n. □

Theorem 2.4.21. *Let G be a graph without pendent vertices and let $D(G,x)$ be its domination polynomial. If $D(G,x)$ has exactly three distinct roots, then G is* $\mathtt{d}$*-stable.*

Proof. By (12) in Results 2.1.2, we have

$$\mathbb{Z}(D(G,x)) \subset \left\{0, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

This implies that G is $\mathtt{d}$-stable. □

Theorem 2.4.22. *For any graph G with three distinct domination roots is* $\mathtt{d}$*-stable.*

Proof. By (13) in Results 2.1.2, we have

$$\mathbb{Z}(D(G,x)) \subset \left\{-2, 0, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

This implies that G is $\mathtt{d}$-stable. □

Theorem 2.4.23. *The Dutch windmill graph G_3^n is not $\mathtt{d}$-stable graph for all but finite values of n.*

Proof. Using maple, we find that the Dutch windmill graph G_3^n is $\mathtt{d}$-stable for $n \leq 6$. We have $D(G_3^n, x) = x(1+x)^{2n} + (2x+x^2)^n$. We rewrite $f_n(x) = D(G_3^n, x)$ as

$$\begin{aligned} f_n(x) &= x\left((1+x)^2\right)^n + (1)(2x+x^2)^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \end{aligned}$$

where

$$\alpha_1 = x,\ \lambda_1 = (1+x)^2,\ \alpha_2 = 1,\ \lambda_2 = 2x + x^2.$$

Clearly, 1 and x are not identically zero and $\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now, for $z = a+ib \in \mathbb{C}$, $|\lambda_1(z)| = |\lambda_2(z)|$ holds if and only if $|(1+z)^2| = |2z+z^2|$. That is, $|(1+a+ib)^2| = |2(a+ib)+(a+ib)^2|$. By a simple calculation we have $(a+1)^2 + b^2 = \frac{1}{2}$. Therefore 0 and the complex numbers z such that $(1+\mathcal{R}(z))^2 + (\mathcal{I}(z))^2 = \frac{1}{2}$ are limits of domination roots of G_3^n. This implies that the domination roots of G_3^n have unbounded positive real part. Therefore the Dutch windmill graph G_3^n is not $\mathtt{d}$-stable for all but finite values of n. □

The domination roots of the Dutch windmill graph G_3^n for $1 \leq n \leq 6$ and for $1 \leq n \leq 30$ are shown in Figures 2.18 and 2.19 respectively.

Theorem 2.4.24. *The bipartite cocktail party graph B_n is $\mathtt{d}$-unstable graph for $n \geq 10$.*

Proof. From (2) in Results 2.1.3, we have the bipartite cocktail party graph B_n have domination roots in the right half-plane for $n \geq 10$. □

The domination roots of the bipartite cocktail party graph B_n for $1 \leq n \leq 9$ and for $1 \leq n \leq 30$ are shown in Figures 2.20 and 2.21 respectively.

Next we consider some graphs and check whether its square is either $\mathtt{d}$-stable or $\mathtt{d}$-unstable.

Theorem 2.4.25. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G^2 is $\mathtt{d}$-stable if and if each G_i^2 is $\mathtt{d}$-stable.*

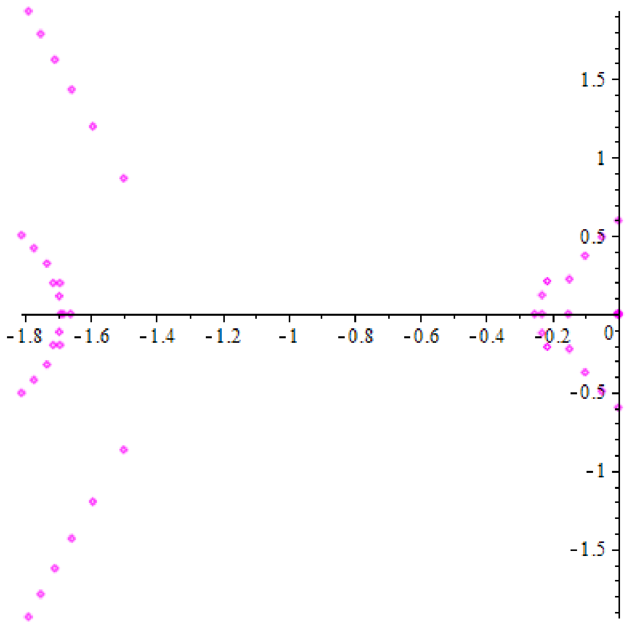

Figure 2.18: Domination roots of G_3^n for $1 \leq n \leq 6$.

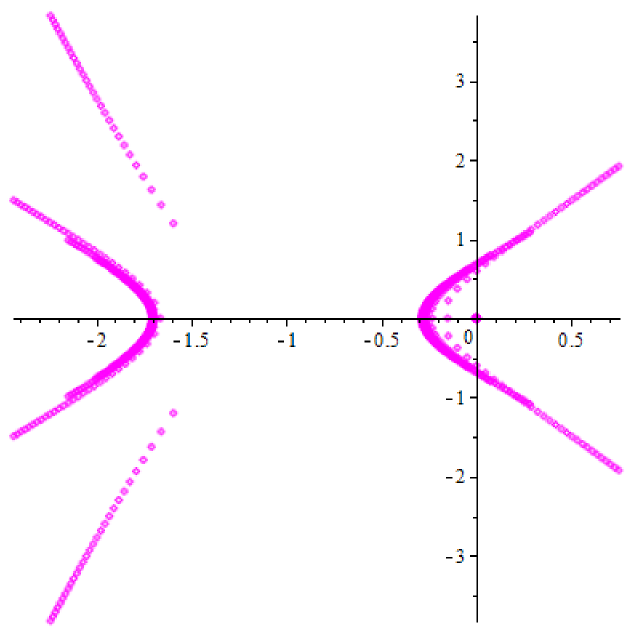

Figure 2.19: Domination roots of G_3^n for $1 \leq n \leq 30$.

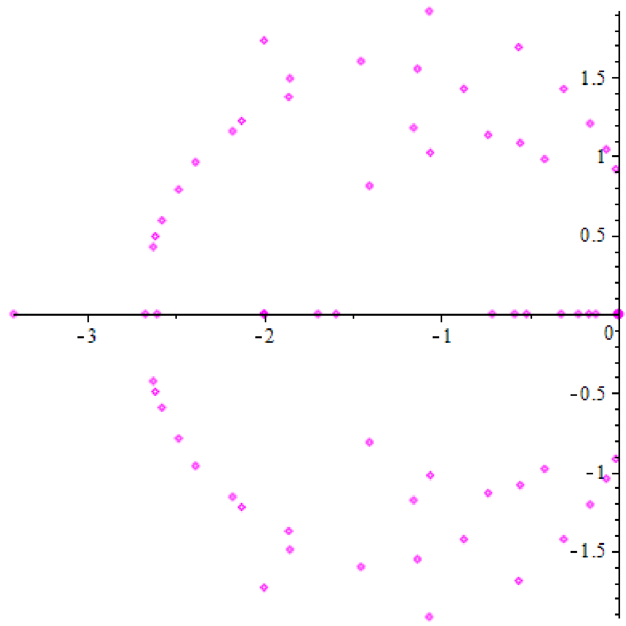

Figure 2.20: Domination roots of B_n for $1 \leq n \leq 9$.

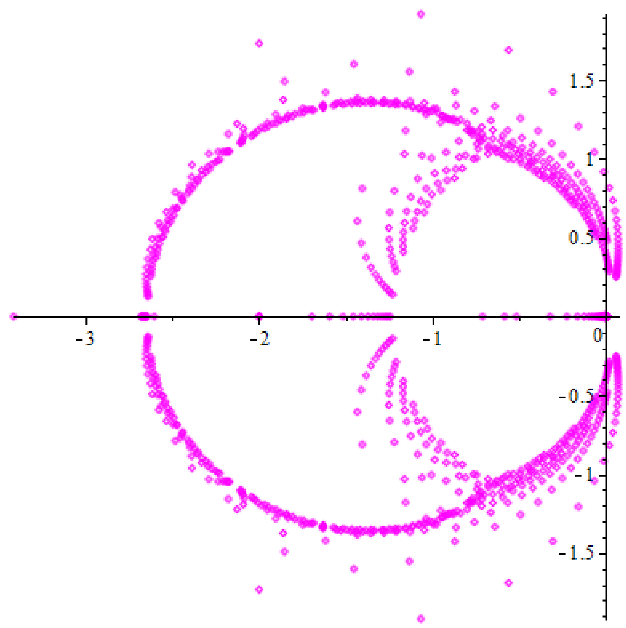

Figure 2.21: Domination roots of B_n for $1 \leq n \leq 30$.

Proof. It follows from the fact that $D(G^2, x) = \prod_{i=1}^{m} D(G_i^2, x)$. $\square$

Corollary 2.4.26. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G^2 is* d*-unstable if and if one of the G_i^2 is* d*-unstable.*

Theorem 2.4.27. *Let G be a graph of order n. If $D(G) \leq 2$ then G^2 is* d*-stable for all n.*

Proof. It follows from the fact that if $D(G) \leq 2$ then G^2 is isomorphic to the complete graph K_n, which is d-stable. $\square$

Corollary 2.4.28. *The square of the complete graph K_n is* d*-stable for all n.*

Corollary 2.4.29. *The square of the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$ is* d*-stable for all $n_1, n_2, \ldots, n_m$.*

Corollary 2.4.30. *The square of the complete bipartite graph $K_{m,n}$ is* d*-stable for all m and n.*

Corollary 2.4.31. *The square of the star graph S_n is* d*-stable for all n.*

Corollary 2.4.32. *The square of the wheel graph W_n is* d*-stable for all n.*

Corollary 2.4.33. *Let H and G be two graphs. Then the square of $H \vee G$ is* d*-stable.*

Corollary 2.4.34. *For the complete graph K_m and K_n, the square of $K_m \square K_n$ is* d*-stable for all m and n.*

Corollary 2.4.35. *The square of the Petersen graph P is* d*-stable.*

Corollary 2.4.36. *The square of the Dutch windmill graph G_3^n is* d*-stable for all n.*

Corollary 2.4.37. *The square of the lollipop graph $L_{n,1}$ is* d*-stable for all n.*

Remark 2.4.38. *Using Maple, we find that square of $K_m \circ K_n$ has domination roots in the right-half plane for $m = 5$ and $3 \leq n \leq 30$. Therefore the square of the corona $K_m \circ K_n$ is not* d*-stable for all but finite values of m and n. But limits of domination roots of the square of the corona $K_m \circ K_n$ are the unit circle centered* *at* $(-1,0)$. *For, we have* $D((K_m \circ K_n)^2, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1$. *We rewrite* $f_n(x) = D((K_m \circ K_n)^2, x)$ *as*

$$\begin{aligned} f_n(x) &= (1+x)^m \left((1+x)^m\right)^n + (-m)(1+x)^n + (m-1)(1)^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \end{aligned}$$

where

$$\alpha_1 = (1+x)^m,\ \lambda_1 = (1+x)^m,\ \alpha_2 = -m,\ \lambda_2 = 1+x,\ \alpha_3 = m-1,\ \lambda_3 = 1.$$

Clearly, α_1*,* α_2 *and* α_3 *are not identically zero and for* $i \neq j$*,* $\lambda_i \neq \omega\lambda_j$ *for any complex number* ω *of modulus* 1. *Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now applying part (i) of the Theorem 1.2.9, we consider the following four different cases :*

(i) $|\lambda_1| = |\lambda_2| = |\lambda_3|$

(ii) $|\lambda_1| = |\lambda_2| > |\lambda_3|$

(iii) $|\lambda_1| = |\lambda_3| > |\lambda_2|$

(iv) $|\lambda_2| = |\lambda_3| > |\lambda_1|$

Assume that $|\lambda_1| = |\lambda_2| > |\lambda_3|$*, that is,* $|(1+x)^m| = |1+x| > 1$. *This implies that either* $x = -1$ *or* $|1+x| = 1$*, both of which contradict* $|1+x| > 1$. *Thus there is no* $x \in \mathbb{C}$ *satisfy* $|\lambda_1| = |\lambda_2| > |\lambda_3|$. *So we need not consider the case (ii). By similar argument, we can rule out the cases (iii) and (iv). So we need to consider the case (i),* $|\lambda_1| = |\lambda_2| = |\lambda_3|$. *Assume that* $|(1+x)^m| = |1+x| = 1$. *This implies that* $|x - (-1)| = 1$*, that is,* x *lies on the unit circle centered at* $(-1,0)$. *So we can conclude that limits of domination roots of the square of the corona* $K_m \circ K_n$ *are the unit circle centered at* $(-1,0)$.

Remark 2.4.39. *Using maple, we found that the square of the bipartite cocktail party graph* B_n *is* d*-stable for* $n \leq 7$ *and is* d*-unstable for* $8 \leq n \leq 30$. *We have the domination polynomial of* B_n^2 *is*

$$D(B_n^2, x) = (1+x)^{2n} - 2nx - 1.$$

Put $y = 1+x$ *and consider* $f(y) = y^{2n} - 2ny + 2n - 1$. *Then* $y = 1$ *is a double* *root of $f(y)$. Therefore $f(y) = (y-1)^2 g(y)$, where*

$$g(y) = y^{2n-2} + 2y^{2n-3} + 3y^{2n-4} + \ldots + (2n-2)y + 2n - 1.$$

We have if $f(z) = a_n z^n + a_{n-1} z^n + \ldots + a_0$ is a polynomial with real coefficient satisfying $a_0 \geq a_1 \geq \ldots \geq a_n > 0$, then no roots of $f(z)$ lie in $\{z \in \mathbb{C} : |z| < 1\}$ [45]. Therefore all the roots z of $g(y)$ satisfies $|z| > 1$. This implies that all nonzero roots of $D(B_n^2, x)$ are out side the unit circle centered at $(-1, 0)$. So we conjectured that the square of the bipartite cocktail party graph B_n is not d*-stable for all but finite values of n.*

The domination roots of the square of the bipartite cocktail party graph B_n for $1 \leq n \leq 7$ and for $1 \leq n \leq 30$ are shown in Figures 2.22 and 2.23 respectively.

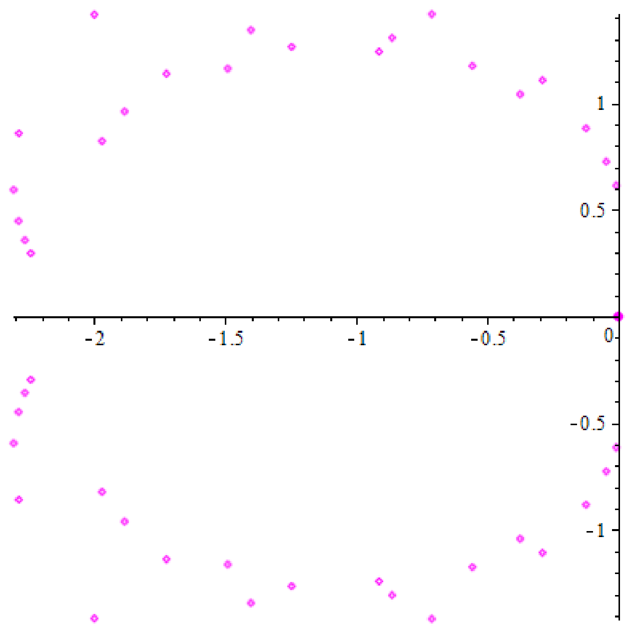

Figure 2.22: Domination roots of B_n^2 for $1 \leq n \leq 7$.

Definition 2.4.2. *Let G and H be graphs, with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The graph $G[H]$ formed by substituting a copy of H for every vertex of G, is formally defined by taking a disjoint copy of H, H_v for every vertex v of G, and joining every vertex in H_u to every vertex in H_v if and only if u is adjacent to v in G.*

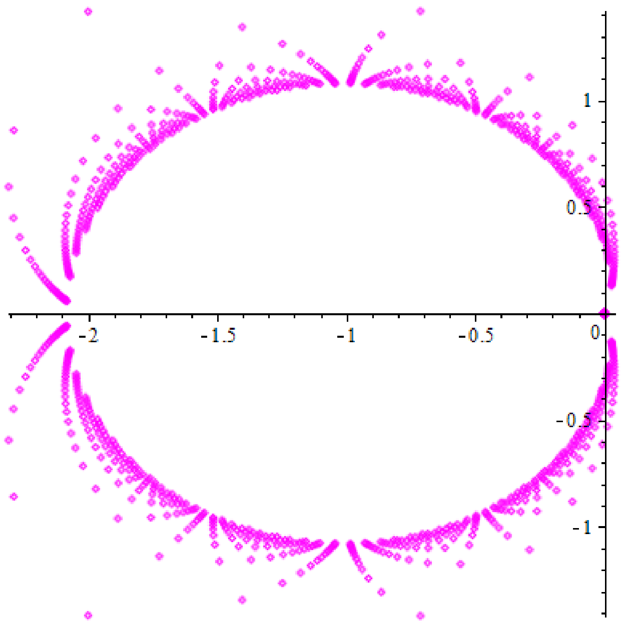

Figure 2.23: Domination roots of B_n^2 for $1 \leq n \leq 30$.

Lemma 2.4.40 (see [16]). *Let G be any graph and let K_n be the complete graph. Then*

$$D(G[K_n], x) = D(G, (1+x)^n - 1).$$

Theorem 2.4.41. *There are infinitely many* d*-unstable graphs.*

Proof. Let z be a domination root of G, then $z \neq 1$. By Lemma 2.4.40 implies the n solutions of $(1+x)^n - 1 = z$ are domination roots of $G[K_n]$. If the modulus of $z+1$ is greater than 1, then the modulus of the n^{th} roots of $z+1$ will also be of modulus greater than 1. So for n is large enough, at least one solution of $(1+x)^n - 1 = z$ will lies in the right half plane. By this argument we have the result. $\square$

CHAPTER 3

DISTANCE-K DOMINATION STABLE GRAPHS

In this chapter we introduce distance-k domination polynomial of graphs. In Section 3.1 we define distance-k domination polynomial of graphs and derive distance-k domination polynomial of some graphs. In Section 3.2, we define distance-k domination root and introduce the concept, $\mathtt{d}^k$-number of a graph and also find $\mathtt{d}^k$-number of some graphs. Bounds for distance-k domination roots of some graphs are included in Section 3.3. We introduce $\mathtt{d}^k$-stable and $\mathtt{d}^k$-unstable graphs in Section 3.4 and provide some examples of $\mathtt{d}^k$-stable and $\mathtt{d}^k$-unstable graphs.

3.1 Distance-k domination polynomial of graphs

We begin this section by state the definition of distance-k domination polynomial of graphs.

Definition 3.1.1. *Let k be a positive integer and let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V$ is a distance-k dominating set if each vertex $v \in V - S$ is with in distance k from some vertex of S. The distance-k domination number of G, denoted by $\gamma^k(G)$, is the minimum cardinality of the distance-k dominating sets in G. Let $\mathcal{D}^k(G, i)$ be the family of distance-k dominating sets of G with* *cardinality i and let $\mathtt{d}^k(G,i) = |\mathcal{D}^k(G,i)|$. The polynomial*

$$D^k(G,x) = \sum_{i=\gamma^k(G)}^{|V(G)|} \mathtt{d}^k(G,i)x^i$$

is defined as distance-k domination polynomial of G.

Observe that the distance-k domination polynomial can be considered as a generalization of the domination polynomial.

Example 3.1.1. *Consider the graph G in Figure 2.1. The distance-2 domination number of G is 1. Also $\mathtt{d}^2(G,1) = 2$, $\mathtt{d}^2(G,2) = 13$, $\mathtt{d}^2(G,3) = 20$, $\mathtt{d}^2(G,4) = 15$, $\mathtt{d}^2(G,5) = 6$ and $\mathtt{d}^2(G,6) = 1$. Therefore the distance-2 domination polynomial of G is $D^2(G,x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 13x^2 + 2x$.*

The following two theorems follows from the fact that every distance-m dominating sets of any graph G is also a distance-k dominating set of G for $m < k$.

Theorem 3.1.2. *For any graph G, $\gamma^k(G) \leq \gamma^m(G)$ when $m < k$.*

Theorem 3.1.3. *For any graph G, $d^m(G,i) \leq \mathtt{d}^k(G,i)$ when $m < k$.*

Theorem 3.1.4. *Let G be a graph of order n with m isolated vertices. Then for all k, $\mathtt{d}^k(G,n-1) = n - m$.*

Proof. Let V be the set of all vertices of G and let N be the set of all isolated vertices of G. It is clear that for any vertex $v \in V - N$, the set $V - v$ is a distance-k dominating set of G. Therefore $\mathtt{d}^k(G,n-1) = |V - N| = n - m$. □

Corollary 3.1.5. *Let G be a connected graph of order $n > 1$. Then for all k, $\mathtt{d}^k(G,n-1) = n$.*

Theorem 3.1.6. *Let G be a graph of order n with m isolated vertices and s K_2-components. Then for $k > 1$,*

$$\mathtt{d}^k(G,n-2) = \frac{1}{2}(n-m)(n-m-1) - s.$$

Proof. Let V be the set of all vertices of G and let N be the set of all isolated vertices of G. Suppose that $S \subseteq V$ is a set of cardinality $n-2$. If S is not a distance-k dominating set of G, then $S \subseteq V - \{v, w\}$, where either $v \in N$ or vw is a K_2 component of G. Therefore

$$\mathtt{d}^k(G, n-2) = \binom{n}{2} - \left(m(n-1) - \binom{m}{2}\right) - s.$$

This implies that $\mathtt{d}^k(G, n-2) = \frac{1}{2}(m-n)(m-n+1) - s$. □

Corollary 3.1.7. *Let G be a connected graph of order $n > 2$. Then for $k > 1$,*

$$\mathtt{d}^k(G, n-2) = \frac{1}{2}n(n-1).$$

Theorem 3.1.8. *If G and H are isomorphic, then $D^k(G, x) = D^k(H, x)$.*

Theorem 3.1.9. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$D^k(G, x) = D^k(G_1, x)D^k(G_2, x)\ldots D^k(G_m, x).$$

Proof. It suffices to prove this theorem for $m = 2$. For $l \geq \gamma^k(G)$, a distance-k dominating set of l vertices in G arises by choosing a distance-k dominating set of j vertices in G_1 for some $j \in \{\gamma^k(G_1), \gamma^k(G_1) + 1, \ldots, |V(G)|\}$ and a distance-k dominating set of $l - j$ vertices of G_2. The number of ways of doing this over all $j = \gamma^k(G_1), \gamma^k(G_1) + 1, \ldots, |V(G)|$ is exactly the coefficient of x^l in $D^k(G_1, x)D^k(G_2, x)$. Hence both side of the above equation have the same coefficient, so they are identical polynomial. □

Theorem 3.1.10. *Let G be a graph and let k be any positive integer, then $D^k(G, x) = D(G^k, x)$.*

Proof. It follows from the fact that every distance-k dominating set of G with cardinality i is exactly the dominating set of G^k with cardinality i. □

Next theorem follows from the definitions of domination polynomial and distance-k domination polynomial.

Theorem 3.1.11. *Let G be a graph with domination polynomial $D(G, x)$, then $D^1(G, x) = D(G, x)$.*

From Theorem 3.1.11 it follows that when $k = 1$, the distance-k domination polynomial coincide with the domination polynomial. So throughout this chapter we assume that k is a positive integer greater than one.

Theorem 3.1.12. *Let G be a graph of order n and diameter D. Then $D^k(G,x) = (1+x)^n - 1$ if and only if $k \geq D$.*

Proof. Suppose $k \geq D$, then all vertices of G are with in a distance k. This implies that for $1 \leq i \leq n$ any subset of vertices of G of cardinality i is a distance-k dominating set. Therefore $D^k(G,x) = (1+x)^n - 1$. Conversely, suppose that $D^k(G,x) = (1+x)^n - 1$. Then $\gamma^k(G) = 1$ and $\mathtt{d}^k(G,1) = n$. This implies that all vertices of G are with in a distance k, that is, $k \geq d$. ☐

Corollary 3.1.13. *For the complete graph K_n,*

$$D^k(K_n,x) = (1+x)^n - 1.$$

Corollary 3.1.14. *For the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$,*

$$D^k(K_{n_1,n_2,\ldots,n_m},x) = (1+x)^N - 1,$$

where $N = n_1 + n_2 + \ldots + n_m$.

Corollary 3.1.15. *For the complete bipartite graph $K_{m,n}$,*

$$D^k(K_{m,n},x) = (1+x)^{m+n} - 1.$$

Corollary 3.1.16. *For the star graph S_n,*

$$D^k(S_n,x) = (1+x)^{n+1} - 1.$$

Corollary 3.1.17. *For the wheel graph W_n,*

$$D^k(W_n,x) = (1+x)^n - 1.$$

Corollary 3.1.18. *For $i = 1,2$, let G_i be a graph of order n_i, then*

$$D^k(G_1 \vee G_2,x) = (1+x)^{n_1+n_2} - 1.$$

Corollary 3.1.19. *For the complete graphs K_m and K_n,*

$$D^k(K_m \Box K_n,x) = (1+x)^{mn} - 1.$$

Corollary 3.1.20. *Let P be the Petersen graph, then*

$$D^k(P,x) = (1+x)^{10} - 1.$$

Corollary 3.1.21. *The distance-k domination polynomial of the Dutch windmill graph G_3^n is*

$$D^k(G_3^n, x) = (1+x)^{2n+1} - 1.$$

Corollary 3.1.22. *The distance-k domination polynomial of lollipop graph $L_{n,1}$ is*

$$D^k(L_{n,1}, x) = (1+x)^{n+1} - 1.$$

Theorem 3.1.23. *Let B_n be the bipartite cocktail party graph. Then for $n \geq 3$,*

$$\begin{aligned} D^2(B_n, x) &= (1+x)^{2n} - 2nx - 1 \quad && \textit{and} \\ D^k(B_n, x) &= (1+x)^{2n} - 1 && \textit{for } k \neq 2. \end{aligned}$$

Proof. Clearly, the diameter of B_n is 3. Therefore for $k \neq 2$ the proof is trivial. For $k = 2$ observe that $\gamma^2(B_n) = 2$. For $2 \leq i \leq n$ any subset of vertices of B_n of cardinality i is a distance-2 dominating set. Therefore we have the result. □

Corollary 3.1.24. *Let B_n be the bipartite cocktail party graph. Then for $n \geq 3$, $D(B_n^2, x) = (1+x)^{2n} - 2nx - 1$.*

Remark 3.1.25. *Note that $B_1 = 2K_1$ and $B_2 = 2K_2$. So $D^2(B_1, x) = x^2$ and $D^2(B_2, x) = x^2(x+2)^2$.*

Theorem 3.1.26. *Let $B_{m,n,1}$ be the generalized barbell graph. Then for all m, n*

$$\begin{aligned} D^2(B_{m,n,1}, x) &= (1+x)^{m+n} - (1+x)^{n-1} - (1+x)^{m-1} + 1 \quad && \textit{and} \\ D^k(B_{m,n,1}, x) &= (1+x)^{m+n} - (m+n)x - 1 && \textit{for } k \neq 2. \end{aligned}$$

Proof. Clearly, the diameter of $B_{m,n,1}$ is 3. Therefore for $k \neq 2$ the proof is trivial. Without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_m and u_n are adjacent. Then $\{v_m\}$ and $\{u_n\}$ are the only distance-2 dominating sets of cardinality 1 of $B_{m,n,1}$. Therefore $\gamma^2(B_{m,n,1}) = 1$ and $\mathtt{d}^2(B_{m,n,1}, 1) = 2$. For $2 \leq i \leq m+n$, a subset S of vertices of $B_{m,n,1}$ of cardinality i is not a distance-2 dominating set if either $S \subset V - \{v_m\}$ or $S \subset U - \{u_n\}$. Therefore $\mathtt{d}^2(B_{m,n,1}, i) = \binom{m+n}{i} - \binom{n-1}{i} - \binom{m-1}{i}$; for $2 \leq i \leq m-1$, $\mathtt{d}^2(B_{m,n,1}, i) = \binom{m+n}{i} - \binom{n-1}{i}$; for $m \leq i \leq n-1$ and $\mathtt{d}^2(B_{m,n,1}, i) = \binom{m+n}{i}$; for $n \leq i \leq m+n$. This implies that $D^2(B_{m,n,1}, x) = (1+x)^{m+n} - (1+x)^{n-1} - (1+x)^{m-1} + 1$. □

Corollary 3.1.27. *Let $B_{m,n,1}$ be generalized barbell graph. Then for all m, n, $D(B^2_{m,n,1}, x) = (1+x)^{m+n} - (1+x)^{n-1} - (1+x)^{m-1} + 1$.*

Theorem 3.1.28. *Let $B_{n,1}$ be n-barbell graph. Then for all n,*

$$\begin{aligned} D^2(B_{n,1}, x) &= (1+x)^{2n} - 2(1+x)^{n-1} + 1 && \text{and} \\ D^k(B_{n,1}, x) &= (1+x)^{2n} - 1 && \text{for } k \neq 2. \end{aligned}$$

Proof. Clearly, the diameter of $B_{n,1}$ is 3. Therefore for $k \neq 2$ the proof is trivial. For $k = 2$, let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_n and u_n are adjacent. Then $\{v_n\}$ and $\{u_n\}$ are the only distance-2 dominating sets of cardinality 1 of $B_{n,1}$. Therefore $\gamma^2(B_{n,1}) = 1$ and $\mathtt{d}^2(B_{n,1}, 1) = 2$. For $2 \leq i \leq 2n$, a subset S of vertices of $B_{n,1}$ of cardinality i is not a distance-2 dominating set if either $S \subset V - \{v_n\}$ or $S \subset U - \{u_n\}$. Therefore $\mathtt{d}^2(B_{n,1}, i) = \binom{2n}{i} - 2\binom{n-1}{i}$ for $2 \leq i \leq n-1$ and $\mathtt{d}^2(B_{n,1}, i) = \binom{2n}{i}$, for $n \leq i \leq 2n$. This implies that $D^2(B_{n,1}, x) = (1+x)^{2n} - 2(1+x)^{n-1} + 1$. $\square$

Corollary 3.1.29. *Let $B_{n,1}$ be the n-barbell graph. Then the domination polynomial of the square of $B_{n,1}$ is $D(B^2_{n,1}, x) = (1+x)^{2n} - 2(1+x)^{n-1} + 1$.*

Theorem 3.1.30. *Let $B_{(m,n)}$ be the bi-star graph. Then for all m, n*

$$\begin{aligned} D^2(B_{(m,n)}, x) &= (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1 && \text{and} \\ D^k(B_{(m,n)}, x) &= (1+x)^{m+n+2} - 1 && \text{for } k \neq 2. \end{aligned}$$

Proof. It follows from the fact that the square of the bi-star graph $B_{(m,n)}$ and the square of the generalized barbell graph $B_{m+1,n+1,1}$ are isomorphic. $\square$

Corollary 3.1.31. *Let $B_{(m,n)}$ be the bi-star graph. Then the domination polynomial of the square of $B_{(m,n)}$ is $D(B^2_{(m,n)}, x) = (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1$.*

Theorem 3.1.32. *If K_m and K_n are the complete graphs, then for $m \geq 2$*

$$\begin{aligned} D^2(K_m \circ K_n, x) &= (1+x)^{m(n+1)} - m(1+x)^n + m - 1 && \text{and} \\ D^k(K_m \circ K_n, x) &= (1+x)^{m(n+1)} - 1 && \text{for } k \neq 2. \end{aligned}$$

Proof. The proof is similar to the proof of the Theorem 3.1.26. $\square$

Corollary 3.1.33. *If K_m and K_n are the complete graphs, then for $m \geq 2$, $D((K_m \circ K_n)^2, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1$.*

Corollary 3.1.34. *For $m \geq 2$, the distance-k domination polynomial of $Q(m,n)$ is*

$$D^2(Q(m,n),x) = (1+x)^{mn} - m(1+x)^{n-1} + m - 1 \quad \text{and}$$
$$D^k(Q(m,n),x) = (1+x)^{mn} - 1 \quad \text{for } k \neq 2.$$

Proof. It follows from the fact that $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Corollary 3.1.35. *For $m \geq 2$, $D(Q^2(m,n),x) = (1+x)^{mn} - m(1+x)^{n-1} + m - 1$.*

3.2 $\mathtt{d}^k$-number of graphs

In this section we focus on the number of the real roots of distance-k domination polynomial of some graphs. First we define distance-k domination root of a graph.

Definition 3.2.1. *Let G be a graph with distance-k domination polynomial $D^k(G,x)$. A root of $D^k(G,x)$ is called a distance-k domination root of G and the set of all the distance-k domination roots of G is denoted by $\mathbb{Z}(D^k(G,x))$.*

Remark 3.2.1. *Let G be a graph with distance-k domination polynomial $D^k(G,x)$. Since the coefficients of $D^k(G,x)$ are positive, $(0,\infty)$ is a zero-free interval for $D^k(G,x)$.*

In this section we study the number of real distance-k domination roots of some specific graphs. So we need the following :

Definition 3.2.2. *Let G be a graph. The number of distinct real distance-k domination roots of the graph G is called $\mathtt{d}^k$-number of G and is denoted by $\mathtt{d}^k(G)$.*

Example 3.2.2. *The distance-2 domination polynomial of the graph G in Figure 2.1 is*

$$D^2(G,x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 13x^2 + 2x.$$

The distance-2 domination roots of G are

$$\mathbb{Z}(D^2(G,x)) = \{-2, -1.7861514, -0.213849, 0, -1 - 1.27201965i, -1 + 1.27201965i\}.$$

Therefore $\mathtt{d}^2(G) = 4$.

Next theorem follows from the fact that 0 is a distance-k domination root of any graph.

Theorem 3.2.3. *For any graph G, $\mathtt{d}^k(G) \geq 1$.*

Theorem 3.2.4. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}^k(G) \leq \sum_{i=1}^{m} \mathtt{d}^k(G_i) - m + 1.$$

Proof. It follows from the fact that $D^k(G, x) = \prod_{i=1}^{m} D^k(G_i, x)$. $\square$

Theorem 3.2.5. *If G and H are isomorphic, then $\mathtt{d}^k(G, x) = \mathtt{d}^k(H, x)$.*

Proof. It follows from the fact that if G and H are isomorphic, then $D^k(G, x) = D^k(H, x)$. $\square$

Theorem 3.2.6. *Let G be a graph and let k be any positive integer, then $\mathtt{d}^k(G) = m$ if and only if $\mathtt{d}(G^k) = m$.*

Proof. It follows from Theorem 3.1.10. $\square$

Theorem 3.2.7. *Let G be a graph of order n and diameter D. If $D \leq k$, then*

$$\mathtt{d}^k(G) = \begin{cases} 1 & ;\ \text{if } n \text{ is odd,} \\ 2 & ;\ \text{if } n \text{ is even.} \end{cases}$$

Proof. The result follows from the transformation $y = 1 + x$ in the distance-k domination polynomial $D^k(G, x)$. $\square$

Corollary 3.2.8. *For all n we have the following :*

$$\mathtt{d}^k(K_n) = \begin{cases} 1 & ;\ \text{if } n \text{ is odd,} \\ 2 & ;\ \text{if } n \text{ is even.} \end{cases}$$

Corollary 3.2.9. *For all n we have the following :*

$$\mathtt{d}^k(S_n) = \begin{cases} 1 & ;\ \text{if } n \text{ is even,} \\ 2 & ;\ \text{if } n \text{ is odd.} \end{cases}$$

Corollary 3.2.10. *For all n we have the following :*

$$\mathtt{d}^k(W_n) = \begin{cases} 1 & ;\ \text{if } n \text{ is odd,} \\ 2 & ;\ \text{if } n \text{ is even.} \end{cases}$$

Corollary 3.2.11. *For all n we have the following :*

$$\mathtt{d}^k(L_{n,1}) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is even,} \\ 2 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 3.2.12. *Let H and G be two graphs of order m and n respectively. Then the $\mathtt{d}^k$-number of $H \vee G$ is 1 if m and n have opposite parity and 2 otherwise.*

Corollary 3.2.13. *For all m, n, we have the following :*

$$\mathtt{d}^k(K_m \square K_n) = \begin{cases} 1 & ;\ \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 2 & ;\ \textit{if } n \textit{ otherwise.} \end{cases}$$

Corollary 3.2.14. *The $\mathtt{d}^k$-number of the Petersen graph P is 2.*

Corollary 3.2.15. *For all m, n the $\mathtt{d}^k$-number of the complete bipartite graph $K_{m,n}$ is 1 if m and n have opposite parity and 2 otherwise.*

Corollary 3.2.16. *For all n the $\mathtt{d}^k$-number of the complete bipartite graph $K_{n,n}$ is 2.*

Corollary 3.2.17. *For all n the $\mathtt{d}^k$-number of the Dutch windmill graph G_3^n is 1.*

Theorem 3.2.18. *The $\mathtt{d}^2$-number of the bipartite cocktail party graph B_n is 1 for $n \geq 3$.*

Proof. We have distance-2 domination polynomial of the bipartite cocktail party graph B_n is

$$D^2(B_n, x) = (1 + x)^{2n} - 2nx - 1.$$

Put $y = 1 + x$, then $D^2(B_n, y - 1) = f(y) = y^{2n} - 2ny + 2n - 1$. The remaining proof is similar to the proof of Theorem 2.2.41. □

Theorem 3.2.19. *For the generalized barbell graph $B_{m,n,1}$; $m, n \geq 2$ we have the following :*

$$\mathtt{d}^2(B_{m,n,1}) = \begin{cases} 4 & ;\ \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 2 & ;\ \textit{if both } m \textit{ and } n \textit{ are even,} \\ 3 & ;\ \textit{if } m \textit{ and } n \textit{ have opposite parity.} \end{cases}$$

Proof. We have distance-2 domination polynomial of generalized barbell graph $B_{m,n,1}$ is

$$D^2(B_{m,n,1}, x) = (1+x)^{m+n} - (1+x)^{n-1} - (1+x)^{m-1} + 1.$$

This implies that $D^2(B_{m,n,1}, y-1) = f(y) = y^{m+n} - y^{n-1} - y^{m-1} + 1$. The remaining proof is similar to the proof of Theorem 2.2.42. □

Corollary 3.2.20. *For the n-barbell graph $B_{n,1}$; $n \geq 2$ we have the following :*

$$\mathtt{d}^2(B_{n,1}) = \begin{cases} 4 & ;\textit{ if } n \textit{ is odd,} \\ 2 & ;\textit{ if } n \textit{ is even.} \end{cases}$$

Remark 3.2.21. *Note that -2 is a distance-2 domination root of the generalized barbell graph $B_{m,n,1}$ if either m and n are odd or m and n have opposite parity. Hence -2 is a distance-2 domination root of the n-barbell graph $B_{n,1}$ if n is odd.*

Theorem 3.2.22. *For the bi-star graph $B_{(m,n)}$ we have the following :*

$$\mathtt{d}^2(B_{(m,n)}) = \begin{cases} 2 & ;\textit{ if both } m \textit{ and } n \textit{ are odd,} \\ 4 & ;\textit{ if both } m \textit{ and } n \textit{ are even,} \\ 3 & ;\textit{ if } m \textit{ and } n \textit{ have opposite parity.} \end{cases}$$

Proof. We have $D^2(B_{(m,n)}, x) = (1+x)^{m+n+2} - (1+x)^n - (1+x)^m + 1$. Put $y = 1+x$, then $D^2(B_{(m,n)}, y-1) = f(y) = y^{m+n+2} - y^n - y^m + 1$. The remaining proof is similar to the proof of Theorem 2.2.45. □

Remark 3.2.23. *Note that -2 is a distance-2 domination root of the bi-star graph $B_{(m,n)}$ if either m and n are even or m and n have opposite parity.*

Theorem 3.2.24. *For $m \geq 2$ and $n \geq 1$ we have the following :*

$$\mathtt{d}^2(K_m \circ K_n) = \begin{cases} 2 & ;\textit{ if } n \textit{ is odd,} \\ 3 & ;\textit{ if } m \textit{ is odd and } n \textit{ is even,} \\ 4 & ;\textit{ if both } m \textit{ and } n \textit{ are even.} \end{cases}$$

Proof. The proof is similar to the proof of the Theorem 2.2.42. □

Corollary 3.2.25. *For $m \geq 2$ and $n \geq 1$ we have the following :*

$$\mathtt{d}^2(Q(m,n)) = \begin{cases} 2 & ; \text{if } n \text{ is even,} \\ 3 & ; \text{if both } m \text{ and } n \text{ are odd,} \\ 4 & ; \text{if } m \text{is} \text{even, and } n \text{ is odd.} \end{cases}$$

Remark 3.2.26. *Note that -2 is a distance-2 domination root of $K_m \circ K_n$ if both m and n are even. Hence -2 is a distance-2 domination root of $Q(m,n)$ if m is even and n is odd.*

3.3 Bounds for the distance-k domination roots of some graphs

In this section we estimate the bounds for the distance-k domination roots of some graphs.

Theorem 3.3.1. *Let G be a graph with diameter D. If $D \leq k$, then all the distance-k domination roots of the graph G lie on the unit circle with center $(-1,0)$.*

Proof. It follows from the fact that all n^{th} roots of unity lie on the unit circle centered at $(0,0)$. □

Corollary 3.3.2. *Let G be a graph of order n and diameter D. If $D \leq k$, then*

(1) $D^k(G,x)$ has no nonzero integer root, if n is odd.

(2) -2 is the only nonzero integer root of $D^k(G,x)$, if n is even.

Corollary 3.3.3. *All the distance-k domination roots of the complete graph K_n lie on the unit circle centered at $(-1,0)$.*

Corollary 3.3.4. *All the distance-k domination roots of the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$ lie on the unit circle centered at $(-1,0)$.*

Corollary 3.3.5. *All the distance-k domination roots of the complete bipartite graph $K_{m,n}$ lie on the unit circle centered at $(-1,0)$.*

Corollary 3.3.6. *All the distance-k domination roots of the star graph S_n lie on the unit circle centered at $(-1,0)$.*

Corollary 3.3.7. *All the distance-k domination roots of the wheel graph W_n lie on the unit circle centered at $(-1, 0)$.*

Corollary 3.3.8. *For any two graphs H and G, all the distance-k domination roots of the graph $H \vee G$ lie on the unit circle centered at $(-1, 0)$.*

Corollary 3.3.9. *For the complete graphs K_m and K_n, all the distance-k domination roots of the graph $K_m \square K_n$ lie on the unit circle centered at $(-1, 0)$.*

Corollary 3.3.10. *All the distance-k domination roots of the Petersen graph P lie on the unit circle centered at $(-1, 0)$.*

Corollary 3.3.11. *All the distance-k domination roots of the Dutch windmill graph G_3^n lie on the unit circle centered at $(-1, 0)$.*

Corollary 3.3.12. *All the distance-k domination roots of the lollipop graph of $L_{n,1}$ lie on the unit circle centered at $(-1, 0)$.*

Theorem 3.3.13. *All the nonzero distance-2 domination roots of the bipartite cocktail party graph B_n lie in the annulus $1 < |z + 1| \leq 2$.*

Proof. We have distance-2 domination polynomial of the bipartite cocktail party graph B_n is

$$D^2(B_n, x) = (1 + x)^{2n} - 2nx - 1.$$

Put $y = 1 + x$ and consider $f(y) = y^{2n} - 2ny + 2n - 1$. The remaining proof is similar to the proof of Theorem 2.3.21. $\square$

Corollary 3.3.14. *The bipartite cocktail party graph B_n has no nonzero integer distance-2 domination roots.*

Theorem 3.3.15. *All distance-2 domination roots of the generalized barbell graph $B_{m,n,1}$ lie inside the circle with center $(-1, 0)$ and radius 2.*

Proof. We have $D^2(B_{m,n,1}, y - 1) = f(y) = y^{m+n} - y^{n-1} - y^{m-1} + 1$. In this case $\max_i |a_i| = 1$, where a_i's are the coefficients of $f(y)$ for $i = 0, 1, \ldots, m + n$. Then by Theorem 1.2.5 we have the result. $\square$

The distance-2 domination roots of the generalized barbell graph $B_{m,n,1}$ for $2 \leq m \leq 10$ and $2 \leq n \leq 30$ are shown in Figure 3.1.

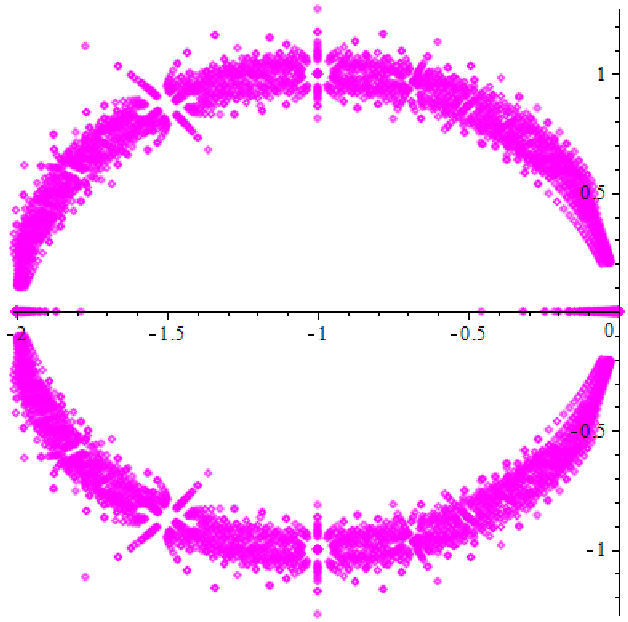

Figure 3.1: Distance-2 domination roots of $B_{m,n,1}$ for $2 \leq m \leq 10$, $2 \leq n \leq 30$.

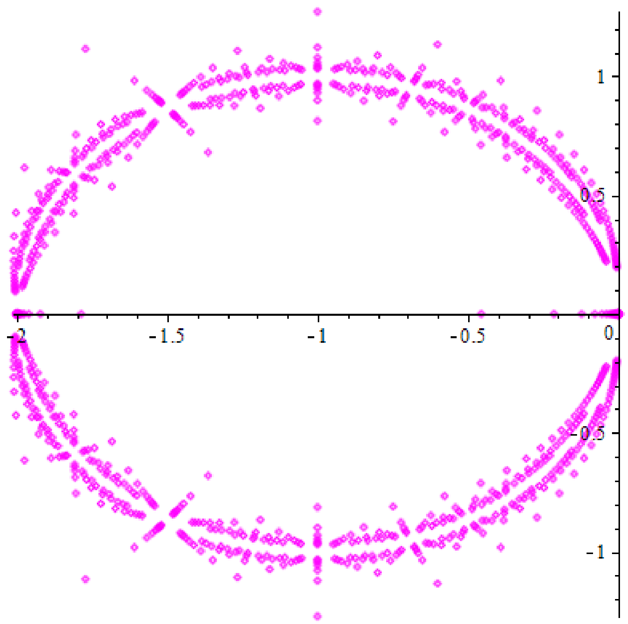

Figure 3.2: Distance-2 domination roots of $B_{n,1}$ for $1 \leq n \leq 30$.

Corollary 3.3.16. *All distance-2 domination roots of the n-barbell graph $B_{n,1}$ lie inside the circle with center $(-1, 0)$ and radius 2.*

The distance-2 domination roots of the n-barbell graph $B_{n,1}$ for $1 \leq n \leq 30$ are shown in Figure 3.2.

Corollary 3.3.17. *We have the following :*

(1) If m and n are odd, then -2 is the only nonzero integer root of $D^2(B_{m,n,1}, x)$.

(2) If m and n are even, then $D^2(B_{m,n,1}, x)$ has no nonzero integer root.

(3) If m and n have opposite parity, then -2 is the only nonzero integer root of $D^2(B_{m,n,1}, x)$.

Corollary 3.3.18. *For $n \geq 2$, we have the following :*

(1) If n is even, then $D^2(B_{n,1}, x)$ has no nonzero integer root.

(2) If n is odd, then -2 is the only nonzero integer root of $D^2(B_{n,1}, x)$.

Theorem 3.3.19. *All the distance-2 domination roots of the bi-star graph $B_{(m,n)}$ lie inside the circle with center $(-1, 0)$ and radius 2.*

Proof. The proof is similar to the proof of the Theorem 3.3.15. □

The distance-2 domination roots of the bi-star graph $B_{(m,n)}$ for $1 \leq m \leq 15$ and $1 \leq n \leq 30$ are shown in Figure 3.3.

Corollary 3.3.20. *We have the following :*

(1) If m and n are even, then -2 is the only nonzero integer root of $D^2(B_{(m,n)}, x)$.

(2) If m and n are odd, then $D^2(B_{(m,n)}, x)$ has no nonzero integer root.

(3) If m and n have opposite parity, then -2 is the only nonzero integer root of $D^2(B_{(m,n)}, x)$.

Theorem 3.3.21. *All the distance-2 domination roots of the corona $K_m \circ K_n$ lie inside the circle with center $(-1, 0)$ and radius m.*

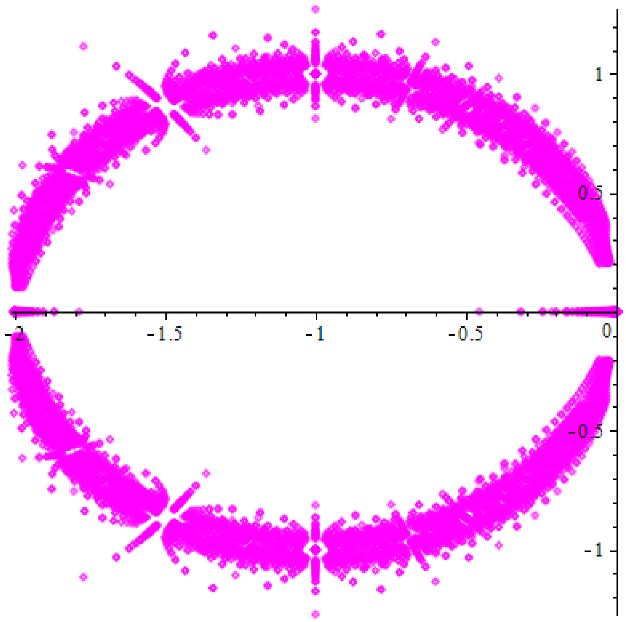

Figure 3.3: Distance-2 domination roots of $B_{(m,n)}$ for $1 \leq m \leq 15$, $1 \leq n \leq 30$.

Proof. We have the distance-2 domination polynomial of the corona $K_m \circ K_n$ is

$$D^2(K_m \circ K_n, x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1. \tag{3.1}$$

Replace $1+x$ by y in equation (3.1) we get

$$D^2(K_m \circ K_n, y-1) = y^{m(n+1)} - my^n + m - 1.$$

We have $y = 1$ is a root of $D^2(K_m \circ K_n, y-1)$. The remaining proof is similar to the proof of Theorem 2.3.28. □

Corollary 3.3.22. *All the distance-2 domination roots of the graph $Q(m,n)$ lie inside the circle with center $(-1,0)$ and radius m.*

Theorem 3.3.23. *All the distance-2 domination roots of the corona $K_n \circ K_n$ lie inside the circle with center $(-1,0)$ and radius $n^{\frac{1}{n}}$.*

Proof. Observe that the distance-2 domination polynomial of the corona $K_n \circ K_n$ is

$$D^2(K_n \circ K_n, x) = (1+x)^{n(n+1)} - n(1+x)^n + n - 1. \tag{3.2}$$

This implies that

$$D^2(K_n \circ K_n, y-1) = y^{n(n+1)} - ny^n + n - 1. \tag{3.3}$$

So it suffices to show that all the roots of

$$f(y) = y^{n+1} - ny + n - 1$$

lie in the circle center at the origin and having radius n. We have $y = 1$ is a root of $f(y)$. The remaining proof is similar to the proof of Theorem 2.3.30. $\square$

The distance-2 domination roots of the corona $K_n \circ K_n$ for $1 \le n \le 10$ are shown in Figure 3.4.

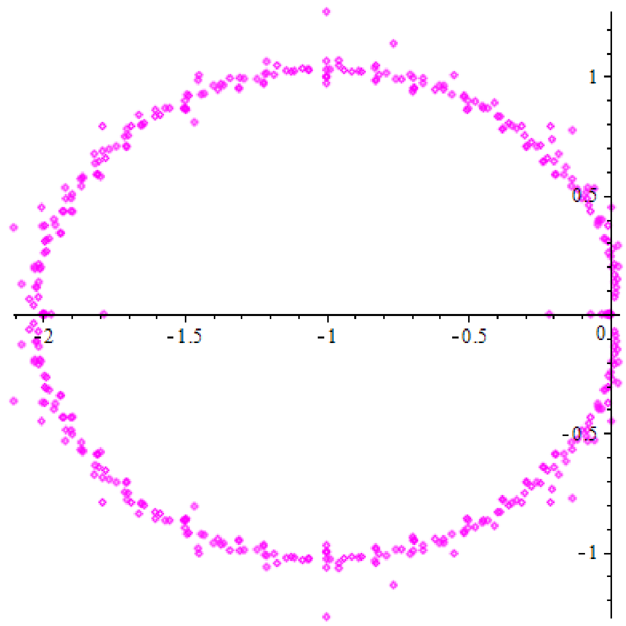

Figure 3.4: Distance-2 domination roots of $K_n \circ K_n$ for $1 \le n \le 10$.

Corollary 3.3.24. *All the distance-2 domination roots of the graph $Q(n, n)$ lie inside the circle with center $(-1, 0)$ and radius $n^{\frac{1}{n}}$.*

3.4 Stable graphs related to distance-k domination polynomial

In this section we introduce $\mathtt{d}^k$-stable and $\mathtt{d}^k$-unstable graphs and provide some examples of $\mathtt{d}^k$-stable graphs and $\mathtt{d}^k$-unstable graphs. We begin this section by defining $\mathtt{d}^k$ stable graphs.

Definition 3.4.1. *Let $G = (V(G), E(G))$ be a graph. For $k \geq 1$ the graph G is said to be a distance-k domination stable graph or simply $\mathtt{d}^k$-stable graph if all the nonzero distance-k domination roots lie in the left open half-plane, that is, if real part of the nonzero distance-k domination roots is negative. If G is not $\mathtt{d}^k$-stable, then G is said to be a distance-k domination unstable or simply $\mathtt{d}^k$-unstable graph.*

Example 3.4.1. *The distance-2 domination polynomial of the graph $K_3 \circ K_4$ is*

$$D(K_3 \circ K_4, x) = (1 + x)^{15} + 3(1 + x)^4 + 2.$$

With the aid of Maple, we find that the distance-2 domination roots of $K_3 \circ K_4$ are :

$\mathbb{Z}(K_3 \circ K_4) = \{-2.0449 - 0.3546i, -2.0449 + 0.3546i, -1.7512 - 0.8554i, -1.7512 + 0.8554i,$
$-1.2006 - 1.0536i, -1.2006 + 1.0536i, -0.9788 - 0.8966i, -0.9788 + 0.8966i,$
$-0.5017 - 1.0054i, -0.5017 + 1.0054i, -0.05896 - 0.63132i, -0.05896 + 0.63132i,$
$-1.88496, -0.04263, 0\}.$

All the nonzero distance-2 domination roots of the graph $K_3 \circ K_4$ lie in the open left half-plane. Hence $K_3 \circ K_4$ is a $\mathtt{d}^2$-stable graph.

Example 3.4.2. *The distance-2 domination polynomial of bipartite cocktail party graph B_8 is*

$$D(B_8, x) = (1 + x)^{16} - 16x - 1.$$

With the aid of Maple, we find that the distance-2 domination roots of B_8 are :

$\mathbb{Z}(B_8) = \{-2.2226 - 0.2524i, -2.2226 + 0.2524i, -2.0198 - 0.7142i, -2.0198 + 0.7142i,$
$-1.6497 - 1.0539i, -1.6497 + 1.0539i, -1.1768 - 1.2141i, -1.1768 + 1.2141i,$
$-0.6842 - 1.16832i, -0.6842 + 1.16832i, -0.2603 - 0.92594i, -0.2603 + 0.92594i,$
$0.13394 - 0.53015i, 0.13394 + 0.53015i, 0, 0\}.$

Observe that the distance-2 domination roots $0.13394 - 0.53015i$ and $0.13394 + 0.53015i$ lie in the open right half-plane. Hence B_8 is a $\mathtt{d}^2$-unstable graph.

Theorem 3.4.3. *If G and H are isomorphic graphs then G is $\mathtt{d}^k$-stable if and if H is $\mathtt{d}^k$-stable.*

Proof. It follows from the fact that if G and H are isomorphic graphs then $D^k(G,x) = D^k(H,x)$. □

Corollary 3.4.4. *If G and H are isomorphic graphs then G is $\mathtt{d}^k$-unstable if and if H is $\mathtt{d}^k$-unstable.*

Theorem 3.4.5. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}^k$-stable if and if each G_i is $\mathtt{d}^k$-stable.*

Proof. It follows from the fact that $D^k(G,x) = \prod_{i=1}^{m} D^k(G_i, x)$. □

Corollary 3.4.6. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}^k$-unstable if and if one of the G_i is $\mathtt{d}^k$-unstable.*

Next theorem follows from Theorem 3.1.10.

Theorem 3.4.7. *Let G be a graph and let k be any positive integer, then G is $\mathtt{d}^k$-stable if and only if G^k is $\mathtt{d}$-stable.*

Corollary 3.4.8. *Let G be a graph and let k be any positive integer, then G is $\mathtt{d}^k$-unstable if and only if G^k is $\mathtt{d}$-unstable.*

Remark 3.4.9. *Using Maple, we find that real part of all the distance-2 domination roots of all graphs of order upto 6 is negative. Therefore there is no $\mathtt{d}^2$-unstable graph of order upto 6.*

Theorem 3.4.10. *Let G be a connected graph of order n. If G is $\mathtt{d}^2$-stable then*

$$n < 1 + 2\,\mathtt{d}^2(G, n-3).$$

Proof. Suppose G is $\mathtt{d}^2$-stable. Then by Routh-Hurwitz criteria 1.2.7, we have Routh-Hurwitz matrix $H_2 > 0$. This implies that

$$\mathtt{d}^2(G, n-1)\mathtt{d}^2(G, n-3) - \mathtt{d}^2(G, n-2) > 0.$$

Since G is connected we have

$$\mathtt{d}^2(G, n-1) = n \text{ and } \mathtt{d}^2(G, n-2) = \frac{1}{2}n(n-1).$$

This completes the proof. □

Theorem 3.4.11. *Let G be a graph of order n with diameter D. If $D \leq k$, then G is $\mathtt{d}^k$-stable.*

Proof. If $D \leq k$, then we have

$$D^k(G,x) = (1+x)^n - 1. \tag{3.4}$$

It follows from the transformation $y = 1+x$ in equation (3.4). □

Corollary 3.4.12. *The complete graph K_n is $\mathtt{d}^k$-stable for all n.*

Corollary 3.4.13. *The complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$ is $\mathtt{d}^k$-stable for all $n_1, n_2, \ldots, n_m$.*

Corollary 3.4.14. *The complete bipartite graph $K_{m,n}$ is $\mathtt{d}^k$-stable for all m, n.*

Corollary 3.4.15. *The star graph S_n is $\mathtt{d}^k$-stable for all n.*

Corollary 3.4.16. *The wheel graph W_n is $\mathtt{d}^k$-stable for all n.*

Corollary 3.4.17. *Let H and G be two graphs. Then $H \vee G$ is $\mathtt{d}^k$-stable.*

Corollary 3.4.18. *For the complete graph K_m and K_n, $K_m \square K_n$ is $\mathtt{d}^k$-stable for all m, n.*

Corollary 3.4.19. *The Petersen graph P is $\mathtt{d}^k$-stable.*

Corollary 3.4.20. *The the Dutch windmill graph G_3^n is $\mathtt{d}^k$-stable for all n.*

Corollary 3.4.21. *The lollipop graph $L_{n,1}$ is $\mathtt{d}^k$-stable for all n.*

Remark 3.4.22. *Using maple, we find that the bipartite cocktail party graph B_n is d^2- stable for $n \leq 7$ and is $\mathtt{d}^2$-unstable for $8 \leq n \leq 30$. We have the distace-2 domination polynomial of B_n is*

$$D^2(B_n,x) = (1+x)^{2n} - 2nx - 1.$$

Put $y = 1+x$ and consider $f(y) = y^{2n} - 2ny + 2n - 1$. Then $y = 1$ is a double root of $f(y)$. Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{2n-2} + 2y^{2n-3} + 3y^{2n-4} + \ldots + (2n-2)y + 2n - 1.$$

We have if $f(z) = a_n z^n + a_{n-1} z^n + \ldots + a_0$ is a polynomial with real coefficient satisfying $a_0 \geq a_1 \geq \ldots \geq a_n > 0$ then no roots of $f(z)$ lie in $\{z \in \mathbb{C} : |z| < 1\}$ [45]. Therefore if $g(z) = 0$, then $|z| > 1$. This implies that all the nonzero roots of $D(B_n, x)$ are out side the unit circle centered at $(-1, 0)$. So we conjectured that the bipartite cocktail party graph B_n is not $\mathtt{d}^2$-stable for all but finite values of n.

The distance-2 domination roots of the bipartite cocktail party graph B_n for $1 \leq n \leq 7$ and $1 \leq n \leq 30$ are shown in Figures 3.5 and 3.6 respectively.

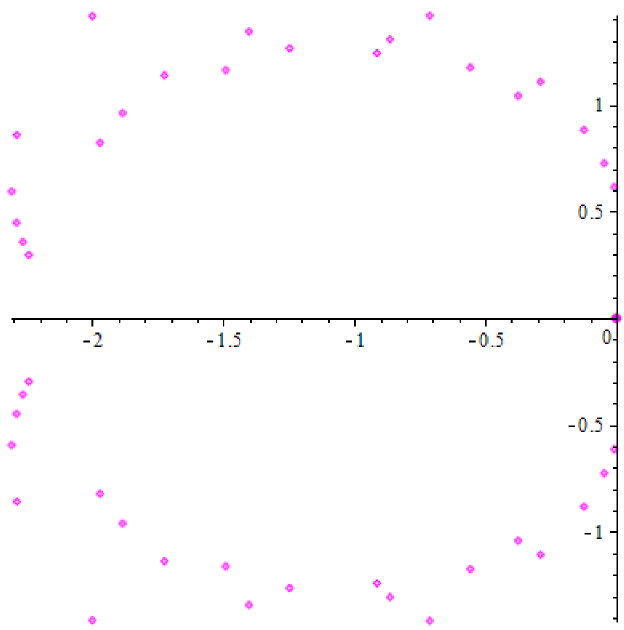

Figure 3.5: Distance-2 domination roots of B_n for $1 \leq n \leq 7$.

Remark 3.4.23. *Using Maple, we find that $K_m \circ K_n$ has distance-2 domination roots in the right-half plane for $m = 5$ and $3 \leq n \leq 30$. Therefore the corona $K_m \circ K_n$ is not $\mathtt{d}^2$-stable for all but finite values of m and n. But the limits of distance-2 domination roots of the corona $K_m \circ K_n$ are the unit circle centered at $(-1, 0)$. For, we have $D^2((K_m \circ K_n), x) = (1+x)^{m(n+1)} - m(1+x)^n + m - 1$. We rewrite $f_n(x) = D^2((K_m \circ K_n), x)$ as*

$$
\begin{aligned}
f_n(x) &= (1+x)^m \left((1+x)^m\right)^n + (-m)(1+x)^n + (m-1)(1)^n \\
&= \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \lambda_3^n,
\end{aligned}
$$

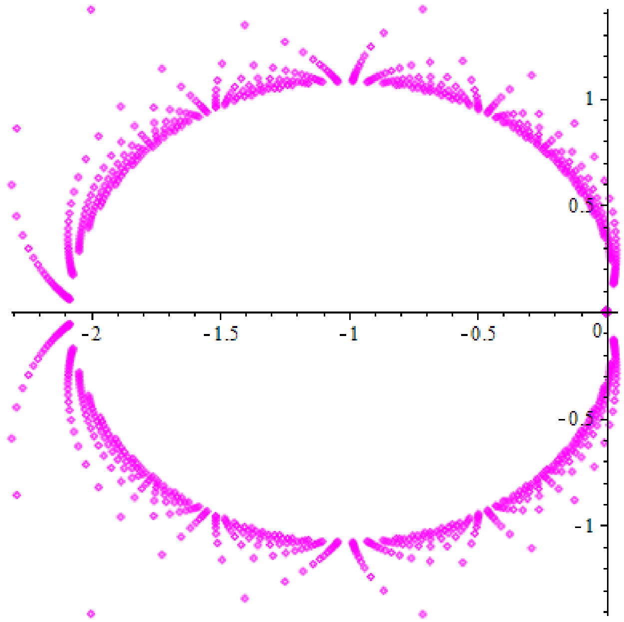

Figure 3.6: Distance-2 domination roots of B_n for $1 \leq n \leq 30$.

where

$$\alpha_1 = (1+x)^m,\ \lambda_1 = (1+x)^m,\ \alpha_2 = -m,\ \lambda_2 = 1+x,\ \alpha_3 = m-1,\ \lambda_3 = 1.$$

Clearly α_1, α_2 and α_3 are not identically zero and for $i \neq j$, $\lambda_i \neq \omega\lambda_j$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now applying part (i) of the Theorem 1.2.9, we consider the following four different cases :

(i) $|\lambda_1| = |\lambda_2| = |\lambda_3|$

(ii) $|\lambda_1| = |\lambda_2| > |\lambda_3|$

(iii) $|\lambda_1| = |\lambda_3| > |\lambda_2|$

(iv) $|\lambda_2| = |\lambda_3| > |\lambda_1|$

Assume that $|\lambda_1| = |\lambda_2| > |\lambda_3|$, that is, $|(1+x)^m| = |1+x| > 1$. This implies that either $x = -1$ or $|1+x| = 1$, both of which contradict $|1+x| > 1$. Thus there is no $x \in \mathbb{C}$ satisfy $|\lambda_1| = |\lambda_2| > |\lambda_3|$, so we need not consider the case (ii). By similar argument, we can rule out the cases (iii) and (iv). So we need *to consider the case (i),* $|\lambda_1| = |\lambda_2| = |\lambda_3|$. *Assume that* $|(1+x)^m| = |1+x| = 1$. *This implies that* $|x-(-1)| = 1$, *that is,* x *lies on the unit circle centered at* $(-1,0)$. *So we can conclude that the limits of distance-*2 *domination roots of the corona* $K_m \circ K_n$ *are the unit circle centered at* $(-1,0)$.

CHAPTER 4

TOTAL DOMINATION STABLE GRAPHS

In this chapter we mainly deals with the total domination polynomial of graphs. In Section 4.1 we define total domination polynomial of a graph and find total domination polynomial of some graphs. In Subsection 4.1.1 we find total domination polynomial of the square of some graphs. In Section 4.2, we define total domination root and introduce a new concept, $\mathtt{d}_t$-number of a graph and also find $\mathtt{d}_t$-number of some graphs. We include bounds for total domination roots of some graphs in Section 4.3. We introduce $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs in Section 4.4 and provide some examples of $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs.

4.1 Total domination polynomial of graphs

In this section we state the definition of total domination polynomial and find the same for some graphs.

Definition 4.1.1 (see [42]). *Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V$ is a total dominating set if every vertex $v \in V$ is adjacent to at least one vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of the total dominating sets in G. Let $\mathcal{D}_t(G, i)$ be the family of total dominating* *sets of G with cardinality i and let $\mathtt{d}_t(G,i) = |\mathcal{D}_t(G,i)|$. The polynomial*

$$D_t(G,x) = \sum_{i=\gamma_t(G)}^{|V(G)|} \mathtt{d}_t(G,i)x^i$$

is defined as total domination polynomial of G.

Example 4.1.1. *Consider the graph G in Figure 2.1. The total domination number of G is $\gamma_t(G) = 2$. Also $\mathtt{d}_t(G,2) = 1$, $\mathtt{d}_t(G,3) = 4$, $\mathtt{d}_t(G,4) = 6$, $\mathtt{d}_t(G,5) = 4$ and $\mathtt{d}_t(G,6) = 1$. Therefore the total domination polynomial of G is $D_t(G,x) = x^6 + 4x^5 + 6x^4 + 4x^3 + x^2$.*

Theorem 4.1.2. *If G and H are isomorphic, then $D_t(G,x) = D_t(H,x)$.*

Theorem 4.1.3. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$D_t(G,x) = \prod_{i=1}^{m} D_t(G_i,x)$$

Theorem 4.1.4. *The total domination polynomial of the complete graph K_n is $(1+x)^n - nx - 1$.*

Lemma 4.1.5. *Let G be a graph of order n without isolated vertices and let H be any graph. Then the total domination number $\gamma_t(G \circ H) = n$.*

Theorem 4.1.6. *Let G be a graph of order n without isolated vertices. Then the total domination polynomial of $G \circ \overline{K_m}$ is*

$$D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}.$$

Proof. By Lemma 4.1.5, we have $\gamma_t(G \circ \overline{K_m}) = n$. If S is a total dominating set of $G \circ \overline{K_m}$, then $V(G) \subset S$, therefore $\mathtt{d}_t(G \circ \overline{K_m}, n) = 1$. For $n+1 \le i \le n(m+1)$, $\mathtt{d}_t(G \circ \overline{K_m}, i) = \binom{mn}{i-n}$. Hence $D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}$. □

Theorem 4.1.7. *Let G be a graph of order n. Then the total domination polynomial of $K_1 \circ G$ is*

$$D_t(K_1 \circ G, x) = D_t(G,x) + x\left((1+x)^n - 1\right).$$

Proof. It follows from the facts that total dominating sets of G is a total dominating set of $K_1 \circ G$ and any set of vertices of $K_1 \circ G$ containing the vertex of K_1 is also a total dominating set. □

Corollary 4.1.8. *Let G be a graph of order n. Then the total domination polynomial of $\overline{K_m} \circ G$ is*

$$D_t(\overline{K_m} \circ G, x) = \left(D_t(G, x) + x\left((1+x)^n - 1\right)\right)^m.$$

Proof. It follows from the fact that $\overline{K_m} \circ G = \bigcup_{i=1}^{m} K_1 \circ G$. □

Corollary 4.1.9. *The total domination polynomial of the Dutch windmill graph G_3^n is*

$$D_t(G_3^n, x) = x^{2n} + x\left((1+x)^{2n} - 1\right).$$

Proof. It follows from the fact that G_3^n and $K_1 \circ nK_2$ are isomorphic. □

Theorem 4.1.10. *The total domination polynomial of the spider graph Sp_{2n+1} is*

$$D_t(Sp_{2n+1}, x) = x^n\left((1+x)^{n+1} - 1\right).$$

Proof. Let v, $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of Sp_{2n+1} such that v is adjacent to v_i for every $i = 1, 2, \ldots, n$ and v_i and u_i are adjacent for every $i = 1, 2, \ldots, n$. It is clear that the total dominating sets of Sp_{2n+1} are exactly the sets of vertices of Sp_{2n+1} properly containing V. Hence $\gamma_t(Sp_{2n+1}) = n + 1$ and $\mathtt{d}_t(Sp_{2n+1}, n + i) = \binom{n+1}{i}$ for $i = 1, 2, \ldots, n + 1$. □

Theorem 4.1.11. *The total domination polynomial of the lollipop graph $L_{n,1}$ is*

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right).$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the complete graph K_n and v be the path P_1 and let v is adjacent to v_1. Clearly the total dominating sets of $L_{n,1}$ are exactly the set of vertices of $L_{n,1}$ properly containing v_1. Therefore $\gamma_t(L_{n,1}) = 2$ and $2 \le i \le n + 1$, $\mathtt{d}_t(L_{n,1}, i) = \binom{n}{i-1}$. □

Theorem 4.1.12. *The total domination polynomial of the bipartite cocktail party graph B_n is*

$$D_t(B_n, x) = \left((1+x)^n - nx - 1\right)^2.$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. Any total dominating set S of B_n contains at least two v_i and at least two u_i. Note that sets of this form are of size greater than or equal to 4. Therefore $\gamma_t(B_n) = 4$. Also for $4 \leq i \leq n$, $\mathtt{d}_t(B_n, i) = \binom{2n}{i} - 2\binom{n}{i} - 2\binom{n}{i-1}$, $\mathtt{d}_t(B_n, n+1) = \binom{2n}{n+1} - 2n$ and for $n+2 \leq i \leq 2n$, $\mathtt{d}_t(B_n, i) = \binom{2n}{i}$. □

Theorem 4.1.13. *The total domination polynomial of the generalized barbell graph $B_{m,n,1}$ is*

$$D_t(B_{m,n,1}, x) = \left[(1+x)^m - (m-1)x - 1\right]\left[(1+x)^n - (n-1)x - 1\right].$$

Proof. Without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_m and u_n are adjacent. The only two element total dominating set of $B_{m,n,1}$ is $\{v_m, u_n\}$. Therefore $\gamma_t(B_{m,n,1}) = 2$ and $\mathtt{d}_t(B_{m,n,1}, 2) = 1$. Also observe that for $2 \leq i \leq m+n$, a subset S of vertices $B_{m,n,1}$ of cardinality i is not a total dominating set if and only if (i) $S \subset V$ or (ii) $S \subset U$ or (iii) S contains one element from $V - \{v_n\}$ and $i-1$ elements from U or (iv) S contains one element from $U - \{u_n\}$ and $i-1$ elements from V. Therefore

$$\mathtt{d}_t(B_{m,n,1}, i) = \begin{cases} 1 & \text{if } i = 2, \\ \binom{m+n}{i} - \binom{n}{i} - \binom{m}{i} - (n-1)\binom{m}{i-1} - (m-1)\binom{n}{i-1} & \text{if } 3 \leq i \leq m, \\ \binom{m+n}{m+1} - \binom{n}{m+1} - (n-1) - (m-1)\binom{n}{m} & \text{if } i = m+1, \\ \binom{m+n}{i} - \binom{n}{i} - (m-1)\binom{n}{i-1} & \text{if } m+2 \leq i \leq n, \\ \binom{m+n}{n+1} - (m-1) & \text{if } i = n+1, \\ \binom{m+n}{i} & \text{if } n+2 \leq i \leq m+n. \end{cases}$$

Hence $D_t(B_{m,n,1}, x) = \left[(1+x)^m - (m-1)x - 1\right]\left[(1+x)^n - (n-1)x - 1\right]$. □

Corollary 4.1.14. *The total domination polynomial of the n-barbell graph $B_{n,1}$ is*

$$D_t(B_{n,1}) = \left((1+x)^n - (n-1)x - 1\right)^2.$$

Proof. It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Theorem 4.1.15. *The total domination polynomial of the bi-star graph $B_{(m,n)}$ is*

$$D_t(B_{(m,n)}, x) = x^2(1+x)^{m+n}.$$

Proof. Let $\{u, v\}$, $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_m\}$ be the vertices of $B_{(m,n)}$ such that u and v are adjacent, every vertices in U are adjacent to u and every vertices in V are adjacent to v. It is clear that any total dominating set of $B_{(m,n)}$ must contain $\{u, v\}$. Therefore $\gamma_t(B_{(m,n)}) = 2$ and $\mathtt{d}_t(B_{(m,n)}, 2) = 1$. For $3 \le i \le m+n+2$, the $i-$element dominating set of $B_{(m,n)}$ must contain $\{u, v\}$, and the $i-2$ elements can have $\binom{m+n}{i-2}$ choices. □

Corollary 4.1.16. *The total domination polynomial of the bi-star graph $B_{(n,n)}$ is*

$$D_t(B_{(n,n)}, x) = x^2(1+x)^{2n}.$$

Next we study the total domination polynomial of $m-$partite graph $K_{n_1,\ldots,n_m}$ with $N = n_1 + n_2 + \ldots + n_m$. Clearly $\gamma_t(K_{n_1,n_2,\ldots,n_m}) > 1$. For all distinct i, j in $\{1, 2, \ldots, m\}$, $\{a, b\}$ is a total dominating set of $K_{n_1,n_2,\ldots,n_m}$, where $a \in V_i$ and $b \in V_j$. Therefore $\gamma_t(K_{n_1,n_2,\ldots,n_m}) = 2$. If $B \subseteq V_i$ for some i in $\{1, 2, \ldots, m\}$, then any vertices in B is not adjacent. Therefore B is not a total dominating set of $K_{n_1,n_2,\ldots,n_m}$. That is, the subset B of vertices of $K_{n_1,n_2,\ldots,n_m}$ is a non-total dominating set of $K_{n_1,n_2,\ldots,n_m}$ if and only if $B \subseteq V_i$ for some i in $\{1, 2, \ldots, m\}$. Without loss of generality we assume that $n_1 \le n_2 \le \ldots \le n_m$.

Theorem 4.1.17. *We have the following :*

$$\mathtt{d}_t(K_{n_1,n_2,\ldots,n_m}, i) = \begin{cases} \binom{N}{i} - \sum\limits_{j=1}^{m} \binom{n_j}{i} & \text{if } i \le n_1, \\ \binom{N}{i} - \sum\limits_{j=k}^{m} \binom{n_j}{i} & \text{if } n_{k-1} < i \le n_k, k \in \{2, 3, \ldots, m\}, \\ \binom{N}{i} & \text{otherwise.} \end{cases}$$

Proof. Let $i \le n_1$ and let B be a non-total dominating subset of vertices of $K_{n_1,n_2,\ldots,n_m}$ with cardinality i. Then we can choose such B from V_j's in $\binom{n_j}{i}$ ways and j varies from $1, 2, \ldots, m$. This implies that the total number of ways to choose such B is $\sum\limits_{j=1}^{m} \binom{n_j}{i}$. Hence $\mathtt{d}_t(K_{n_1,n_2,\ldots,n_m}, i) = \binom{N}{i} - \sum\limits_{j=1}^{m} \binom{n_j}{i}$ for all $i \le n_1$. Let i such that $n_{k-1} < i \le n_k$ where $k \in \{2, 3, \ldots, m\}$, then the non-total dominating set with cardinality i does not contains elements from V_j for all

$j < k$. Let B be a non-total dominating set of cardinality i, then elements in B from $V_j, (j \geq k)$ can be choosen in $\binom{n_j}{i}$ ways and j varies from $k, k+1, \ldots, m$. This implies that the total number of ways to choose such B is $\sum\limits_{j=k}^{m} \binom{n_j}{i}$. Hence $\mathtt{d}_t(K_{n_1,n_2,\ldots,n_m}, i) = \binom{N}{i} - \sum\limits_{j=k}^{m} \binom{n_j}{i}$ for all $n_{k-1} < i \leq n_k$, where $k \in \{2, 3, \ldots, m\}$. Let $i > n_m$ then every subset of vertices of G having i elements contains vertices from V_i and V_j for distinct i, j. This implies that every subset of vertices of $K_{n_1,n_2,\ldots,n_m}$ having i elements are total dominating sets of G. Therefore $\mathtt{d}_t(K_{n_1,n_2,\ldots,n_m}, i) = \binom{N}{i}$ for all $i > n_m$. $\square$

Corollary 4.1.18. $\mathtt{d}_t(K_{n[m]}, i) = \begin{cases} \binom{mn}{i} - m\binom{n}{i} & \text{if } i \leq n, \\ \binom{mn}{i} & \text{otherwise.} \end{cases}$

Theorem 4.1.19. *The total domination polynomial of $K_{n_1,n_2,n_3,\ldots,n_m}$ is :*
$D_t(K_{n_1,n_2,n_3,\ldots,n_m}, x) = \sum\limits_{i=2}^{m} [(1+x)^{n_i} - 1][(1+x)^{n_1+n_2+\ldots+n_{i-1}} - 1]$.

Corollary 4.1.20. *The total domination polynomial of $K_{n[m]}$ is*

$$D_t(K_{n[m]}, x) = (1+x)^{mn} - m(1+x)^n + (m-1).$$

Corollary 4.1.21. *The total domination polynomial of $K_{m,n}$ is*

$$D_t(K_{m,n}, x) = [(1+x)^m - 1][(1+x)^n - 1].$$

4.1.1 Total domination polynomial of square of some graphs

In this section we obtain an explicit formula for the total domination polynomial of the square of some specific graphs. Next two theorems will give total domination polynomial of the square of some graphs with some specified properties.

Theorem 4.1.22. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$D_t(G^2, x) = \prod_{i=1}^{m} D_t(G_i^2, x)$$

Proof. We have $G = G_1 \cup G_2 \cup \ldots \cup G_m$, then $G^2 = G_1^2 \cup G_2^2 \cup \ldots \cup G_m^2$. Therefore $D_t(G^2, x) = \prod\limits_{i=1}^{m} D_t(G_i^2, x)$. $\square$

Theorem 4.1.23. *Let G be a graph of order n. Then the total domination polynomial of G^2 is $(1+x)^n - nx - 1$ if and only if $D(G) \leq 2$.*

Proof. It follows from the facts that K_n is $\mathcal{D}_t-$unique [42] and the graphs G^2 and the complete graph K_n are isomorphic if and only if $D(G) \leq 2$. □

Corollary 4.1.24. *For the complete graph K_n,*

$$D_t(K_n^2, x) = (1+x)^n - nx - 1.$$

Corollary 4.1.25. *For the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$,*

$$D_t(K_{n_1,n_2,\ldots,n_m}^2, x) = (1+x)^N - Nx - 1,$$

where $N = n_1 + n_2 + \ldots + n_m$.

Corollary 4.1.26. *For the complete bipartite graph $K_{m,n}$,*

$$D_t(K_{m,n}^2, x) = (1+x)^{m+n} - (m+n)x - 1.$$

Corollary 4.1.27. *For the star graph S_n,*

$$D_t(S_n^2, x) = (1+x)^{n+1} - (n+1)x - 1.$$

Corollary 4.1.28. *For the wheel graph W_n,*

$$D_t(W_n^2, x) = (1+x)^n - nx - 1.$$

Corollary 4.1.29. *Let H and G be two graphs of order m and n respectively. Then the total domination polynomial of the square of $H \vee G$ is*

$$D_t((H \vee G)^2, x) = (1+x)^{m+n} - (m+n)x - 1.$$

Corollary 4.1.30. *For the complete graphs K_m and K_n,*

$$D_t((K_m \Box K_n)^2, x) = (1+x)^{mn} - mnx - 1.$$

Corollary 4.1.31. *Let P be the Petersen graph P, then*

$$D_t(P^2, x) = (1+x)^{10} - 10x - 1.$$

Corollary 4.1.32. *The total domination polynomial of the square of the Dutch windmill graph G_3^n is*

$$D_t(G_3^{n^2}, x) = (1+x)^{2n+1} - (2n+1)x - 1.$$

Corollary 4.1.33. *The total domination polynomial of the square of the lollipop graph $L_{n,1}$ is*

$$D_t(L_{n,1}^2, x) = (1+x)^{n+1} - (n+1)x - 1.$$

Theorem 4.1.34. *Let H and G be two graphs. Then the total domination polynomial of $(H \vee G)^2$ and $H^2 \vee G^2$ are equal if and only if $D(H)$ and $D(G)$ are less than or equal to two.*

Proof. It follows from Lemma 2.1.24. □

Corollary 4.1.35. *Let G be a graph of order n such that G^2 is a complete graph. Then $D_t(K_m \vee G^2, x) = D_t((K_m \vee G)^2, x) = (1+x)^{m+n} - (m+n)x - 1$.*

Theorem 4.1.36. *For $n \geq 3$ the total domination polynomial of the square of the bipartite cocktail party graph B_n is*

$$D_t(B_n^2, x) = (1+x)^{2n} - n(1+x)^2 + (n-1).$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. Clearly, any subset of vertices of B_n of cardinality 2 forms a total dominating set of B_n^2 excluding $\{v_i, u_i\}$ for $i = 1, 2, \ldots, n$. Therefore $\gamma_t(B_n^2) = 2$, $\mathtt{d}_t(B_n^2, 2) = \binom{2n}{2} - n$ and $\mathtt{d}_t(B_n^2, i) = \binom{2n}{i}$, for all $3 \leq i \leq 2n$. □

Remark 4.1.37. *Note that $B_1 = 2K_1$ and $B_2 = 2K_2$, so $D_t(B_1^2, x) = 0$ and $D_t(B_2^2, x) = x^4$.*

Theorem 4.1.38. *The total domination polynomial $D_t(B_{m,n,1}^2, x)$ of the square of the generalized barbell graph $B_{m,n,1}$ is*

$$\left[(1+x)^{m-1} - (m-1)x - 1\right]\left[(1+x)^{n-1} - (n-1)x - 1\right] + (1+x)^{m+n-2}(x^2+2x) - 2x.$$

Proof. Without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_m and u_n are adjacent.

Any set of vertices of $B_{m,n,1}$ properly containing v_m or u_n are total dominating set of $B^2_{m,n,1}$. Therefore $\gamma_t(B^2_{m,n,1}) = 2$ and $\mathtt{d}_t(B^2_{m,n,1}, 2) = 2m + 2n - 3$. Also observe that for $2 \leq i \leq 2n$, a subset S of vertices $B_{m,n,1}$ of cardinality i is not a total dominating set of $B^2_{m,n,1}$ if and only if (i) $S \subset V - \{v_n\}$ or (ii) $S \subset U - \{u_n\}$ or (iii) S contains one element from $V - \{v_n\}$ and $i - 1$ elements from $U - \{u_n\}$ or (iv)S contains one element from $U - \{u_n\}$ and $i - 1$ elements from $V - \{v_n\}$. Therefore

$$\mathtt{d}_t(B^2_{m,n,1}, i) = \begin{cases} 2m+2n-3 & \text{if } i = 2, \\ \binom{m+n}{i} - \binom{n-1}{i} - \binom{m-1}{i} - (n-1)\binom{m-1}{i-1} - (m-1)\binom{n-1}{i-1} & \text{if } 3 \leq i \leq m-1, \\ \binom{m+n}{m} - \binom{n-1}{m} - (n-1) - (m-1)\binom{n-1}{m-1} & \text{if } i = m, \\ \binom{m+n}{i} - \binom{n-1}{i} - (m-1)\binom{n-1}{i-1} & \text{if } m+1 \leq i \leq n-1, \\ \binom{m+n}{n} - (m-1) & \text{if } i = n, \\ \binom{m+n}{i} & \text{if } n+1 \leq i \leq m+n. \end{cases}$$

Hence $D_t(B^2_{m,n,1}, x) = [(1+x)^{m-1} - (m-1)x - 1]\,[(1+x)^{n-1} - (n-1)x - 1] + (1+x)^{m+n-2}(x^2+2x) - 2x.$ □

Theorem 4.1.39. *The total domination polynomial of the square of the n-barbell graph $B_{n,1}$ is*

$$D_t(B^2_{n,1}, x) = \left[(1+x)^{n-1} - (n-1)x - 1\right]^2 + (1+x)^{2(n-1)}(x^2+2x) - 2x.$$

Proof. It follows from the fact that the square of the n-barbell graph $B_{n,1}$ and the square of the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Theorem 4.1.40. *The total domination polynomial of the square of the bi-star graph $B_{(m,n)}$ is*

$$D_t(B^2_{(m,n)}, x) = [(1+x)^m - mx - 1]\,[(1+x)^n - nx - 1] + (1+x)^{m+n}(x^2+2x) - 2x.$$

Proof. It follows from the fact that the square of the bi-star graph $B_{(m,n)}$ and the square of the generalized barbell graph $B_{m+1,n+1,1}$ are isomorphic. □

4.2 $\mathtt{d}_t$-number of graphs

In this section we find the number of real roots of the total domination polynomial of some graphs. First we define total domination root of a graph.

Definition 4.2.1. *Let G be a graph with total domination polynomial $D_t(G,x)$. A root of $D_t(G,x)$ is called a total domination root of G and set of all total domination roots of G is denoted by $\mathbb{Z}(D_t(G,x))$.*

Remark 4.2.1. *Let G be a graph with total domination polynomial $D_t(G,x)$. Since the coefficients of $D_t(G,x)$ are positive, $(0,\infty)$ is a zero-free interval for $D_t(G,x)$.*

We are interested to find the number of real total domination roots of graphs. We define $\mathtt{d}_t$-number of a graph G as follows :

Definition 4.2.2. *Let G be a graph. The number of distinct real total domination roots of the graph G is called $\mathtt{d}_t$-number of G and is denoted by $\mathtt{d}_t(G)$.*

Example 4.2.2. *The total domination polynomial of the graph G in Figure 2.1 is*

$$D_t(G,x) = x^6 + 4x^5 + 6x^4 + 4x^3 + x^2.$$

The total domination roots of G are $\mathbb{Z}(D_t(G,x)) = \{-1,-1,-1,-1,0,0\}$, hence $\mathtt{d}_t(G) = 2$.

Theorem 4.2.3. *For any graph G, $\mathtt{d}_t(G) \geq 1$.*

Proof. It follows from the fact that 0 is a total domination root of any graph. □

Theorem 4.2.4. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}_t(G) \leq \sum_{i=1}^{m} \mathtt{d}_t(G_i) - m + 1.$$

Proof. It follows from the fact that $D_t(G,x) = \prod_{i=1}^{m} D_t(G_i,x)$. □

Theorem 4.2.5. *If G and H are isomorphic, then $\mathtt{d}_t(G,x) = \mathtt{d}_t(H,x)$.*

Proof. It follows from the fact that if G and H are isomorphic, then $D_t(G,x) = D_t(H,x)$. □

Theorem 4.2.6. *For $n \geq 2$ the $\mathtt{d}_t$-number of the complete graph K_n is 1 for even n and 2 for odd n.*

Proof. We have the total domination polynomial of K_n is

$$D_t(K_n, x) = (1+x)^n - nx - 1.$$

From the above equation it follows that $D_t(K_n, y-1) = y^n - ny + n - 1$.. Clearly, $y = 1$ is a double root of $D_t(K_n, y-1)$. By De Gua's rule 1.2.2 for imaginary roots, there are at least $n-2$ complex roots if n is even and there are at least $n-3$ complex roots if n is odd. This give the result. $\square$

Remark 4.2.7. *By the intermediate value theorem 1.2.3 and by Theorem 4.2.6 we have the complete graph K_n has exactly one nonzero total domination real root in $[-3, -2)$.*

Theorem 4.2.8. *For all n the $\mathtt{d}_t$-number of the star graph S_n is 1 if n is odd and 2 if n is even.*

Proof. We have the total domination polynomial of S_n is

$$D_t(S_n, x) = x((1+x)^n - 1). \tag{4.1}$$

The result follows from the transformation $y = 1 + x$ in equation (4.1). $\square$

Remark 4.2.9. *The nonzero total domination root of the star graph S_n is -2 for even n.*

The total domination roots of the star graph S_n for $1 \leq n \leq 60$ are shown in Figure 4.1.

Theorem 4.2.10. *Let G be a graph of order n. Then for all m, n the $\mathtt{d}_t$-number of $G \circ \overline{K_m}$ is 2.*

Proof. We have the total domination polynomial of $G \circ \overline{K_m}$ is

$$D_t(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}. \tag{4.2}$$

By equation (4.2) all the total domination roots of $G \circ \overline{K_m}$ are real, namely, 0 with multiplicity n and -1 with multiplicity mn, thus $\mathtt{d}_t(G \circ \overline{K_m}) = 2$. $\square$

Theorem 4.2.11. *For all n the $\mathtt{d}_t$-number of the Dutch windmill graph G_3^n is greater than or equal to 2.*

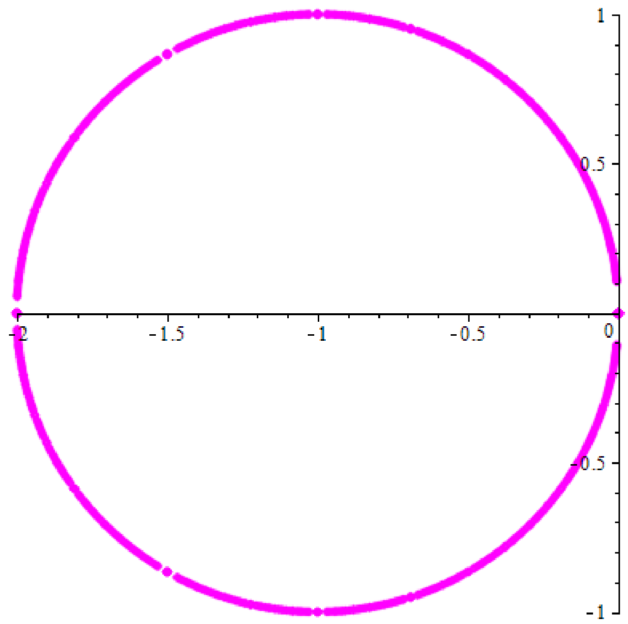

Figure 4.1: Total domination roots of S_n for $1 \le n \le 60$.

Proof. We have the total domination polynomial of the Dutch windmill graph G_3^n is

$$D_t(G_3^n, x) = x^{2n} + x((1+x)^{2n} - 1).$$

Consider,

$$\begin{aligned} D_t(G_3^n, -\ln n) &= (-\ln n)^{2n} + (-\ln n)((1-\ln n)^{2n} - 1) \\ &= (\ln n)^{2n} \left(1 - \ln n (\tfrac{1-\ln n}{\ln n})^{2n} + \ln n \tfrac{1}{(\ln n)^{2n}}\right). \end{aligned}$$

From Theorem 1.2.4, we have for large n, $D_t(G_3^n, -\ln n) > 0$. Next we show that $D_t(G_3^n, -n) < 0$. Consider $f(x) = x^{2n-1} + (2n+1)x^{2n-2} + \binom{2n}{2}x^{2n-3} + \ldots + 2n$. Then

$$\begin{aligned} f(-n) &= (-1)^{2n-1}n^{2n-1} + (2n+1)n^{2n-2} + (-1)^{2n-3}\tbinom{2n}{2}n^{2n-3} + \ldots + 2n \\ &= (-1)^{2n-1}n^{2n-1}\left(1 - \tfrac{2n+1}{n} + \tfrac{\binom{2n}{2}}{n^2} - \ldots - \tfrac{2n}{n^{2n-1}}\right). \end{aligned}$$

But for sufficiently large n,

$$1 - \frac{2n+1}{n} + \frac{\binom{2n}{2}}{n^2} - \ldots - \frac{2n}{n^{2n-1}} < 0.$$

That is, $D_t(G_3^n, -n) < 0$ for sufficiently large n. By the intermediate value theorem 1.2.3, for sufficiently large n, $D_t(G_3^n, x)$ has a real root in the interval $(-n, -\ln n)$. Therefore the Dutch windmill graph G_3^n has at least two real total domination root and hence $\mathtt{d}_t(G_3^n) \geq 2$. $\square$

Remark 4.2.12. *Using Maple, we observe that G_{2n}^3 has exactly two distinct real total domination roots for $1 \leq n \leq 1000$. So we conjectured that $\mathtt{d}(G_{2n}^3) = 2$ for all n.*

Theorem 4.2.13. *For $n \geq 2$, the $\mathtt{d}_t$-number of the spider graph Sp_{2n+1} is 1 for even n and 2 for odd n.*

Proof. By Theorem 4.1.10 we have the total domination polynomial of the spider graph Sp_{2n+1} is

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right). \tag{4.3}$$

The result follows from the transformation $y = 1 + x$ in 4.3. $\square$

Remark 4.2.14. *The nonzero total domination root of the spider graph Sp_{2n+1} is -2 for odd n.*

Theorem 4.2.15. *The $\mathtt{d}_t$-number of the lollipop graph $L_{n,1}$ is 1 for odd n and 2 for even n.*

Proof. By Theorem 4.1.11 we have the total domination polynomial of the lollipop graph $L_{n,1}$ is

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right). \tag{4.4}$$

The result follows from the transformation $y = 1 + x$ in equation (4.2.15). $\square$

Remark 4.2.16. *The nonzero total domination root of the lollipop graph $L_{n,1}$ is -2 with multiplicity 1 for even n.*

Theorem 4.2.17. *For $n \geq 2$ the $\mathtt{d}_t$-number of the bipartite cocktail party graph B_n is 1 for even n and 2 for odd n.*

Proof. The proof is similar to the proof of 4.2.6. $\square$

Remark 4.2.18. *By the intermediate value theorem 1.2.3 and by Theorem 4.2.17 we have the bipartite cocktail party graph B_n has exactly one nonzero total domination real root in $[-3, -2)$ with multiplicity 2.*

Theorem 4.2.19. *For $m, n \geq 2$; $m \neq n$, the $\mathtt{d}_t$-number of the generalized barbell graph $B_{m,n,1}$ is*

$$\mathtt{d}_t(B_{m,n,1}) = \begin{cases} 3 & \text{; if both } m \text{ and } n \text{ are even,} \\ 5 & \text{; if both } m \text{ and } n \text{ are odd,} \\ 4 & \text{; if } m \text{ and } n \text{ have opposite parity.} \end{cases}$$

Proof. By Theorem 4.1.13 we have the total domination polynomial of generalized barbell graph $B_{m,n,1}$ is

$$D_t(B_{m,n,1}, x) = [(1+x)^m - (m-1)x - 1]\,[(1+x)^n - (n-1)x - 1]\,.$$

Since there is no real number satisfying both the following equations

$$(1+x)^m - (m-1)x - 1 = 0$$
$$(1+x)^n - (n-1)x - 1 = 0$$

simultaneously. So it is enough to show that $f(x) = x^n - (n-1)x + n - 2$ has exactly one nonzero real root for even n and has exactly two nonzero real roots for odd n. Clearly $x = 1$ is a simple root of $f(x)$. For even n, by De Gua's rule 1.2.2 for imaginary roots, there are atleast $n-2$ complex roots. Therefore the remaining root is real number different from 1. For odd n by De Gua's rule for imaginary roots, there are atleast $n-3$ complex roots. Observe that $f(-1) > 0$ and $f(-2) < 0$. By the intermediate value theorem 1.2.3, we have $f(x)$ has a root in the interval $(-2, -1)$. Therefore the remaining roots are real numbers different from 1. It remains to show that $f(x)$ has no double roots. Suppose $a \in \mathbb{R}$ is a double root of $f(x)$. Then

$$a^n - (n-1)a + n - 2 = 0 \tag{4.5}$$
$$na^{n-1} - (n-1) = 0 \tag{4.6}$$

Solving these equations we get $a = \frac{n(n-2)}{(n-1)^2}$. This implies that $a \geq 0$, a contradiction, since $a < 0$. So we have the result. □

Corollary 4.2.20. *For $n \geq 2$ the $\mathtt{d}_t$-number of the n-barbell graph $B_{n,1}$, is*

$$\mathtt{d}_t(B_{n,1}) = \begin{cases} 2 & ; \text{ if } n \text{ is even,} \\ 3 & ; \text{ if } n \text{ is odd.} \end{cases}$$

Theorem 4.2.21. *For all m, n the $\mathtt{d}_t$-number of the complete bipartite graph $K_{m,n}$ is*

$$\mathtt{d}_t(K_{m,n}) = \begin{cases} 1 & ; \text{ if both } m \text{ and } n \text{ are odd,} \\ 2 & ; \text{ otherwise.} \end{cases}$$

Proof. By Corollary 4.1.21 we have the total domination polynomial of $K_{m,n}$ is

$$D_t(K_{m,n}, x) = [(1+x)^m - 1][(1+x)^n - 1]. \tag{4.7}$$

The result follows from the transformation $y = 1 + x$ in equation (4.7). □

Remark 4.2.22. *The complete bipartite graph $K_{m,n}$ has a nonzero real total domination root if and only if either m or n is even. Moreover, -2 is the only nonzero real total domination root with multiplicity 1 or 2 according as exactly one of m or n is even or both m and n are even.*

Theorem 4.2.23. *For $m, n \geq 2$ the $\mathtt{d}_t$-number of the complete partite graph $K_{n[m]}$ is*

$$\mathtt{d}_t(K_{n[m]}) = \begin{cases} 2 & ; \text{ if } n \text{ is even,} \\ 1 & ; \text{ if } m \text{ is even and } n \text{ is odd,} \\ 2 & ; \text{ if both } m \text{ and } n \text{ are odd.} \end{cases}$$

Proof. From Corollary 4.1.20, we have

$$D_t(K_{n[m]}, x) = (1+x)^{mn} - m(1+x)^n + m - 1. \tag{4.8}$$

From the equation (4.8), it follows that $D_t(K_{n[m]}, y-1) = y^{mn} - my^n + m - 1$. To find the real roots of $y^{mn} - my^n + m - 1 = 0$, it is enough to find the real roots of $f_m(z) = z^m - mz + m - 1 = 0$. Clearly, $z = 1$ is a double root of $f_m(z)$. If m is even, then by De Gua's rule 1.2.2 for imaginary roots, there are at least $m-2$ complex roots. Therefore $z = 1$ is the only real root of $f_m(z)$. But $y^n = 1$ has exactly two real solutions, namely $y = \pm 1$ for even n and has exactly one solution, namely $y = 1$ for odd n. If m is odd, then by De Gua's rule for imaginary roots, there are at least $m - 3$ complex roots. By the intermediate value theorem 1.2.3, $f_m(z)$ has at least one real root in $(-3,-1)$. So the roots of $f_m(z)$ are 1 and $c \in (-3,-1)$. But $y^n = c$ has a real solution only for odd n and that solution is unique. Therefore $K_{n[m]}$ has only one nonzero real total domination root for even n and if m is even and n is odd, then $K_{n[m]}$ has no nonzero real total domination root. Finally, if both m and n are odd $K_{n[m]}$ has exactly one nonzero total domination root. $\square$

Remark 4.2.24. *If n is even, then $K_{n[m]}$ has exactly one nonzero real total domination root, namely -2 with multiplicity 2. If m and n are odd, then $K_{n[m]}$ has exactly one nonzero real total domination root that lies in $(-3,-1)$ with multiplicity 1.*

Theorem 4.2.25. *Let G be a graph of order n and diameter D. If $D \leq 2$, then*

$$\mathtt{d}_t(G^2) = \begin{cases} 1 & ; \textit{if } n \textit{ is even,} \\ 2 & ; \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. It follows from the fact that if $D \leq 2$, then G^2 and the complete graph K_n are isomorphic. $\square$

Remark 4.2.26. *Let G be a graph of order $2n+1$. Then G^2 has exactly one nonzero real total domination root c with multiplicity 1 where $c \in [-3,-2)$.*

Corollary 4.2.27. *For all n the $\mathtt{d}_t$-number of the square of the complete graph K_n is*

$$\mathtt{d}_t(K_n^2) = \begin{cases} 1 & ; \textit{if } n \textit{ is even,} \\ 2 & ; \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 4.2.28. *For all m,n we have the following :*

$$\mathtt{d}_t(K_{m,n}^2) = \begin{cases} 1 & ; \textit{if } m \textit{ and } n \textit{ have same parity,} \\ 2 & \textit{otherwise.} \end{cases}$$

Corollary 4.2.29. *For all n the $\mathtt{d}_t$-number of the square of the star graph S_n is*

$$\mathtt{d}_t(S_n^2) = \begin{cases} 2 & ; \textit{if } n \textit{ is even,} \\ 1 & ; \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 4.2.30. *For all n the $\mathtt{d}_t$-number of the square of the wheel graph W_n is*

$$\mathtt{d}_t(W_n^2) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is even,} \\ 2 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 4.2.31. *Let H and G be two graphs with order m and n respectively, then*

$$\mathtt{d}_t((H \vee G)^2) = \begin{cases} 1 & ;\ \textit{if } m \textit{ and } n \textit{ have same parity,} \\ 2 & ;\ \textit{if } m \textit{ and } n \textit{ have opposite parity.} \end{cases}$$

Corollary 4.2.32. *For all m, n we have the following :*

$$\mathtt{d}_t((K_m \square K_n)^2) = \begin{cases} 2 & ;\ \textit{if } m \textit{ and } n \textit{ are odd,} \\ 1 & \ \textit{otherwise.} \end{cases}$$

Corollary 4.2.33. *For all n the $\mathtt{d}_t$-number of the square of the Dutch windmill G_3^n graph is 2.*

Corollary 4.2.34. *For all n the $\mathtt{d}_t$-number of the square of the lollipop graph $L_{n,1}$ is*

$$\mathtt{d}_t(L_{n,1}^2) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is even,} \\ 1 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Theorem 4.2.35. *For all n, $\mathtt{d}_t((K_n \circ K_1)^2) = 1$.*

Proof. We have $D_t((K_n \circ K_1)^2, y-1) = y^{2n} - y^n - ny + n$. Let $f(y) = y^{2n} - y^n - ny + n$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes rule 1.2.1, it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Now consider, $f(-y)$.

Case 1 : If n is odd.

$f(-y) = y^{2n} + y^n + ny + n$. There is no sign changes, $f(y)$ has no negative real roots. Therefore the only possible real root of $D_t((K_n \circ K_1)^2, x)$ is zero.

Case 2 : If n is even.

$f(-y) = y^{2n} - y^n + ny + n$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 2, by Descartes rule, it has at most two negative real roots. We claim that $f(-y)$ has no positive real roots. Let $z > 0$ be a real root of $f(-y)$. Then $z^{2n} - z^n + nz + n = 0$. That is, $z^{2n} - z^n = -n(z+1)$. If $z \geq 1$, $z^{2n} - z^n \geq 0$, but right side is negative. Therefore $z \geq 1$ is not possible. If $0 < z < 1$, then $-1 \leq z^{2n} - z^n \leq 0$, but right side is greater than -1. Therefore

$0 < z < 1$ is also not possible.
In both cases the only possible real roots of $D_t((K_n \circ K_1)^2, x)$ is zero. Hence the result. $\square$

Theorem 4.2.36. *The* $\mathtt{d}_t$*-number of the square of the bipartite cocktail party graph* B_n *is* 2 *for* $n \geq 3$.

Proof. We have $D_t(B_n^2, y-1) = y^{2n} - ny^2 + n - 1$. Then by De Gua's rule 1.2.2 for imaginary roots, there are at least $2n-4$ complex roots. Clearly, $y = 1$ and $y = -1$ are double roots of $D_t(B_n^2, y-1)$. Therefore $x = 0$ and $x = -2$ are the only real roots. $\square$

4.3 Bounds for the total domination roots of some graphs

In this section we estimate the bounds for the total domination roots of some graphs.

Theorem 4.3.1. *All the nonzero total domination roots of the complete graph* K_n *lie in the annulus* $1 < |z+1| \leq 2$.

Proof. We have total domination polynomial of the complete graph K_n is

$$D_t(K_n, x) = (1+x)^n - nx - 1.$$

Put $y = 1 + x$ and consider $f(y) = y^n - ny + n - 1$. Then $y = 1$ is a double root of $f(y)$. Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{n-2} + 2y^{n-3} + 3y^{n-4} + \ldots + (n-2)y + n - 1.$$

It suffices to show that all the roots of $g(y)$ lie in the annulus $1 < |z| \leq 2$. By Enestrom-Kakeya theorem 1.2.6 if $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ has positive real coefficients, then all roots of f lie in the annulus $r \leq |z| \leq R$, where

$$r = \min\left\{\frac{a_i}{a_{i+1}} \;: 0 \leq i \leq n-1\right\} \text{ and } R = \max\left\{\frac{a_i}{a_{i+1}} \;: 0 \leq i \leq n-1\right\}.$$

In this case

$$r = \min\left\{\frac{n-1}{n-2}, \frac{n-2}{n-3}, \ldots, 2\right\} \text{ and } R = \max\left\{\frac{n-1}{n-2}, \frac{n-2}{n-3}, \ldots, 2\right\}.$$

So we have the result. $\square$

The total domination roots of the complete graph K_n for $2 \leq n \leq 60$ are shown in Figure 4.2.

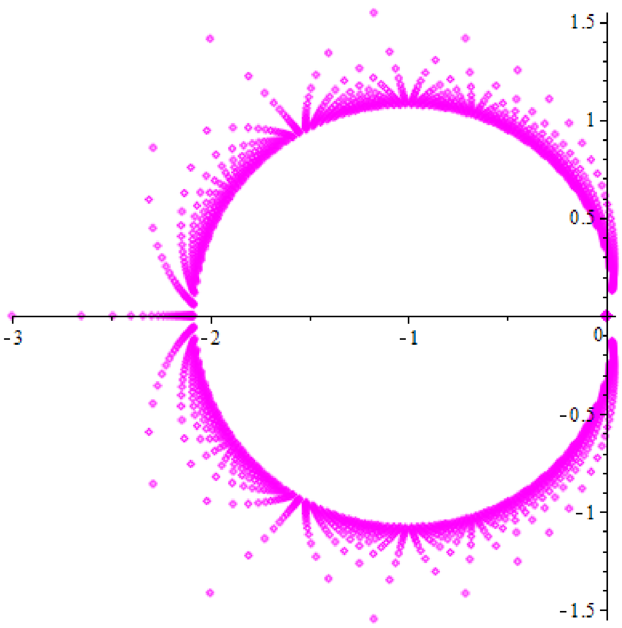

Figure 4.2: Total domination roots of K_n for $2 \leq n \leq 60$.

Theorem 4.3.2. *All the nonzero total domination roots of the spider graph Sp_{2n+1} lies on the unit circle centered at $(-1, 0)$.*

Proof. It follows from the fact that $n + 1^{\text{th}}$ roots of unity lies on the unit circle centered at the origin. $\square$

The total domination roots of the spider graph Sp_{2n+1} for $1 \leq n \leq 60$ are shown in Figure 4.3.

Theorem 4.3.3. *All the nonzero total domination roots of the lollipop graph $L_{n,1}$ lies on the unit circle centered at $(-1, 0)$.*

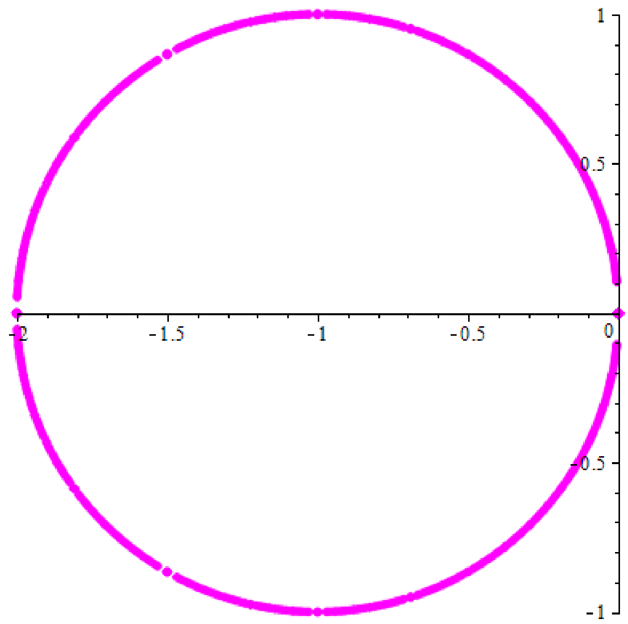

Figure 4.3: Total domination roots of Sp_{2n+1} for $1 \leq n \leq 60$.

Proof. It follows from the fact that n^{th} roots of unity lies on the unit circle centered at the origin. □

The total domination roots of the lollipop graph $L_{n,1}$ for $1 \leq n \leq 30$ are shown in Figure 4.4.

Theorem 4.3.4. *All the nonzero total domination roots of the bipartite cocktail party graph B_n lie in the annulus $1 < |z+1| \leq 2$.*

Proof. The proof is similar to the proof of Theorem 4.3.1. □

The total domination roots of the bipartite cocktail party graph B_n for $1 \leq n \leq 30$ are shown in Figure 4.5.

Theorem 4.3.5. *All the total domination roots of the n-barbell graph $B_{n,1}$ lie inside the circle with center $(-1,0)$ and radius n.*

Remark 4.3.6. *Using Maple, we find that all the roots of $D(B_{n,1},x)$ lie inside the circle centered at $(-1,0)$ and has radius 2 for $2 \leq n \leq 60$.*

The total domination roots of the n-barbell graph $B_{n,1}$ for $1 \leq n \leq 60$ are shown in Figure 4.6.

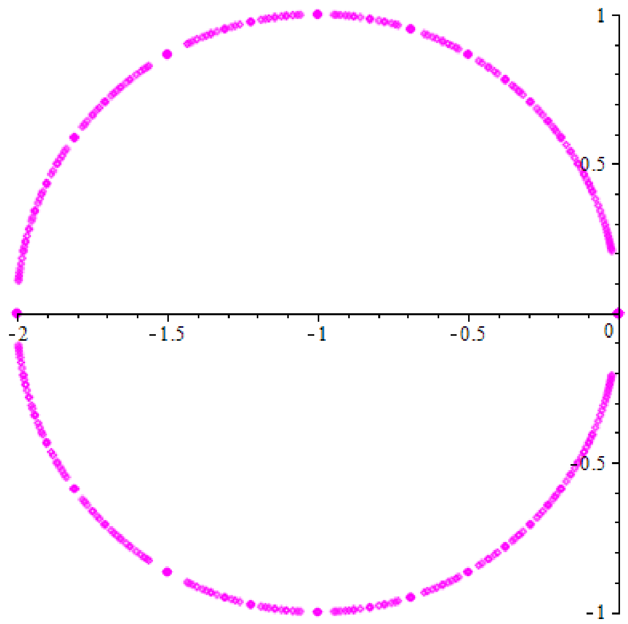

Figure 4.4: Total domination roots of $L_{n,1}$ for $1 \leq n \leq 30$.

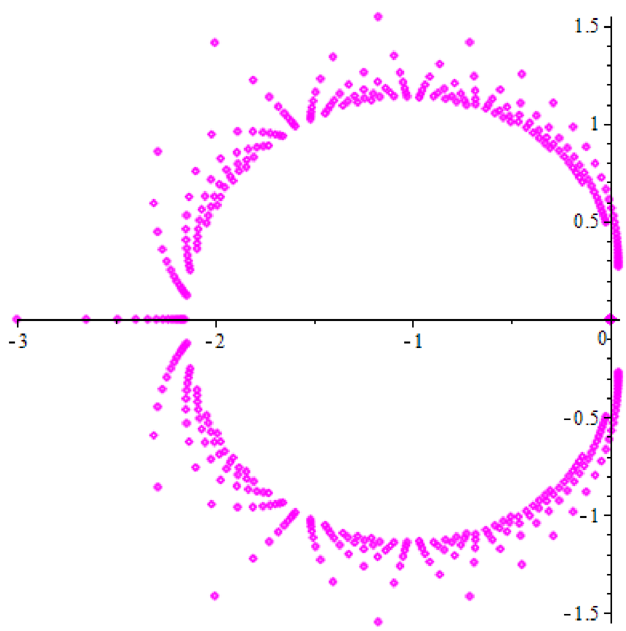

Figure 4.5: Total domination roots of B_n for $1 \leq n \leq 30$.

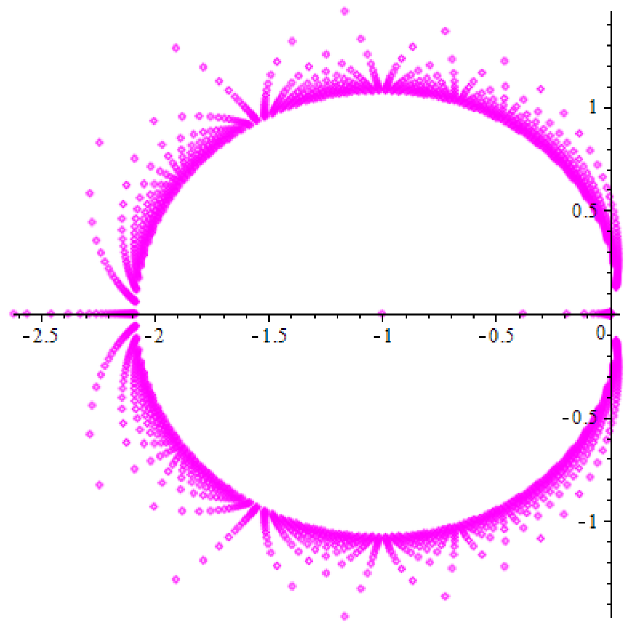

Figure 4.6: Total domination roots of $B_{n,1}$ for $1 \leq n \leq 60$.

Theorem 4.3.7. *All the nonzero total domination roots of the complete bipartite graph $K_{m,n}$ lies on the unit circle centered at $(-1, 0)$.*

Proof. It follows from the fact that n^{th} roots of unity lies on the unit circle centered at the origin. □

Theorem 4.3.8. *All the nonzero total domination roots of the complete partite graph $K_{n[m]}$ lie in the annulus $1 < |z + 1| \leq 2^{\frac{1}{n}}$.*

Proof. From Corollary 4.1.20, we have

$$D_t(K_{n[m]}, x) = (1 + x)^{mn} - m(1 + x)^n + m - 1.$$

From the above equation it follows that $D_t(K_{n[m]}, y - 1) = y^{mn} - my^n + m - 1$. It suffices to show that the roots of $f_m(z) = z^m - mz + m - 1 = 0$ lie in the annulus $1 < |z| \leq 2$. Clearly, $z = 1$ is a double root of $f_m(z)$. Therefore $f_m(z) = (z - 1)^2 g(z)$, where

$$g(z) = z^{n-2} + 2z^{n-3} + 3z^{n-4} + \ldots + (n - 2)z + n - 1.$$

By Enestrom-Kakeya theorem 1.2.6, we get that all the roots of $g(z)$ lie in the annulus $1 < |z| \leq 2$. So we have the result. $\square$

The total domination roots of the complete 3-partite graph $K_{n[3]}$ for $1 \leq n \leq 20$ are shown in Figure 4.7.

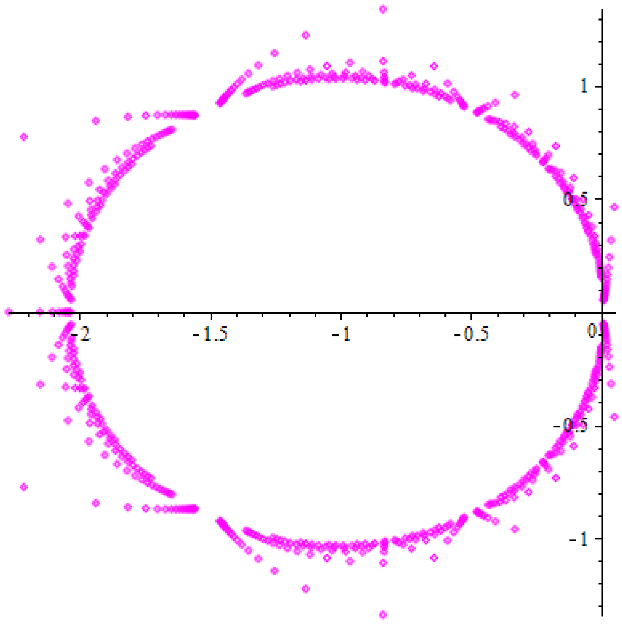

Figure 4.7: Total domination roots of $K_{n[3]}$ for $1 \leq n \leq 20$.

Theorem 4.3.9. *The Dutch windmill graph G_3^n has a real total domination root in the interval $(-n, -\ln n)$, for n sufficiently large.*

Proof. The proof is similar to the proof of Theorem 4.2.11. $\square$

Theorem 4.3.10. *Let G be a graph of order n and diameter D. If $D \leq 2$, then all the nonzero total domination roots of G^2 lie in the annulus $1 < |z+1| \leq 2$.*

Proof. The proof is similar to the proof of Theorem 4.3.1. $\square$

Corollary 4.3.11. *All the nonzero total domination roots of the square of the complete graph K_n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.12. *All the nonzero total domination roots of the square of the complete bipartite graph $K_{m,n}$ lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.13. *All the nonzero total domination roots of the square of the star graph S_n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.14. *All the nonzero total domination roots of the square of the wheel graph W_n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.15. *Let H and G be two graphs of order m and n respectively. Then all the nonzero total domination roots of the square of $H \vee G$ lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.16. *All the nonzero total domination roots of the square of the graph $K_m \square K_n$ lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.17. *All the nonzero total domination roots of the square of the Dutch windmill graph G_3^n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 4.3.18. *All the nonzero total domination roots of the square of the lollipop graph $L_{n,1}$ lie in the annulus $1 < |z+1| \leq 2$.*

Theorem 4.3.19. *All the total domination roots of the square of $K_n \circ K_1$ lie in the annulus $1 \leq |z+1| \leq 2$.*

Proof. We have $D_t((K_n \circ K_1)^2, y-1) = y^{2n} - y^n - ny + n$. It suffices to show that the roots of $f(y) = y^{2n} - y^n - ny + n$ lie in the annulus $1 \leq |z| \leq 2$. Clearly, $y = 1$ is a double root of $f(y)$. Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{2n-2} + 2y^{2n-3} + 3y^{2n-4} + \ldots + (n-1)y^n + n\left(y^{n-1} + y^{n-2} + \ldots + y + 1\right).$$

By Enestrom-Kakeya theorem 1.2.6, we obtain that all the roots of $g(y)$ lie in the annulus $1 \leq |z| \leq 2$. So we have the result. □

The total domination roots of the square of the corona $K_n \circ K_1$ for $1 \leq n \leq 30$ are shown in Figure 4.8.

Theorem 4.3.20. *All the nonzero total domination roots of the square of the bipartite cocktail party graph B_n lie in the annulus $1 < |z+1| \leq 2^{\frac{1}{2}}$.*

Proof. From Theorem 4.1.36, we have for $n \geq 3$, the total domination polynomial of the square of the bipartite cocktail party graph B_n is

$$D_t(B_n^2, x) = (1+x)^{2n} - n(1+x)^2 + (n-1).$$

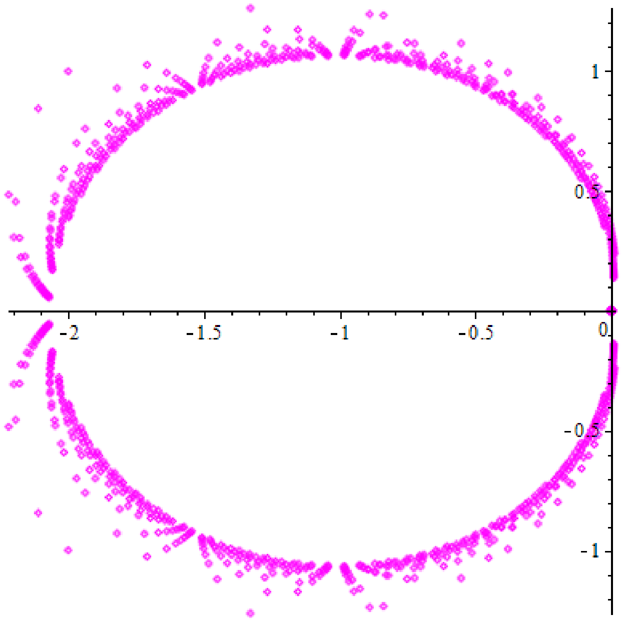

Figure 4.8: Total domination roots of $(K_n \circ K_1)^2$ for $1 \leq n \leq 30$.

From the above equation it follows that $D_t(B_n^2, y-1) = y^{2n} - ny^2 + n - 1$. It suffices to show that the roots of $f(y) = y^n - ny + n - 1$ lie in the annulus $1 < |z| \leq 2$. The remaining proof is similar to the proof of Theorem 4.3.1. $\square$

The total domination roots of the square of the bipartite cocktail party graph B_n for $1 \leq n \leq 30$ are shown in Figure 4.9.

4.4 Stable graphs related to total domination polynomial

In this section we introduce $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs. Some examples of $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs are obtained. First we define $\mathtt{d}_t$-stable and $\mathtt{d}_t$-unstable graphs as follows :

Definition 4.4.1. *Let $G = (V(G), E(G))$ be a graph. The graph G is said to be a total domination stable graph or simply $\mathtt{d}_t$-stable graph if all the nonzero total domination roots lie in the left open half-plane, that is, if real part of the nonzero*

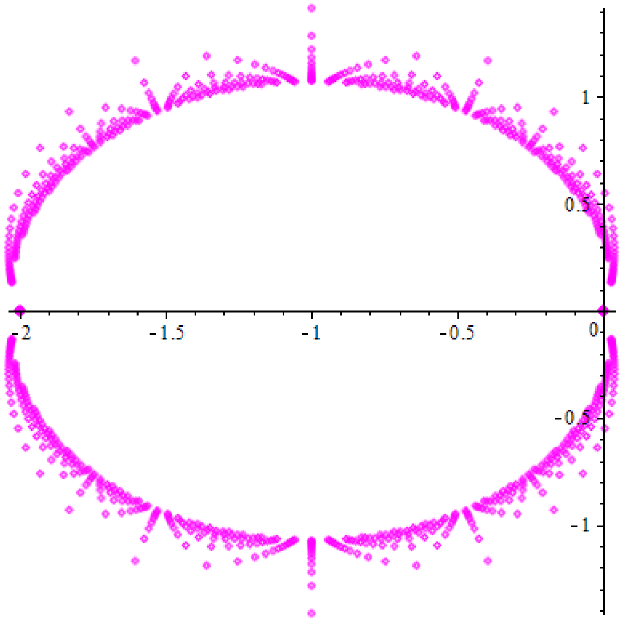

Figure 4.9: Total domination roots of B_n^2 for $1 \leq n \leq 30$.

total domination roots are negative. If G is not $\mathtt{d}_t$-stable graph, then G is said to be a total domination unstable graph or simply $\mathtt{d}_t$-unstable graph.

Example 4.4.1. *The total domination polynomial of the Dutch windmill graph G_3^{10} is*

$$D(G_3^{10}, x) = x((1 + x)^{20} - 1) + x^{20}.$$

With the aid of Maple, we find that the total domination roots of G_3^{10} are :
$\mathbb{Z}(G_3^{10}) = \{-9.4247, -1.4870 - 3.6366i, -1.4870 + 3.6366i, -0.98636 - 0.16459i,$
$-0.98636 + 0.16459i, -0.87947 - .47595i, -0.87947 + 0.47595i, -0.67790 - 0.73551i,$
$-0.67790 + 0.73551i, -0.59499 - 1.7129i, -0.59499 + 1.7129i, -0.51330 - 1.0552i,$
$-0.51330 + 1.0552i, -0.40867 - 0.80289i, -0.40867 + .80289i, -0.19098 - 0.58779i,$
$-0.19098 + 0.58779i, -0.48943 - 0.30902i, -0.48943 + 0.30902i, 0, 0\}.$
All the nonzero total domination roots of the graph G_3^{10} lie in the open left half-plane. Hence G_3^{10} is a $\mathtt{d}_t$-stable graph.

Example 4.4.2. *The total domination polynomial of complete graph K_{20} is*

$$D(K_{20}, x) = (1 + x)^{20} - 20x - 1.$$

With the aid of Maple, we find that the total domination roots of K_{20} *are :*
$\mathbb{Z}(K_{20}) = \{-2.1912 - 0.19419i, -2.1912 + 0.19419i, -2.0664 - 0.56173i, -2.0664 + 0.56173i, -1.8304-0.86900i, -1.8304+0.86900i, -1.5090-1.0832i, -1.5090+1.0832i, -1.1374-1.1815i, -1.1374 + 1.1815i, -0.75630 - 1.1538i, -0.75630 + 1.1538i, -0.40825 - 1.0039i, -0.40825 + 1.0039i, -0.13329 - 0.74859i, -0.13329 + 0.74859i, 0, 0, 0.32186 - 0.41579i, 0.32186 + 0.41579i\}$.
The total domination roots $0.32186 - 0.41579i, 0.32186 + 0.41579i$ *lie in the open right half-plane. Hence* K_{20} *is a* $\mathtt{d}_t$*-unstable graph.*

Theorem 4.4.3. *If* G *and* H *are isomorphic graphs then* G *is* $\mathtt{d}_t$*-stable if and if* H *is* $\mathtt{d}_t$*-stable.*

Proof. It follows from the fact that if G and H are isomorphic graphs then $D_t(G, x) = D_t(H, x)$. □

Corollary 4.4.4. *If* G *and* H *are isomorphic graphs then* G *is* $\mathtt{d}_t$*-unstable if and if* H *is* $\mathtt{d}_t$*-unstable.*

Theorem 4.4.5. *If a graph* G *consists of* m *components* $G_1, G_2, \ldots, G_m$*, then* G *is* $\mathtt{d}_t$*-stable if and if each* G_i *is* $\mathtt{d}_t$*-stable.*

Proof. It follows from the fact that $D_t(G, x) = \prod_{i=1}^{m} D_t(G_i, x)$. □

Corollary 4.4.6. *If a graph* G *consists of* m *components* $G_1, G_2, \ldots, G_m$*, then* G *is* $\mathtt{d}_t$*-unstable if and if one of the* G_i *is* $\mathtt{d}_t$*-unstable.*

Remark 4.4.7. *Using Maple, we find that real part of all the total domination roots of all graphs of order upto* 5 *is negative. Therefore there is no* $\mathtt{d}_t$*-unstable graph of order upto* 5.

Theorem 4.4.8. *Let* G *be a graph of order* n *without isolated vertices. Then* $G \circ \overline{K_m}$ *is* $\mathtt{d}_t$*-stable for all* m, n.

Proof. We have the total domination polynomial of $G \circ \overline{K_m}$ is

$$D_t(G \circ \overline{K_m}, x) = x^n(1 + x)^{mn}.$$

Therefore $\mathbb{Z}(D_t(G \circ \overline{K_m}, x)) = \{0, -1\}$, hence $G \circ \overline{K_m}$ is $\mathtt{d}_t$-stable for all m, n. □

Theorem 4.4.9. *The spider graph* Sp_{2n+1} *is* $\mathtt{d}_t$*-stable for all* n.

Proof. We have the total domination polynomial of the spider graph Sp_{2n+1} is

$$D_t(Sp_{2n+1}, x) = x^n \left((1+x)^{n+1} - 1\right).$$

Therefore

$$\mathbb{Z}(D_t(Sp_{2n+1}, x)) = \left\{\exp\left(\frac{2k\pi i}{n+1}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the spider graph Sp_{2n+1} is $\mathtt{d}_t$-stable for all n. □

Theorem 4.4.10. *The lollipop graph $L_{n,1}$ is $\mathtt{d}_t$-stable for all n.*

Proof. We have the total domination polynomial of the lollipop graph $L_{n,1}$ is

$$D_t(L_{n,1}, x) = x\left((1+x)^n - 1\right).$$

Therefore

$$\mathbb{Z}(D_t(L_{n,1}, x)) = \left\{\exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, 1, \ldots, n\right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the lollipop graph $L_{n,1}$ is $\mathtt{d}_t$-stable for all n. □

Theorem 4.4.11. *The bi-star graph $B_{(m,n)}$ is $\mathtt{d}_t$-stable for all m, n.*

Proof. We have the total domination polynomial of the bi-star graph $B_{(m,n)}$ is

$$D_t(B_{(m,n)}, x) = x^2(1+x)^{m+n}.$$

Therefore

$$\mathbb{Z}(D_t(B_{(m,n)}, x) = \{0, -1\},$$

hence the bi-star graph $B_{(m,n)}$ is $\mathtt{d}_t$-stable for all m, n. □

Corollary 4.4.12. *The corona graph $K_2 \circ \overline{K_n}$ is $\mathtt{d}_t$-stable for all n.*

Proof. It follows from the fact that the corona graph $K_2 \circ \overline{K_n}$ and the bi-star graph $B_{(n,n)}$ are isomorphic. □

Remark 4.4.13. *Using maple, we find that the complete graph K_n is $\mathtt{d}_t$-stable for $1 \leq n \leq 14$ and is $\mathtt{d}_t$-unstable for $15 \leq n \leq 30$. We have the total domination polynomial of K_n is*

$$D_t(K_n, x) = (1+x)^n - nx - 1.$$

Put $y = 1+x$ and consider $f(y) = y^n - ny + n - 1$. Then $y = 1$ is a double root of $f(y)$. Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{n-2} + 2y^{n-3} + 3y^{n-4} + \ldots + (n-2)y + n - 1.$$

We have if $f(z) = a_n z^n + a_{n-1} z^n + \ldots + a_0$ is a polynomial with real coefficient satisfying $a_0 \geq a_1 \geq \ldots \geq a_n > 0$ then no roots of $f(z)$ lie in $\{z \in \mathbb{C} : |z| < 1\}$ [45]. Therefore all the roots z of $g(y)$ satisfy $|z| > 1$. This implies that all the nonzero roots of $D_t(K_n, x)$ are out side the unit circle centered at $(-1, 0)$. So we conjectured that the complete graph K_n is not $\mathtt{d}_t$-stable for all but finite values of n.

The total domination roots of the complete graph K_n for $1 \leq n \leq 14$ and $1 \leq n \leq 30$ are shown in Figures 4.10 and 4.11 respectively.

Remark 4.4.14. *We have the total domination polynomial of G_3^n is*

$$D_t(G_3^n, x) = x(1+x)^{2n} - x + x^{2n}.$$

Rewrite $D(G_3^n, x)$ as

$$\begin{aligned} D_t(G_3^n, x) = f_{2n}(x) &= x(1+x)^{2n} + (-x)(1)^{2n} + (1)x^{2n}. \\ &= \alpha_1 \lambda_1^{2n} + \alpha_2 \lambda_2^{2n} + \alpha_3 \lambda_3^{2n}, \end{aligned}$$

where $\alpha_1 = x$, $\lambda_1 = 1+x$, $\alpha_2 = -x$, $\lambda_2 = 1$, $\alpha_3 = 1$ and $\lambda_3 = x$. Clearly α_1,α_2 and α_3 are not identically zero and $\lambda_i \neq \omega\lambda_j$ for $i \neq j$ and any complex number ω of modulus 1. Therefore the initial conditions of Theorem 1.2.9 are satisfied. Now, applying part(i) of Theorem 1.2.9, we consider the following four different cases :

(i) $|\lambda_1| = |\lambda_2| = |\lambda_3|$

(ii) $|\lambda_1| = |\lambda_2| > |\lambda_3|$

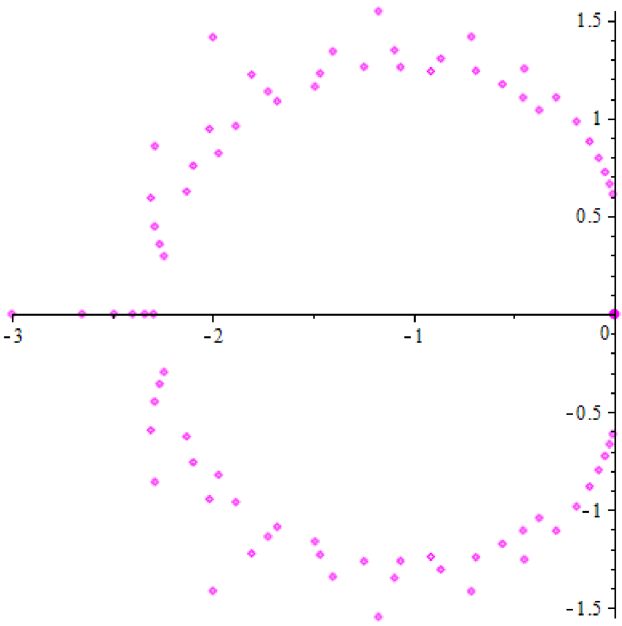

Figure 4.10: Total domination roots of K_n for $1 \leq n \leq 14$.

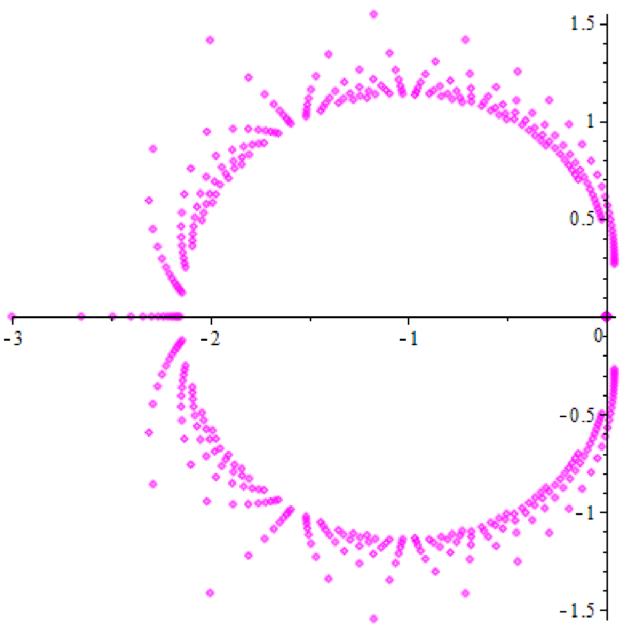

Figure 4.11: Total domination roots of K_n for $1 \leq n \leq 30$.

(iii) $|\lambda_1| = |\lambda_3| > |\lambda_2|$

(iv) $|\lambda_2| = |\lambda_3| > |\lambda_1|$

Case (i) : *Assume that* $|1+x| = |1| = |x|$. *Then* $|x-(-1)| = |x-0|$ *implies that* x *lies on the vertical line* $z = -\frac{1}{2}$, $|x-(-1)| = 1$ *implies that* x *lies on the unit circle centered at* $(-1,0)$ *and* $1 = |x-0|$ *implies that* x *lies on the unit circle centered at the origin. Therefore the two points of intersection,* $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ *are the limits of roots.*

Case (ii) : *Assume that* $|1+x| = |1| > |x|$. *Then* $|x-(-1)| = 1$ *implies that* x *lies on the unit circle centered at* $(-1,0)$, $|x-(-1)| > |x-0|$ *implies that* x *lies to the right of the vertical line* $z = -\frac{1}{2}$. *Therefore the complex numbers* x *that satisfy* $|x-(-1)| = 1$ *and* $\mathcal{R}(x) > -\frac{1}{2}$ *are the limits of roots.*

Case (iii) : *Assume that* $|1+x| = |x| > |1|$. *Then* $|x-(-1)| = |x-0|$ *implies that* x *lies on the vertical line* $z = -\frac{1}{2}$ *and* $|x-0| > 1$ *implies that* x *lies outside the unit circle centered at the origin. Therefore the complex numbers* x *that satisfy* $|x| > 1$ *and* $\mathcal{R}(x) > -\frac{1}{2}$ *are the limits of roots.*

Case (iv) : *Assume that* $|1| = |x| > |1+x|$. *Then* $1 = |x-0|$ *implies that* x *lies on the unit circle centered at the origin and* $|x-0| > |x-(-1)|$ *implies that* x *lies to the left of the vertical line* $z = -\frac{1}{2}$. *Therefore the complex numbers* x *that satisfy* $|x| = 1$ *and* $\mathcal{R}(x) < -\frac{1}{2}$ *are the limits of roots.*

The union of the curves and points above yield that, the limits of roots of the total domination polynomial of the Dutch windmill graph G_3^n *consists of the part of the circle* $|z| = 1$ *with real part at most* $-\frac{1}{2}$, *the part of the circle* $|z+1| = 1$ *with real part at least* $-\frac{1}{2}$ *and the part of the line* $\mathcal{R}(z) = -\frac{1}{2}$ *with modulus at least* 1. *So we conjectured that the Dutch windmill graph* G_3^n *is* $\mathtt{d}_t$*-stable for all* n.

The total domination roots of the Dutch windmill graph G_3^n for $1 \leq n \leq 30$ are shown in Figure 4.12.

Remark 4.4.15. *We have the total domination polynomial of* B_n *is*

$$D_t(B_n, x) = \left((1+x)^n - nx - 1\right)^2.$$

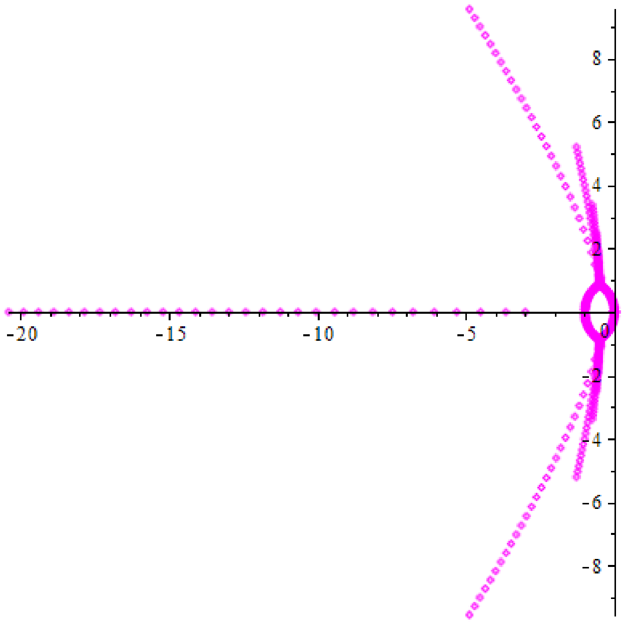

Figure 4.12: Total domination roots of G_3^n for $1 \leq n \leq 30$.

Because of the same reason as mentioned in Remark 4.4.13, we conjectured that the bipartite cocktail party graph B_n is not a $\mathtt{d}_t$-stable for all but finite values of n.

Next we consider some graphs and check whether its square is either $\mathtt{d}_t$-stable or $\mathtt{d}_t$-unstable.

Theorem 4.4.16. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G^2 is $\mathtt{d}_t$-stable if and if each G_i^2 is $\mathtt{d}_t$-stable.*

Proof. It follows from the fact that $D_t(G^2, x) = \prod_{i=1}^{m} D_t(G_i^2, x)$. □

Corollary 4.4.17. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G^2 is $\mathtt{d}_t$-unstable if and if one of the G_i^2 is $\mathtt{d}_t$-unstable.*

Theorem 4.4.18. *The square of the Petersen graph P is $\mathtt{d}_t$-stable.*

Proof. We have the total domination polynomial of the square of the Petersen graph P is

$$D_t(P^2, x) = x^{10} + 10x^9 + 45^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2.$$

The total domination roots are :
$\mathbb{Z}(D_t(P^2, x)) = \{-2.289 - 0.4477i, -2.289 + 0.4477i, -1.72436053 - 1.137i, -1.72436053+1.137i, -0.8636147-1.304953i, -0.8636147+1.304953i, -0.12327-0.8814i, -0.12327 + 0.8814i, 0, 0\}$. Hence the square of the Petersen graph P is $\mathtt{d}_t$-stable. $\square$

Remark 4.4.19. *Let G be a graph of order n. If $D(G) \leq 2$ then G^2 is isomorphic to the complete graph K_n. Because of the same reason as mentioned in Remark 4.4.13, we conjectured that G^2 is not $\mathtt{d}_t$-stable for all but finite values of n. Because of the same reason we conjectured that the square of the following graphs G_n are not $\mathtt{d}_t$-stable for all but finite values of n, where n is the order of G.*

(1) Complete graph K_n.

(2) Complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$.

(3) Complete bipartite graph $K_{m,n}$.

(4) Star graph S_n.

(5) Wheel graph W_n.

(6) $H \vee G$.

(7) $K_m \square K_n$.

(8) Dutch windmill graph G_3^n.

(9) Lollipop graph $L_{n,1}$.

CHAPTER 5

DISTANCE-K TOTAL DOMINATION STABLE GRAPHS

Distance-k total domination polynomial is introduced in this chapter. In Section 5.1 we define distance-k total domination polynomial of graphs and find distance-k total domination polynomial of some graphs. In Section 5.2, we define distance-k total domination root and introduce a new concept, $\mathtt{d}_t^k$-number of a graph and also find $\mathtt{d}_t^k$-number of some graphs. We obtained bounds for distance-k total domination roots of some graphs in Section 5.3. We introduce $\mathtt{d}_t^k$-stable and $\mathtt{d}^k$-unstable graphs in Section 5.4 and find some examples of $\mathtt{d}_t^k$-stable and $\mathtt{d}_t^k$-unstable graphs.

5.1 Distance-k total domination polynomial of graphs

In this section we state the definition of distance-k total domination polynomial and find this polynomial for some graphs.

Definition 5.1.1. *Let k be a positive integer and let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V$ is a distance-k total dominating set if each vertex $v \in V$ is with in distance k from some vertex of S. The distance-k total domination number of G, denoted by $\gamma_t^k(G)$, is the minimum cardinality of the distance-k total dominating sets in G. Let $\mathcal{D}_t^k(G, i)$ be the family of distance-k total dominating* *sets of G with cardinality i and let $\mathtt{d}_t^k(G,i) = |\mathcal{D}_t^k(G,i)|$. The polynomial*

$$D_t^k(G,x) = \sum_{i=\gamma_t^k(G)}^{|V(G)|} \mathtt{d}_t^k(G,i)x^i$$

is defined as distance-k total domination polynomial of G.

Observe that the distance-k total domination polynomial is a generalization of the total domination polynomial.

Example 5.1.1. *Consider the graph G in Figure 2.1. The distance-2 total domination number of G is $\gamma_t^2(G) = 2$. Also $\mathtt{d}_t^2(G,2) = 9$, $\mathtt{d}_t^2(G,3) = 16$, $\mathtt{d}_t^2(G,4) = 15$, $\mathtt{d}_t^2(G,5) = 6$ and $\mathtt{d}_t^2(G,6) = 1$. Therefore the total domination polynomial of G is $D_t^2(G,x) = x^6 + 6x^5 + 15x^4 + 16x^3 + 9x^2$.*

The following two theorems follows from the fact that every distance-m total dominating of any graph G is also a distance-k total dominating set of G for $m < k$.

Theorem 5.1.2. *For any graph G, $\gamma_t^k \leq \gamma_t^m$, when $m < k$.*

Theorem 5.1.3. *For any graph G, $\mathtt{d}_t^m(G,i) \leq \mathtt{d}_t^k(G,i)$, when $m < k$.*

Theorem 5.1.4. *If G and H are isomorphic, then $D_t^k(G,x) = D_t^k(H,x)$.*

Theorem 5.1.5. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$D_t^k(G,x) = D_t^k(G_1,x)D_t^k(G_2,x)\ldots D_t^k(G_m,x).$$

Proof. It suffices to prove this theorem for $m = 2$. For $l \geq \gamma_t^k(G)$, a distance-k total dominating set of l vertices in G arises by choosing a distance-k total dominating set of j vertices in G_1, for some $j \in \{\gamma_t^k(G_1), \gamma_t^k(G_1)+1, \ldots, |V(G)|\}$ and a distance-k total dominating set of $l-j$ vertices of G_2. The number of way of doing this over all $j = \gamma_t^k(G_1), \gamma_t^k(G_1)+1, \ldots, |V(G)|$ is exactly the coefficient of x^l in $D_t^k(G_1,x)D_t^k(G_2,x)$. Hence both side of the above equation have the same coefficient, so they are identical polynomial. □

Theorem 5.1.6. *Let G be a graph and let k be any positive integer, then $D_t^k(G,x) = D_t(G^k,x)$.*

Proof. It follows from the fact that every distance-k total dominating set of G with cardinality i is exactly the total dominating set of G^k with cardinality i. $\square$

The following theorem follows from the definitions of total domination polynomial and distance-k total domination polynomial.

Theorem 5.1.7. *Let G be a graph with total domination polynomial $D_t(G,x)$, then $D_t^1(G,x) = D_t(G,x)$.*

From Theorem 5.1.7 it follows that when $k = 1$, the distance-k domination polynomial coincide with the domination polynomial. So throughout this chapter we assume that k is a positive integer greater than one.

Theorem 5.1.8. *Let G be a graph of order n and diameter D. Then $D_t^k(G,x) = (1+x)^n - nx - 1$ if and only if $k \geq D$.*

Proof. Suppose $k \geq D$, then all the vertices of G are with in a distance k. This implies that for $2 \leq i \leq n$, any subset of vertices of G of cardinality i is a distance-k total dominating set. Therefore $D_t^k(G,x) = (1+x)^n - nx - 1$. Conversely, suppose that $D_t^k(G,x) = (1+x)^n - nx - 1$. Then $\gamma_t^k(G) = 2$ and $\mathsf{d}_t^k(G,2) = \binom{n}{2}$, the number of edges in G. This implies that all vertices of G are with in a distance k, hence $k \geq D$. $\square$

Corollary 5.1.9. *For the complete graph K_n,*

$$D_t^k(K_n,x) = (1+x)^n - nx - 1.$$

Corollary 5.1.10. *For the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$,*

$$D_t^k(K_{n_1,n_2,\ldots,n_m},x) = (1+x)^N - Nx - 1,$$

where $N = n_1 + n_2 + \ldots + n_m$.

Corollary 5.1.11. *For the complete bipartite graph $K_{m,n}$,*

$$D_t^k(K_{m,n},x) = (1+x)^{m+n} - (m+n)x - 1.$$

Corollary 5.1.12. *For the star graph S_n,*

$$D_t^k(S_n,x) = (1+x)^{n+1} - (n+1)x - 1.$$

Corollary 5.1.13. *For the wheel graph W_n,*

$$D_t^k(W_n, x) = (1+x)^n - nx - 1.$$

Corollary 5.1.14. *For $i = 1, 2$, let G_i be a graph of order n_i, then*

$$D_t^k(G_1 \vee G_2, x) = (1+x)^{n_1+n_2} - (n_1+n_2)x - 1.$$

Corollary 5.1.15. *For the complete graphs K_m and K_n, $D^k((K_m \Box K_n), x) = (1+x)^{mn} - mnx - 1$.*

Corollary 5.1.16. *Let P be the Petersen graph, then*

$$D_t^k(P, x) = (1+x)^{10} - 10x - 1.$$

Corollary 5.1.17. *The distance-k total domination polynomial of the Dutch windmill graph G_3^n is*

$$D_t^k(G_3^n, x) = (1+x)^{2n+1} - (2n+1)x - 1.$$

Corollary 5.1.18. *The distance-k total domination polynomial of lollipop graph $L_{n,1}$ is*

$$D_t^k(L_{n,1}, x) = (1+x)^{n+1} - (n+1)x - 1.$$

Theorem 5.1.19. *Let B_n be the bipartite cocktail party graph. Then for $n \geq 3$,*

$$\begin{aligned} D_t^2(B_n, x) &= (1+x)^{2n} - n(1+x)^2 + (n-1) && \textit{and} \\ D_t^k(B_n, x) &= (1+x)^{2n} - 2nx - 1 && \textit{for } k \neq 2. \end{aligned}$$

Proof. Clearly, the diameter of B_n is 3. Therefore for $k \neq 2$ the proof is trivial. For $k = 2$, let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of B_n such that every vertex v_i in V and every vertex u_i in U are adjacent if $i \neq j$. Clearly, any subset of vertices of B_n of cardinality 2 forms a distance-2 total dominating set excluding $\{v_i, u_i\}$ for all $i = 1, 2, \ldots, n$. Therefore $\gamma_t^2(B_n) = 2$, $\mathtt{d}_t^2(B_n, 2) = \binom{2n}{2} - n$ and $\mathtt{d}_t^2(B_n, i) = \binom{2n}{i}$; for all $3 \leq i \leq 2n$. Hence the result. $\square$

Corollary 5.1.20. *Let B_n be the bipartite cocktail party graph. Then for $n \geq 3$, $D_t(B_n^2, x) = (1+x)^{2n} - n(1+x)^2 + (n-1)$.*

Remark 5.1.21. *Observe that $B_1 = 2K_1$ and $B_2 = 2K_2$. So $D_t^2(B_1, x) = 0$ and $D_t^2(B_2, x) = x^4$.*

Theorem 5.1.22. *Let $B_{m,n,1}$ be the generalized barbell graph. Then for all m, n, $D_t^2(B_{m,n,1}, x) = [(1 + x)^{m-1} - (m - 1)x - 1]\,[(1 + x)^{n-1} - (n - 1)x - 1] + (1 + x)^{m+n-2}(x^2 + 2x) - 2x$ and $D_t^k(B_{m,n,1}, x) = (1 + x)^{m+n} - (m + n)x - 1$ for $k \neq 2$.*

Proof. Clearly, the diameter of $B_{m,n,1}$ is 3. Therefore for $k \neq 2$ the proof is trivial. For $k = 2$, without loss of generality, we assume $m \leq n$. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices in V are adjacent, every vertices in U are adjacent and v_m and u_n are adjacent. Any set of vertices of $B_{m,n,1}$ properly containing v_m or u_n are distance-2 total dominating set of $B_{m,n,1}$. Therefore $\gamma_t^2(B_{m,n,1}) = 2$ and $\mathtt{d}_t^2(B_{m,n,1}, 2) = 2m + 2n - 3$. Also observe that for $2 \leq i \leq m + n$, a subset S of vertices of $B_{m,n,1}$ of cardinality i is not a distance-2 total dominating set of $B_{m,n,1}$ if and only if one of the following condition is true.

(i) $S \subset V - \{v_n\}$.

(ii) $S \subset U - \{u_n\}$.

(iii) S contains one element from $V - \{v_n\}$ and $i - 1$ elements from $U - \{u_n\}$.

(iv) S contains one element from $U - \{u_n\}$ and $i - 1$ elements from $V - \{v_n\}$.

This implies that,

$$\mathtt{d}_t^2(B_{m,n,1}, i) = \begin{cases} 2m + 2n - 3 & \text{if } i = 2, \\ \binom{m+n}{i} - \binom{n-1}{i} - \binom{m-1}{i} - (n-1)\binom{m-1}{i-1} - (m-1)\binom{n-1}{i-1} & \text{if } 3 \leq i \leq m-1, \\ \binom{m+n}{m} - \binom{n-1}{m} - (n-1) - (m-1)\binom{n-1}{m-1} & \text{if } i = m, \\ \binom{m+n}{i} - \binom{n-1}{i} - (m-1)\binom{n-1}{i-1} & \text{if } m+1 \leq i \leq n-1, \\ \binom{m+n}{n} - (m-1) & \text{if } i = n, \\ \binom{m+n}{i} & \text{if } n+1 \leq i \leq m+n. \end{cases}$$

Hence $D_t^2(B_{m,n,1}, x) = [(1 + x)^{m-1} - (m - 1)x - 1]\,[(1 + x)^{n-1} - (n - 1)x - 1] + (1 + x)^{m+n-2}(x^2 + 2x) - 2x$. □

Corollary 5.1.23. *Let $B_{m,n,1}$ be the generalized barbell graph. Then for all m, n, $D_t(B_{m,n,1}^2, x) = [(1 + x)^{m-1} - (m - 1)x - 1]\,[(1 + x)^{n-1} - (n - 1)x - 1] + (1 + x)^{m+n-2}(x^2 + 2x) - 2x$.*

Theorem 5.1.24. *Let $B_{n,1}$ be n-barbell graph. Then for all n,*

$$D_t^2(B_{n,1},x) = (1+x)^{2n} - 2(1+(n-1)x)(1+x)^{n-1} + ((n-1)x)^2 + 2(n-2)x + 1 \text{ and}$$
$$D_t^k(B_{n,1},x) = (1+x)^{2n} - 2nx - 1 \text{ for } k \neq 2.$$

Proof. Clearly, the diameter of $B_{n,1}$ is 3. Therefore for $k \neq 2$ the proof is trivial. For $k = 2$, let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be the vertices of $B_{n,1}$ such that v_n and u_n joining by a bridge. For $2 \leq i \leq 2n$, a subset S of vertices $B_{n,1}$ of cardinality i is not a distance-2 total dominating set if and only if (i) $S \subset V - \{v_n\}$ or (ii) $S \subset U - \{u_n\}$ or (iii) S contains one element from $V - \{v_n\}$ and $i-1$ elements from $U - \{u_n\}$ or (iv) S contains one element from $U - \{u_n\}$ and $i-1$ elements from $V - \{v_n\}$. Therefore $\gamma_t^2(B_{n,1}) = 2$, $\mathtt{d}^2(B_{n,1},2) = \binom{2n}{i} - 2\binom{n-1}{2}$, $\mathtt{d}^2(B_{n,1},i) = \binom{2n}{i} - 2\binom{n-1}{i} - 2(n-1)\binom{n-1}{i-1}$ for $3 \leq i \leq n$ and $\mathtt{d}^2(B_{n,1},i) = \binom{2n}{i}$ for $n+1 \leq i \leq 2n$. This implies that $D_t^2(B_{n,1},x) = (1+x)^{2n} - 2(1+(n-1)x)(1+x)^{n-1} + ((n-1)x)^2 + 2(n-2)x + 1$. □

Corollary 5.1.25. *Let $B_{n,1}$ be the n-barbell graph. Then the domination polynomial of square of $B_{n,1}$ is $D_t(B_{n,1}^2,x) = (1+x)^{2n} - 2(1+(n-1)x)(1+x)^{n-1} + ((n-1)x)^2 + 2(n-2)x + 1$.*

Theorem 5.1.26. *Let $B_{(m,n)}$ be the bi-star graph. Then for all m,n,*

$$D_t^2(B_{(m,n)},x) = (1+x)^{m+n+2} - (1+mx)(1+x)^n - (1+nx)(1+x)^m + (m+n-2)x + 1 \text{ and}$$
$$D_t^k(B_{(m,n)},x) = (1+x)^{m+n+2} - (m+n+2)x - 1 \text{ for } k \neq 2.$$

Proof. The proof is similar to the proof of the Theorem 5.1.22. □

Corollary 5.1.27. *Let $B_{(m,n)}$ be the bi-star graph. Then the domination polynomial of square of $B_{(m,n)}$ is $D_t(B_{(m,n)}^2,x) = (1+x)^{m+n+2} - (1+mx)(1+x)^n - (1+nx)(1+x)^m + (m+n-2)x + 1$.*

Theorem 5.1.28. *Let K_n be the complete graph. Then*

$$\begin{aligned} D_t^2(K_n \circ K_1,x) &= (1+x)^{2n} - (1+x)^n - nx && \text{and} \\ D_t^k(K_n \circ K_1,x) &= (1+x)^{2n} - 2nx - 1 && \text{for } k \neq 2. \end{aligned}$$

Proof. The proof is similar to the proof of the Theorem 5.1.24. □

Corollary 5.1.29. *Let K_n be the complete graph. Then $D_t((K_n \circ K_1)^2,x) = (1+x)^{2n} - (1+x)^n - nx$.*

Corollary 5.1.30. *The distance-k total domination polynomial of $Q(n,2)$ is*

$$\begin{aligned} D_t^2(Q(n,2),x) &= (1+x)^{2n} - (1+x)^n - nx \quad && \textit{and} \\ D_t^k(Q(n,2),x) &= (1+x)^{2n} - 2nx - 1 && \textit{for } k \neq 2. \end{aligned}$$

Theorem 5.1.31. *If K_n is the complete graph, then*

$$\begin{aligned} D_t^2(K_n \circ K_2,x) &= (1+x)^{3n} - (1+x)^{2n} + x^{2n} - nx \quad && \textit{and} \\ D_t^k(K_n \circ K_2,x) &= (1+x)^{3n} - 3nx - 1 && \textit{for } k \neq 2. \end{aligned}$$

Proof. The proof is similar to the proof of the Theorem 5.1.24. □

Corollary 5.1.32. *Let K_n be the complete graph. Then $D_t((K_n \circ K_2)^2,x) = (1+x)^{3n} - (1+x)^{2n} + x^{2n} - nx$.*

Corollary 5.1.33. *The distance-k total domination polynomial of $Q(n,3)$ is*

$$\begin{aligned} D_t^2(Q(n,3),x) &= (1+x)^{3n} - (1+x)^{2n} + x^{2n} - nx \quad && \textit{and} \\ D_t^k(Q(n,3),x) &= (1+x)^{3n} - 3nx - 1 && \textit{for } k \neq 2. \end{aligned}$$

5.2 $\mathtt{d}_t^k$-number of graphs

In this section we find the number of the real roots of the distance-k total domination polynomial of some graphs. First we define domination root of a graph.

Definition 5.2.1. *Let G be a graph with distance-k total domination polynomial $D_t^k(G,x)$. A root of $D_t^k(G,x)$ is called a distance-k total domination root of G and set of all distance-k total domination roots of G is denoted by $\mathbb{Z}(D_t^k(G,x))$.*

Remark 5.2.1. *Let G be a graph with distance-k total domination polynomial $D_t^k(G,x)$. Since the coefficients of $D_t^k(G,x)$ are positive, $(0,\infty)$ is a zero-free interval for $D_t^k(G,x)$.*

We mainly find the number of real distance-k total domination roots of some specific graphs. So we introduce a new definition as follows.

Definition 5.2.2. *Let G be a graph. The number of distinct real distance-k total domination roots of the graph G is called $\mathtt{d}_t^k$-number of G and is denoted by $\mathtt{d}_t^k(G)$.*

Example 5.2.2. *The distance-2 total domination polynomial of the graph G in Figure 2.1 is*

$$D_t^2(G,x) = x^6 + 6x^5 + 15x^4 + 16x^3 + 9x^2.$$

The distance-2 total domination roots of G are

$$\mathbb{Z}(D_t^2(G,x)) = \{-2.288 - 1.4161i, -2.288 + 1.4161i - 0.7122 - 0.8579i, -0.7122 + 0.8579i, 0, 0\}.$$

Therefore $\mathtt{d}_t(G) = 1$.

Theorem 5.2.3. *For any graph G, $\mathtt{d}_t^k(G) \geq 1$.*

Proof. It follows from the fact that 0 is a distance-k total domination root of any graph. □

Theorem 5.2.4. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}_t^k(G) \leq \sum_{i=1}^{m} \mathtt{d}_t^k(G_i) - m + 1.$$

Proof. It follows from the fact that $D_t^k(G,x) = \prod_{i=1}^{m} D_t^k(G_i,x)$. □

Theorem 5.2.5. *If G and H are isomorphic, then $\mathtt{d}_t^k(G,x) = \mathtt{d}_t^k(H,x)$.*

Proof. It follows from the fact that if G and H are isomorphic, then $D_t^k(G,x) = D_t^k(H,x)$. □

Next theorem follows from Theorem 5.1.6.

Theorem 5.2.6. *Let G be a graph and let k be any positive integer, then $\mathtt{d}_t^k(G) = m$ if and only if $\mathtt{d}_t(G^k) = m$.*

Next result follows from the transformation $y = 1 + x$ in the distance-k total domination polynomial $D_t^k(G,x)$.

Theorem 5.2.7. *Let G be a graph of order n and diameter D. If $D \leq k$, then*

$$\mathtt{d}_t^k(G) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is even}, \\ 2 & ;\ \textit{if } n \textit{ is odd}. \end{cases}$$

Remark 5.2.8. *Let G be a graph of order $2n+1$. Then G has exactly one nonzero real distance-k total domination root c with multiplicity 1 where $c \in [-3, -2)$.*

Corollary 5.2.9. *For all n the $\mathtt{d}_t^k$-number of the complete graph K_n is*

$$\mathtt{d}_t^k(K_n) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is even,} \\ 2 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 5.2.10. *For all m, n we have the following :*

$$\mathtt{d}_t^2(K_{m,n}) = \begin{cases} 1 & ;\ \textit{if } m \textit{ and } n \textit{ have same parity,} \\ 2 & \ \textit{otherwise.} \end{cases}$$

Corollary 5.2.11. *For all n the $\mathtt{d}_t$-number of the star graph S_n is*

$$\mathtt{d}_t^k(S_n) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is even,} \\ 1 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 5.2.12. *For all n the $\mathtt{d}_t$-number of the wheel graph W_n is*

$$\mathtt{d}_t^k(W_n) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is even,} \\ 2 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 5.2.13. *Let H and G be two graphs with order m and n respectively, then*

$$\mathtt{d}_t^k(H \vee G) = \begin{cases} 1 & ;\ \textit{if } m \textit{ and } n \textit{ have same parity,} \\ 2 & ;\ \textit{otherwise.} \end{cases}$$

Corollary 5.2.14. *For all m, n, we have the following :*

$$\mathtt{d}_t^k(K_m \square K_n) = \begin{cases} 2 & ;\ \textit{if } m \textit{ and } n \textit{ are odd,} \\ 1 & ;\ \textit{otherwise.} \end{cases}$$

Corollary 5.2.15. *The $\mathtt{d}_t^k$-number of the Dutch windmill graph G_3^n is 2 for all n.*

Corollary 5.2.16. *For all n the $\mathtt{d}_t^k$-number of the lollipop graph $L_{n,1}$ is*

$$\mathtt{d}_t^k(L_{n,1}) = \begin{cases} 2 & ;\ \textit{if } n \textit{ is even,} \\ 1 & ;\ \textit{if } n \textit{ is odd.} \end{cases}$$

Theorem 5.2.17. *For all n, $\mathtt{d}_t^2(K_n \circ K_1) = 1$.*

Proof. We have $D_t^2(K_n \circ K_1, y-1) = y^{2n} - y^n - ny + n$. Let $f(y) = y^{2n} - y^n - ny + n$. Since the number of variations of the signs of the coefficients of $f(y)$ is 2, by Descartes rule 1.2.1, it has at most two positive real roots. Clearly, $y = 1$ is a double root of $f(y)$. Now consider, $f(-y)$.

Case 1 : If n is odd.

$f(-y) = y^{2n} + y^n + ny + n$. There is no sign changes, $f(y)$ has no negative real roots. Therefore the only possible real roots of $D_t^2(K_n \circ K_1, x)$ is zero.

Case 2 : If n is even.

$f(-y) = y^{2n} - y^n + ny + n$. Since the number of variations of the signs of the coefficients of $f(-y)$ is 2, by Descartes rule, it has at most two negative real roots. We claim that $f(-y)$ has no positive real roots. Let $z > 0$ be a real root of $f(-y)$. Then $z^{2n} - z^n + nz + n = 0$. That is, $z^{2n} - z^n = -n(z+1)$. If $z \geq 1$, $z^{2n} - z^n \geq 0$, but right side is negative. Therefore $z \geq 1$ is not possible. If $0 < z < 1$, then $-1 \leq z^{2n} - z^n \leq 0$, but right side is greater than -1. Therefore $0 < z < 1$ is also not possible.

In both cases the only possible real roots of $D_t^2(K_n \circ K_1, x)$ is zero. Hence the result. □

Theorem 5.2.18. *The* d_t^2*-number of the bipartite cocktail party graph* B_n *is* 2 *for* $n \geq 3$.

Proof. We have $D_t^2(B_n, y-1) = y^{2n} - ny^2 + n - 1$. Then by De Gua's rule 1.2.2 for imaginary roots, there are at least $2n - 4$ complex roots. Clearly, $y = 1$ and $y = -1$ are double roots of $D_t(B_n^2, y-1)$. Therefore $x = 0$ and $x = -2$ are the only real roots. □

5.3 Bounds for the distance-k total domination roots of some graphs

In this section we estimate the bounds for the distance-k total domination roots of some graphs.

Theorem 5.3.1. *Let G be a graph of order n and diameter D. If $D \leq k$, then all the nonzero distance-k total domination roots of G lie in the annulus* $1 < |z+1| \leq 2$.

Proof. The proof is similar to the proof of Theorem 4.3.1. □

Corollary 5.3.2. *All the nonzero distance-k total domination roots of the complete graph K_n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.3. *All the nonzero distance-k total domination roots of the complete bipartite graph $K_{m,n}$ lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.4. *All the nonzero distance-k total domination roots of the star graph S_n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.5. *All the nonzero distance-k total domination roots of the wheel graph W_n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.6. *Let H and G be two graphs of order m and n respectively, then all the nonzero distance-k total domination roots of $H \vee G$ lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.7. *All the nonzero distance-k total domination roots of the graph $K_m \square K_n$ lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.8. *All the nonzero distance-k total domination roots of the Dutch windmill graph G_3^n lie in the annulus $1 < |z+1| \leq 2$.*

Corollary 5.3.9. *All the nonzero distance-k total domination roots of the lollipop graph $L_{n,1}$ lie in the annulus $1 < |z+1| \leq 2$.*

Theorem 5.3.10. *All the distance-2 total domination roots of $K_n \circ K_1$ lie in the annulus $1 \leq |z+1| \leq 2$.*

Proof. We have $D_t^2(K_n \circ K_1, y-1) = y^{2n} - y^n - ny + n$. It suffices to show that the roots of $f(y) = y^{2n} - y^n - ny + n$ lie in the annulus $1 \leq |z| \leq 2$. Clearly, $y = 1$ is a double root of $f(y)$. Therefore $f(y) = (y-1)^2 g(y)$, where

$$g(y) = y^{2n-2} + 2y^{2n-3} + 3y^{2n-4} + \ldots + (n-1)y^n + n\left(y^{n-1} + y^{n-2} + \ldots + y + 1\right).$$

By Enestrom-Kakeya theorem 1.2.6, we get that all the roots of $g(y)$ lie in the annulus $1 < |z| \leq 2$. So we have the result. □

The distance-2 total domination roots of the corona $K_n \circ K_1$ for $1 \leq n \leq 30$ are shown in Figure 5.1.

Theorem 5.3.11. *All the nonzero distance-2 total domination roots of the bipartite cocktail party graph B_n lie in the annulus $1 < |z+1| \leq \sqrt{2}$.*

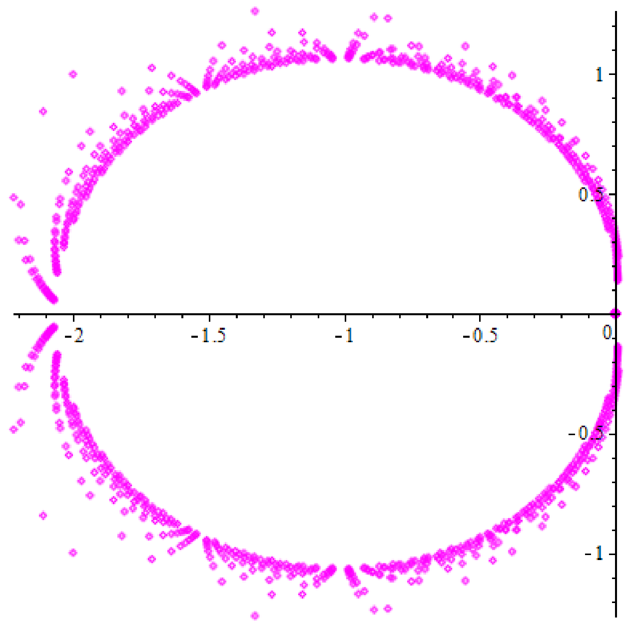

Figure 5.1: Distance-2 total domination roots of $K_n \circ K_1$ for $1 \leq n \leq 30$.

Proof. We have for $n \geq 3$, the distance-2 total domination polynomial of the bipartite cocktail party graph B_n is

$$D_t^2(B_n, x) = (1+x)^{2n} - n(1+x)^2 + (n-1). \tag{5.1}$$

Put $(1+x)^2 = y$ in equation (5.1), then we get

$$D_t^2(B_n, y-1) = y^{2n} - ny^2 + n - 1.$$

Clearly, $y = 1$ is a double root of $D_t^2(B_n, y-1)$. Therefore $D_t^2(B_n, y-1) = (y-1)^2 g(y)$, where

$$g(y) = y^{n-2} + 2y^{n-3} + 3y^{n-4} + \ldots + (n-2)y + n - 1.$$

By Enestrom-Kakeya theorem 1.2.6, we get that all the roots of $g(y)$ lie in the annulus $1 < |z| \leq 2$. This implies that the roots of $D_t^2(B_n, x)$ lie in the annulus $1 < |z+1| \leq \sqrt{2}$. So we have the result. $\square$

The distance-2 total domination roots of the bipartite cocktail party graph

B_n for $1 \leq n \leq 30$ are shown in Figure 5.2.

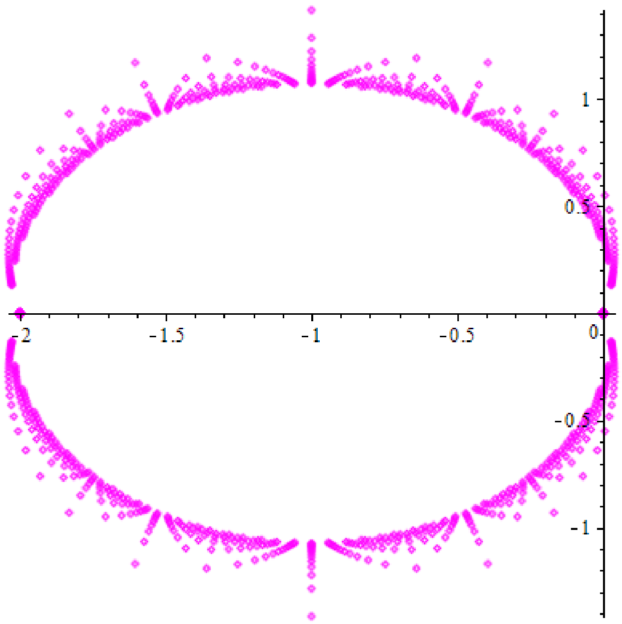

Figure 5.2: Distance-2 total domination roots of B_n for $1 \leq n \leq 30$.

5.4 Stable graphs related to distance-k total domination polynomial

In this section we introduce $\mathtt{d}_t^k$-stable and $\mathtt{d}_t^k$-unstable graphs. We obtain some examples of $\mathtt{d}_t^k$-stable graphs and $\mathtt{d}_t^k$-unstable graphs. We begin this section by defining $\mathtt{d}_t^k$ domination stable graph.

Definition 5.4.1. *Let $G = (V(G), E(G))$ be a graph. For $k \geq 1$, the graph G is said to be a distance-k total domination stable graph or simply $\mathtt{d}_t^k$-stable graph if all the nonzero distance-k total domination roots lie in the left open half-plane, that is, if real part of the nonzero distance-k total domination roots are negative. If G is not distance-k total domination stable graph, then G is said to be a distance-k domination unstable graph or simply $\mathtt{d}_t^k$-unstable graph.*

Example 5.4.1. *The distance-2 total domination polynomial of the Petersen* *graph P is*

$$D(P,x) = x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2.$$

With the aid of Maple, we find that the distance-2 total domination roots of P are :

$\mathbb{Z}(B_{10}) = \{-2.2888-.44768i, -2.2888+.44768i, -1.7244-1.1370i, -1.7244+1.1370i, -.86361-1.3050i, -.86361+1.3050i, -.12327-.88137i, -.12327+.88137i, 0, 0\}$.

All the nonzero distance-2 total domination roots of the graph P lie in the open left half-plane. Hence P is a $\mathtt{d}_t^2$-stable graph.

Example 5.4.2. *The distance-2 total domination polynomial of the bipartite cocktail party graph B_{10} is*

$$D(B_{10}, x) = (1+x)^{20} - 10(1+x)^2 + 19.$$

With the aid of Maple, we find that the distance-2 total domination roots of B_{10} are :

$\mathbb{Z}(B_{10}) = \{-2.0295 - 0.42804i, -2.0295 + 0.42804i, -2, -2, -1.8510 - 0.76671i, -1.8510 + 0.76671i, -1.5585-1.0180i, -1.5585+1.0180i, -1.1943-1.1517i, -1.1943+1.1517i, -.80565-1.1517i, -.80565+1.1517i, -0.44154-1.0180i, -0.44154+1.0180i, -0.14899-0.76671i, -0.14899+0.76671i, 0, 0, 0.29540-1-.42804i, 0.29540-1+.42804i\}$.

The distance-2 total domination roots $0.29540-1-.42804i, 0.29540-1+.42804i$ lie in the open right half-plane. Hence B_{10} is a $\mathtt{d}_t^2$-unstable graph.

Theorem 5.4.3. *If G and H are isomorphic graphs then G is $\mathtt{d}_t^k$-stable if and if H is $\mathtt{d}_t^k$-stable.*

Proof. It follows from the fact that if G and H are isomorphic graphs then $D_t^k(G,x) = D_t^k(H,x)$. □

Corollary 5.4.4. *If G and H are isomorphic graphs then G is $\mathtt{d}_t^k$-unstable if and if H is $\mathtt{d}_t^k$-unstable.*

Theorem 5.4.5. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}_t^k$-stable if and if each G_i is $\mathtt{d}_t^k$-stable.*

Proof. It follows from the fact that $D_t^k(G,x) = \prod_{i=1}^{m} D_t^k(G_i,x)$. □

Corollary 5.4.6. *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}_t^k$-unstable if and if one of the G_i is $\mathtt{d}_t^k$-unstable.*

Next theorem follows from Theorem 5.1.6.

Theorem 5.4.7. *Let G be a graph and let k be any positive integer, then G is $\mathtt{d}_t^k$-stable if and only if G^k is $\mathtt{d}_t$-stable.*

Corollary 5.4.8. *Let G be a graph and let k be any positive integer, then G is $\mathtt{d}_t^k$-unstable if and only if G^k is $\mathtt{d}_t$-unstable.*

Remark 5.4.9. *Using Maple, we find that real part of all the distance-2 total domination roots of all graphs of order upto 5 is negative. Therefore there is no $\mathtt{d}_t^2$-unstable graph of order upto 5.*

Remark 5.4.10. *Let G be a graph of order n. If $D(G) \leq 2$, then the distance-2 total domination polynomial of G is*

$$D_t^2(G, x) = (1 + x)^n - nx - 1.$$

Because of the same reason as mentioned in Remark 4.4.13, we conjectured that G is not $\mathtt{d}_t^2$-stable for all but finite values of n. Because of the same reason we conjectured that the following graphs G_n are not $\mathtt{d}_t^2$-stable for all but finite values of n, where n in the order of G_n.

(1) Complete graph K_n.

(2) Complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$.

(3) Complete bipartite graph $K_{m,n}$.

(4) Star graph S_n.

(5) Wheel graph W_n.

(6) $H \vee G$.

(7) $K_m \square K_n$.

(8) Dutch windmill graph G_3^n.

(9) Lollipop graph $L_{n,1}$.

CHAPTER 6

HOSOYA STABLE GRAPHS

This chapter mainly deals with the Hosoya polynomial of graphs. In Section 6.1 we define Hosoya polynomial of a graph and find Hosoya polynomial of some graphs. In Subsection 6.1.1 we find Hosoya polynomial of the square of some graphs. In Section 6.2, we define Hosoya root and introduce a new concept, $\mathtt{h}$-number of a graph and also find $\mathtt{h}$-number of some graphs. We estimate bounds for Hosoya roots of some graphs in Section 6.3. We introduce $\mathtt{h}$-stable and $\mathtt{h}$-unstable graphs in Section 6.4 and provide some examples of $\mathtt{h}$-stable and $\mathtt{h}$-unstable graphs.

6.1 Hosoya polynomial of graphs

In this section we state the definition of Hosoya polynomial and find Hosoya polynomial for some well known graphs.

Definition 6.1.1. *Let G be a connected graph of diameter D and let $\mathtt{h}(G, i)$; $i \geq 1$, be the number of vertex pairs of G at distance i. The Hosoya polynomial of G is defined as*

$$\mathtt{H}(G, x) := \sum_{i=1}^{D} \mathtt{h}(G, i)x^i.$$

Example 6.1.1. *Consider the graph G in Figure 2.1. It is clear that $\mathtt{h}(G, 1) = 5$ $\mathtt{h}(G, 2) = 6$ and $\mathtt{h}(G, 3) = 4$. Therefore the Hosoya polynomial of G is $\mathtt{H}(G, x) = 4x^3 + 6x^3 + 5x^2$.*

Remark 6.1.2. *For any graph G, $\mathtt{h}(G,1)$ is equal to the size of G.*

Theorem 6.1.3. *If G_1 and G_2 are isomorphic graphs, then $\mathtt{H}(G_1,x) = \mathtt{H}(G_2,x)$.*

Theorem 6.1.4. *For the complete graph K_n, $\mathtt{H}(K_n,x) = \frac{1}{2}n(n-1)x$.*

Proof. It follows from the fact that diameter of K_n is 1. □

Theorem 6.1.5. *Let G be a non-complete graph of order n and size m. Then the Hosoya polynomial of G is $\binom{n}{2}x^2 - m\,x(x-1)$ if and only if $D(G) = 2$.*

Proof. Suppose $D(G) = 2$, then the result follows from the fact that $\mathtt{d}(u,v) = 1$ if u and v are adjacent in G, and $\mathtt{d}(u,v) = 2$ if u and v are adjacent in $\overline{G}$. Conversely, suppose that $\mathtt{H}(G,x) = \binom{n}{2}x^2 - m\,x(x-1)$. Since G is not a complete graph, $\binom{n}{2} - m \neq 0$. This implies that $D(G) = 2$. □

Corollary 6.1.6. *The Hosoya polynomial of the star graph S_n is*

$$\mathtt{H}(S_n,x) = \frac{1}{2}n(n-1)x^2 + nx.$$

Corollary 6.1.7. *For $i = 1,2$ let G_i be a graph of order n_i and size m_i. Then the Hosoya polynomial of $G_1 \vee G_2$ is*

$$\mathtt{H}(G_1 \vee G_2,x) = \binom{n_1+n_2}{2}x^2 - (n_1\,n_2 + m_1 + m_2)\;x(x-1).$$

Corollary 6.1.8. *The Hosoya polynomial of complete $r-$partite graph $K_{n_1,n_2,\ldots,n_r}$ is*

$$\mathtt{H}(K_{n_1,n_2,\ldots,n_r},x) = \binom{n}{2}x^2 - m\,x(x-1),$$

where $n = n_1 + n_2 + \ldots + n_r$ and $m = \sum\limits_{i=1}^{r-1} n_i \left(\sum\limits_{j=i}^{r-1} n_{j+1}\right)$.

Corollary 6.1.9. *The Hosoya polynomial of Petersen graph P is*

$$\mathtt{H}(P,x) = 15x(2x+1).$$

Corollary 6.1.10. *The Hosoya polynomial of the Dutch windmill graph G_3^n is*

$$\mathtt{H}(G_3^n,x) = nx\left((2n+1)x - 3(x-1)\right).$$

Corollary 6.1.11. *The Hosoya polynomial of $L_{n,1}$ is*

$$\mathtt{H}(L_{n,1},x)=\binom{n+1}{2}x^2-\left(\binom{n}{2}+1\right)x(x-1).$$

Theorem 6.1.12. *Let B_n be the bipartite cocktail party graph. Then for $n\geq 3$ the Hosoya polynomial of B_n is*

$$\mathtt{H}(B_n,x)=n\,x\,\left(x^2+(n-1)(x+1)\right).$$

Proof. Clearly, the diameter of B_n is 3. Since the size of B_n is $n(n-1)$, $\mathtt{h}(B_n,1)=n(n-1)$. Let V_1 and V_2 be the two partitions of vertex set of B_n. Then any two vertices in the same partition sets are at distance two. Therefore $\mathtt{h}(B_n,2)=2\times\binom{n}{2}=n(n-1)$. Finally, since the pair of vertices having at distance 3 are exactly the perfect matching pairs of B_n, $\mathtt{h}(B_n,3)=n$. □

Theorem 6.1.13. *The Hosoya polynomial of the n-barbell graph $B_{n,1}$ is*

$$\mathtt{H}(B_{n,1},x)=x\,\left(n(n-1)+((n-1)x+1)^2\right).$$

Proof. Clearly, the diameter of $B_{n,1}$ is 3. Since the size of $B_{n,1}$ is $2\binom{n}{2}+1$, $\mathtt{h}(B_{n,1},1)=n\,(n-1)+1$. Let $V=\{v_1,v_2,\ldots,v_n\}$ and $U=\{u_1,u_2,\ldots,u_n\}$ be the verices of two copies of K_n such that v_n and u_n joining by a bridge. Then any vertex v_i and the vertex u_n for $i\neq n$ are at distance 2. Similarly, any vertex u_i and the vertex v_n for $i\neq n$ are at distance 2. Therefore $\mathtt{h}(B_{n,1},2)=2(n-1)$. Finally, since $\{v_i,u_j\}$ are at distance 3 for every $i,j\neq n$. Therefore $\mathtt{h}(B_{n,1},3)=(n-1)^2$. This completes the proof. □

Theorem 6.1.14. *The Hosoya polynomial $\mathtt{H}(S_{(n_1,n_2,\ldots,n_r)},x)$ of star-like tree graph $S_{(n_1,n_2,\ldots,n_r)}$ is*

$$\left(\binom{n-r}{2}-\sum_{i=1}^{r}\binom{n_i-1}{2}\right)x^4+(n-r)(r-1)x^3+\left(\sum_{i=1}^{r}\binom{n_i}{2}+\binom{r}{2}\right)x^2+nx,$$

where $n=n_1+n_2+\ldots+n_r$.

Proof. Clearly diameter of $S_{(n_1,n_2,\ldots,n_r)}$ is 4. Since the size of $S_{(n_1,n_2,\ldots,n_r)}$ is $n=n_1+n_2+\ldots+n_r$, $\mathtt{h}(S_{(n_1,n_2,\ldots,n_r)},1)=n$. Let v be the vertex in common and let

v_i, be the vertex with degree n_i for $i = 1, 2, \ldots, r$. The vertices are at distance 2 are as follows :

(i) any two vertices v_i, v_j $i \neq j$

(ii) any two vertices in S_{n_i+1} except v_i.

Any vertex in S_{n_i+1} except v_i and v and any v_j, $i \neq j$ are at distance 3 for $i = 1, 2, \ldots, r$. Finally, for $i \neq j$ any vertex in S_{n_i+1} except v_i and v and any vertex in S_{n_j+1} except v_j and v are at distance 4. This will give the result. $\square$

Theorem 6.1.15. *The Hosoya polynomial of bi-star graph $B_{(m,n)}$ is*

$$\mathtt{H}(B_{(m,n)}, x) = mnx^3 + \frac{1}{2}\left(m(m+1) + n(n+1)\right)x^2 + (m+n+1)x.$$

Proof. Clearly, the diameter of $B_{(m,n)}$ is 3. Since the size of $B_{(m,n)}$ is m+n+1, $\mathtt{h}(B_{(m,n)}, 1) = m+n+1$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices which give the m and n pendent edges by attaching u and v respectively, where u and v be the verices of K_2. Then any vertices of U and v are at distance 2. Similarly, any vertices of V and u are at distance 2. Also any two vertices of U are at distance 2. Similarly any two vertices of V are at distance 2. Therefore $\mathtt{h}(B_{(m,n)}, 2) = \frac{1}{2}\left(m(m+1) + n(n+1)\right)$. Finally, $\{u_i, v_j\}$ are at distance 3 for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. This completes the proof. $\square$

Theorem 6.1.16. *The Hosoya polynomial of the path graph P_n is*

$$\mathtt{H}(P_n, x) = \sum_{i=1}^{n-1}(n-i)x^i.$$

Proof. Clearly, the diameter of P_n is $n-1$. A simple observation we have $\mathtt{h}(P_n, i) = n - i$ for $1 \leq i \leq n-1$ and hence the result. $\square$

Theorem 6.1.17. *The Hosoya polynomial of the cycle graph C_n is*

$$\mathtt{H}(C_n) = \begin{cases} \frac{n}{2}x^{\frac{n}{2}} + n\sum\limits_{i=1}^{\frac{n}{2}-1} x^i & \textit{; if } n \textit{ is even,} \\ n\sum\limits_{i=1}^{\frac{n-1}{2}} x^i & \textit{; if } n \textit{ is odd.} \end{cases}$$

Proof. If n is even, then $D(C_n) = \frac{n}{2}$. Observe that $\mathtt{h}(C_n, i) = n$ for $1 \leq i \leq \frac{n}{2} - 1$ and $\mathtt{h}(C_n, \frac{n}{2}) = \frac{n}{2}$. If n is odd, then $D(C_n) = \frac{n-1}{2}$. Clearly, $\mathtt{h}(C_n, i) = n$ for $1 \leq i \leq \frac{n-1}{2}$. $\square$

Theorem 6.1.18. *Let T_n^* be the tree consisting of a path on n vertices and two vertices adjacent to one of the endpoints of the path. Then the Hosoya polynomial of T_n^* is*

$$\mathtt{H}(T_n^*, x) = \mathtt{H}(P_{n+2}, x) + x^2(1 - x^{n-1}).$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the n vertices of the path P_n and u and v be the vertices adjacent to one of the endpoints of P_n, say v_1. Then $\{v, v_1, v_2, \ldots, v_n\}$ is the path P_{n+1}, $\mathtt{d}(u, v_i) = i$, $i = 1, 2, \ldots n$ and $\mathtt{d}(u, v) = 2$. Therefore

$$\begin{aligned}
\mathtt{H}(T_n^*, x) &= x^n + 2x^{n-1} + 3x^{n-2} + \ldots + (n-2)x^3 + (n-1)x^2 + nx \\
&\quad + x^n + x^{n-1} + x^{n-2} + \ldots + x^3 + x^2 + x + x^2 \\
&= 2x^n + 3x^{n-1} + 4x^{n-2} + \ldots + (n-1)x^3 + nx^2 + (n+1)x + x^2 \\
&= \mathtt{H}(P_{n+2}, x) + x^2 - x^{n+1}.
\end{aligned}$$

Therefore $\mathtt{H}(T_n^*, x) = \mathtt{H}(P_{n+2}, x) + x^2(1 - x^{n-1})$. $\square$

Theorem 6.1.19. *The Hosoya polynomial $\mathtt{H}(T_{l,m,n}^*, x)$ of $T_{l,m,n}^*$ is*

$$\mathtt{H}(S_l, x) + \mathtt{H}(S_m, x) + \mathtt{H}(P_n, x) + (l + m - 2)x^2 \frac{x^n - 1}{x - 1} + (lm - 2(l + m) + 3)\, x^{n+1}.$$

Proof. The proof is similar to the proof of the Theorem 6.1.18. $\square$

Theorem 6.1.20. *The Hosoya polynomial $\mathtt{H}(B_{l,m,n}, x)$ of $B_{l,m,n}$ is*

$$\mathtt{H}(K_l, x) + \mathtt{H}(K_m, x) + \mathtt{H}(P_n, x) + (l + m - 2)x^2 \frac{x^n - 1}{x - 1} + (lm - 2(l + m) + 3)\, x^{n+1}.$$

Proof. The proof is similar to the proof of the Theorem 6.1.13. $\square$

Theorem 6.1.21. *The Hosoya polynomial of the corona $K_m \circ K_n$ is*

$$\mathtt{H}(K_m \circ K_n, x) = \frac{1}{2}m(m + n^2 + n - 1)x + m(m-1)nx^2 + \frac{1}{2}m(m-1)n^2x^3.$$

Proof. Clearly, the diameter of $K_m \circ K_n$ is 3. Since the size of $K_m \circ K_n$ is $\binom{m}{2} + m\binom{n}{2} + mn$, $\mathtt{h}(K_m \circ K_n, 1) = \frac{1}{2}m(m + n^2 + n - 1)$. Let $V = \{v_1, v_2, \ldots, v_m\}$ be the vertices of K_m and $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ be the vertices of i^{th} copy of K_n for $i = 1, 2, \ldots, m$. Then any vertices of V_i and any vertex $v_j \in V$ for $i \neq j$ are at distance 2. Therefore $\mathtt{h}(K_m \circ K_n, 2) = m(m-1)n$. Finally, any vertices of V_i and any vertices of V_j are at distance 3, for $i \neq j$. Therefore $\mathtt{h}(K_m \circ K_n, 3) = \frac{1}{2}m(m-1)n^2$. This completes the proof. □

Corollary 6.1.22. *The Hosoya polynomial* $\mathtt{H}(Q(m,n), x)$ *of* $Q(m,n)$ *is*

$$\frac{1}{2}m(m+n^2-n-1)x + m(m-1)(n-1)x^2 + \frac{1}{2}m(m-1)(n-1)^2x^3.$$

Proof. It follows from the fact that $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Theorem 6.1.23. *The Hosoya polynomial* $\mathtt{H}(G_{n,m}, x)$ *of the square grid graph* $G_{n,m}$ *is*

$$\mathtt{H}(G_{n,m}, x) = n\mathtt{H}(P_m, x) + m\mathtt{H}(P_n, x) + \sum_{i=1}^{n-1} 2(n-i)\left(\sum_{j=1+i}^{m+i} x^{j-1}\right).$$

Proof. The proof is similar to the proof of the Theorem 6.1.18. □

Theorem 6.1.24. *The Hosoya polynomial of ladder graph* L_n *is*

$$\mathtt{H}(L_n, x) = 2\left(\mathtt{H}(P_{n+1}, x) + \mathtt{H}(P_n, x)\right) - nx.$$

Proof. It follows from the fact that $L_n = G_{n,2}$. □

6.1.1 Hosoya polynomial of square of some graphs

In this section we find some relation between coefficients of Hosoya polynomial of a graph G and its square G^2. Then we consider some specific graphs and obtain an explicit formula for the Hosoya polynomial of the square of these graphs. To prove the main results we need the following result.

Lemma 6.1.25. *Let* G *be a graph with diameter* D. *Then* $D(G^2) = \frac{D}{2}$ *when* D *is even and* $D(G^2) = \frac{D+1}{2}$ *when* D *is odd.*

Now we state a relation between Hosoya polynomials of graph G and its square G^2 in the following theorem.

Theorem 6.1.26. *Let G be a graph with diameter D. Then the Hosoya polynomial of the square of G^2 is*

$$\begin{aligned} \mathtt{H}(G^2,x) &= \sum_{k=1}^{n}\left[\mathtt{h}(G,2k-1)+\mathtt{h}(G,2k)\right]x^k && \text{if } D=2n \\ \mathtt{H}(G^2,x) &= \sum_{k=1}^{n-1}\left[\mathtt{h}(G,2k-1)+\mathtt{h}(G,2k)\right]x^k+\mathtt{h}(G,2n-1)x^n && \text{if } D=2n-1. \end{aligned}$$

Next theorem will give Hosoya polynomial of power of some graphs with some specified properties.

Theorem 6.1.27. *Let G be a graph of order n and diameter D and let m be any positive integer. If $m \geq D$, then the Hosoya polynomial of G^m is*

$$\mathtt{H}(G^m,x)=\frac{1}{2}n(n-1)x.$$

Corollary 6.1.28. *If G is a graph of n vertices with diameter less than or equal to 2 then Hosoya polynomial of the square of G is*

$$\mathtt{H}(G^2,x)=\frac{1}{2}n(n-1)x.$$

If G is a graph of n vertices with diameter 1 or 2, then from Corollary 6.1.28 we have the Hosoya polynomial of the square G is $\mathtt{H}(G^2,x)=\frac{1}{2}n(n-1)x$. Therefore we consider only the graphs whose diameter greater than 2.

Theorem 6.1.29. *Let B_n be the bipartite cocktail party graph. Then for $n \geq 3$, the Hosoya polynomial of the square of B_n is*

$$\mathtt{H}(B_n^2,x)=nx^2+2n(n-1)x.$$

Theorem 6.1.30. *The Hosoya polynomial of the square of $B_{n,1}$ is*

$$\mathtt{H}(B_{n,1}^2,x)=(n-1)^2x^2+(n^2+n-1)x.$$

Theorem 6.1.31.

$$\mathtt{H}(B_{(m,n)}^2,x)=mnx^2+\frac{1}{2}\left[m\left(m+3\right)+n\left(n+3\right)+2\right]x.$$

Theorem 6.1.32. *The Hosoya polynomial of the square of the corona $K_m \circ K_n$*

is

$$\mathtt{H}((K_m \circ K_n)^2, x) = \frac{1}{2}m(m-1)n^2x^2 + \frac{1}{2}m(2mn + m + n^2 - n - 1)x.$$

Theorem 6.1.33. *The Hosoya polynomial of the square of $Q(m,n)$ is*

$$\mathtt{H}(Q(m,n)^2, x) = \frac{1}{2}m(2mn - m + n^2 - 3n + 1)x^2 + \frac{1}{2}m(m-1)(n-1)^2x.$$

Theorem 6.1.34. *The square Hosoya polynomial $\mathtt{H}(S^2_{(n_1,n_2,\ldots,n_r)}, x)$ of $S_{(n_1,n_2,\ldots,n_r)}$ is*

$$\left[\left(\binom{n-r}{2} - \sum_{i=1}^{r}\binom{n_i-1}{2}\right) + (n-r)(r-1)\right]x^2 + \left(\sum_{i=1}^{r}\binom{n_i}{2} + \binom{r}{2} + n\right)x,$$

where $n = n_1 + n_2 + \ldots + n_r$.

Proof. It follows from the fact that the diameter of $S^2_{(n_1,n_2,\ldots,n_r)}$ is 2,

$$\mathtt{h}(S^2_{(n_1,n_2,\ldots,n_r)}, 1) = \mathtt{h}(S_{(n_1,n_2,\ldots,n_r)}, 1) + \mathtt{h}(S_{(n_1,n_2,\ldots,n_r)}, 2)$$

and

$$\mathtt{h}(S^2_{(n_1,n_2,\ldots,n_r)}, 2) = \mathtt{h}(S_{(n_1,n_2,\ldots,n_r)}, 3) + \mathtt{h}(S_{(n_1,n_2,\ldots,n_r)}, 4).$$

This completes the proof. □

Theorem 6.1.35. *The Hosoya polynomial of the square of the path graph P_{2n-1} is*

$$\mathtt{H}(P^2_{2n-1}, x) = 3x^{n-1} + 7x^{n-2} + 11x^{n-2} + \ldots + (4n-9)x^2 + (4n-5)x.$$

Theorem 6.1.36. *The Hosoya polynomial of the square of the path graph P_{2n} is*

$$\mathtt{H}(P^2_{2n}, x) = x^n + 5x^{n-1} + 9x^{n-2} + \ldots + (4n-7)x^2 + (4n-3)x.$$

Theorem 6.1.37. *The Hosoya polynomial of the square of the cycle graph C_{4n-1} is*

$$\mathtt{H}(C^2_{4n-1}, x) = (4n-1)\left[x^n + 2(x^{n-1} + x^{n-2} + \ldots + x)\right].$$

Theorem 6.1.38. *The Hosoya polynomial of the square of the cycle graph C_{4n}*

is

$$\mathtt{H}(C_{4n}^2, x) = 2n\left[3x^n + 4(x^{n-1} + x^{n-2} + \ldots + x)\right].$$

Theorem 6.1.39. *The Hosoya polynomial of the square of the cycle graph C_{4n+1} is*

$$\mathtt{H}(C_{4n+1}^2, x) = 2(4n+1)\left(x^n + x^{n-1} + x^{n-2} + \ldots + x\right).$$

Theorem 6.1.40. *The Hosoya polynomial of the square of the cycle graph C_{4n+2} is*

$$\mathtt{H}(C_{4n+2}^2, x) = (2n+1)\left[x^{n+1} + 4(x^n + x^{n-1} + \ldots + x)\right].$$

Theorem 6.1.41. *Let T_{2n-1}^* be the tree consisting of a path on $2n-1$ vertices and two vertices adjacent to one of the endpoints of the path. Then the Hosoya polynomial of the square of T_{2n-1}^* is*

$$\mathtt{H}(T_{2n-1}^{*^2}, x) = \mathtt{H}(P_{2n+1}^2, x) + x(1 - x^{n-1}).$$

Theorem 6.1.42. *Let T_{2n}^* be the tree consisting of a path on $2n$ vertices and two vertices adjacent to one of the endpoints of the path. Then the Hosoya polynomial of the square of T_{2n}^* is*

$$\mathtt{H}(T_{2n}^{*^2}, x) = \mathtt{H}(P_{2n+2}^2, x) + x(1 - x^n).$$

Theorem 6.1.43. *The Hosoya polynomial $\mathtt{H}(T_{l,m,2n-1}^{*^2}, x)$ of the square of $T_{l,m,2n-1}^*$ is*

$$\mathtt{H}(S_l^2, x) + \mathtt{H}(S_m^2, x) + \mathtt{H}(P_{2n-1}^2, x) + (l+m-2)\left[2(x^{n-1} + x^{n-2} + \ldots + x^2) + x\right] + (lm-1)x^n.$$

Theorem 6.1.44. *The Hosoya polynomial $\mathtt{H}(T_{l,m,2n}^{*^2}, x)$ of the square of $T_{l,m,2n}^*$ is*

$$\mathtt{H}(S_l^2, x) + \mathtt{H}(S_m^2, x) + \mathtt{H}(P_{2n}^2, x) + (l+m-2)\left[2(x^n + x^{n-1} + \ldots + x^2) + x\right] + (l-1)(m-1)x^{n+1}.$$

Theorem 6.1.45. *The Hosoya polynomial $\mathtt{H}(B_{l,m,2n-1}^2, x)$ of the square of $B_{l,m,2n-1}$ is*

$$\mathtt{H}(K_l, x) + \mathtt{H}(K_m, x) + \mathtt{H}(P_{2n-1}^2, x) + (l+m-2)\left[2(x^{n-1} + x^{n-2} + \ldots + x^2) + x\right] + (lm-1)x^n.$$

Theorem 6.1.46. *The Hosoya polynomial* $\mathtt{H}(B^2_{l,m,2n},x)$ *of the square of* $B_{l,m,2n}$ *is*

$$\mathtt{H}(K_l,x)+\mathtt{H}(K_m,x)+\mathtt{H}(P^2_{2n},x)+(l+m-2)\left[2(x^n+x^{n-1}+\ldots+x^2)+x\right]+(l-1)(m-1)x^{n+1}.$$

Theorem 6.1.47. *The Hosoya polynomial of the square of* $L_{m,2n-1}$ *is*

$$\mathtt{H}(L^2_{m,2n-1},x)=\frac{1}{2}(m^2+m+2)x+\mathtt{H}(P^2_{2n-1},x)+2m(x^{n-1}+x^{n-2}+\ldots+x^2)+(2m-1)x^n.$$

Theorem 6.1.48. *The Hosoya polynomial of the square of* $L_{m,2n}$ *is*

$$\mathtt{H}(L^2_{m,2n},x)=\frac{1}{2}(m^2+m+2)x+\mathtt{H}(P^2_{2n},x)+2m(x^n+x^{n-1}+\ldots+x^2)+(m-1)x^{n+1}.$$

Theorem 6.1.49. *The Hosoya polynomial of the square ladder graph* L_n *is*

$$\mathtt{H}(L^2_n,x)=2\left[\mathtt{H}(P^2_{n+1},x)+\mathtt{H}(P^2_n,x)\right]-nx.$$

6.2 $\mathtt{h}$-number of graphs

In this section we find the number of the real roots of Hosoya polynomial of some graphs. First we define Hosoya root of a graph.

Definition 6.2.1. *Let G be a connected graph with Hosoya polynomial $\mathtt{H}(G,x)$. A root of $\mathtt{H}(G,x)$ is called a Hosoya root of G and set of all Hosoya roots of G is denoted by $\mathbb{Z}(\mathtt{H}(G,x))$.*

We mainly find the number of real Hosoya roots of some specific graphs. So we introduce a new concept, $\mathtt{h}$-number of a graph.

Definition 6.2.2. *Let G a connected graph. The number of distinct real Hosoya roots of the graph G is called $\mathtt{h}$-number of G and is denoted by $\mathtt{h}(G)$.*

Example 6.2.1. *The Hosoya polynomial of the graph G in Figure 2.1 is*

$$\mathtt{H}(G,x)=4x^3+6x^2+5x^2.$$

Therefore the Hosoya roots of G are

$$\mathbb{Z}(\mathtt{H}(G,x))=\{-0.75-0.83i,-0.75+0.83i,0\}.$$

Hence $\mathtt{h}(G) = 1$.

Next theorem follows from the fact that 0 is a Hosoya root of any graph.

Theorem 6.2.2. *For any graph* G*,* $\mathtt{h}(G) \geq 1$.

Theorem 6.2.3. *If* G_1 *and* G_2 *are isomorphic graphs then* $\mathtt{h}(G_1) = \mathtt{h}(G_2)$.

Proof. It follows from the fact that if G_1 and G_2 are isomorphic, then $\mathtt{H}(G_1, x) = \mathtt{H}(G_2, x)$. □

Theorem 6.2.4. *Let* G *be a graph with diameter* 2*, then* $\mathtt{h}(G) = 2$.

Proof. Let G be a graph of order n and size m. Then by Theorem 6.1.5, we have $\mathbb{Z}(\mathtt{H}(G, x)) = \left\{0, \frac{m}{m - \binom{n}{2}}\right\}$, hence $\mathtt{h}(G) = 2$. □

Theorem 6.2.5. *Let* G *be a graph of diameter* 2*, then* $\mathtt{h}(G^2) = 1$.

Proof. Result follows from the Corollary 6.1.28. □

Theorem 6.2.6. *For* $n \geq 6$ *the* $\mathtt{h}$*-number of the bipartite cocktail party graph* B_n *is* 3.

Proof. By Theorem 6.1.12 we have the following quadratic equation :

$$x^2 + (n-1)x + (n-1) = 0.$$

It is easy to see that $\Delta = (n-1)(n-5)$, where Δ is the discriminant of the quadratic equation. Since $n \geq 6$, we have $\Delta > 0$. Therefore the Hosoya roots of bipartite cocktail party graph B_n are real and distinct for $n \geq 6$. □

The Hosoya roots of the bipartite cocktail party graph B_n for $1 \leq n \leq 100$ are shown in Figure 6.1.

Theorem 6.2.7. *The* $\mathtt{h}$*-number of bi-star graph* $B_{(n,n)}$ *is*

$$\mathtt{h}(B_{(n,n)}) = \begin{cases} 1 & \textit{; for } n \leq 6, \\ 3 & \textit{; otherwise.} \end{cases}$$

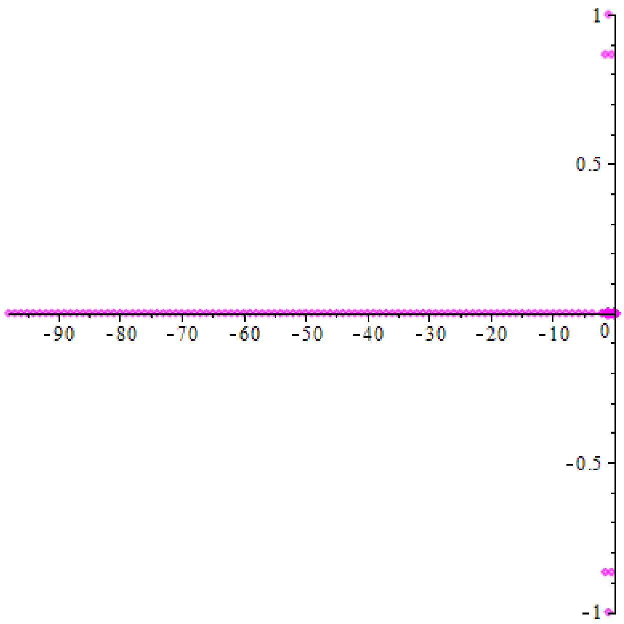

Figure 6.1: Hosoya roots of B_n for $1 \leq n \leq 100$.

Proof. $\mathtt{H}(B_{(n,n)}, x) = n^2x^3 + n(n+1)x^2 + (2n+1)x$. It suffices to prove that all the roots of $P(x) = n^2x^2 + n(n+1)x + (2n+1)$ are complex for $n \leq 6$ and are real for $n > 6$. The discriminant of $P(x)$ is $\Delta = n^2(n^2 - 6n - 3)$. If $n \leq 6$, $\Delta < 0$, therefore all the roots are complex. If $n > 6$, then $\Delta > 0$. Therefore all the nonzero Hosoya roots of the bi-star graph $B_{(n,n)}$ are real and distinct for $n > 6$, this completes the proof. □

The Hosoya roots of the bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 6$ and $7 \leq n \leq 100$ are shown in Figures 6.2 and 6.3 respectively.

Theorem 6.2.8. *The* h*-number of the corona* $K_m \circ K_n$ *is* 1.

Proof. By Theorem 6.1.21 we have the following quadratic equation :

$$(m-1)n^2x^2 + 2(m-1)nx + n^2 + m + n - 1 = 0.$$

It is easy to see that the discriminant of the quadratic equation

$$\Delta = -4n^3(n+1)(m-1).$$

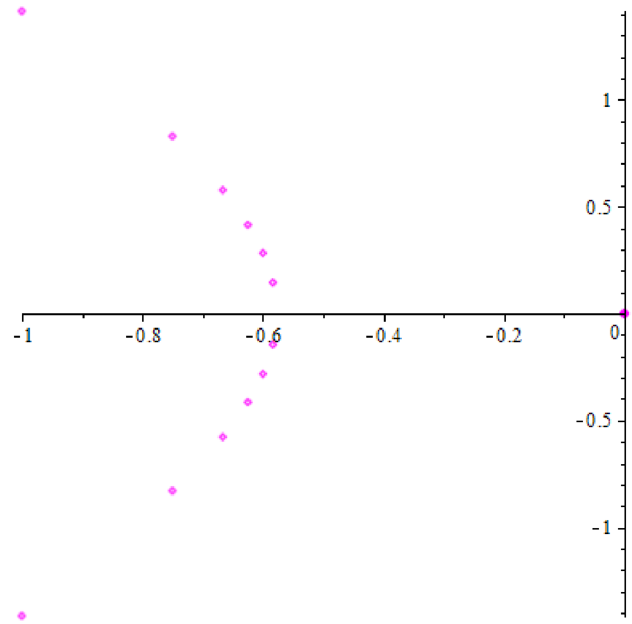

Figure 6.2: Hosoya roots of $B_{(n,n)}$ for $1 \leq n \leq 6$.

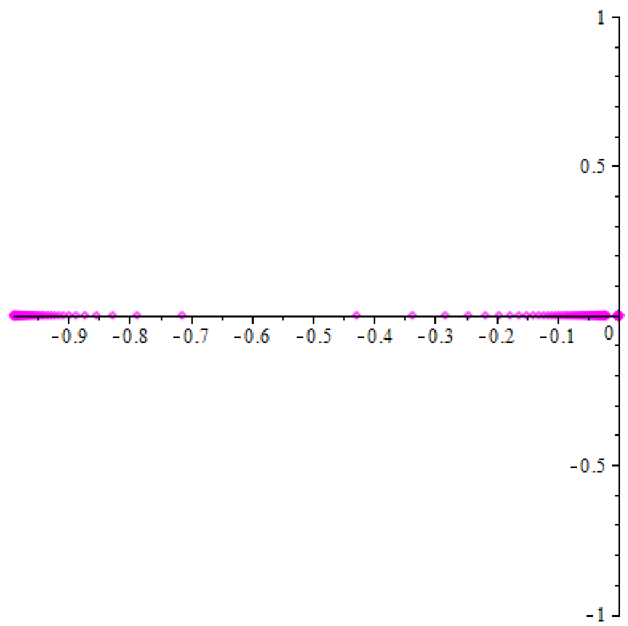

Figure 6.3: Hosoya roots of $B_{(n,n)}$ for $7 \leq n \leq 100$.

Since $K_m \circ K_n$ has nonzero Hosoya root only if $m > 1$, hence $\Delta < 0$. Therefore all the nonzero Hosoya roots of $K_m \circ K_n$ are complex, that is, $\mathtt{h}(K_m \circ K_n) = 1$. $\square$

The Hosoya roots of the corona $K_n \circ K_n$ for $1 \leq n \leq 200$ are shown in Figure 6.4.

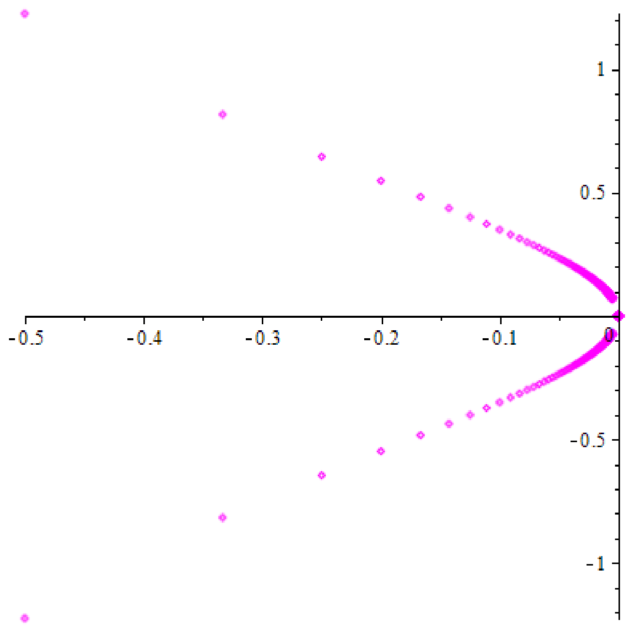

Figure 6.4: Hosoya roots of $K_n \circ K_n$ for $1 \leq n \leq 200$.

Corollary 6.2.9. *The* $\mathtt{h}$*-number of* $Q(m, n)$ *is 1.*

Proof. It follows from the fact that $Q(m, n)$ and $K_m \circ K_{n-1}$ are isomorphic. $\square$

Theorem 6.2.10. *The* $\mathtt{h}$*-number of the* n*-barbell graph* $B_{n,1}$ *is* 1.

Proof. We have to show that all the nonzero Hosoya roots of the n-barbell graph $B_{n,1}$ are complex. The n-barbell graph $B_{1,1}$ has no nonzero Hosoya roots. For $n > 1$, by theorem 6.1.13 we have the following quadratic polynomial :

$$P(x) = (n-1)^2 x^2 + 2(n-1)x + n(n-1) + 1.$$

It is easy to see that $\Delta = -4n(n-1)^3$, where Δ is the discriminant of the quadratic equation. Since $n \geq 2$, we have $\Delta < 0$. Therefore all the roots are complex, that is, $\mathtt{h}(B_{n,1}) = 1$. $\square$

The Hosoya roots of the n-barbell $B_{n,1}$ for $1 \leq n \leq 200$ are shown in Figure 6.5.

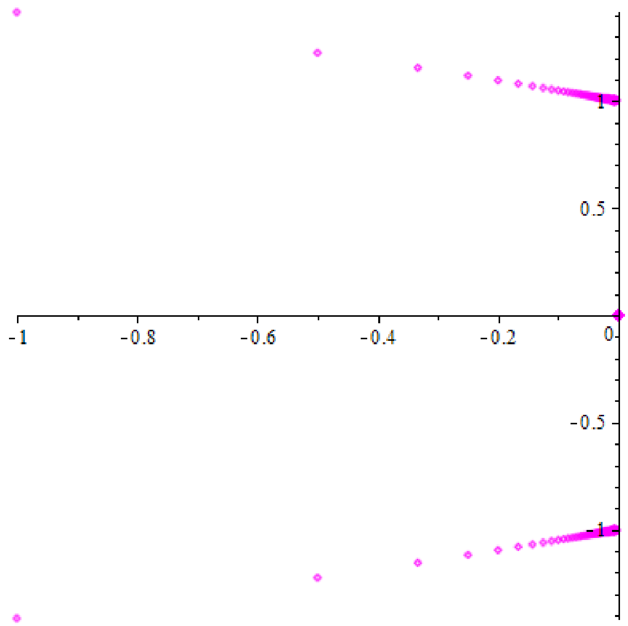

Figure 6.5: Hosoya roots of $B_{n,1}$ for $1 \leq n \leq 200$.

Theorem 6.2.11. *Let G be a graph with diameter 3. Then either $\mathtt{h}(G) = 3$ or $\mathtt{h}(G) = 1$*

Proof. We have 0 is a Hosoya root of any graph and number of complex roots are even. Therefore all the nonzero Hosoya roots of G are either complex or real. This implies that $\mathtt{h}(G) = 3$ or $\mathtt{h}(G) = 1$. □

Theorem 6.2.12. *The $\mathtt{h}$-number of the square of the generalized barbell graph $B_{m,m,5}$ is 1.*

Proof. The the square of the generalized barbell graph $B_{1,1,5}$ has no nonzero Hosoya roots. By Theorem 6.1.45 we have the following quadratic polynomial :

$$P(x) = (m^2 - 1)x^2 + (4m - 1)x + m^2 + m + 5.$$

The discriminant of the equation $P(x) = 0$ is $\Delta = 21 - 4m(m^3 + m^2 + 1)$. Since $m \geq 2$, we have $\Delta < 0$. Therefore all the roots are complex, that is, $\mathtt{h}(B_{m,m,5}) = 1$. □

The Hosoya roots of the square of the generalized barbell graph $B_{m,m,5}$ for $2 \leq m \leq 200$ are shown in Figure 6.6.

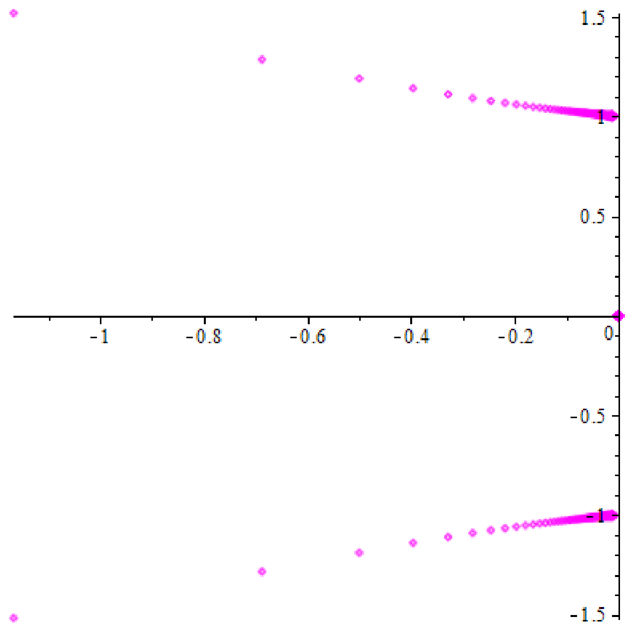

Figure 6.6: Hosoya roots of $B^2_{m,m,5}$ for $2 \leq m \leq 200$.

Theorem 6.2.13. *Let G be a graph of diameter 3, order n and size m. Then $\mathtt{h}(G) = 3$ if and only if*

$$\mathtt{h}(G,2) + 4m > \frac{2m}{\mathtt{h}(G,2)}\left(n(n-1) - 2m\right).$$

Proof. We have the Hosoya polynomial of G is

$$\mathtt{H}(G,x) = \mathtt{h}(G,3)x^3 + \mathtt{h}(G,2)x^2 + \mathtt{h}(G,1)x.$$

It is clear that the nonzero Hosoya roots of G and the roots of the quadratic polynomial

$$P(x) = \mathtt{h}(G,3)x^2 + \mathtt{h}(G,2)x + \mathtt{h}(G,1)$$

are equal. But we have $\mathtt{h}(G,1) = m$ and $\mathtt{h}(G,3) = \binom{n}{2} - m - \mathtt{h}(G,2)$. Then the discriminant of $P(x)$ is

$$\begin{aligned}\Delta &= (\mathtt{h}(G,2))^2 - 4m\left(\binom{n}{2} - m - \mathtt{h}(G,2)\right)\\ &= \mathtt{h}(G,2)\left(\mathtt{h}(G,2)+4m\right) - 2m\left(n(n-1)-2m\right).\end{aligned}$$

Therefore the result follows from the fact that all the roots of quadratic polynomial are real and distinct if and only if $\Delta > 0$. □

Corollary 6.2.14. *Let G be a graph of diameter 3, order n and size m. Then $\mathtt{h}(G) = 2$ if and only if*

$$\mathtt{h}(G,2) + 4m = \frac{2m}{\mathtt{h}(G,2)}\left(n(n-1)-2m\right).$$

Corollary 6.2.15. *Let G be a graph of diameter 3, order n and size m. Then $\mathtt{h}(G) = 1$ if and only if*

$$\mathtt{h}(G,2) + 4m < \frac{2m}{\mathtt{h}(G,2)}\left(n(n-1)-2m\right).$$

Theorem 6.2.16. *For $n \geq 2$, the h-number of the path graph P_n is :*

$$\mathtt{h}(P_n) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is even,}\\ 2 & ;\ \textit{if } n \textit{ is odd.}\end{cases}$$

Proof. We have the Hosoya polynomial of the path graph P_n is

$$\mathtt{H}(P_n, x) = x^{n-1} + 2x^{n-2} + \ldots + (n-1)x.$$

But

$$\mathtt{H}(P_n, x)(x-1)^2 = x(x^n - nx + n - 1).$$

Clearly, $x = 1$ is a double root of $f(x) = x^n - nx + n - 1$. By De Gua's rule 1.2.2 for imaginary roots, there are at least $n-2$ complex roots for even n and there are at least $n-3$ complex roots for odd n. This give the result. □

Theorem 6.2.17. *We have the following :*

$$\mathtt{h}(C_{2n+1}) = \begin{cases} 1 & ;\ \textit{if } n \textit{ is odd,}\\ 2 & ;\ \textit{if } n \textit{ is even.}\end{cases}$$

Proof. We have the Hosoya polynomial of C_{2n+1} is

$$\mathtt{H}(C_{2n+1}, x) = (2n+1)(x + x^2 + \ldots + x^n).$$

Therefore it is enough to show that $f(x) = 1 + x + x^2 + \ldots + x^{n-1}$ has no real root when n is odd and it has only one real root when n is even. This follows from the fact that $x^n - 1 = (x-1)f(x)$. $\square$

$\mathtt{h}$-Number of some well known graphs and its square are shown in Table 6.1.

Graph G	Nonzero Hosoya roots of G	$\mathtt{h}(G)$	$\mathtt{h}(G^2)$
Star graph S_n	$-\frac{2}{n-1}$	2	1
$G_1 \vee G_2$	$\frac{n_1n_2+m_1+m_2}{(n_1n_2+m_1+m_2)-\binom{n_1+n_2}{2}}$	2	1
Dutch windmill graph G_3^n	$\frac{3}{2(1-n)}$	2	1
Petersen graph P	$-\frac{1}{2}$	2	1
Lollipop graph $L_{n,1}$	$-\frac{1}{1-n} - \frac{n}{2}$	2	1
Bipartite cocktail party graph B_n for $n \geq 6$	$\frac{1-n\pm\sqrt{(n-1)(n-5)}}{2}$	3	2
n-barbell graph $B_{n,1}$	$\frac{-1\pm\sqrt{(n-n^2)}}{n-1}$	1	2
Bi-star graph $B_{(n,n)}$ for $n \leq 6$	$\frac{-(n+1)\pm\sqrt{(n)^2-6n-3}}{2n}$	1	2
Bi-star graph $B_{(n,n)}$ for $n > 6$	$\frac{-(n+1)\pm\sqrt{(n)^2-6n-3}}{2n}$	3	2
Corona $K_m \circ K_n$	$\frac{1-m\pm\sqrt{n(n+1)(m-1)}}{n(m-1)}$	1	2
$Q(m,n)$	$\frac{1-m\pm\sqrt{(n-1)n(m-1)}}{(n-1)(m-1)}$	1	2

Table 6.1: $\mathtt{h}$-number of graph G and G^2

6.3 Bounds for the Hosoya roots of some graphs

In this section we estimate the bounds for the Hosoya roots of the path graph P_n and the cycle graph C_n. Also find the bounds for the Hosoya roots of the square of the path graph P_n and the cycle graph C_n.

Theorem 6.3.1. *All the nonzero Hosoya roots of the path graph P_n lie in the annulus* $1 < |z| \leq 2$.

Proof. We have $\mathtt{H}(P_n, x) = \sum\limits_{i=1}^{n-1}(n-i)x^i$. Therefore it suffices to show that all the roots of $f(x) = x^{n-2} + 2x^{n-3} + 3x^{n-4} + \ldots + (n-2)x + n - 1$ lie in the annulus

$1 < |z| \leq 2$. The result follows from Enestrom-Kakeya theorem 1.2.6. $\square$

The Hosoya roots of the path graph P_n for $1 \leq n \leq 100$ are shown in Figure 6.7.

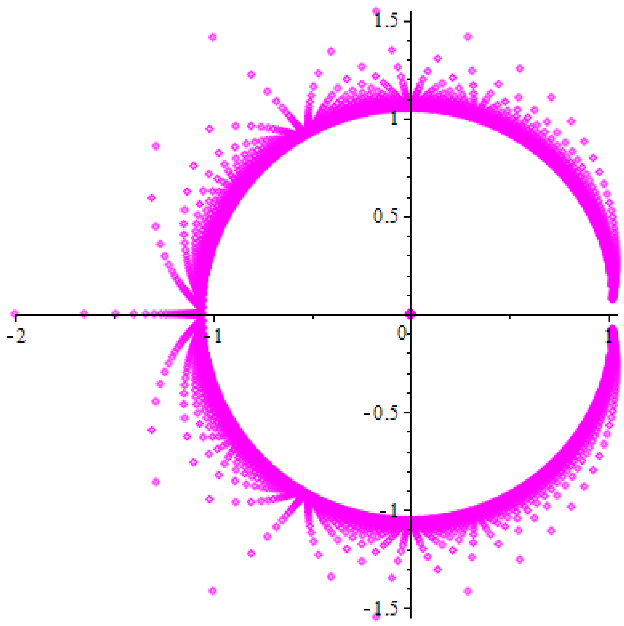

Figure 6.7: Hosoya roots of P_n for $1 \leq n \leq 100$.

Theorem 6.3.2. *All the nonzero Hosoya roots of the square of the path graph P_{2n-1} lie in the annulus $1 < |z| \leq 2.333$.*

Proof. The proof is similar to the proof of the Theorem 6.3.1. $\square$

The Hosoya roots of the square of the path graph P_{2n-1} for $1 \leq n \leq 100$ are shown in Figure 6.8.

Theorem 6.3.3. *All the nonzero Hosoya roots of the square of the path graph P_{2n} lie in the annulus $1 < |z| \leq 5$.*

Proof. The proof is similar to the proof of the Theorem 6.3.1. $\square$

The Hosoya roots of the square of the path graph P_{2n} for $1 \leq n \leq 100$ are shown in Figure 6.9.

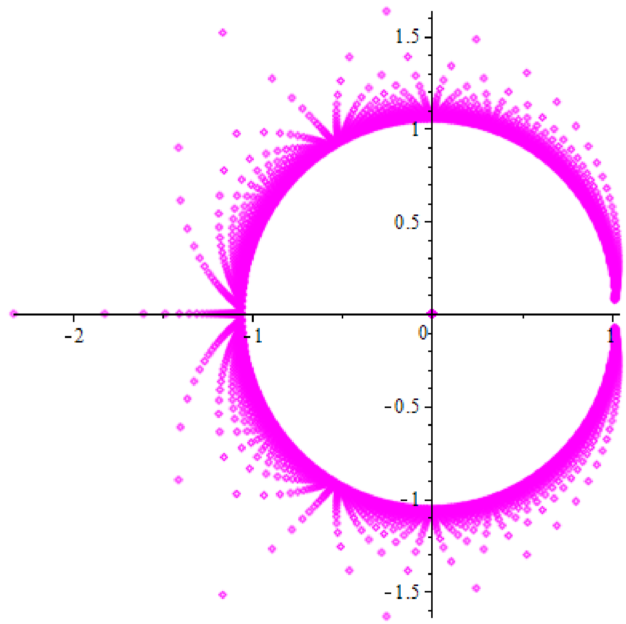

Figure 6.8: Hosoya roots of P^2_{2n-1} for $1 \leq n \leq 100$.

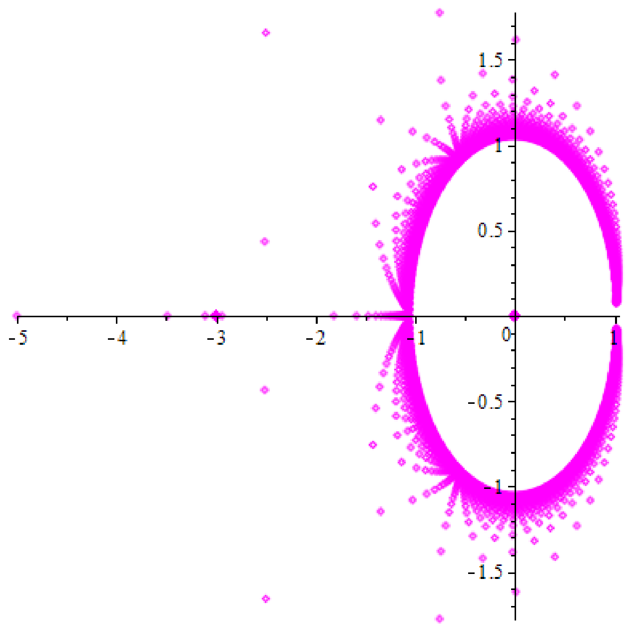

Figure 6.9: Hosoya roots of P^2_{2n} for $1 \leq n \leq 100$.

Theorem 6.3.4. *All the nonzero Hosoya roots of the cycle graph C_{2n} lie in the annulus* $1 \leq |z| \leq 2$.

Proof. We have $\mathtt{H}(C_{2n}, x) = \sum_{i=1}^{n-1} 2nx^i + nx^n$. Therefore it suffices to show that all the roots of $f(x) = 2n(x^{n-2} + x^{n-3} + x^{n-4} + \ldots + x + 1) + nx^{n-1}$ lie in the annulus $1 \leq |z| \leq 2$. The result follows from Enestrom-Kakeya theorem 1.2.6. $\square$

The Hosoya roots of the cycle graph C_{2n} for $1 \leq n \leq 100$ are shown in Figure 6.10.

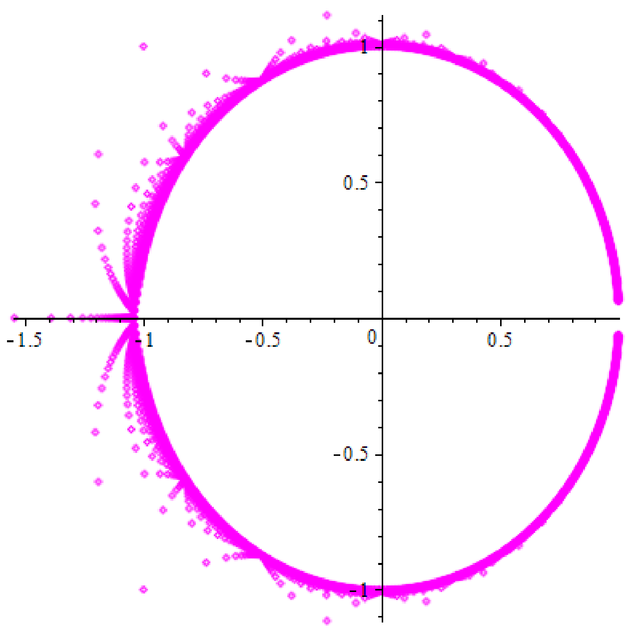

Figure 6.10: Hosoya roots of C_{2n} for $1 \leq n \leq 100$.

Theorem 6.3.5. *All the nonzero Hosoya roots of the cycle graph C_{2n+1} lie on the unit circle centered at the origin.*

Proof. We have $\mathtt{H}(C_{2n+1}, x) = (2n+1)(x + x^2 + \ldots + x^n)$. Therefore it is enough to show that all the roots of $f(x) = 1 + x + x^2 + \ldots + x^{n-1}$ lie on the unit circle centered at the origin. This is followed from the fact that $x^n - 1 = (x-1)f(x)$ and n^{th} roots of unity lie on the unit circle centered at the origin. $\square$

The Hosoya roots of the cycle graph C_{2n+1} for $1 \leq n \leq 100$ are shown in Figure 6.11.

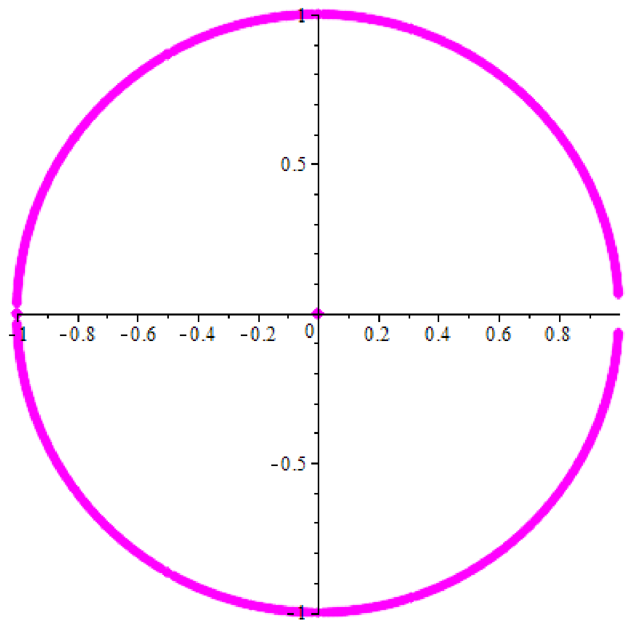

Figure 6.11: Hosoya roots of C_{2n+1} for $1 \leq n \leq 100$.

Theorem 6.3.6. *All the nonzero Hosoya roots of the square of the cycle graph C_{4n-1} lie in the annulus $1 \leq |z| \leq 2$.*

Proof. The proof is similar to the proof of the Theorem 6.3.4. □

The Hosoya roots of the square of the cycle graph C_{4n-1} for $1 \leq n \leq 100$ are shown in Figure 6.12.

Theorem 6.3.7. *All the nonzero Hosoya roots of the square of the cycle graph C_{4n} lie in the annulus $1 \leq |z| \leq 1.333$.*

Proof. The proof is similar to the proof of the Theorem 6.3.4. □

The Hosoya roots of the square of the cycle graph C_{4n} for $1 \leq n \leq 100$ are shown in Figure 6.13.

Theorem 6.3.8. *All the nonzero Hosoya roots of the square of the cycle graph C_{4n+1} lie on the unit circle centered at the origin.*

Proof. The proof is similar to the proof of the Theorem 6.3.5. □

The Hosoya roots of the square of the cycle graph C_{4n+1} for $1 \leq n \leq 100$ are shown in Figure 6.14.

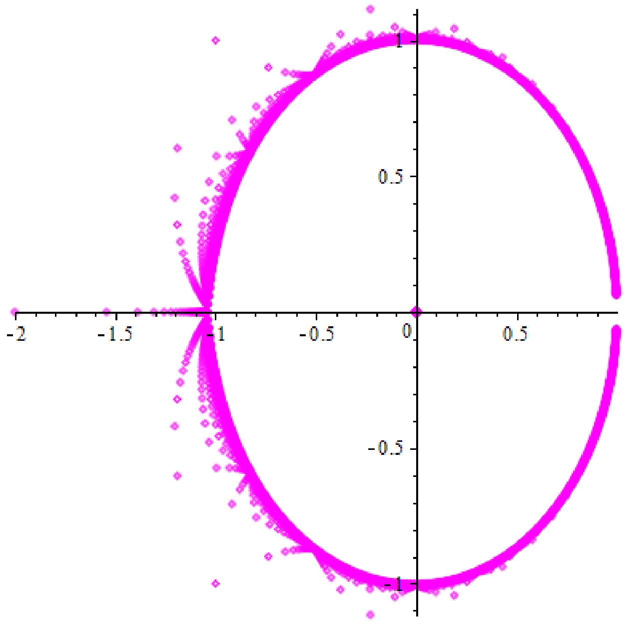

Figure 6.12: Hosoya roots of C_{4n-1}^2 for $1 \leq n \leq 100$.

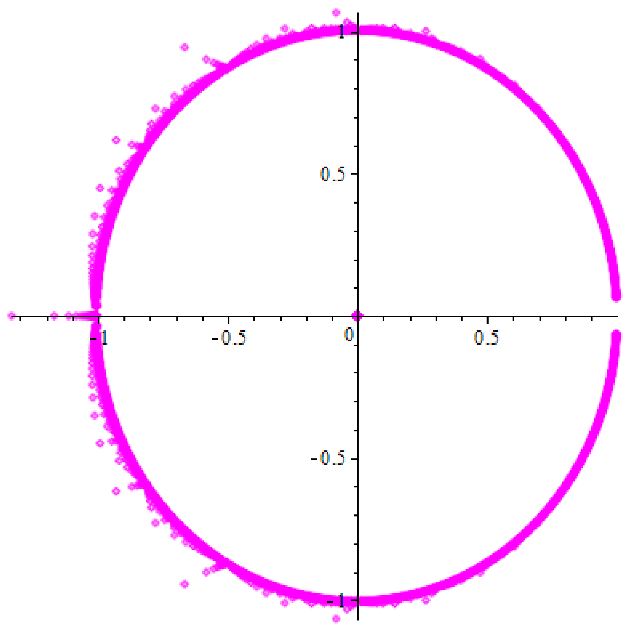

Figure 6.13: Hosoya roots of C_{4n}^2 for $1 \leq n \leq 100$.

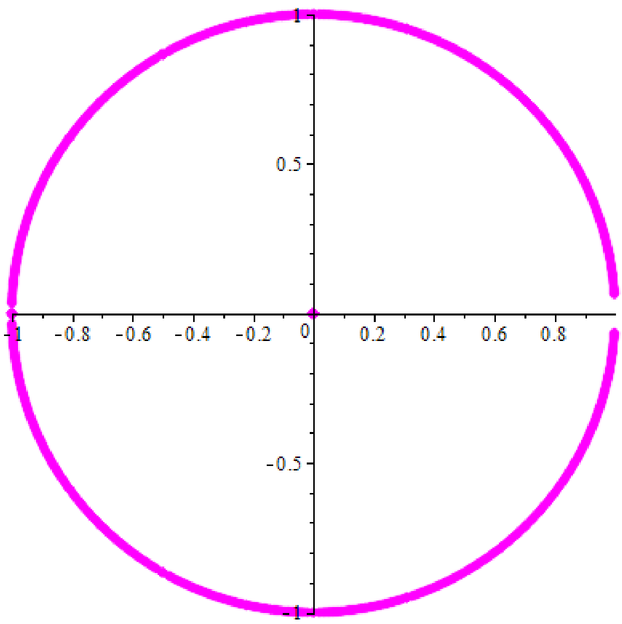

Figure 6.14: Hosoya roots of C^2_{4n+1} for $1 \leq n \leq 100$.

Theorem 6.3.9. *All the nonzero Hosoya roots of the square of the cycle graph C_{4n+2} lie in the annulus $1 \leq |z| \leq 4$.*

Proof. The proof is similar to the proof of the Theorem 6.3.4. □

The Hosoya roots of the square of the cycle graph C_{4n+2} for $1 \leq n \leq 100$ are shown in Figure 6.15.

6.4 Stable graphs related to Hosoya polynomial

In this section we introduce h-stable and h-unstable graphs. We obtain some examples of h-stable graphs and h-unstable graphs. We begin this section by defining h-stable graph.

Definition 6.4.1. *Let $G = (V(G), E(G))$ be a connected graph. The graph G is said to be a Hosoya stable graph or simply* h*-stable graph if all the nonzero Hosoya roots of G lie in the left open half-plane, that is, if real part of the nonzero Hosoya roots is negative. If G is not* h*-stable graph, then G is said to be a Hosoya unstable graph or simply* h*-unstable graph.*

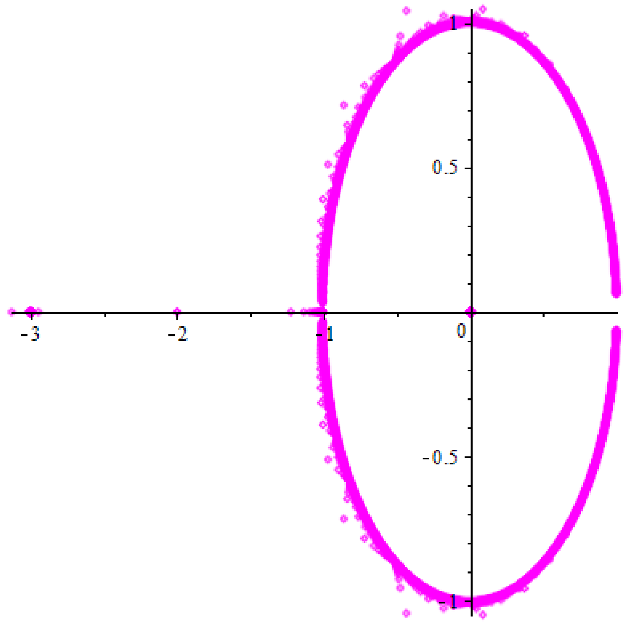

Figure 6.15: Hosoya roots of C^2_{4n+2} for $1 \leq n \leq 100$.

Example 6.4.1. *The Hosoya polynomial of the cycle graph C_8 is*

$$\mathtt{H}(C_8, x) = 4x^4 + 8x^3 + 8x^2 + 8x.$$

With the aid of Maple, the Hosoya roots of C_8 are :
$\mathbb{Z}(\mathtt{H}(C_8, x)) = \{-1.544, -0.2282 - 1.115i, -0.2282 + 1.115i, 0\}$. *All the nonzero Hosoya roots of C_8 lie in the left half-plane. Hence C_8 is a* h*-stable graph.*

Example 6.4.2. *The Hosoya polynomial of the path graph P_7 is*

$$\mathtt{H}(P_7, x) = x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x.$$

With the aid of Maple, the Hosoya roots of P_7 are :
$\mathbb{Z}(\mathtt{H}(P_7, x)) = \{-1.492, -0.8058 - 1.223i, -.8058 + 1.223i, 0, 0.5517 - 1.253i, 0.5517 + 1.253i\}$. *The Hosoya roots $0.5517 - 1.253i$ and $0.5517 + 1.253i$ lie in the right half-plane. Hence P_7 is a* h*-unstable graph.*

Theorem 6.4.3. *Let G be any connected graph of diameter D. If $D \leq 2$, then G is* h*-stable.*

Proof. The result follows from the fact that all the Hosoya roots of G are real. $\square$

Corollary 6.4.4. *The complete graph K_n and its square K_n^2 are* h*-stable for all n.*

Corollary 6.4.5. *The Petersen graph P and its square P^2 are* h*-stable.*

Corollary 6.4.6. *Let H and G be any two connected graphs. Then $H \vee G$ and $(H \vee G)^2$ are* h*-stable.*

Corollary 6.4.7. *$r-$partite graph $K_{n_1,n_2,\ldots,n_r}$ and its square $(K_{n_1,n_2,\ldots,n_r})^2$ are* h*-stable for all $n_1, n_2, \ldots, n_r$.*

Corollary 6.4.8. *The star graph S_n and its square S_n^2 are* h*-stable for all n.*

Corollary 6.4.9. *The Dutch windmill graph G_3^n and its square $G_3^{n^2}$ are* h*-stable for all n.*

Corollary 6.4.10. *The lollipop graph $L_{n,1}$ and its square $L_{n,1}^2$ are* h*-stable for all n.*

Theorem 6.4.11. *The bipartite cocktail party graph B_n and its square B_n^2 are* h*-stable for all n.*

Proof. By Theorem 6.2.6 we have all roots of B_n are real for $n \geq 6$. Using Maple, we observe that for $1 \leq n \leq 5$ all the nonzero Hosoya roots lie in the left half-plane. Therefore the bipartite cocktail party graph B_n is h-stable for all n. Also we have diameter of B_n^2 is 2, therefore by Theorem 6.4.3 B_n^2 is h-stable for all n. $\square$

Theorem 6.4.12. *The bi-star graph $B_{(m,n)}$ and its square $B_{(m,n)}^2$ are* h*-stable for all m, n.*

Proof. By the Theorem 6.1.15, the Hosoya polynomial of bi-star graph $B_{(m,n)}$ is

$$\mathtt{H}(B_{(m,n)}, x) = mnx^3 + \frac{1}{2}(m(m+1) + n(n+1))x^2 + (m+n+1)x.$$

It suffices to prove that all the roots of $P(x) = 2mnx^2 + m(m+1) + n(n+1)x + 2m + 2n + 2$ lie in the left half-plane. Observe that the real part of all the roots of $P(x)$ is $\frac{-(m+1)}{4n} - \frac{(n+1)}{4m}$. Therefore $B_{(m,n)}$ is h-stable for all m, n. Also we have diameter of $B_{(m,n)}^2$ is 2, therefore by Theorem 6.4.3 $B_{(m,n)}^2$ is h-stable for all m, n. $\square$

Theorem 6.4.13. *The graph $K_m \circ K_n$ and its square $(K_m \circ K_n)^2$ are* h*-stable for all m, n.*

Proof. We have diameter of $(K_m \circ K_n)^2$ is 2, therefore by Theorem 6.4.3 $(K_m \circ K_n)^2$ is h-stable for all m, n. To prove $K_m \circ K_n$ is h-stable, by Theorem 6.1.21, it suffices to prove that all the roots of $P(x) = (m-1)n^2x^2 + 2(m-1)nx + n^2 + m + n - 1$ lie in the left half-plane. If $m = 1$, $K_m \circ K_n$ has no nonzero roots. If $m > 1$, we have by Theorem 6.2.8 all the nonzero Hosoya roots of $K_m \circ K_n$ are complex. Observe that real part of all the roots of $P(x)$ is $-\frac{1}{n}$, that is, $K_m \circ K_n$ is h-stable for all m, n. □

Theorem 6.4.14. *The graph $Q(m, n)$ and its square $Q^2(m, n)$ are* h*-stable.*

Proof. We have if $Q(m, n)$ has a nonzero Hosoya root, then $m, n \geq 2$. By Theorem 6.1.22 we have the following quadratic equation :

$$(m-1)(n-1)^2x^2 + 2(m-1)(n-1)x + n^2 + m - n - 1.$$

It is easy to see that the discriminant Δ of the quadratic equation is

$$\Delta = -4n(n-1)^3(m-1).$$

This implies that $n - 1$, $m - 1$ are positive. Therefore all the nonzero roots are complex and observe that real part of all the roots of the quadratic equation is $\frac{1}{1-n}$, this implies that $Q(m, n)$ is h-stable for all m, n. Finally, we have diameter of $Q^2(m, n)$is 2, therefore by Theorem 6.4.3 $Q^2(m, n)$ is h-stable for all m, n. □

Theorem 6.4.15. *The n-barbell graph $B_{n,1}$ and its square $B^2_{n,1}$ are* h*-stable for all n.*

Proof. By Theorem 6.2.10, we have all the Hosoya roots of $B_{n,1}$ are complex. Observe that real part of all these Hosoya roots is $\frac{1}{1-n}$. This implies that $B_{n,1}$ is h-stable for all n. Finally, we have diameter of $B^2_{n,1}$is 2, therefore by Theorem 6.4.3 $B^2_{n,1}$ is h-stable for all n. □

All the graphs discussed above have diameter less than 4 and nonzero Hosoya roots of these graphs are negative or have negative real part, that is, lies in the left half plane.

Theorem 6.4.16. *Let G be a graph with diameter D. If $D \leq 3$, then G is* $\mathtt{h}$*-stable.*

Proof. If $D = 1$, then zero is the only Hosoya root. If $D = 2$, then $\mathtt{H}(G, x) = \binom{n}{2}x^2 - m\,x(x-1)$, where n and m are the order and the size of G respectively. Therefore the only nonzero Hosoya root is $\frac{-m}{\binom{n}{2}-m}$. Since m, $\binom{n}{2} - m$ are positive, this Hosoya root is negative. Finally it suffices to prove that, if $D = 3$, then the roots of $P(x) = x^2 + \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,3)}x + \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,3)}$ are negative or have negative real part. Consider the Hurwitz matrix $H_2 = \begin{bmatrix} \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,3)} & 1 \\ 0 & \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,3)} \end{bmatrix}$, whose determinant is always positive. Therefore by Routh-Hurwitz criteria 1.2.7 we have the result. □

Corollary 6.4.17. *For any natural number l and m, the graph $B_{l,m,n}$ is* $\mathtt{h}$*-stable for $n = 1, 2$.*

Corollary 6.4.18. *For any natural number l and m, the graph $T^*_{l,m,n}$ is* $\mathtt{h}$*-stable for $n = 1, 2$.*

Theorem 6.4.19. *The square of $B_{m,m,5}$ is* $\mathtt{h}$*-stable for all m.*

Proof. For $m = 1$, the diameter of $B^2_{m,m,5}$ is 2, then by Theorem 6.4.3 we have the result. For $m > 1$, by Theorem 6.1.45 we have the following quadratic polynomial :

$$P(x) = (m^2 - 1)x^2 + (4m - 1)x + m^2 + m + 5.$$

The discriminant of the equation $P(x) = 0$ is $\Delta = 21 - 4m(m^3 + m^2 + 1)$. Since $m \geq 2$, we have $\Delta < 0$. Therefore all the roots are complex and observe that the real part of all the roots of $P(x)$ is $\frac{1-4m}{2(m^2+m+5)}$. So we have the result. □

All the graphs discussed above have diameter less than 7 and its square are $\mathtt{h}$-stable.

Theorem 6.4.20. *Let G be a graph with diameter D. If $D \leq 6$, then the square of G is* $\mathtt{h}$*-stable.*

Proof. By Lemma 6.1.25 the diameter of $G^2 \leq 3$ and by Theorem 6.4.16 we have the result. □

Corollary 6.4.21. *The square of the star-like tree graph $S_{(n,n,\ldots,n)_{m\text{-times}}}$ is* $\mathtt{h}$*-stable for all m, n.*

Corollary 6.4.22. *For any natural number l and m, the square of the graph $B_{l,m,n}$ is* $\mathtt{h}$*-stable for $n = 1, 2, 3, 4, 5$.*

Corollary 6.4.23. *For any natural number l and m, the square of the graph $T^*_{l,m,n}$ is* $\mathtt{h}$*-stable for $n = 1, 2, 3, 4, 5$.*

Next theorem gives a necessary and sufficient condition for graphs having diameter 4 is $\mathtt{h}$-stable.

Theorem 6.4.24. *Let G be a graph of diameter 4. Then the graph G is* $\mathtt{h}$*-stable if and only if*

$$\mathtt{h}(G,2)\mathtt{h}(G,3) > \mathtt{h}(G,1)\mathtt{h}(G,4).$$

Proof. Let G be a graph of diameter 4. To study the location of the nonzero roots of the Hosoya polynomial $\mathtt{H}(G,x)$ of G, it suffices to study the behavior of the polynomial $P(x) = x^3 + \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,4)}x^2 + \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,4)}x + \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,4)}$. The Hurwitz matrices of $P(x)$ are :

$$H_1 = \begin{bmatrix} \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,4)} \end{bmatrix} \qquad H_2 = \begin{bmatrix} \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,4)} & 1 \\ \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,4)} & \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,4)} \end{bmatrix} \qquad H_3 = \begin{bmatrix} \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,4)} & 1 & 0 \\ \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,4)} & \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,4)} & \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,4)} \\ 0 & 0 & \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,4)} \end{bmatrix}.$$

$det\, H_2 = \frac{\mathtt{h}(G,2)\mathtt{h}(G,3)-\mathtt{h}(G,1)\mathtt{h}(G,4)}{(\mathtt{h}(G,4))^2}$ and $det\, H_3 = \mathtt{h}(G,1)\frac{\mathtt{h}(G,2)\mathtt{h}(G,3)-\mathtt{h}(G,1)\mathtt{h}(G,4)}{(\mathtt{h}(G,4))^3}$. It is clear that H_2 and H_3 has positive determinant if and only if

$$\mathtt{h}(G,2)\mathtt{h}(G,3) - \mathtt{h}(G,1)\mathtt{h}(G,4) > 0.$$

Therefore by Routh-Hurwitz criteria 1.2.7 all the roots of the polynomial $P(x)$ are negative or have negative real part if and only if

$$\mathtt{h}(G,2)\mathtt{h}(G,3) > \mathtt{h}(G,1)\mathtt{h}(G,4).$$

This completes the proof. $\square$

Corollary 6.4.25. *The star-like tree graph $S_{(n,n,\ldots,n)_{m\text{-times}}}$ is* $\mathtt{h}$*-stable for every natural number n, m.*

Proof. For $m, n = 1$, the only Hosoya root is zero. For $m, n \geq 2$, the diameter of the graph $S_{(n,n,\ldots,n)_{\text{m-times}}}$ is 4. Therefore by Theorem 6.4.24, it suffices to prove that $\mathtt{h}(G,2)\mathtt{h}(G,3) > \mathtt{h}(G,1)\mathtt{h}(G,4)$. By a simple calculation we have

$$\begin{aligned} \mathtt{h}(G,2)\mathtt{h}(G,3) - \mathtt{h}(G,1)\mathtt{h}(G,4) &= \tfrac{m^2(n-1)(m-1)^2}{2} \\ &> 0 \qquad \text{(since } m,n \geq 2\text{)}. \end{aligned}$$

Therefore for every natural number n and m, the nonzero Hosoya roots of the star-like tree graph $S_{(n,n,\ldots,n)_{\text{m-times}}}$ lie in the left half-plane, that is, $S_{(n,n,\ldots,n)_{\text{m-times}}}$ is $\mathtt{h}$-stable. □

Theorem 6.4.26. *Let G be a graph of diameter 7. Then G^2 is $\mathtt{h}$-stable if and only if*

$$[\mathtt{h}(G,3) + \mathtt{h}(G,4)]\,[\mathtt{h}(G,5) + \mathtt{h}(G,6)] > [\mathtt{h}(G,1) + \mathtt{h}(G,2)]\,\mathtt{h}(G,7).$$

Proof. By Lemma 6.1.25 the diameter of G^2 is 4. By Theorem 6.1.26 we have $\mathtt{h}(G^2,1) = \mathtt{h}(G,1) + \mathtt{h}(G,2)$, $\mathtt{h}(G^2,2) = \mathtt{h}(G,3) + \mathtt{h}(G,4)$, $\mathtt{h}(G^2,3) = \mathtt{h}(G,5) + \mathtt{h}(G,6)$, $\mathtt{h}(G^2,4) = \mathtt{h}(G,7)$. So the result follows from Theorem 6.4.24. □

Theorem 6.4.27. *Let G be a graph of diameter 8. Then G^2 is $\mathtt{h}$-stable if and only if*

$$[\mathtt{h}(G,3) + \mathtt{h}(G,4)]\,[\mathtt{h}(G,5) + \mathtt{h}(G,6)] > [\mathtt{h}(G,1) + \mathtt{h}(G,2)]\,[\mathtt{h}(G,7) + \mathtt{h}(G,8)]\,.$$

Proof. The proof is similar to the proof the Theorem 6.4.26. □

Next theorem gives a necessary and sufficient condition for graphs having diameter 5 is $\mathtt{h}$-stable

Theorem 6.4.28. *Let G be a graph of diameter 5. Then the graph G is $\mathtt{h}$-stable if and only if*

$$\mathtt{h}(G,2)\,[\mathtt{h}(G,3)\mathtt{h}(G,4) - \mathtt{h}(G,2)\mathtt{h}(G,5)] > \mathtt{h}(G,1)(\mathtt{h}(G,4))^2.$$

Proof. Suppose the graph G is $\mathtt{h}$-stable. Then by Routh-Hurwitz criteria 1.2.7, the Hurwitz matrices of $P(x) = x^4 + \frac{\mathtt{h}(G,4)}{\mathtt{h}(G,5)}x + \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,5)}x + \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,5)}x + \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,5)}$ have positive determinant. In particular,

$$\det H_3 = \begin{vmatrix} \frac{\mathtt{h}(G,4)}{\mathtt{h}(G,5)} & 1 & 0 \\ \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,4)}{\mathtt{h}(G,5)} \\ 0 & \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,5)} \end{vmatrix} > 0$$

This implies that $\mathtt{h}(G,2)\left[\mathtt{h}(G,3)\mathtt{h}(G,4)-\mathtt{h}(G,2)\mathtt{h}(G,5)\right] > \mathtt{h}(G,1)(\mathtt{h}(G,4))^2$. Conversely, suppose that the inequality

$$\mathtt{h}(G,2)\left[\mathtt{h}(G,3)\mathtt{h}(G,4)-\mathtt{h}(G,2)\mathtt{h}(G,5)\right] > \mathtt{h}(G,1)(\mathtt{h}(G,4))^2 \tag{6.1}$$

is hold. Then

$$\frac{\mathtt{h}(G,2)\left[\mathtt{h}(G,3)\mathtt{h}(G,4)-\mathtt{h}(G,2)\mathtt{h}(G,5)\right]-\mathtt{h}(G,1)(\mathtt{h}(G,4))^2}{(\mathtt{h}(G,5))^3} > 0.$$

That is, det $H_3 > 0$. Also, the Hurwitz matrix $H_4 = \begin{bmatrix} \frac{\mathtt{h}(G,4)}{\mathtt{h}(G,5)} & 1 & 0 & 0 \\ \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,4)}{\mathtt{h}(G,5)} & 1 \\ 0 & \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,5)} \\ 0 & 0 & 0 & \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,5)} \end{bmatrix}$
and

$$\det\ H_4 = \frac{\mathtt{h}(G,1)}{\mathtt{h}(G,5)}\ \det\ H_3.$$

Therefore det $H_4 > 0$. Finally, by the inequality (6.1), we have

$$\left[\mathtt{h}(G,3)\mathtt{h}(G,4)-\mathtt{h}(G,2)\mathtt{h}(G,5)\right] > 0.$$

This implies that

$$\det\ H_2 = \begin{vmatrix} \frac{\mathtt{h}(G,4)}{\mathtt{h}(G,5)} & 1 \\ \frac{\mathtt{h}(G,2)}{\mathtt{h}(G,5)} & \frac{\mathtt{h}(G,3)}{\mathtt{h}(G,5)} \end{vmatrix} = \frac{\mathtt{h}(G,3)\mathtt{h}(G,4)-\mathtt{h}(G,2)\mathtt{h}(G,5)}{(\mathtt{h}(G,5))^2} > 0.$$

Therefore by Routh-Hurwitz criteria 1.2.7, all the nonzero Hosoya roots of G lie in the left half-plane, that is the graph G is h-stable. $\square$

Next two theorems follows from Lemma 6.1.25 and Theorem 6.4.28.

Theorem 6.4.29. *Let G be a graph of diameter 9. Then G^2 is* h*-stable if and only if*

$$[\mathtt{h}(G,5)+\mathtt{h}(G,6)][\mathtt{h}(G,7)+\mathtt{h}(G,8)-[\mathtt{h}(G,3)+\mathtt{h}(G,4)]\mathtt{h}(G,9) > \frac{[\mathtt{h}(G,1)+\mathtt{h}(G,2)][(\mathtt{h}(G,7)+\mathtt{h}(G,8)]^2}{\mathtt{h}(G,3)+\mathtt{h}(G,4)}.$$

Theorem 6.4.30. *Let G be a graph of diameter 10. Then G^2 is* h*-stable if and* *only if*

$$[\mathtt{h}(G,5)+\mathtt{h}(G,6)][\mathtt{h}(G,7)+\mathtt{h}(G,8)-[\mathtt{h}(G,3)+\mathtt{h}(G,4)][\mathtt{h}(G,9)+\mathtt{h}(G,10)] > \frac{[\mathtt{h}(G,1)+\mathtt{h}(G,2)][(\mathtt{h}(G,7)+\mathtt{h}(G,8)]^2}{\mathtt{h}(G,3)+\mathtt{h}(G,4)}.$$

Theorem 6.4.31. *Let G be a graph of diameter D. If the graph G is* h*-stable, then*

$$\mathtt{h}(G,D-1)\mathtt{h}(G,D-2) > \mathtt{h}(G,D-3)\mathtt{h}(G,D).$$

Proof. Proof follows by Routh-Hurwitz criteria 1.2.7. □

Remark 6.4.32. *Consider the Hosoya polynomial $\mathtt{H}(B_{2,3,4},x) = 2x^5 + 3x^4 + 4x^3 + 5x^2 + 7x$. $\mathtt{h}(G,D-1)\mathtt{h}(G,D-2) = 3\times 4 > 5\times 2 = \mathtt{h}(G,D-3)\mathtt{h}(G,D)$. But $\mathtt{H}(B_{2,3,4},x)$ has roots in the right half-plane. That is, the generalized barbell graph $B_{2,3,4}$ is* h*-unstable, that is, the converse of Theorem 6.4.31 need not be true.*

Next two theorems follows by Theorem 6.1.26 and Routh-Hurwitz criteria 1.2.7.

Theorem 6.4.33. *Let G be a graph of diameter $2D-1$. If the graph G^2 is* h*-stable, then*

$$\frac{\mathtt{h}(G,2D-2)+\mathtt{h}(G,2D-3)}{\mathtt{h}(G,2D-6)+\mathtt{h}(G,2D-7)} > \frac{\mathtt{h}(G,2D-1)}{\mathtt{h}(G,2D-4)+\mathtt{h}(G,2D-5)}.$$

Theorem 6.4.34. *Let G be a graph of diameter $2D$. If the graph G^2 is* h*-stable, then*

$$\frac{\mathtt{h}(G,2D-2)+\mathtt{h}(G,2D-3)}{\mathtt{h}(G,2D-6)+\mathtt{h}(G,2D-7)} > \frac{\mathtt{h}(G,2D)+\mathtt{h}(G,2D-1)}{\mathtt{h}(G,2D-4)+\mathtt{h}(G,2D-5)}.$$

Now we discuss some h-unstable graphs.

Theorem 6.4.35. *The path graph P_n is* h*-unstable for $n \geq 6$.*

Proof. We have the Hosoya polynomial of the path graph P_n is

$$\mathtt{H}(P_n,x) = x^{n-1} + 2x^{n-2} + \ldots + (n-1)x.$$

It suffices to prove that for $n \geq 6$, $P(x) = x^{n-2} + 2x^{n-3} + \ldots + (n-1)$ has roots in right half-plane. It is easy to see that, the determinant of Hurwitz matrix H_3

is negative if $n = 6$ and zero if $n \geq 7$. Therefore by Routh-Hurwitz criteria 1.2.7, the path P_n has roots in right half-plane for $n \geq 6$. That is, the path graph P_n is h-unstable for $n \geq 6$. $\square$

Theorem 6.4.36. *The square of the path graphs P_{2n} and P_{2n+1} are* h*-unstable for $n \geq 5$.*

Proof. From Theorems 6.1.36 and 6.1.35 it is enough to prove that

$$\begin{aligned} P(x) = &\quad x^{n-1} + 5x^{n-2} + 9x^{n-3} + \ldots + (4n-7)x + 4n - 3 \text{ and} \\ Q(x) = &\quad x^{n-1} + \tfrac{7}{3}x^{n-2} + \tfrac{11}{3}x^{n-3} + \ldots + \tfrac{4n-5}{3}x + \tfrac{4n-1}{3} \end{aligned}$$

have roots in the right half plane. In the case of $P(x)$, the determinant of Hurwitz matrix H_3 is negative if $n = 5$, the determinant of Hurwitz matrix H_4 is negative if $n = 6, 7$ and the determinant of Hurwitz matrix H_4 is zero if $n \geq 8$. Similarly in the case of $Q(x)$, the determinant of Hurwitz matrix H_3 is negative if $n = 5$, the determinant of Hurwitz matrix H_4 is negative if $n = 6$, the determinant of Hurwitz matrix H_5 is negative if $n = 7$ and the determinant of Hurwitz matrix H_4 is zero if $n \geq 8$. Therefore by Routh-Hurwitz criteria 1.2.7, we have the result. $\square$

Theorem 6.4.37. *The cycle graph C_n is* h*-unstable for $n \geq 10$.*

Proof. We have the Hosoya polynomials of the cycle graphs C_{2n} and C_{2n+1} are $\mathtt{H}(C_{2n}, x) = 2n(x + x^2 + \ldots + x^{n-1}) + nx^n$ and $\mathtt{H}(C_{2n+1}, x) = (2n+1)(x + x^2 + \ldots + x^n)$. It suffices to prove that for $n \geq 5$, $P(x) = x^{n-1} + 2x^{n-2} + \ldots + 2x + 2$ and $Q(x) = x^{n-2} + x^{n-3} + \ldots + x + 1$ has Hosoya roots in the right half-plane. If $n = 5$, the determinant of the Hurwitz matrix H_3 of $P(x)$ is negative and the determinant of the Hurwitz matrix H_3 of $Q(x)$ is zero. If $n > 5$, the determinant of Hurwitz matrix H_3 of $P(x)$ is zero and the determinant of Hurwitz matrix H_2 of $Q(x)$ is zero. Therefore by Routh-Hurwitz criteria 1.2.7, the cycle graph C_n has Hosoya roots in the right half-plane for $n \geq 10$. That is, the cycle graph C_n is h-unstable for $n \geq 10$. $\square$

Theorem 6.4.38. *The square of the cycle graphs C_{4n-1} and C_{4n} are* h*-unstable for $n \geq 5$.*

Proof. We have by Theorem 6.1.37 and Theorem 6.1.38 it is enough to prove that

$$\begin{aligned} P(x) = &\quad x^{n-1} + 2\left(x^{n-2} + x^{n-3} + \ldots + x + 1\right) \text{ and} \\ Q(x) = &\quad x^{n-1} + \tfrac{4}{3}\left(x^{n-2} + x^{n-3} + \ldots + x + 1\right) \end{aligned}$$

have roots in the right half plane. In the both case the determinant of Hurwitz matrix H_3 is negative if $n = 5$ and zero if $n > 5$. Therefore by Routh-Hurwitz criteria 1.2.7, we have the result. $\square$

Theorem 6.4.39. *The square of the cycle graphs C_{4n+1} and C_{4n+2} are* h*-unstable for $n \geq 4$.*

Proof. We have by Theorem 6.1.39 and Theorem 6.1.40 it is enough to prove that

$$\begin{aligned} P(x) = &\quad x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x + 1 \text{ and} \\ Q(x) = &\quad x^n + 4\left(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x + 1\right) \end{aligned}$$

have roots in the right half plane. If $n \geq 4$ the determinant of Hurwitz matrix H_2 of $P(x)$ is zero. Also the determinant of Hurwitz matrix H_3 of $Q(x)$ is negative if $n = 4$ and zero if $n > 4$. Therefore by Routh-Hurwitz criteria 1.2.7, we have the result. $\square$

Theorem 6.4.40. *The graph $B_{m,m,n}$ is* h*-unstable for $n \geq 6$.*

Proof. We have $D(B_{m,m,n}) \geq 6$ for $n \geq 6$. If $m = 1, 2$, $B_{1,1,n}$ and $B_{2,2,n}$ are the path graphs P_n and P_{n+2} respectively. Then by Theorem 6.4.35 we have the result. Suppose $m > 2$, by Theorem 6.4.31, it is enough to show that

$$\mathtt{h}(G,n)\mathtt{h}(G,n-1) - \mathtt{h}(G,n-2)\mathtt{h}(G,n+1) < 0.$$

By a simple calculation, we have

$$\begin{aligned} \mathtt{h}(G,n)\mathtt{h}(G,n-1) - \mathtt{h}(G,n-2)\mathtt{h}(G,n+1) &= -2(m^3 - 4m^2 + 4m - 1) \\ &= -2(m-1)\left(m(m-3)+1\right) \\ &< 0 \qquad \text{for } m > 2. \end{aligned}$$

This completes the proof. $\square$

Theorem 6.4.41. *The graph $B_{3,m,n}$is* h*-unstable for $m, n \geq 3$.*

Proof. By Theorem 6.1.20, the Hosoya polynomial $\mathtt{H}(B_{3,m,n}, x) = \mathtt{H}(K_3, x) + \mathtt{H}(K_m, x) + \mathtt{H}(P_n, x) + (m+1)\sum_{i=2}^{n} x^i + 2(m-1)x^{n+1}$. If $m, n \geq 3$, then the diameter of the graph $B_{3,m,n}$ is greater than 3. By Theorem 6.4.31, it is enough to show that $\mathtt{h}(G,n)\mathtt{h}(G,n-1) - \mathtt{h}(G,n-2)\mathtt{h}(G,n+1) < 0$. If $n \geq 4$, by a simple calculation, we have

$$\begin{aligned}\mathtt{h}(G,n)\mathtt{h}(G,n-1)-\mathtt{h}(G,n-2)\mathtt{h}(G,n+1) &= (m+1)(m+2)-2(m-1)(m+3)\\ &= -m(m+1)+8\\ &< 0 \qquad \text{for } m\geq 3.\end{aligned}$$

If $n=3$, we have

$$\begin{aligned}\mathtt{h}(G,n)\mathtt{h}(G,n-1)-\mathtt{h}(G,n-2)\mathtt{h}(G,n+1) &= (m+1)(m+2)-m(m-1)^2-10(m-1)\\ &= -(m-2)\,(m(m-1)+6)\\ &< 0 \qquad \text{for } m\geq 3.\end{aligned}$$

This completes the proof. □

Theorem 6.4.42. *The square of the graph $B_{3,3,2n-1}$ is* h*-unstable for $n\geq 4$..*

Proof. By Theorem 6.1.45 it is enough to prove that

$$P(x)=x^{n-1}+\frac{11}{8}x^{n-2}+\frac{15}{8}x^{n-3}+\ldots+\frac{4n-1}{8}x+\frac{4n+5}{8}$$

has roots in the right half plane for $n\geq 4$. It is easy to see that the determinant of Hurwitz matrix H_3 is negative for all $n\geq 4$ except 6 and the determinant of Hurwitz matrix H_4 is negative for $n=6$. Therefore by Routh-Hurwitz criteria 1.2.7, we have the result. □

Theorem 6.4.43. *The square of the graph $B_{m,m,2n-1}$ is* h*-unstable for $m,n\geq 4$..*

Proof. By Theorem 6.4.31, it is enough to show that

$$\mathtt{h}(G,n-1)\mathtt{h}(G,n-2)-\mathtt{h}(G,n-3)\mathtt{h}(G,n)<0.$$

If $n=4$ we have

$$\begin{aligned}\mathtt{h}(G^2,3)\mathtt{h}(G^2,2)-\mathtt{h}(G^2,1)\mathtt{h}(G^2,4) &= (4m-1)(4m+3)-(m^2+3m+7)(m^2-1)\\ &= m^2(10-m^2)+m(11-3m^2)+4\\ &< 0 \qquad (\text{since } m\geq 4).\end{aligned}$$

If $n>4$ we have

$$\begin{aligned}\mathtt{h}(G^2,n)\mathtt{h}(G^2,n-1)-\mathtt{h}(G^2,n-2)\mathtt{h}(G^2,n+1) &= (4m-1)(4m+3)-(4m+7)(m^2-1)\\ &= m^2(9-4m)+4(3m+1)\\ &< 0 \qquad (\text{since } m\geq 4).\end{aligned}$$

This completes the proof. □

Theorem 6.4.44. *The square of the graph $B_{4,4,2n}$ is $\mathtt{h}$-unstable for $n \geq 4$..*

Proof. By Theorem 6.1.46, it is enough to show that

$$P(x) = x^n + \frac{13}{9}x^{n-1} + \ldots + \frac{4n+5}{9}x + \frac{4n+15}{9}$$

has roots in the right half plane for $n \geq 4$. It is easy to see that the determinant of Hurwitz matrix H_3 is negative for all $n \geq 4$ except 5 and the determinant of Hurwitz matrix H_4 is negative for $n = 5$. Therefore by Routh-Hurwitz criteria 1.2.7, we have the result. □

Theorem 6.4.45. *The square of the graph $B_{m,m,2n}$ is $\mathtt{h}$-unstable for $m \geq 5$ for $n \geq 3$.*

Proof. By Theorem 6.4.31, it is enough to show that

$$\mathtt{h}(G^2, n)\mathtt{h}(G^2, n-1) - \mathtt{h}(G^2, n-2)\mathtt{h}(G^2, n+1) < 0.$$

If $n = 3$ we have

$$\begin{aligned} \mathtt{h}(G^2,3)\mathtt{h}(G^2,2) - \mathtt{h}(G^2,1)\mathtt{h}(G^2,4) &= (4m-3)(4m+4) - (m^2+3m+5)(m-1)^2 \\ &= m^2(16-m^2) + m(1-m^2) - 8 \\ &< 0 \qquad \text{(since } m \geq 5\text{)}. \end{aligned}$$

If $n > 3$ we have

$$\begin{aligned} \mathtt{h}(G^2,n)\mathtt{h}(G^2,n-1) - \mathtt{h}(G^2,n-2)\mathtt{h}(G^2,n+1) &= (4m-3)(4m+4) - (4m+5)(m-1)^2 \\ &= -4(m^3+2) + m(19m-2) \\ &< 0 \qquad \text{(since } m \geq 5\text{)}. \end{aligned}$$

This completes the proof. □

Theorem 6.4.46. *The square of the graph $B_{4,m,2n}$ is $\mathtt{h}$-unstable for $m \geq 5$, $n \geq 3$.*

Proof. By Theorem 6.4.31, it is enough to show that

$$\mathtt{h}(G^2, n)\mathtt{h}(G^2, n-1) - \mathtt{h}(G^2, n-2)\mathtt{h}(G^2, n+1) < 0.$$

If $n = 3$ we have

$$\begin{aligned}
\mathtt{h}(G^2,3)\mathtt{h}(G^2,2) - \mathtt{h}(G^2,1)\mathtt{h}(G^2,4) &= (2m+5)(2m+9) - 3(m^2+3m+38)(m-1)\\
&= m^2(2-m) - 7m + 83\\
&< 0 \qquad (\text{since } m \geq 5).
\end{aligned}$$

If $n > 3$ we have

$$\begin{aligned}
\mathtt{h}(G^2,n)\mathtt{h}(G^2,n-1) - \mathtt{h}(G^2,n-2)\mathtt{h}(G^2,n+1) &= (2m+5)(2m+9) - 3(2m+13)(m-1)\\
&= 44 - 38m - 2m^2\\
&< 0 \qquad (\text{since } m \geq 5).
\end{aligned}$$

This completes the proof. □

Remark 6.4.47. *The Theorem 6.4.41 shows that for any natural number $n \geq 4$ there is a graph G with diameter n such that G is* h*-unstable.*

Theorem 6.4.48. *The lollipop graph $L_{m,n}$is* h*-unstable for $n \geq 6$.*

Proof. We have

$$\mathtt{H}(L_{m,n},x) = (\binom{m}{2} + 1)x + \mathtt{H}(P_n,x) + m\sum_{i=2}^{n} x^i + (m-1)x^{n+1}.$$

Observe that $L_{1,n}$ is the path graph P_{n+1}. Then by Theorem 6.4.35 we have the result. Suppose $m \geq 2$, then the Hurwitz matrix H_3 is

$$\begin{aligned}
|H_3| &= \begin{vmatrix} \frac{m}{m-1} & 1 & 0\\ \frac{m+2}{m-1} & \frac{m+1}{m-1} & \frac{m}{m-1}\\ \frac{m+4}{m-1} & \frac{m+3}{m-1} & \frac{m+2}{m-1} \end{vmatrix}\\
&= \frac{-2m}{m-1}\\
&< 0 \qquad \text{for } m \geq 2.
\end{aligned}$$

This completes the proof. □

Theorem 6.4.49. *The square of the lollipop graph $L_{m,2n+1}$ is* h*-unstable for $n \geq 9$.*

Proof. $L_{1,n}$ is the path graph P_{n+1}. Then by Theorem 6.4.36 we have the result. If $m \geq 2$, by theorem 6.1.47, it is enough to show that

$$P(x) = x^n + \frac{2m+3}{2m-1}x^{n-1} + \frac{2m+7}{2m-1}x^{n-2} + \ldots + \frac{2m+4n-5}{2m-1}x + \frac{m^2+m+8n}{2(2m-1)}$$

has roots in the right half plane for $n \geq 9$. If $n \geq 9$, the determinant of the Hurwitz matrix H_4 of $P(x)$ is

$$\begin{aligned} |H_4| &= \begin{vmatrix} \frac{2m+3}{2m-1} & 1 & 0 & 0 \\ \frac{2m+11}{2m-1} & \frac{2m+7}{2m-1} & \frac{2m+3}{2m-1} & 1 \\ \frac{2m+19}{2m-1} & \frac{2m+15}{2m-1} & \frac{2m+11}{2m-1} & \frac{2m+7}{2m-1} \\ \frac{2m+27}{2(2m-1)} & \frac{2m+23}{2m-1} & \frac{2m+19}{2m-1} & \frac{2m+15}{2m-1} \end{vmatrix} \\ &= 0 \qquad \text{for } m \geq 2. \end{aligned}$$

This completes the proof. ☐

Theorem 6.4.50. *The square of the lollipop graph $L_{m,2n}$ is* h*-unstable for $n \geq 11$.*

Proof. The proof is similar to the proof of Theorem 6.4.49. ☐

Theorem 6.4.51. *The graph $T^*_{m,m,n}$ is* h*-unstable for $m, n \geq 4$.*

Proof. The proof is similar to the proof of the Theorem 6.4.40. ☐

Theorem 6.4.52. *The graph $T^*_{3,3,n}$ is* h*-unstable for $n \geq 4$.*

Proof. It suffices to prove that

$$P(x) = x^n + x^{n-1} + \frac{5}{4}x^{n-2} + \frac{6}{4}x^{n-3} + \ldots + \frac{n+1}{4}x^2 + \frac{n+4}{4}x + \frac{n+3}{4}$$

has roots in right half-plane for $n \geq 4$. It is easy to see that, the determinant of Hurwitz matrix H_2 is negative for all $n \geq 5$ and the determinant of Hurwitz matrix H_3 is negative if $n = 4$. Therefore by Routh-Hurwitz criteria 1.2.7, the graph $T^*_{3,3,n}$ has Hosoya roots in the right half-plane for $n \geq 4$. That is, the graph $T^*_{3,3,n}$ is h-unstable for $n \geq 4$. ☐

From the facts that the square of the generalized barbell graph $B_{l,m,n}$ and the square of the generalized star-tree graph $T^*_{l,m,n}$ are isomorphic we have the following five theorems.

Theorem 6.4.53. *The square of the graph $T^*_{3,3,2n-1}$ is* h*-unstable for $n \geq 4$.*

Theorem 6.4.54. *The square of the graph $T^*_{m,m,2n-1}$ is* h*-unstable for $m, n \geq 4$.*

Theorem 6.4.55. *The square of the graph $T^*_{4,4,2n}$ is* h*-unstable for $n \geq 3$.*

Theorem 6.4.56. *The square of the graph* $T^*_{m,m,2n}$ *is* h*-unstable for* $m \geq 5$, $n \geq 3$.

Theorem 6.4.57. *The square of the graph* $T^*_{4,m,2n}$ *is* h*-unstable for* $m \geq 5$ *adn* $n \geq 3$.

Theorem 6.4.58. *Let* T^*_n *as in Theoerem 6.1.18. Then* T^*_n *is* h*-unstable for* $n \geq 5$.

Proof. It suffices to prove that

$$P(x) = x^{n-1} + \frac{3}{2}x^{n-2} + \frac{4}{2}x^{n-3} + \ldots + \frac{n-1}{2}x^2 + \frac{n+1}{2}x + \frac{n+1}{2}$$

has roots in right half-plane for $n \geq 5$. It is easy to see that, the determinant of Hurwitz matrix H_3 is negative for all $n \geq 5$ except 7 and the determinant of Hurwitz matrix H_5 is negative if $n = 7$. Therefore by Routh-Hurwitz criteria 1.2.7, the graph T^*_n has roots in the right half-plane for $n \geq 5$. That is, T^*_n is h-unstable for $n \geq 5$. $\square$

The Hosoya roots of the graph T^*_n for $1 \leq n \leq 100$ are shown in Figure 6.16.

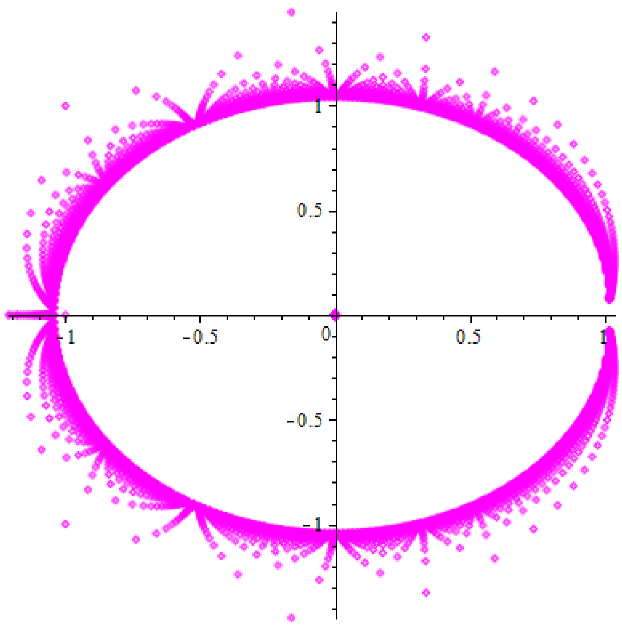

Figure 6.16: Hosoya roots of T^*_n for $1 \leq n \leq 100$.

Theorem 6.4.59. *The ladder graph L_n is* h*-unstable for $n \geq 5$.*

Proof. It suffices to prove that for $n \geq 5$, $P(x) = x^{n-1} + 3x^{n-2} + 5x^{n-3} + \ldots + (2n-5)x^2 + (2n-3)x + \frac{3n-2}{2}$ has Hosoya roots in the right half-plane. Observe that the determinant of Hurwitz matrix H_3 is negative if $n = 5, 6$ and the determinant of Hurwitz matrices H_5 and H_6 are negative if $n = 7$ and $n = 8$ respectively. Also we can observe that the determinant of the Hurwitz matrix H_4 is zero if $n \geq 9$. Therefore by Routh-Hurwitz criteria 1.2.7, for $n \geq 5$, the ladder graph L_n has Hosoya roots in the right half-plane, that is, the ladder graph L_n is h-unstable. □

Theorem 6.4.60. *The square of the ladder graph L_n is* h*-unstable for $n \geq 10$.*

Proof. The proof is similar to the proof of the Theorem 6.4.59. □

Remark 6.4.61. *We conjectured that all the graphs with diameter greater than 6 are* h*-unstable.*

CHAPTER 7

MISCELLANEOUS RESULTS

This chapter has two sections. In the first section we state some general properties of graph polynomials which are discussed in the previous chapters. In the second section we state some properties of distance-k domination polynomials.

7.1 General results on graph polynomials

Observe that $(0, \infty)$ is a zero-free interval for graph polynomials which we have discussed in the previous chapters.

Lemma 7.1.1. *Let $f(x)$ be a polynomial with non-negative integer coefficients. Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square. If $x = a + b\sqrt{r}$ is a root of $f(x)$, then so is $x^* = a - b\sqrt{r}$.*

Lemma 7.1.2. *Let $f(x)$ be a polynomial with non-negative integer coefficients. Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be a root of $f(x)$.*

Theorem 7.1.3. *Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be a domination root.*

Corollary 7.1.4. *Let b be a rational number, and let r be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a domination root.*

Theorem 7.1.5. *Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be a distance-k domination root.*

Corollary 7.1.6. *Let b be a rational number, and let r be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a distance-k domination root.*

Theorem 7.1.7. *Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be a total domination root.*

Corollary 7.1.8. *Let b be a rational number, and let r be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a total domination root.*

Theorem 7.1.9. *Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be a distance-k total domination root.*

Corollary 7.1.10. *Let b be a rational number, and let r be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a distance-k total domination root.*

Theorem 7.1.11. *Suppose that a, b are rational numbers, $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 0$. Then $-a - |b|\sqrt{r}$ can not be Hosoya root.*

Corollary 7.1.12. *Let b be a rational number, and let r be a positive rational number such that $\sqrt{r}$ is irrational. Then $-|b|\sqrt{r}$ can not be a Hosoya root.*

Let τ be the golden ratio. Next we will prove that $-\tau^n$ for odd n, can not be a domination or distance-k domination or total domination or distance-k total domination or Hosoya root. Here we need some relations between golden ratio τ and Fibonacci numbers F_n.

Theorem 7.1.13 (see [43]). *For every natural number n,*

$$F_n = \frac{1}{\sqrt{5}}(\tau^n - (1-\tau)^n).$$

Theorem 7.1.14 (see [43])**.** *For every natural number n, $\frac{F_n}{F_{n-1}} < \tau$, if n is even and $\frac{F_n}{F_{n-1}} > \tau$, if n is odd.*

Theorem 7.1.15 (Cassini's formula [43])**.** *. For every natural number n,*

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

Theorem 7.1.16 (see [43])**.** *For every $n \geq 2$, $\tau^n = F_n\tau + F_{n-1}$.*

Now we are ready to prove the following lemma.

Lemma 7.1.17. *Let n be an odd natural number and let $f(x)$ be a polynomial with non-negative integer coefficients. Then $-\tau^n$ can not be a root of $f(x)$.*

Proof. Since coefficients of $f(x)$ are non-negative, $f(x)$ has no roots in $(0, \infty)$. For $n = 1$, it follows from the fact that $\frac{1-\sqrt{5}}{2} < 0$. For odd $n \geq 2$,

$$\tau^n = F_n\tau + F_{n-1} = \left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right).$$

Let G be a graph with $-\tau^n$ be a distance-k domination root. Then,

$$f\left(G, -\left(\frac{F_n + 2F_{n-1}}{2}\right) - \left(\frac{\sqrt{5}F_n}{2}\right)\right) = 0.$$

Then,

$$f\left(G, -\left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right)\right) = 0.$$

But

$$\begin{aligned} -\left(\frac{F_n + 2F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right) &= \left(\frac{F_{n+1} + F_{n-1}}{2}\right) + \left(\frac{\sqrt{5}F_n}{2}\right) \\ &= \tau^{-1}F_n - F_{n-1}. \end{aligned}$$

As n is odd,

$$\begin{aligned} \frac{F_{n-1}}{F_{n-2}} &< \tau < \frac{F_n}{F_{n-1}} \\ \frac{F_{n-1}}{F_n} &< \tau^{-1} < \frac{F_{n-2}}{F_{n-1}} \end{aligned}$$

$$
\begin{aligned}
&F_{n-1} < \tau^{-1}F_n < \frac{F_{n-2}F_n}{F_{n-1}}\\
&0 < \tau^{-1}F_n - F_{n-1} < \frac{F_{n-2}F_n - F_{n-1}^2}{F_{n-1}}\\
&0 < \tau^{-1}F_n - F_{n-1} < \frac{1}{F_{n-1}}\\
&\tau^{-1}F_n - F_{n-1} \in \left(0, \frac{1}{F_{n-1}}\right),
\end{aligned}
$$

this is a contradiction. □

Theorem 7.1.18. *Let n be an odd natural number. Then $-\tau^n$ can not be a domination root.*

Theorem 7.1.19. *Let n be an odd natural number. Then $-\tau^n$ can not be a distance-k domination root.*

Theorem 7.1.20. *Let n be an odd natural number. Then $-\tau^n$ can not be a total domination root.*

Theorem 7.1.21. *Let n be an odd natural number. Then $-\tau^n$ can not be a distance-k total domination root.*

Theorem 7.1.22. *Let n be an odd natural number. Then $-\tau^n$ can not be a Hosoya root.*

7.2 Some properties of distance-k domination polynomial

We conclude this chapter by stating some properties of distance-k domination polynomial. A.E Brouwer [1] has shown that the number of dominating sets of any graph is odd. Thus by Theorem 3.1.10, we have the following theorem.

Theorem 7.2.1. *For every graph G the number of distance-k dominating set is odd. That is, $D^k(G, 1)$ is odd.*

Corollary 7.2.2. *Let G be a graph. Then for every odd integer n, $D^k(G, n)$ is odd.*

Proof. It follows from the fact that $D^k(G,m) \equiv D^k(G,n)(mod\ 2)$, for every odd integers m and n. $\square$

Corollary 7.2.3. *Every integer distance-k domination root of a graph is even.*

Theorem 7.2.4. *Let G be a graph. Then zero is the only distance-k domination root of G if and only if G is a null graph.*

Theorem 7.2.5. *There is no connected graph G such that*

$$\mathbb{Z}(D^k(G,x)) = \left\{0, \frac{-3 \pm \sqrt{5}}{2}\right\}.$$

Proof. Let G be a connected graph such that $\mathbb{Z}(D^k(G,x)) = \{0, \frac{-3\pm\sqrt{5}}{2}\}$. Then by Theorem 3.1.10, $\mathbb{Z}(D(G^k,x)) = \{0, \frac{-3\pm\sqrt{5}}{2}\}$. Then by (11) in Results 2.1.2 we have $G^k = H \circ \overline{K_2}$ for some graph H. But G^k has no leaf except when $G = K_1$ or K_2. So there exists no graph H such that $G^k = H \circ \overline{K_2}$, this is a contradiction. Therefore there is no such connected graph. $\square$

CHAPTER 8

GENERAL CONCLUSION AND OPEN PROBLEMS

In this chapter we summaries the thesis and we state some of the open problems and conjectures.

Summary and conclusion

In this thesis, we consider some graph polynomials, namely domination polynomial, total domination polynomial and Hosoya polynomial of a graph. We introduce two graph polynomials, namely distance-k domination polynomial and distance-k total domination polynomial of a graph which are analogue to domination polynomial and total domination polynomial respectively.

We prove that distance-k domination polynomial of a graph G is same as the domination polynomial of k^{th} power of G. Similarly we prove that distance-k total domination polynomial of a graph G is same as the total domination polynomial of k^{th} power of G. We find independently the distance-2 domination polynomial of some graphs and domination polynomial of the square of these graphs. Similarly we find independently the distance-2 total domination polynomial of some graphs and total domination polynomial of the square of these graphs.

We find the number of real domination roots, distance-k domination roots, total domination roots, distance-k total domination roots and Hosoya roots of some graphs. We identified some graphs such that all their domination roots, distance-k domination roots, total domination roots, distance-k total domination roots and Hosoya roots except zero are not real.

We obtained some bounds for domination roots, distance-k domination roots, total domination roots, distance-k total domination roots and Hosoya roots of some graphs.

We introduced a new concept, namely stability. This did not attract much attension in the literature. In various problems on stability one has to investigate whether all the roots of a given polynomial belong to the left half-plane, that is, whether the real parts of the roots are negative. The polynomials with this property are said to be stable. Using this stability concept we define the following:

(1) $\mathtt{d}$-stable graph,

(2) $\mathtt{d}^k$-stable graph,

(3) $\mathtt{d}_t$-stable graph,

(4) $\mathtt{d}_t^k$-stable graph,

(5) $\mathtt{h}$-stable graph.

We find some examples of $\mathtt{d}$-stable graph, $\mathtt{d}^k$-stable graph, $\mathtt{d}_t$-stable graph, $\mathtt{d}_t^k$-stable graph, $\mathtt{h}$-stable graph. Also we find some examples of $\mathtt{d}$-unstable graph, $\mathtt{d}^k$-unstable graph, $\mathtt{d}_t$-unstable graph, $\mathtt{d}_t^k$-unstable graph, $\mathtt{h}$-unstable graph. We prove some graphs are not $\mathtt{d}$-stable graph or $\mathtt{d}^k$-stable graph or $\mathtt{d}_t$-stable graph or $\mathtt{d}_t^k$-stable graph for all but finite values of n, where n is the order of graph by finding limits of roots of domination polynomial or distance-k domination polynomial or total domination polynomial or distance-k total domination polynomial of these graphs respectively. Using Routh-Hurwitz criteria 1.2.7, we prove some of the graphs are not $\mathtt{h}$-stable graphs.

Also we obtained some general properties of the graph polynomials which are studied in this thesis. We find some interesting results like "for odd n, $-\tau^n$ can not be a domination or distance-k domination or total domination or distance-k total domination or Hosoya root, where τ is the golden ratio." We prove that all the integer distance-k domination roots are even. Also we prove that there is no connected graphs G such that

$$\mathbb{Z}(D^k(G,x)) = \left\{0, \frac{-3 \pm \sqrt{5}}{2}\right\}.$$

Further scope of research

Now we state some of the open problems and conjectures.

Domination polynomial

First we state some the open problems and conjectures related to domination polynomial and domination roots of a graph.

Open problem 1. *What are the sharp bounds for the domination roots of the family of all graphs?*

Open problem 2. *Characterize all graphs with* $\mathtt{d}$*-number* 1.

Open problem 3. *Characterize all* $\mathtt{d}$*-stable graphs.*

Conjecture 1. *If* r *is an nonzero integer domination root of a graph* G*, then* $r = -2$.

Conjecture 2. *For all* n, $\mathtt{d}(K_{2n+1,2n+1}) = 3$.

Conjecture 3. *For all* n, $\mathtt{d}(G^3_{2n}) = 3$.

Conjecture 4. *The lollipop graph* $L_{n,1}$ *is* $\mathtt{d}$*-stable graph for all* n.

Conjecture 5. *The star graph* S_n *is* $\mathtt{d}$*-stable for all* n.

Conjecture 6. *The complete bipartite graph* $K_{m,n}$ *is* $\mathtt{d}$*-stable for all* m, n.

Conjecture 7. *The complete bipartite graph* $K_{n,n}$ *is* $\mathtt{d}$*-stable for all* n.

Conjecture 8. *The square of the generalized barbel graph* $B_{m,n,1}$ *is* $\mathtt{d}$*-stable graph for all* m, n.

Conjecture 9. *The square of the* n*-barbell graph* $B_{n,1}$ *is* $\mathtt{d}$*-stable graph for all* n.

Conjecture 10. *The square of the bi-star graph* $B_{(m,n)}$ *is* $\mathtt{d}$*-stable for all* m, n.

Conjecture 11. *The square of the corona* $K_n \circ K_n$ *is not* $\mathtt{d}$*-stable for all but finite values of* n.

Conjecture 12. *The square of the bipartite cocktail party graph* B_n *is not* $\mathtt{d}$*-stable for all but finite values of* n.

Distance-k domination polynomial

We state some open problems and conjectures related to distance-k domination polynomial and distance-k domination roots of a graph.

Open problem 4. *What are the sharp bounds for the distance-k domination roots of the family of all graphs?*

Open problem 5. *Characterize all graphs with* $\mathtt{d}^k$*-number* 1.

Open problem 6. *Characterize all* $\mathtt{d}^k$*-stable graphs.*

Conjecture 13. *If* r *is an nonzero integer distance-k domination root of a graph* G*, then* $r = -2$.

Conjecture 14. *The generalized barbel graph* $B_{m,n,1}$ *is* $\mathtt{d}^2$*-stable graph for all* m, n.

Conjecture 15. *The n-barbell graph* $B_{n,1}$ *is* $\mathtt{d}^2$*-stable graph for all* n.

Conjecture 16. *The bi-star graph* $B_{(m,n)}$ *is* $\mathtt{d}^2$*-stable for all* m, n.

Conjecture 17. *The corona* $K_n \circ K_n$ *is not* $\mathtt{d}^2$*-stable for all but finite values of* n.

Conjecture 18. *The bipartite cocktail party graph* B_n *is not* $\mathtt{d}^2$*-stable for all but finite values of* n.

Total domination polynomial

We state some open problems and conjectures related to total domination polynomial and total domination roots of a graph.

Open problem 7. *What are the sharp bounds for the total domination roots of the family of all graphs?*

Open problem 8. *Characterize all graphs with* $\mathtt{d}_t$*-number* 1.

Open problem 9. *Characterize all* $\mathtt{d}_t$*-stable graphs.*

Conjecture 19. *If* r *is an nonzero integer total domination root of a graph* G*, then* $r = -1$ *or* -2 *or* -3.

Conjecture 20. *For all n, $\mathtt{d}_t(G^3_{2n}) = 2$.*

Conjecture 21. *The Dutch windmill graph G^n_3 is $\mathtt{d}_t$-stable for all n.*

Conjecture 22. *The complete graph K_n is not $\mathtt{d}_t$-stable for all but finite values of n.*

Conjecture 23. *The bipartite cocktail party graph B_n is not $\mathtt{d}_t$-stable for all but finite values of n.*

Conjecture 24. *The square of the following graphs G_n are not $\mathtt{d}_t$-stable graphs for all but finite values of n, where n is the order of G_n.*

(1) Complete graph K_n.

(2) Complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$.

(3) Complete bipartite graph $K_{m,n}$.

(4) Star graph S_n.

(5) Wheel graph W_n.

(6) $H \vee G$.

(7) $K_m \square K_n$.

(8) Dutch windmill graph G^n_3.

(9) Lollipop graph $L_{n,1}$.

Distance-k total domination polynomial

We state some open problems and conjectures related to distance-k total domination polynomial and distance-k total domination roots of a graph.

Open problem 10. *What are the sharp bounds for the distance-k total domination roots of the family of all graphs?*

Open problem 11. *Characterize all graphs with $\mathtt{d}^k_t$-number 1.*

Open problem 12. *Characterize all $\mathtt{d}^k_t$-stable graphs.*

Conjecture 25. *If r is an nonzero integer distance-k total domination root of a graph G, then $r = -1$ or -2 or -3.*

Conjecture 26. *The following graphs G_n are not $\mathtt{d}_t^k$-stable graphs for all but finite values of n, where n is the order of G_n.*

(1) Complete graph K_n.

(2) Complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$.

(3) Complete bipartite graph $K_{m,n}$.

(4) Star graph S_n.

(5) Wheel graph W_n.

(6) $H \vee G$.

(7) $K_m \square K_n$.

(8) Dutch windmill graph G_3^n.

(9) Lollipop graph $L_{n,1}$.

Hosoya polynomial

We end this thesis by state some open problems and conjectures related to Hosoya polynomial and Hosoya roots of a graph.

Open problem 13. *What are the sharp bounds for the Hosoya roots of the family of all graphs?*

Open problem 14. *Characterize all graphs with* $\mathtt{h}$*-number* 1.

Open problem 15. *Characterize all* $\mathtt{h}$*-stable graphs.*

Conjecture 27. *Let G be a graph with diameter greater than 6. Then G is* $\mathtt{h}$*-unstable graph.*

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