Quasi Particle Models in Quark Gluon Plasma under Magnetic field, Coulomb field and Color field

Thesis submitted to the University of Calicut
in partial fulfillment of the requirements
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Doctor of Philosophy in Physics
by
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Under the guidance of
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Publications

International Journals

1. **K. Arjun**, A. M. Vinodkumar, and Vishnu Mayya Bannur, Running coupling constant in thermal ϕ^4 theory up to two loop order
   Phys. Rev. D **105**, 025023 (2022)

2. P. Jisha, A. M. Vinodkumar, S. Sanila, **K. Arjun**, B. R. S. Babu, J. Gehlot, S. Nath, N. Madhavan, Rohan Biswas, A. Parihari, A. Vinayak, Amritraj Mahato, E. Prasad, and A. C. Visakh, Role of positive transfer Q values in fusion cross sections for $^{18}O+^{182,184,186}W$ reactions
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3. **K. Arjun**, A. M. Vinodkumar, and Vishnu Mayya Bannur, Equation of state of quark gluon plasma in the presence of magnetic field with a modified liquid drop model (Submitted to The European Physical Journal C)

Papers Presented in Conferences

1. **K. Arjun**, A.M. Vinodkumar, and Vishnu Mayya Bannur, Running of Coupling Constant in $\lambda\phi^4$ theory using Finite Temperature Field Theory, XXIII DAE-BRNS High Energy Physics Symposium (December 10-14, 2018) (IIT-Madras)

2. **K. Arjun**, A.M. Vinodkumar, and Vishnu Mayya Bannur, Two Loop Order Running Coupling Constant in Toy Model Using Imaginary Time Formalism, XXIV DAE-BRNS Symposium on High Energy Physics (December 14-18, 2020) (Online, NISER, Odisha)

3. P. Jisha, A.M. Vinodkumar, S. Sanila, **K. Arjun**, B.R.S. Babu, J.Gehlot, S. Nath, N. Madhavan, Biswas Rohan, A. Parihari, A. Vinayak, Mahato Amritraj, A.C. Visakh, E. Prasad, Comparisons of evaporation residue cross-section of ^{16}O and ^{18}O induced reactions on W isotopes DAE Symp. Nucl. Phys. **65**, 271 (2021).

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Date: Arjun. K

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Acronyms and Abbreviations

QGP	Quark Gluon Plasma
QFT	Quantum Field Theory
QED	Quantum Electro Dynamics
ITF	Imaginary Time Formalism
MCE	Mayer's Cluster Expansion
EoS	Equation of State
SEMF	Semi-empirical mass formula
LHC	Large Hadron Collider
RHIC	Relativistic Heavy Ion Collider
Quasi particle	Quasiparticle

Symbols

h	Planck constant
$\hbar$	Reduced Planck constant ($\frac{h}{2\pi}$)
T	Temperature
k_B	Boltzmann constant
β	$(k_B T)^{-1}$
p,$\|p\|$	Magnitude of momentum
$\vec{p}$	Momentum (Vector)
$E^{\text{kin.}}$	Kinetic Energy
$\vec{r}_i$	Radial vector representing position of i^{th} particle
$U(r_{ij})$	Pair potential of particles i and j having relative separation $r_{ij} = \|\vec{r}_i - \vec{r}_j\|$.
$K_n(x)$	Modified Bessel function of second kind

Preface

In recent years, equation of state of quark gluon plasma in the presence of external magnetic field was investigated by many authors. Many of the phenomenological models have successfully described the equation of state of quark gluon plasma in the absence of a magnetic field. Some of these models have shown remarkable agreement with lattice data. In these models, certain quasiparticle phenomenological models have used coupling constants of QCD. Up to two loop order, these coupling constants exhibit a logarithmic dependence on temperature. When the temperature is below the critical temperature, some models fail to explain the lattice data. In our work, we obtain the equation of state of quark gluon plasma in the absence and presence of a magnetic field, from temperature lower than the critical temperature, to higher temperature range. Furthermore, our research introduces a novel method for determining the coupling constant in ϕ^4 theory.

The Chapters in the thesis can be classified into three. Chapter 2 to 4 is based on Mayer's cluster expansion. Chapter 5 is based on the quasiparticle model of VM Bannur in the context of quark matter at zero temperature. Chapter 6, is based on quantum field theory and thermal field theory.

In Chapter 1, we give a brief introduction to quark gluon plasma (QGP). In Chapter 2, we provide a general overview of Mayer's cluster expansion, which forms the foundation for the equations utilized in Chapter 3 and Chapter 4. .

We draw inspiration from Bannur's introduction of Mayer's cluster expansion to explain the equation of state (EoS) of the QGP. In Chapter 3, we combined this approach with the mathematical tools from the dimensional regularization method and developed a generalized formula for central potential of the polynomial form. Based on the famous semiempirical mass formula of nuclear physics, we developed a modified liquid drop model. Chapter 3 has focused on employing this modified liquid drop model to explain the equation of state of QGP across a wide temperature range.

In Chapter 4, we have extended the model to incorporate the influence of magnetic fields. This allowed us to compare the magnetized quarks lattice data with our model.

In Chapter 5 we use the quasiparticle model developed by VM Bannur to determine the equation of state of a quark star in the presence of a magnetic field at zero temperature. We compare the pressure, number and energy density ratios of quasi particles with that of free particles.

Furthermore, Chapter 6 introduces a new method, named as "Same Mass Scale and Coupling" (SMC) method, for deriving coupling constants in ϕ^4 theory. In this approach, the coupling constant and mass scale in both the imaginary time formalism (ITF) and non-thermal quantum field theory (QFT) are considered to be identical. By employing the conventional renormalization method and applying renormalization group equations (RGE) simultaneously to both ITF and QFT, we obtain the running mass and running coupling constant. The running coupling constant and running mass values obtained through SMC are in agreement with the expected behavior.

Finally, in Chapter 7, we outline our future plans and provide a comprehensive summary of the thesis.

Chapter 1

Introduction

1.1 General description of quasiparticles, plasma and screening

1.1.1 Quasiparticles

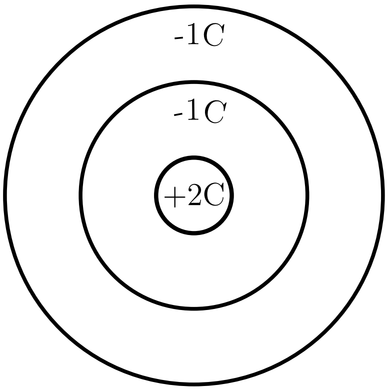
A quasiparticle is not an actual particle, like protons or electrons, but is more like a mathematical model that approximates a whole phenomenon as contributions from quasiparticles whose behaviour is represented by simple mathematical equations.
For example, in semiconductor physics, a hole is actually the absence of an electron. It is not a particle, but from a mathematical point of view, the hole can be considered a quasiparticle having the opposite charge of an electron. In solid state physics, the term electron-quasiparticle is widely used; it is essentially a particle-like mathematical approximation of an electron that is interacting with other forces within the solid. Similarly "*phonon*" is a quasiparticle associated with quantum mechanical vibrations in solids.

1.1.2 Plasma, Screening and Anti-Screening

It is possible to classify plasma as a quasi-neutral gas since it is a gas of charged and neutral particles that exhibits collective behavior. Assume that two charged spheres are immersed in plasma. The balls start to attract opposite charges around them. Assume that a layer of opposite charges accumulates around the spheres, which reduces the influence of the effective charge of the sphere over a finite distance. In order to understand the idea, let's take an exaggerated example, as shown below.

1.1.2. (a) Binary charges example (+ and - charges)

Consider a sphere, as shown in the figure below, having a centre charge of +2 C. Assume that two layers of particles with opposite charges are formed around the centre charge. Let the first and second layers contain the opposite charge of -1C.

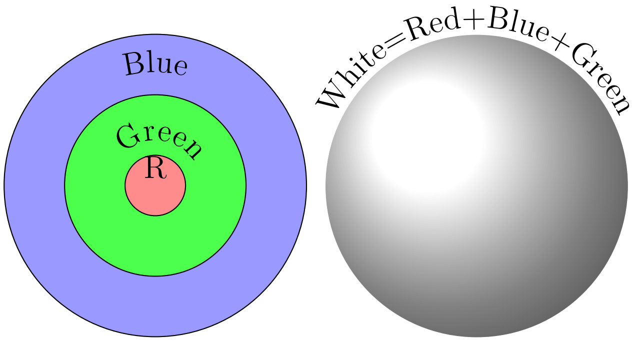


As one looks from outside, one sees a zero charge. In other words, the +2C charge at the centre is screened by the two -1C charges. After penetrating the first layer of charge -1C, one could observe a net charge of +2C - C = +1 C. Penetrating the next layer gives the centre charge +2C.

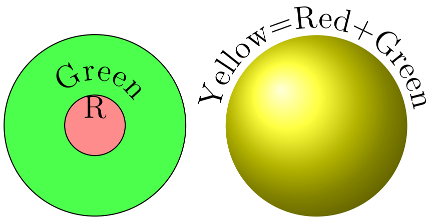
1.1.2. (b) Ternary charges example (red, blue, and green charges)

The notion of antiscreening can be exemplified in layman's terms as shown below. Assume a system where the charges aren't binary but are ternary charges. The labels red, green, and blue are taken from the analogy of primary colours. So say green, red, and blue are the colour charges, and the combined form gives a neutral white colour charge.

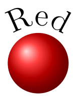
In the figure below, the left side figure is a cross section of sphere, and the right side is the view from outside. The viewer from outside will not see blue, green, or red, but the observer will see a white colour.



If observer penetrate the first layer of Blue, he will see a colour composite that's neither red nor green but yellow formed by the combination of red and green.



Penetrating the layer of green gives red.

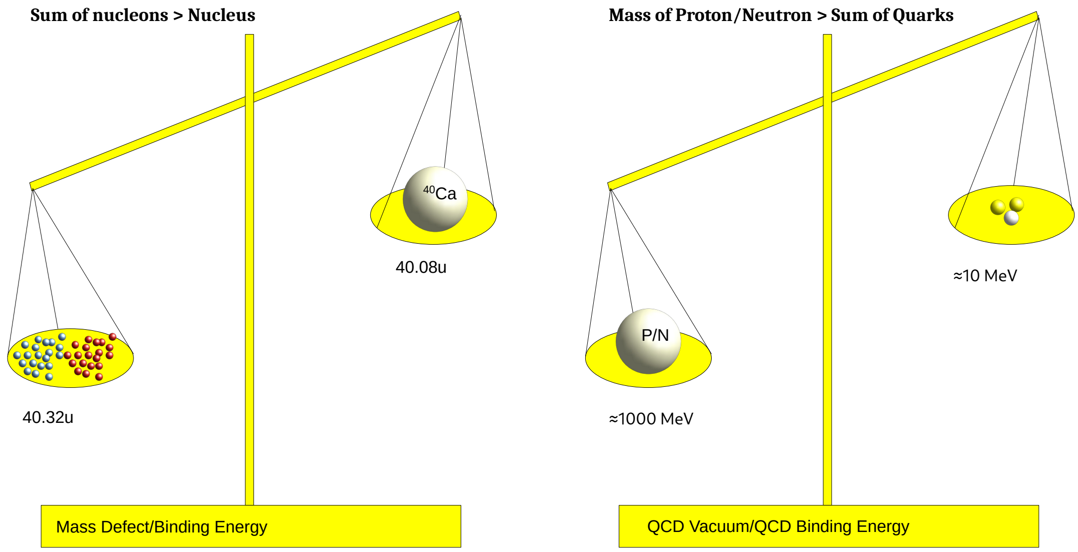


The anti-screening effect in quantum chromo dynamics (QCD) is much more complex. Both the quarks and gluons have colour degrees of freedom, while the quarks additionaly have a flavour degree of freedom. The spherical distribution of colour charges as layers and the penetration of such a spherical colour layer are not physically feasible as of now, but the above example can be taken to understand the general notion of colour charge.

1.1.3 Quark Gluon Plasma

Consider the chalk we use to write on the dark/green board, which is made up of white limestone composed of calcium carbonate ($CaCO_3$). Calcium, carbon, and oxygen are the components of $CaCO_3$. An isolated atom of ^{40}Ca has a weight of 40.078 u. 1.0073 u is the mass of a proton, and 1.0087 u is the mass of a neutron, adding up to 40.32 u in the case of calcium (20 proton+ 20 neutron). The fact that the components are heavier than the nuclei suggests that an atom lost some mass when it formed. All the nucleons in the nucleus are glued together by the missing mass or mass defect, which functions as a binding energy.

The nucleus is made up of protons and neutrons, thus protons and neutrons are collectively known as nucleons. The proton is composed of two up quarks and one down quark, while the neutron is composed of two down quarks and one up quark. The nucleons have a mass of about 1000 MeV. The mass of the quarks that made up the proton was roughly 10 MeV in total, meaning that they only made up 1% of the proton's mass. Through mass-less gluon interaction and chromodynamic binding energy, the remaining mass is attained.



Both quarks and electrons belong to the class of particles known as fermions that have a half-integer spin, but quarks are fermions with flavor degrees of freedom and color degrees of freedom.

Up	Top	Strange
Down	Bottom	Charm

×

Red	Green	Blue

Experimentally, isolated color has never been seen [1], but this suggests that quarks are constantly bonded together to create the color-white composite objects known as hadrons. Nambu made the first attempt to define color in 1966 (QCD) [1]. Quantum field theories that separately describe the electromagnetism and color fields are known as quantum electro dynamics (QED) and quantum chromo dynamics (QCD) respectively. In QCD, gluons mediate the strong force field, whereas in QED, photons mediate the E-M field. In terms of the variety of mediators, QCD has eight gluon mediators while QED only has one photon. The Quantum Chromo-Dynamical Lagrangian (QCD) [2] is

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{sym}} + \mathcal{L}_m \tag{1.1}$$

with

$$\begin{aligned}
\mathcal{L}_{\text{sym}} &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \sum_f \bar{\psi}^f_\alpha \left(i\gamma^\mu \partial_\mu + g\gamma^\mu A_\mu{}^a F^a \right)^{\alpha\beta} \psi^f_\beta \\
\mathcal{L}_{\text{m}} &= -\sum_f \bar{\psi}^f_\alpha m^{\alpha\beta}_f \psi^f_\beta \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu
\end{aligned} \tag{1.2}$$

A^a_μ denotes the gluon field with $a \in [1, 8]$ and ψ^f_α is quark field color with $\alpha \in [1, 3]$ and f denotes the flavor with quark masses being m_f. F^a are the SU(3) generators satisfying the commutation relation $[F^a, F^b] = F^aF^b - F^bF^a = if^{abc}F^c$. There exist other conventions where g is changed to $-g$ in Eq. (1.2) [3]. At temperatures $T \ll T_c$, the quarks and gluons together form hadrons (mesons plus baryons), with quarks having masses near 300 MeV. But as temperatures go way beyond T_c, i.e., $T \gg T_c$, the hadrons melt and the effective mass of quarks goes to zero.
So from Eqs. (1.1) and (1.2)

$$T \gg T_c \implies m_f \to 0 \implies \mathcal{L}_m \to 0 \implies \mathcal{L} = \mathcal{L}_{\text{sym}} \tag{1.3}$$

i.e., at temperatures greater than the critical temperature, due to the loss of effective quark mass, the Lagrangian becomes symmetric, known as chiral symmetry restoration. Similarly, as temperature becomes less than critical temperature, the Lagrangian becomes non-symmetric $\mathcal{L} = \mathcal{L}_{\text{sym}} + \mathcal{L}_m$. This is known as spontaneous chiral symmetry breaking [2]. Asymptotic freedom is a distinctive property of the QCD, which involves non-abelian gauge theory (SU3). The coupling constant [4] that represent the asymptotic freedom can be expressed as

$$\begin{aligned}
\alpha_s(\mu) &= \frac{12\pi}{(33 - 2N_f)\ln\left(\frac{\mu^2}{\Lambda^2_{QCD}}\right)} \quad \text{(One loop order)} \\
\alpha_s(\mu) &= \frac{12\pi}{(33 - 2N_f)\ln\left(\frac{\mu^2}{\Lambda^2_{QCD}}\right)} \left(1 - \frac{6(153 - 19N_f)}{(33 - 2N_f)^2} \frac{\ln\left(\ln\left(\frac{\mu^2}{\Lambda^2_{QCD}}\right)\right)}{\ln\left(\frac{\mu^2}{\Lambda^2_{QCD}}\right)} \right) \\
&\quad \text{(Two loop order)}
\end{aligned} \tag{1.4}$$

in which N_f is the number of flavors.
Λ_{QCD} is the QCD scale parameter. The parameter μ corresponds to energy,

i.e., μ can be approximated as a function of temperature or momentum transfer depending on the interaction and environment. As $\mu \to \infty$, the coupling constant $\alpha_s(\mu) \to 0$. Similarly $\mu \to \Lambda_{QCD}$, the coupling constant diverges.

In other words, quarks and gluons interact weakly at high energies at short distances but strongly at low energies, resulting in the confinement of quarks and gluons within the composite hadrons. Quarks cannot be observed in free states in the natural world. Photons do, however, exist in free states in the natural world, and QED is an Abelian gauge theory. In contrast to QCD, which also involves potentials other than the Coulomb potential, QED heavily relies on the Coulomb potential [1].

In case of QED screening, the vacuum becomes polarized when a charge is present, attracting virtual particles with opposite charges and repelling virtual particles with similar charges. Overall, the field at any fixed distance is only partially canceled. The effect of the vacuum becomes less and less noticeable as one gets closer to the core charge, but the effective charge rises. However, QCD has an anti-screening nature, meaning that its force-carrying particles, the gluons, carry color charge in their own particular way. A color charge and an anti-color magnetic moment are both carried by each gluon. In the vacuum, the polarization of virtual gluons has the overall effect of enhancing and altering the field rather than screening it. This effect would contribute to a weakening of the effective charge with decreasing distance since getting closer to a quark reduces the antiscreening impact of the nearby virtual gluons.

Consider heating the QCD vacuum within a box. The vacuum excites hadrons. The hadrons begin to overlap as the temperature climbs toward the critical temperature T_c, which is between 150 and 200 MeV. The hadronic system disintegrates into quarks and gluons (QGP) as the temperature rises further. In comparison the temperature at the center of the sun is 1.5×10^7 kelvins, or 0.0013 MeV. Nuclear matter has a density of about 0.16 fm^{-3}; if the density is increased by a factor of many, the hadronic system breaks down into quarks and gluons.

The universe was expanding and had an origin involving a high temperature transition, according to Alpher and Gamow's paper *The Origin of Chemical Elements* [5], Hubble's law of galaxy redshift, and Friedmann's solution of Einstein's gravitational equation in 1922 [6]. The cosmic hot era period was established by Penzias and Wilson's discovery of the cosmic microwave background in 1965 [7].

There is a good chance that the neutrons will melt into the cold quark matter if the center density of the neutron stars exceeds $5-10\rho_{\mathrm{n}}$. The strange matter theory proposes that the quark matter (the strange matter), which has almost equal amounts of *up*, *down*, and *strange* quarks, may represent a stable ground state of matter. i.e., theoretically, there exists a finite probability for the existence of strange quark stars. We must solve the Oppenheimer-Volkoff (TOV) equation (Oppenheimer and Volkoff, 1939), which is derived from the Einstein equation, along with the equation of states for the super-dense matter, in order to understand the structure of these compact stars.

But as one can combine the density and temperature effects simultaneously on a hadronic matter, QGP can be produced in the lab, which is done in the Large Hadron Collider (LHC) and Relativistic Heavy Ion Collider (RHIC) [1].

1.2 Important phenomenological models

1.2.1 MIT Bag Model

Harald Fritzsch, Heinrich Leutwyler, and Murray Gell-Mann developed the idea of colour into the theory of quantum chromodynamics in 1973 [8]. In the very next year, Chodos et al. [9] of the Massachusetts Institute of Technology (MIT) introduced a simple phenemenological model to study the equation of state of hadrons and QGP.
The potential in the MIT bag model is defined as the the sum of the volume term and the inverse radial term. The MIT bag model can be explained in the following manner from a qualitative standpoint. The potential energy can be written as

$$E_H = BV + \frac{C}{r} \tag{1.5}$$

The vacuum energy density, which holds quarks and gluons inside the bag, is related to the constant B. The quarks in the bag behave as free particles (fermions), according to the concept and the number of quarks decreases to zero (confined) outside the bag. Eq. (1.5) can be rewritten for a spherical drop-like form as

$$E_H = \frac{4}{3}\pi r^3 B + \frac{C}{r} \tag{1.6}$$

The kinetic energy term combined with the uncertainty principle gives rise to the inverse radial term. In the quantum mechanics approximation, the parameter C can be expressed as a function of the quantum number and the quark number. Under stable conditions, the external pressure is compensated by the confined quark pressure.

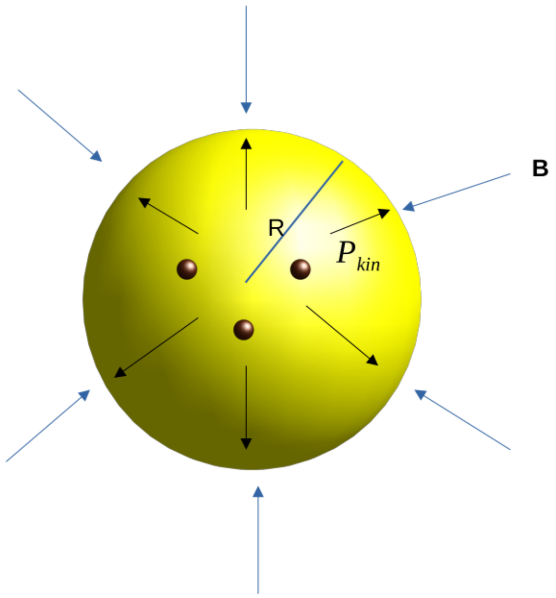


Figure 1.1: The quarks within the bag exert kinetic pressure. As long as the bag pressure B is greater than the kinetic pressure P_{kin}, the bag holds the quarks. When bag pressure is overwhelmed by the kinetic pressure P_{kin} of quarks, deconfinement occurs.

By minimizing Eq. (1.6), the radius of the bag can be determined.

$$\left.\frac{\partial E_H}{\partial r}\right|_{r=R} = 0 \implies R = \left(\frac{C}{4\pi B}\right)^{\frac{1}{4}} \tag{1.7}$$

Therefore, the total energy at the ground state equilibrium radius can be obtained as

$$\begin{aligned} E_H(R) &= \frac{4}{3}\pi R^3 B + \frac{C}{R} = \frac{4\pi R^4 B + 3C}{3R} = \frac{4C}{3R} \\ &= \frac{4}{3R}\left(4\pi B R^4\right) = 4\left(\frac{4}{3}\pi R^3 B\right) = 4V_0 B \end{aligned} \tag{1.8}$$

According to the bag model [10], if the particle is a boson in a free particle state, then the energy density and pressure relation at zero chemical potential (when the fugacity is equal to unity) are

$$\begin{aligned}
\langle\varepsilon\rangle_B =& g_B \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p}{e^{\beta p}-1} = g_B\ \frac{1}{2\pi^2}\int_0^\infty p^3\sum_{n=1}^{\infty} e^{-n\beta p} \\
=& \frac{3g_B}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{3g_B}{\pi^2\beta^4}\zeta(4) \\
P_B =& -\frac{1}{\beta}\int \langle\varepsilon_B\rangle\ \mathrm{d}\beta = g_B\ \frac{\zeta(4)}{\pi^2\beta^4} = \frac{\langle\varepsilon_B\rangle}{3}
\end{aligned} \tag{1.9}$$

If the distribution is fermionic, then the results would be

$$\begin{aligned}
\langle\varepsilon_f\rangle =& g_f \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p}{e^{\beta p}+1} = g_f\ \frac{1}{2\pi^2}\int_0^\infty p^3\sum_{n=1}^{\infty}(-1)^{n-1} e^{-n\beta p} \\
=& \frac{3g_f}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = \frac{3g_f}{\pi^2\beta^4}\eta(4) \\
P_f =& -\frac{1}{\beta}\int \langle\varepsilon_f\rangle\ \mathrm{d}\beta = g_f\ \frac{\eta(4)}{\pi^2\beta^4} = \frac{\langle\varepsilon_f\rangle}{3}
\end{aligned} \tag{1.10}$$

The Riemann zeta function ζ, and the Dirichlet eta function η, are both used, where

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\
\eta(s) &= \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} \\
\eta(s) &= \left(1-2^{1-s}\right)\zeta(s)
\end{aligned} \tag{1.11}$$

Degenerative factors for fermionic and bosonic particles are g_B and g_I. $\beta = 1/T$ is in natural units.

Consider a scenario where a transition occurs from a hadronic phase (massless pions) to a QGP phase (composed of massless quarks and gluons) as the temperature increases beyond the critical temperature T_c. From Eqs. (1.9) and (1.10), the pressure of pionic stage is

$$P_\pi = 3\frac{\zeta(4)}{\pi^2\beta^4} \tag{1.12}$$

Three degrees of freedom comes from the three charge states of pion (π^+, π^0, π^-). The kinetic pressure of the free particle of quarks and gluons are

$$P_{q+g} = \frac{24\eta(4)+16\zeta(4)}{\pi^2\beta^4} \tag{1.13}$$

The two spin and eight colour degrees of freedom of the gluons give the factor 16 (2×8). For quarks, the factor 24 $(2\times 2\times 2\times 3)$ comes as a contribution of two particle-antiparticle degrees of freedom, two spin, two flavour, and three colour degrees of freedom.

At critical temperature T_c, the confinement pressure (pionic pressure + bag pressure) becomes equal to the deconfinement pressure (quarks and gluons kinetic pressure). i.e.,

$$B + P_\pi\Big|_{T_c} = P_{q+g}\Big|_{T_c} \tag{1.14}$$

Using Eqs. (1.11) and (1.12) and $\zeta(4) = \pi^4/90$, bag constant can be derived as

$$B = \left(P_{q+g} - P_\pi\right)\Big|_{T_c} = \frac{(37-3)}{\pi^2\beta_c^4}\zeta(4) = \frac{17}{45}\pi^2 T_c^4 \tag{1.15}$$

Thus the critical temperature is $T_c = \left(\frac{45B}{17\pi^2}\right)^{\frac{1}{4}}$.

For values such as $B^{\frac{1}{4}} = 200$ MeV and $T_c \approx 144$ MeV, this model is rather straightforward and fits the mass spectra of light hadrons. The pion-to-QGP transition temperature is around 144 MeV according to the bag model. However, the bag model falls short in explaining a number of crucial aspects of strong interactions, including chiral symmetry and others [11].

1.2.2 Relativistic Harmonic Oscillator (RHO) Models

The Hamiltonian of confined quarks and gluons is taken to be

$$\begin{aligned} H_q &= \sqrt{p^2 + M_q^2 + \Omega_q^2 r^2} \text{ (quarks)} \\ H_g &= \sqrt{p^2 + C_g^4 r^2} \text{ (gluons)} \end{aligned} \tag{1.16}$$

in the RHO model, which was first set forth by Khadkikar and Gupta [12] and later expanded by Khadkikar and Vinodkumar [13]; with C_g, Ω_g and M_q being the frequency of gluon fields, quark fields and mass of quark respectively. The Hamiltonians are replaced by the eigenvalues of the corresponding quantum mechanical operators. i.e., a harmonic oscillator of the form

$$\hat{H} = \alpha_p \hat{\vec{\mathbf{p}}}^2 + \alpha_q \hat{\vec{\mathbf{r}}}^2 = \left(2\hat{N} + 3\right)\hbar\alpha_p\alpha_q \tag{1.17}$$

has an energy eigenvalue of $(2n + 1)\,\hbar\alpha_p\alpha_q$ with $n \in \mathbb{W}$. If one moves the range of n from whole number to natural number $\mathbb{N}$, then the Hamiltonian becomes $(2n + 1)\hbar\alpha_p\alpha_q$. Thus the result of Eq. (1.16) in natural units can be written as

$$
\begin{aligned}
H_q &= \sqrt{(2n+1)\,\Omega_q + M_q^2} \\
H_g &= \sqrt{(2n+1)\,C_g^2}
\end{aligned}
\tag{1.18}
$$

The RHO model became successful in explaining various hadron spectroscopy results and various experimental values such as mesonic mass having open flavours, baryon magnetic moment, nucleon polarizability experimental values, and leptonic decay width [12, 13, 14]. Even though the model failed to fit the equation of state lattice data, especially near the critical temperature region.

1.2.3 Quasiparticle Model of QGP

The quasiparticle model proposed by VM Bannur [15, 16, 17] consistently explains the lattice data obtained. According to this model, the number density of fermionic quarks and bosonic gluons at zero chemical potential (fugacity = 1) can be derived by integrating the appropriate distribution function in the relativistic limit, as shown below.

$$
\begin{aligned}
\langle n \rangle_{\mathrm{F.D}} &= \mathrm{g}_{\mathrm{F.D}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\exp\left(\beta\sqrt{p^2+m^2}\right)+1} \\
\langle n \rangle_{\mathrm{B.E}} &= \mathrm{g}_{\mathrm{B.E}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\exp\left(\beta\sqrt{p^2+m^2}\right)-1}
\end{aligned}
\tag{1.19}
$$

with β being inverse of temperature in natural units.
The model is thermodynamically consistent in such a way that the integration of the distribution function over energy gives the energy density value

$$
\begin{aligned}
\langle \varepsilon \rangle_{\mathrm{B.E}} &= \mathrm{g}_{\mathrm{B.E}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\sqrt{p^2+m^2}}{\exp\left(\beta\sqrt{p^2+m^2}\right)-1} \\
\langle \varepsilon \rangle_{\mathrm{F.D}} &= \mathrm{g}_{\mathrm{F.D}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\sqrt{p^2+m^2}}{\exp\left(\beta\sqrt{p^2+m^2}\right)+1}
\end{aligned}
\tag{1.20}
$$

Both Eqs. (1.19) and (1.20), can be simplified by substituting $p = m\sinh(x)$, which leads to

$$\begin{aligned}\langle n\rangle_{\text{F.D/B.E}} &= g_{\text{F.D/B.E}}\frac{1}{2\pi^2}\int_0^\infty \frac{m^3\sinh^2(x)\cosh(x)}{\exp\left(\beta m\cosh(x)\right)\pm 1}\,\mathrm{d}x \\ \langle \varepsilon\rangle_{\text{F.D/B.E}} &= g_{\text{F.D/B.E}}\frac{1}{2\pi^2}\int_0^\infty \frac{m^4\sinh^2(x)\cosh^2(x)}{\exp\left(\beta m\cosh(x)\right)\pm 1}\,\mathrm{d}x\end{aligned} \tag{1.21}$$

using the results

$$\begin{aligned}\sinh^2(x)\cosh(x) &= \frac{\cosh(3x)-\cosh(x)}{4} \\ \sinh^2(x)\cosh^2(x) &= \frac{\cosh(4x)-1}{8} \\ \frac{1}{Y\pm 1} &= \sum_{n=1}^{\infty}(\mp)^{n-1}Y^{-n},\ \text{for } Y>1 \\ K_n(\beta) &= \int_0^\infty \cosh(nx)\exp\left(-\beta\cosh(x)\right)\mathrm{d}x\end{aligned} \tag{1.22}$$

where $K_n(x)$ is the modified Bessel function of second kind.

Eq. (1.21) becomes

$$\begin{aligned}\langle n\rangle_{\text{F.D/B.E}} &= g_{\text{F.D/B.E}}\frac{m^3}{8\pi^2}\sum_{n=1}^{\infty}(\mp)^{n-1}\Big[K_3(n\beta m)-K_1(n\beta m)\Big] \\ &= g_{\text{F.D/B.E}}\frac{m^3}{2\pi^2}\sum_{n=1}^{\infty}(\mp)^{n-1}\frac{K_2(n\beta m)}{n\beta m} \\ \langle \varepsilon\rangle_{\text{F.D/B.E}} &= g_{\text{F.D/B.E}}\frac{m^4}{16\pi^2}\sum_{n=1}^{\infty}(\mp)^{n-1}\Big[K_4(n\beta m)-K_0(n\beta m)\Big]\end{aligned} \tag{1.23}$$

Pressure as a function of temperature can be derived using the thermodynamic relation

$$\frac{P}{T}-\frac{P_0}{T_0} = \int_{T_0}^{T}\frac{\varepsilon(T)}{T^2}\mathrm{d}T = -\int_{\beta_0}^{\beta}\varepsilon(\beta)\mathrm{d}\beta \tag{1.24}$$

The model proposed by VM Bannur [15, 16, 17], has $m^2(T) = g^2(T)T^2$, where $g^2(T) = 4\pi\alpha_s(T)$ is the coupling constant in QCD. This QPM model fits well with lattice data[18] and is one of the successful models in QGP.

1.3 Present study

There are several quasiparticle models [19, 20, 15, 16, 17] that involve quasiparticle mass as a function of coupling constants. There are studies that extend these models to the magnetic field regime [21, 22]. But these models approximate the coupling constant as a function of $\ln(T/T_c)$, with T_c being the critical temperature. As $T/T_c \gg 1$, many of theses model fits with lattice data. But when $T/T_c \approx 1$ and $0 < T/T_c < 1$, $\ln(T/T_c)$ becomes negative. This affects the coupling constant, which causes a deviation between lattice data and quasiparticle model predictions. So the quasiparticle model involving coupling constants is successful only in the regime $T/T_c > 1$. In some extended quasiparticle models involving magnetic fields, the same problem persists due to their dependence on the coupling constant having a dependence on $\ln\left(T/T_c\right)$.

So we developed a model that is independent of the coupling constant but still able to fit the lattice data for $T/T_c \leq 1$ and $T/T_c > 1$. Bannur [23] introduced the idea of using Mayer's cluster expansion (MCE) to explain equation of state (EoS) of QGP. We borrow the same idea and combine it with the idea of the dimensional regularization method, and the modified liquid drop model enables us to successfully explain the EoS of QGP for a wide range of temperatures. We extended the model to the magnetic field regime to explain and fit the QCD lattice data in the presence of an external magnetic field.

In addition to this, we have developed a new method for deriving coupling constants known as the same mass scale and coupling (SMC) method. In which, the coupling constant and mass scale for both imaginary time formalism and non-thermal quantum field theory are considered to be the same. Then, using the usual renormalization method and applying RGE equations to both ITF and QFT simultaneously, we get the running mass and running coupling constant. This running mass and coupling constant fit with the expected behaviour of the equation of state of ϕ^4 theory.

1.4 Plan of the thesis

In Chapter 2, we introduce the Mayer's cluster expansion formulation step by step. We also give an idea of the correction factor that needs to be multiplied to make the distribution of distinguishable particles to the distribution of indistinguishable particles of different kinds. Finally, we show the equation of state of pressure and energy density for neutral systems having an arbitrary potential that is proportional to the product of the charges of particles in the system. This Chapter is mainly based on the works of Mayer [24, 25] and Balescu [26].

In Chapter 3, we introduce the core part of the thesis, i.e., the modified liquid drop model in QGP. The Fourier transforms of central potentials are discussed. Examples are given to understand the cluster expansion idea and are compared with the related published works. Various integrals and their results are derived using the contour integration method. Methods to avoid poles in an integral by introducing an infinitesimal complex term to the integrand are discussed.

In Chapter 4, the modified liquid drop potential is extended to the magnetic field regime. The idea of a harmonic oscillator is used with a magnetic vector potential, giving rise to modifications in the integral equation. A new integral table with the modified integrating technique in the presence of a magnetic field is also derived.

In Chapter 5, we have taken the idea of Bannur's deconfined quark matter quasiparticle model [27] and extended it to the magnetic field regime. The results are compared with the EoS of free particles. The formulas are derived in a systematic way.

In Chapter 6, the quasiparticle model in thermal ϕ^4 theory is discussed. A temperature-dependent coupling constant is derived for that purpose. Running mass, mass scale, and constant temperature relations are found using both imaginary time formalism and non-thermal quantum field theory. For that, a new approach, named as Same Mass Scale and Coupling (SMC), is introduced. Applying this result to the quasiparticle model provides us with the equation of state of the system. In Chapter 7, summary and future plans are discussed.

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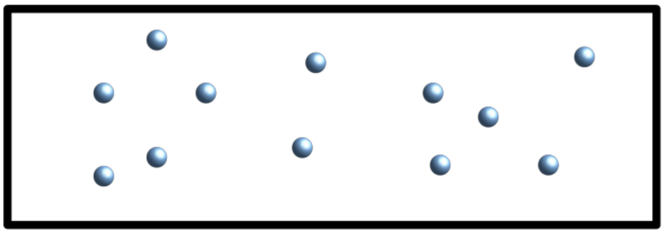
Chapter 2

Mayer's Cluster Expansion

2.1 Cluster expansion for Distinguishable particles



Consider a closed system containing particles. Consider a particle in such a system, say B. Either the particle B can interact with no other particles or it can interact with at least some of the particles in the system. The former is known as an *ideal gas case*, while the latter is known as a *non-ideal gas case*. In the non-ideal gas case, the total interaction can be divided into several clusters of interactions[1, 2, 3] .

Consider a monoatomic gas system having a volume V and containing N identical particles, each with mass m at equilibrium temperature. Then we can express the Hamiltonian as

$$\mathrm{H} = \sum_{i=1}^{N} E^{\mathrm{kin}}(\vec{\mathbf{p}}_i, m_i) + \sum_{1 \le i < j \le N} U(r_{ij}) \tag{2.1}$$

with $E^{\mathrm{kin}}(\vec{\mathbf{p}}_i, m_i)$ representing the kinetic energy contribution of i^{th} particle and $U(r_{ij})$ is the pair potential of particles i and j having relative separation $r_{ij} = |\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j|$. The partition function of the gas including the Gibb's correction factor [1] is

$$Q_N(V,T) = \frac{1}{N!h^{3N}} \int \exp(-\beta H)\ d^{3N}p\ d^{3N}x = Z_m Z_c \tag{2.2}$$

with $\beta = (k_B T)^{-1}$, h is the Planck's constant. Z_c and Z_m are the integrals in radial coordinate and momentum coordinate respectively.
At the non-relativistic limit $E^{\text{kin}} = \frac{p_i^2}{2m_i}$

$$\begin{aligned} Z_m^{\text{NR}} &= \frac{1}{h^{3N}} \prod_{i=1}^{N} \left[\int \exp\left(-\beta \frac{p_i^2}{2m_i}\right) \mathrm{d}^3 p \right] \\ &= \left(\frac{2\pi}{h^2\beta}\right)^{\frac{3N}{2}} \prod_{i=1}^{N} m_i^{\frac{3}{2}}. \end{aligned} \tag{2.3}$$

At the relativistic limit $E^{\text{kin}} = \sqrt{p_i^2 + m_i^2}$,

$$\begin{aligned} Z_m^{\text{R}} &= \frac{1}{h^{3N}} \prod_{i=1}^{N} \left[\int \exp\left(-\beta \sqrt{p_i^2 + m_i^2}\right) \mathrm{d}^3 p_i \right] \\ &= \left(\frac{4\pi}{h^3\beta}\right)^{N} \prod_{i=1}^{N} \left[m_i^2 K_2(\beta m_i) \right] \end{aligned} \tag{2.4}$$

with K_2 being modified Bessel function of the second kind and m_i being the mass of i^{th} particle in the system.
When all the particles have the same mass;

$$Z_m = \begin{cases} \left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{3N}{2}} & \text{, Non-Relativistic limit} \\ \left(\frac{m^2}{2\pi^2\hbar^3\beta} K_2(\beta m)\right)^{N} & \text{, Relativistic limit.} \end{cases} \tag{2.5}$$

Now the remaining part of integral in Eq. (2.2) involving the potential part is

$$Z_c = \frac{1}{N!} \int \prod_{i<j} \exp\left[-\beta U_{ij}\right] d^{3N} r. \tag{2.6}$$

The evaluation of above integral is difficult in the present form. One could use the cluster expansion method to simplify the integral. In cluster expansion method we define $\eta_{ij} = \exp\left[-\beta U_{ij}\right] - 1$.

2.1.1 Distinguishable particles

Let us assume that all the particles are distinguishable, then let us remove the $N!$ (We will consider the indistinguishability factor later). Then

$$Z_c = \int \prod_{i<j} (1+\eta_{ij})\, \mathrm{d}^3r_1 \;\; \mathrm{d}^3r_2 \;\; \mathrm{d}^3r_3 \;\;\; \ldots \mathrm{d}^3r_N \tag{2.7}$$

where the expansion of the function is

$$\prod_{i<j} (1+\eta_{ij}) = 1 + \sum_{i<j} \eta_{ij} + \sum_{\substack{i<j \\ k<l \\ i,j\neq k,l}} \eta_{ij}\eta_{kl} + \ldots \tag{2.8}$$

Consider a subsystem with j particles within the N particle system. The integral in Eq. (2.10) involves $\eta(1,2,3,...,j)$ as an integrand and $\prod_{i=1}^{N} \mathrm{d}^3r_i$ as integrating variables. Only the $\prod_{i=1}^{j} \mathrm{d}^3r_i$ variable interacts with the $\eta(1,2,3,...,j)$ integrand. The remaining $\prod_{i=j}^{N} \mathrm{d}^3r_i$ radial coordinates provide the volume contribution of V^{N-j} when integrated. In addition to that, the multiplicity factor will be $\binom{N}{j}$ which corresponds to choosing j particles from N particles. Since we use the small letter c to denote the concentration of particles in general and the capital letter C to denote concentration in the context of the MLDM model of Chapter 3, we will be using $\binom{N}{j}$ to represent the combination representation of ${}^{N}C_j$. i.e.,

$$\binom{N}{j} = {}^{N}C_j = \frac{N!}{(N-j)!j!} \tag{2.9}$$

$$\begin{aligned} \therefore \mathrm{I_j} &= \binom{N}{j} \times \int \eta(1,2,...,j)\, \mathrm{d}^3r_1 \;\; \mathrm{d}^3r_2 \;\; \ldots \mathrm{d}^3r_j \ldots \mathrm{d}^3r_N \\ &= \binom{N}{j} \times V^{N-j} \int \eta(1,2,3...,j)\, \mathrm{d}^3r_1\, \mathrm{d}^3r_2 \ldots \mathrm{d}^3r_j \end{aligned} \tag{2.10}$$

As the volume goes to infinity

$$\mathop{\mathrm{Lt}}_{V\to\infty} \frac{\mathrm{I_j}}{V^N} = \mathop{\mathrm{Lt}}_{V\to\infty} \binom{N}{j} \frac{1}{V^j} = \mathop{\mathrm{Lt}}_{V\to\infty} \frac{N}{V} \left(\frac{N}{V} - \frac{1}{V}\right) \ldots \left(\frac{N}{V} - \frac{j-1}{V}\right) \frac{1}{j!} \tag{2.11}$$

At fixed number density $c = \dfrac{N}{V}$, at infinite volume

$$\underset{V\to\infty}{\text{Lt}} \frac{\mathrm{I}_j}{V^N} = \frac{c^j}{j!} \times w_f \int \eta(1,2,3,\ldots,j)\, \mathrm{d}^3r_1 \;\; \mathrm{d}^3r_2 \;\ldots \mathrm{d}^3r_n \tag{2.12}$$

2.1.2 Integrals and Diagrammatic representation

2.1.2. (a) Two Particle cluster of the same kind

Writing down the integral as a diagram will ease the calculation. We represent, the two particle cluster integral as

$$\frac{1}{V^N} \sum_{i<j} \textcircled{\scriptsize i}\!\!-\!\!-\!\!-\!\!\textcircled{\scriptsize j} = \frac{1}{V^N} \times \sum_{i<j}^{N} \int \eta_{ij} d\tau \tag{2.13}$$

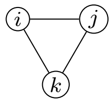
with $d\tau = \mathrm{d}^3r_1 \;\; \mathrm{d}^3r_2 \ldots \mathrm{d}^3r_N$.
This cluster integral of two particle is also known as an irreducible cluster integral, because it cannot be broken down further. The factor $\sum_{\substack{i,j=0\\ i<j}}^{N} \eta_{ij} d\tau$ is equivalent to $[N(N-1)/2]\, \eta_{ij} d\tau$ in the integral. So,

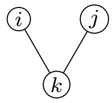
$$\begin{aligned}
\frac{1}{V^N} \sum_{i<j} \textcircled{\scriptsize i}\!\!-\!\!-\!\!-\!\!\textcircled{\scriptsize j} &= \begin{array}{c}\text{Choosing two particles}\\ \text{from N distinguishable}\\ \text{particle}\end{array} \times \frac{1}{V^N} \times \int \eta_{ij} d\tau \\
&= \binom{N}{2} \times \frac{1}{V^N} \times V^{N-2} \times \int \eta_{ij}\, \mathrm{d}^3r_i\, \mathrm{d}^3r_j \\
&= \left(\frac{N}{V}\right)\left(\frac{N}{V} - \frac{1}{V}\right) \times \frac{1}{2} \int \eta_{ij}\, \mathrm{d}^3r_i\, \mathrm{d}^3r_j \\
&\approx c^2 \times \frac{1}{2} \int \eta_{ij}\, \mathrm{d}^3r_{ij}\, \mathrm{d}^3r \\
&\approx c^2 \bar{\beta}_2 = c^2 b_2 V
\end{aligned} \tag{2.14}$$

with $c = N/V$ where N is total number of particles and b_2 is a finite term that doesn't go to infinity as $V \to \infty$. The variables dr_i and dr_j can be rearranged using a Jacobian determinant to produce new variables dr_{ij} and dr.

2.1.2. (b) Three Particle cluster of the same kind

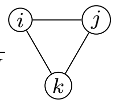
Each integral equation is associated with certain probability factors. We are given N particles, so the probability factor is proportional to the number of ways we can choose the three particles from the given N particles that form the three particle cluster. The next two important diagrams, formed by three particles, are depicted below. As mentioned previously, all these diagrams correspond to some integral equation. Here we have to evaluate mainly two diagrams



and


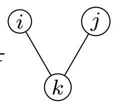
(2.15)

The result can be written as

$$\underset{V\to\infty}{\mathrm{Lt}}\frac{1}{V^N}$$


$$\begin{aligned}
&= \begin{array}{c}\text{Choosing three particles}\\ \text{from N distinguishable}\\ \text{particle}\end{array} \times \frac{1}{V^N} \times \int \eta_{ij}\eta_{jk}\eta_{ki} d\tau \\
&= \underset{V\to\infty}{\mathrm{Lt}} \binom{N}{3} \times \frac{1}{V^N} \times V^{N-3} \times \int \eta_{ij}\eta_{jk}\eta_{ki}\ \mathrm{d}^3 r_i\ \mathrm{d}^3 r_j\ \mathrm{d}^3 r_k \\
&= \frac{c^3}{3!}\int \eta_{ij}\eta_{jk}\eta_{ki}\ \mathrm{d}^3 r_i\ \mathrm{d}^3 r_j\ \mathrm{d}^3 r_k \\
&= c^3\bar{\beta}_3 \approx c^3 b_3 V
\end{aligned} \tag{2.16}$$

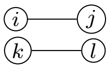
Similarly in this cluster of three particles, $\binom{N}{3}$ is multiplied by the multiplicative factor $\binom{3}{2}$, which corresponds to the number of ways the two bonds are chosen from the three-particle cluster.

$$\frac{1}{V^N}$$


$$\begin{aligned}
&= \begin{array}{c}\text{Choosing three particles}\\ \text{from N distinguishable}\\ \text{particle}\end{array} \times \begin{array}{c}\text{Choosing two bonds}\\ \text{from 3}\\ \text{particle}\end{array} \times \frac{1}{V^N} \times \int \eta_{ik}\eta_{jk} d\tau \\
&= \binom{N}{3} \times \binom{3}{2} \times \frac{1}{V^N} \times V^{N-3} \int \eta_{ij}\eta_{jk}\, \mathrm{d}^3 r_i\, \mathrm{d}^3 r_j\, \mathrm{d}^3 r_k \\
&= \binom{N}{3}\binom{3}{2} \times \frac{1}{V^3} \times V \left(\frac{2\bar{\beta}_2}{V}\right)^2 \\
&= \frac{c^3}{2}\frac{4\bar{\beta}_2^2}{V}
\end{aligned} \tag{2.17}$$

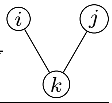
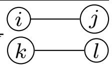
2.1.2. (c) The cluster ratio

Consider the next simple cluster having four particles with two bonds, i.e.,

$\frac{1}{V^N}\sum_{i<j}\sum_{k<l}$  $= \text{Choosing four particles from N distinguishable particle} \times \text{Choosing two bonds from 4 particle} \times \frac{1}{2} \times \frac{1}{V^N} \times \int \eta_{ij}\eta_{kl} d\tau$

$$= \binom{N}{4} \times \binom{4}{2} \times \frac{1}{2} \times \frac{1}{V^4}\left[\int \eta_{ij}\, \mathrm{d}^3 r_i\, \mathrm{d}^3 r_j\right]^2$$
$$= \frac{c^4}{8}\left[2\bar{\beta}_2\right]^2 = \frac{c^4}{2}\bar{\beta}_2{}^2 \tag{2.18}$$

The 1/2 factor comes to remove the repeatability. i.e., $\binom{4}{2}$ will have 6 two particle sets. But two cluster pair will be half.
On comparing Eqs. (2.17) and (2.18), the ratio becomes

Ratio $= \underset{V\to\infty}{\mathrm{Lt}}$ $\frac{1}{V^N}$  / $\frac{1}{V^N}$  $= \underset{V\to\infty}{\mathrm{Lt}}\frac{4}{c}\frac{1}{V} \to 0$ (2.19)

Since the ratio denotes that Eq. (2.18) contribution is greater as compared to Eq. (2.17) at an infinite volume limit; so let us ignore the contribution of Eq. (2.17).

On summing up the relevant diagram results in Tables 2.1 and 2.2, one can rearrange them in such a way that

$$\underset{V\to\infty}{\mathrm{Lt}}\frac{Z_c}{V^N} = 1 + \left(c^2\bar{\beta}_2 + c^3\bar{\beta}_3 + c^4\bar{\beta}_4 + \dots\right) + \frac{\left(c^2\bar{\beta}_2 + c^3\bar{\beta}_3 + c^4\bar{\beta}_4 + \dots\right)^2}{2!} + \dots \tag{2.20}$$

So

$$\underset{V\to\infty}{\mathrm{Lt}}\frac{Z_c}{V^N} \approx \exp\left(\sum_{n\geq 2} c^n \bar{\beta}_n\right) \tag{2.21}$$

2.1.3 Free Energy Perturbation

Consider a system that goes from state A to state B. The free energy difference can be calculated by using the *Zwanzig equation* [4].

$$\Delta F_{A\to B} = F_B - F_A$$
$$= -k_B T \ln\left(\frac{Q_B}{Q_A}\right) \tag{2.22}$$

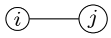
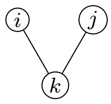
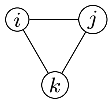
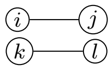
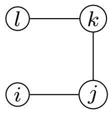
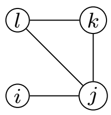
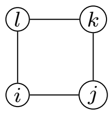
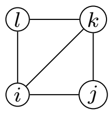
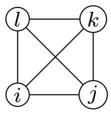
No of particles	Diagram	Integral	At infinite Volume
None	None	1	1
2		$\binom{N}{2}\frac{1}{V^2}\int \eta_{ij}\,\mathrm{d}^3r_i\,\mathrm{d}^3r_j$	$c^2\bar{\beta}_2$
3		$\binom{N}{3}\binom{3}{2}\frac{1}{V^3}V\left(\frac{2\bar{\beta}_2}{V}\right)^2$	$\frac{c^3}{2}\frac{4\bar{\beta}_2^2}{V}$ (neg.)
3		$\binom{N}{3}\frac{1}{V^3}\int \eta_{ij}\eta_{jk}\eta_{ki}\,\mathrm{d}^3r_i\,\mathrm{d}^3r_j\,\mathrm{d}^3r_k$	$c^3\bar{\beta}_3$
4		$\binom{N}{4}3\frac{1}{V^4}\left(2\bar{\beta}_2\right)^2$	$\frac{1}{2}c^4\bar{\beta}_2^2$
4		$\binom{N}{4}\frac{12}{V^4}\frac{8\bar{\beta}_2^3}{V^2}$	$c^4\frac{4\bar{\beta}_2^3}{V^2}$(neg.)
4		$\binom{N}{4}\times\frac{12}{V^4}\times\frac{12\bar{\beta}_2\bar{\beta}_3}{V}$	$c^4\frac{6\bar{\beta}_2\bar{\beta}_3}{V}$(neg.)
4		$\binom{N}{4}\frac{3}{V^4}24\int \eta_{ij}\eta_{jk}\eta_{kl}\eta_{li}\,\mathrm{d}^3r_i\,\mathrm{d}^3r_j\,\mathrm{d}^3r_k\,\mathrm{d}^3r_l$	$c^4\bar{\beta}_4$
4		$\binom{N}{4}\frac{6}{V^4}24\int \eta_{ij}\eta_{jk}\eta_{kl}\eta_{li}\eta_{ki}\,\mathrm{d}^3r_i\,\mathrm{d}^3r_j\,\mathrm{d}^3r_k\,\mathrm{d}^3r_l$	
4		$\binom{N}{4}\frac{24}{V^4}\int \eta_{ij}\eta_{jk}\eta_{kl}\eta_{li}\eta_{ik}\eta_{jl}\,\mathrm{d}^3r_i\,\mathrm{d}^3r_j\,\mathrm{d}^3r_k\,\mathrm{d}^3r_l$	

Table 2.1: The list of different clusters made up of 2, 3, and 4 particles.

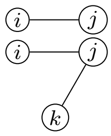
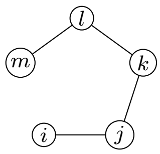
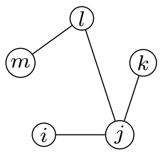
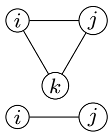
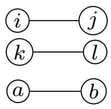
No of particles	Diagram	Integral	At infinite Volume
5		$\binom{N}{5}\frac{30}{V^5}\frac{8\beta_2^3}{V}$	$\frac{2c^5\beta_2^3}{V}$ (neg.)
5		$\binom{N}{5}\frac{60}{V^5}\frac{16\bar{\beta}_2^4}{V^3}$	$\frac{8\bar{\beta}_2^4c^5}{V^3}$ (neg.)
5		$\binom{N}{5}\frac{60}{V^5}\frac{16\bar{\beta}_2^4}{V^3}$	$\frac{8\bar{\beta}_2^4c^5}{V^3}$ (neg.)
5		$\binom{N}{5}\frac{10}{V^5}12\bar{\beta}_2\bar{\beta}_3$	$c^5\bar{\beta}_2\bar{\beta}_3$
	⋮	⋮	⋮
6		$\binom{N}{6}\frac{15}{V^6}8\bar{\beta}_2^3$	$\frac{c^6\bar{\beta}_2^3}{6}$

Table 2.2: (In continuation of Table 2.1). The diagrams shown in the Tables 2.1 and 2.2 with (neg.) denotes that such diagram contributions are negligible compared to those of other diagrams involving the integral [2]. An example can be seen in Eq. (2.19).

If A is an ideal gas state, then ΔF =Free energy of real gas (with potential) - Free energy of ideal gas (zero potential).

$$\begin{aligned}
\Delta F_{Ideal \to Real} &= -k_B T \ln \left(\frac{Z_c Z_m}{V^N Z_m} \right) \\
&= -k_B T \ln \left(V^{-N} Z_c \right) \\
&\text{From Eq. (2.21)} \\
-\frac{\Delta F}{k_B T} &= \sum_{n \geq 2} c^n \bar{\beta}_n
\end{aligned} \tag{2.23}$$

As $V \to \infty$, one can see that ΔF goes to infinity. One can define a finite quantity,

$$\mathfrak{S} = \frac{-\Delta F}{k_B T V} = \sum_{n \geq 2} c^n \frac{\bar{\beta}_n}{V} = \sum_{n \geq 2} c^n b_n \tag{2.24}$$

For example, let's take the first term;

$$\begin{aligned}
c^2 \bar{\beta}_2 &= \frac{c^2}{2} \int \eta_{ij} \, \mathrm{d}^3 r_i \, \mathrm{d}^3 r_j \\
&= \frac{c^2}{2} \int \eta_{ij} \, \mathrm{d}^3 r_{ij} \, \mathrm{d}^3 r \\
&= V \frac{c^2}{2} \int \eta_r \, \mathrm{d}^3 r \\
&= V b_2
\end{aligned} \tag{2.25}$$

So As $V \to \infty$, $\bar{\beta}_2 \to \infty$ but

$$\begin{aligned}
b_2 &= \frac{1}{2} \int \eta_r \, \mathrm{d}^3 r \\
&\text{Let us define an arbitrary } \eta(r) = \exp\left(-\beta \left[\frac{a}{r} + br\right]\right) \\
&= \frac{1}{2} \int \left\{ \exp\left(-\beta \left[\frac{a}{r} + br\right]\right) \right\} \mathrm{d}^3 r \\
&= \frac{1}{2} \int \left\{ \exp\left(-\beta \left[\frac{a}{r} + br\right]\right) \right\} \mathrm{d}^3 r \\
&= \pi \left(\frac{b}{a}\right)^{\frac{3}{2}} K_3 \left[2\beta\sqrt{ab}\right]
\end{aligned} \tag{2.26}$$

goes to a finite result.

2.1.4 Pressure relation

Imagine a closed system that expands at constant temperature (reversible) and goes from state S_1 to S_2, changing the Helmholz energy in the process. The change in Helmholz energy,

$$F_2 - F_1 = -\int_{S_1}^{S_2} P \mathrm{d}V = -\underbrace{\int_{S_1}^{S_2} P_1 \mathrm{d}V}_{\text{Ideal gas}} - \underbrace{\int_{S_1}^{S_2} (P - P_1)\, \mathrm{d}V}_{\text{Real gas - Ideal gas}} \tag{2.27}$$

where P is the pressure for real gas, P_1 is the pressure for ideal gas, V is the volume and N is the number of particle in the system. With

$$-\int_{S_1}^{S_2} P_1 \mathrm{d}V = -\int \frac{N k_B T}{V} \mathrm{d}V \tag{2.28}$$

The free energy expansion integral representing the ideal gas in Eq. (2.28) goes through the same states (c, T) as that of the real gas. c denotes the concentration of particles. i.e., N/V.

Subtracting Eq. (2.28) from Eq. (2.27) will give us the free energy that corresponds to the interactions among the molecules. Since the ideal gas contribution from the momentum integral part is subtracted from integral in Eq. (2.27); the integral in equation Eq. (2.29) is known as the free excess energy.

$$\begin{aligned} F^{\text{Excess}} = -\int_{S_1}^{S_2} (P - P_1)\, \mathrm{d}V &= -\int_{S_1}^{S_2} \left(P - \frac{N k_B T}{V}\right) \mathrm{d}V \\ &= -N \int_{S_1}^{S_2} (P - c k_B T)\, d\left[\frac{1}{c}\right] \end{aligned} \tag{2.29}$$

Let us relate cluster to this as

$$\begin{aligned} \mathfrak{S} = -\frac{F^{\text{Excess}}}{V k_B T} &= \frac{c}{k_B T} \int_{c=0}^{c} [P - c k_B T]\, d\left[\frac{1}{c}\right] \\ \implies \frac{\mathfrak{S}}{c} &= \int_{c=0}^{c} \left(\frac{P}{k_B T} - c\right) d\left[\frac{1}{c}\right] \\ \implies \frac{\partial\left(\frac{\mathfrak{S}}{c}\right)}{\partial\left(\frac{1}{c}\right)} &= \frac{P}{k_B T} - c \\ \implies & \boxed{\frac{P}{k_B T} - c = \mathfrak{S} - c\frac{\partial \mathfrak{S}}{\partial c}} \end{aligned} \tag{2.30}$$

2.1.5 Virial Coefficient

Combining $\mathfrak{S} = \sum_{n\geq 2} c^n b_n$, with $\dfrac{\partial\left(\frac{\mathfrak{S}}{c}\right)}{\partial\left(\frac{1}{c}\right)} = \dfrac{P}{k_B T} - c$ will give us the n^{th} virial coefficient. i.e.,

$$\frac{\partial\left(\frac{\mathfrak{S}}{c}\right)}{\partial\left(\frac{1}{c}\right)} = \sum_{n\geq 2} c^n b_n[1-n] = \frac{P}{k_B T} - c \tag{2.31}$$

The factor $[1-n]b_n = \mathfrak{B}_n$ is the n^{th} Virial coefficient. So

$$\frac{P}{k_B T} - c = \sum_{n\geq 2} c^n \mathfrak{B}_n \tag{2.32}$$

Consider the term

$$\begin{aligned} b_2 &= \frac{1}{2V}\int\int \eta_{ij}\mathrm{d}^3r_i\mathrm{d}^3r_j \\ &= \frac{1}{2V}\int\int \left(\exp\left[-\beta U(r_{ij)}\right] - 1\right)\mathrm{d}^3r_i\mathrm{d}^3r_j \end{aligned}$$

substituting $r = r_i - r_j$ and $R = (r_i + r_j)/2$ leads to

$$\begin{aligned} b_2 &= \frac{1}{2}\int \left(\exp\left[-\beta U(r)\right] - 1\right)\mathrm{d}^3r \\ &= \frac{1}{2}\int \sum_{n=1}^{\infty} \frac{\left[-\beta U(r)\right]^n}{n!}\mathrm{d}^3r \end{aligned}$$

Similar to the b_2 term, when examining terms in the expansion of b_n, an infinite number of terms analogous to b_2 emerge inside the integral with appropriate changes. As the upper limit of the integral tends towards infinity (i.e., infinite volume), not all the terms contribute with the same weight. Mayer's cluster summation becomes relevant at this moment, where we select the appropriate terms of various orders and arrange them in a way that underscores the lowest relevant order.

$$\begin{aligned} \mathfrak{S} &= \sum_n c^n b_n \\ &\approx \sum_{n=2}^{\infty} \frac{c^n}{n!}\frac{(n-1)!}{2}\int \left[\prod_{i=1}^{n}\left(-\beta U_{i,i+1}\mathrm{d}^3r_i\right)\right]\Bigg|_{n+i=i} + \mathfrak{S}_1 \\ &\approx \sum_{n=2}^{\infty} \frac{\mathfrak{S}^{\text{dist.}}}{n!} + \mathfrak{S}_1 \end{aligned}$$

$\mathfrak{S}_1$ is a term proportional to $\int(-\beta U(r))\,\mathrm{d}^3r$, and this term disappears when the system is neutral. This will be explained in detail in the upcoming sections. Despite arising from the selection of specific terms, the $(n!)^{-1}$ can also be interpreted as making the remaining part $\mathfrak{S}^{\text{dist}}$ indistinguishable when multiplied with it. The selection of such factors through diagrammatic representation will be discussed in the upcoming sections.

2.1.6 Fourier Integral in Three dimension

Here we define the coordinate and momentum Fourier transform as follows:

$$\begin{aligned}\langle f\rangle &= \int f(r,\theta,\phi)\exp\left(-\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}}\right)\mathrm{d}^3r\\ f(r,\theta,\phi) &= \int \langle f\rangle \exp\left(\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}\\ \delta^3(\vec{\mathbf{r}}) &= \int \exp\left(\mathtt{i}\vec{\mathbf{r}}.\vec{\mathbf{p}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}\end{aligned} \tag{2.33}$$

If function $f(r)$ is a central potential function that depends only on the radial coordinate and is independent of angles θ and ϕ, then

$$\begin{aligned}\langle f\rangle &= \int f(r)\exp\left(\pm\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}}\right)\mathrm{d}^3r\\ f(r) &= \int \langle f\rangle \exp\left(\mp\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}\end{aligned} \tag{2.34}$$

The reason behind this is

$$\int f(r)\exp\left\{\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}}\right\}\mathrm{d}^3r = 4\pi\int f(r)\times\frac{\sin(pr)}{pr}\times r^2dr \tag{2.35}$$

As $p\to -p$ doesn't change the integrand because the integrand is even w.r.t p.

2.1.7 Evaluation of $\widetilde{\beta}_n/V$

Consider an integral as shown below (it is a part of Z_cV^{-N} shown in the Eq. (2.23)). Then, one can calculate a specific component of $\bar{\beta}_n/V$, denoted as $\widetilde{\beta}_n/V$, by ignoring the permutations of particles, under the approximation $\eta_{ij}\approx g_{ij}=-\beta U_{ij}$ as,

$$\begin{aligned}\frac{\widetilde{\beta}_n}{V} &= \frac{1}{V}\int \eta_{12}\eta_{23}\dots\eta_{n1}\;d\tau\\ &= \int\prod_{j=1}^{n}\left[\int\frac{\mathrm{d}^3k_j}{(2\pi)^3}\;\langle\eta_{k_j}\rangle e^{i\vec{\mathbf{k}}_j.(\vec{\mathbf{r}}_j-\vec{\mathbf{r}}_{j+1})}\right]|_{r_{n+1}=r_1}d\tau\end{aligned} \tag{2.36}$$

with $d\tau = \mathrm{d}^3 r_1 \;\; \mathrm{d}^3 r_2 \;\; \mathrm{d}^3 r_3 \ldots \mathrm{d}^3 r_n$ In Eq. (2.36) we ignore the contribution of permutations of particles within the cluster. It will be considered in the Section 2.1.8. (a).

$$
\begin{aligned}
\frac{\widetilde{\beta}_n}{V} &= \frac{1}{V}\int \prod_{j=1}^{n}\left[\mathrm{d}^3 k_j \;\; \langle \eta_{k_j}\rangle\right] \times \prod_{j=1}^{n}\left[e^{i(\vec{\mathbf{k}}_j-\vec{\mathbf{k}}_{j-1}).\vec{\mathbf{r}}_j}\frac{\mathrm{d}^3 r_j}{(2\pi)^3}\right] \\
&= \frac{1}{V}\int \prod_{j=1}^{n}\left[\langle \eta_{k_j}\rangle \;\; \delta^3(\vec{\mathbf{k}}_j-\vec{\mathbf{k}}_{j-1})\,\mathrm{d}^3 k_j\right] \\
&= \frac{1}{V}\int \left[\prod_{j=1}^{n}\langle \eta_{k_j}\rangle\right]\delta^3(\vec{\mathbf{k}}_1-\vec{\mathbf{k}}_2)\delta^3(\vec{\mathbf{k}}_2-\vec{\mathbf{k}}_3)\ldots\delta^3(\vec{\mathbf{k}}_1-\vec{\mathbf{k}}_n)\,\mathrm{d}^3 k_1\ldots\mathrm{d}^3 k_n \\
&= \frac{1}{V}\int \langle \eta_{k_n}\rangle\langle \eta_{k_1}\rangle^{n-1}\delta^3(\vec{\mathbf{k}}_1-\vec{\mathbf{k}}_n)\delta^3(\vec{\mathbf{k}}_1-\vec{\mathbf{k}}_n)\,\mathrm{d}^3 k_n\,\mathrm{d}^3 k_1 \\
&= \frac{1}{V}\int \langle \eta_{k_1}\rangle^{n}\delta^3(0)\,\mathrm{d}^3 k_1 \\
&= \int \langle \eta_{k_1}\rangle^{n}\;\frac{\mathrm{d}^3 k_1}{(2\pi)^3}\frac{(2\pi)^3\delta^3(0)}{V}
\end{aligned}
\tag{2.37}
$$

This approximation is done by omitting the permutation within each cluster. Using the asymptotic formula

$$
\underset{V\to\infty}{\mathrm{Lt}}\frac{\widetilde{\beta}_n}{V} = \int \langle \eta_p\rangle^n \frac{d^3 p}{(2\pi)^3} \tag{2.38}
$$

where

$$
\eta_p = \int \eta_r e^{i\vec{\mathbf{p}}.\vec{\mathbf{r}}}\,\mathrm{d}^3 r \tag{2.39}
$$

is the momentum representation of η_r.

2.1.8 Evaluation at the lowest order

The evaluation of Eq. (2.21) can be simplified using diagrams. So far we have not considered the number of ways the particles can get arranged within the same cluster. Doing that using algebra is a bit hard. But using some diagram representation will ease the task.

The expansion of $\exp(x) - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$.

$$
\begin{aligned}
\therefore \eta_{ij} &= \exp\left(g_{ij}\right) - 1 \approx g_{ij} + \frac{g_{ij}^2}{2} \qquad (2^{nd} \text{ Order}) \\
\eta_{ij}\eta_{jk}\eta_{ki} &= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \frac{g_{ij}^{n_1}}{n_1!}\frac{g_{jk}^{n_2}}{n_2!}\frac{g_{ki}^{n_3}}{n_3!} \approx g_{ij}g_{jk}g_{ki} \quad (1^{st} \text{ Order}) \\
\eta_{ij}\eta_{jk}\eta_{kl}\eta_{li} &\approx g_{ij}g_{jk}g_{kl}g_{li} \quad (1^{st} \text{ Order}) \\
&\vdots \\
\eta_{12}\eta_{23}\dots\eta_{n-1,n}\eta_{n1} &\approx g_{12}g_{23}\dots g_{n-1,n}g_{n1} \quad (1^{st} \text{ Order})
\end{aligned} \tag{2.40}
$$

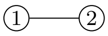
with $g_{ij} = -\beta U_{ij}$.

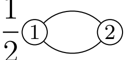
2.1.8. (a) Total ways of arranging the particles within a diagram

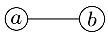
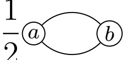
Let us write $\mathfrak{S}_n = \sum_{n\geq 2} c^n b_n \approx \sum_{n\geq 2} c^n S_n$, where S_n is the diagrammatic representation of b_n. We use the diagram with the alphabets as a template, but the actual diagram is the one with the points marked by numbers. The first few terms are

$$
S_2 = \frac{1}{V}\int \mathrm{d}^3r_1\,\mathrm{d}^3r_2\ \left[g_{12} + \frac{g_{12}g_{21}}{2!}\right] \qquad (2^{nd} \text{ Order approximation})
$$

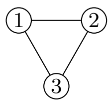
$$
S_2 = \frac{1}{V}\int d\tau_{1,2}\left[g_{12} + \frac{g_{12}^2}{2}\right] =
$$

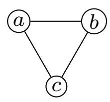


$+ \frac{1}{2}$ 

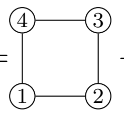
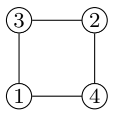
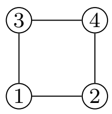
$=$  $+ \frac{1}{2}$  (2.41)

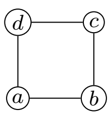
$$
S_3 = \frac{1}{V}\int d\tau_{1,2,3}\ g_{12}\ g_{23}\ g_{31} =
$$



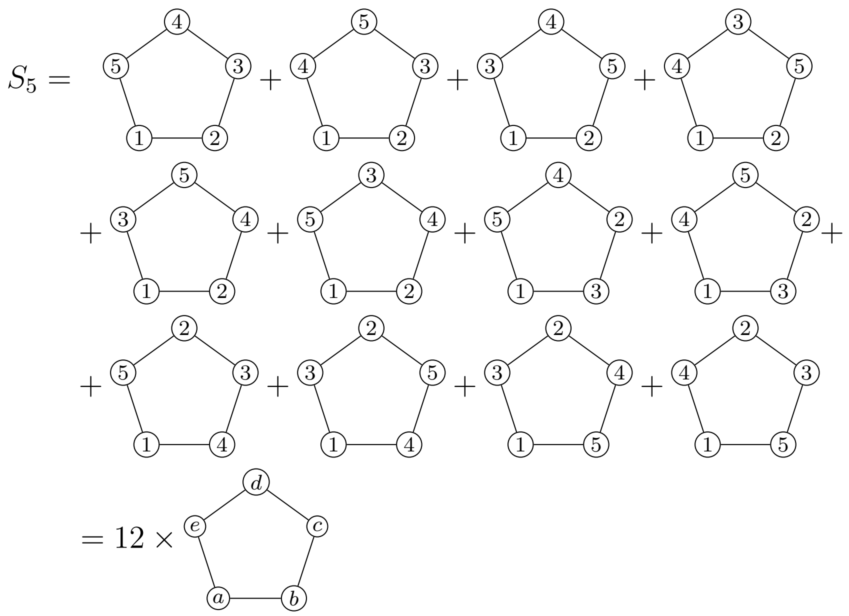
$=$ 

$$
\begin{aligned}
S_4 = &\frac{1}{V}\int d\tau_{1,2,3,4}\ g_{12}\ g_{23}\ g_{34}\ g_{41} + \frac{1}{V}\int d\tau_{1,2,3,4}\ g_{14}\ g_{42}\ g_{23}\ g_{31} \\
&+ \frac{1}{V}\int d\tau_{1,2,3,4}\ g_{12}\ g_{24}\ g_{43}\ g_{31}
\end{aligned}
$$

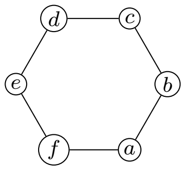
$=$  $+$  $+$  (2.42)

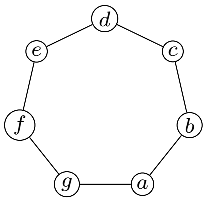
$= 3 \times$ 

$$
\begin{aligned}
S_5 = {} & \frac{1}{V}\int d\tau_{1,2,3,4,5} \;\; g_{12}\, g_{23}\, g_{34}\, g_{45}\, g_{51} + g_{12}\, g_{23}\, g_{35}\, g_{54}\, g_{41} \\
& + \frac{1}{V}\int d\tau_{1,2,3,4,5} \;\; g_{12}\, g_{25}\, g_{54}\, g_{43}\, g_{31} + g_{12}\, g_{25}\, g_{53}\, g_{34}\, g_{41} \\
& + \frac{1}{V}\int d\tau_{1,2,3,4,5} \;\; g_{12}\, g_{24}\, g_{45}\, g_{53}\, g_{31} + g_{12}\, g_{24}\, g_{43}\, g_{35}\, g_{51} \\
& + \frac{1}{V}\int d\tau_{1,2,3,4,5} \;\; g_{13}\, g_{32}\, g_{24}\, g_{45}\, g_{51} + g_{13}\, g_{32}\, g_{25}\, g_{54}\, g_{41} \\
& + \frac{1}{V}\int d\tau_{1,2,3,4,5} \;\; g_{14}\, g_{43}\, g_{32}\, g_{25}\, g_{51} + g_{14}\, g_{45}\, g_{52}\, g_{23}\, g_{31} \\
& + \frac{1}{V}\int d\tau_{1,2,3,4,5} \;\; g_{15}\, g_{54}\, g_{42}\, g_{23}\, g_{31} + g_{15}\, g_{53}\, g_{32}\, g_{24}\, g_{41}
\end{aligned}
\tag{2.43}
$$



(2.44)

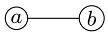
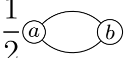
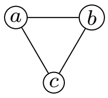
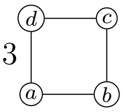
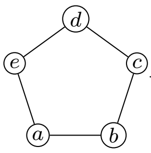
$$S_6 = 60 \times$$ 

$$S_7 = 360 \times$$ 

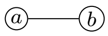
$$\vdots \quad \vdots \quad \vdots$$

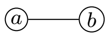
$$S_n = \frac{(n-1)!}{2} \times [\text{n sided polygon}] \tag{2.45}$$

Here $(n-1)!$ is the contribution of permutations and factor $1/2$ to cancel the same order (cycle) values (i.e., to eliminate terms like 1234 and 4321).

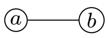
$$\sum_{n=2}^{\infty} S_n =$$  $+ \frac{1}{2}$  $+$  $+ 3$  $+ 12$  $+$

$$\cdots + \frac{(n-1)!}{2} (\text{n sided polygon}) \tag{2.46}$$

$=$  $+ \sum_{n \geq 2} \frac{(n-1)!}{2} (\text{n sided polygon})$

$=$  $+ \sum_{n \geq 2} \frac{(n-1)!}{2} \frac{1}{V} \int g_{12} g_{23} \cdots g_{n-1,n} g_{n1} \; d\tau_{1,2,3,4\ldots,n}$

Now using Eq. (2.38) we get

$$\sum_{n=2}^{\infty} S_n =$$  $$+ \sum_{n \geq 2} \frac{(n-1)!}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \langle g \rangle^n \tag{2.47}$$

where

$$\langle g \rangle = \int g(r) e^{i\vec{\mathbf{p}}.\vec{\mathbf{r}}} \, \mathrm{d}^3 r \tag{2.48}$$

We have from Eq. (2.24)

$$\mathfrak{S} = \sum_{n \geq 2} c^n \frac{\beta_n}{V} = \sum_{n \geq 2} c^n S_n \quad \text{(Here we have considered the permutations.)} \tag{2.49}$$

The difference between Eq. (2.38) and Eq. (2.49) is that in the former we have not considered the permutations within the cluster, but in the latter we have taken care of the permutations. We have used the unbarred β_n to represent the permutations involved in calculations, and barred $\bar{\beta}_n$ to represent the permutations ignored calculations.

2.1.8. (b) Distinguishable particles

Now $\mathfrak{S}$ is

$$\mathfrak{S}^{\text{dist}} = \sum_{n=2}^{\infty} c^n S_n = c^2 \textcircled{a}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{b} + \sum_{n\geq 2} \frac{(n-1)!}{2}\; c^n \int \langle g \rangle^n \, \frac{\mathrm{d}^3 p}{(2\pi)^3} \tag{2.50}$$

2.2 Partition functions, Probability and Physics

2.2.1 Permutation Rule

Consider n objects taken all at once; with
p_1 objects are of 1^{st} kind
p_2 objects are of 2^{nd} kind
$\vdots$
p_k objects are of k^{th} kind
Then

$$\text{Total number of permutations} = \frac{n!}{\mathrm{p_1}! \times \mathrm{p_2}! \times \cdots \times \mathrm{p_k}!} \tag{2.51}$$

2.2.2 N same particles vs. N different particles

In statistical physics, the number of states is evaluated by the integration of the corresponding distribution function in phase space. i.e.,

$$\int f(q,p) \frac{d^{3N}q d^{3N}p}{h^{3N}} \rightarrow \textbf{For N different particle arrangements} \tag{2.52}$$

This integral consider N different particles, say $\{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \ldots, \mathrm{A}_n\}$. So it will give the result for the maximum permutations of the system i.e $N!$ ways. Now if the N particles are same kind such as $A_1 = A_2 = \cdots = A_n = A$. Now the number of permutations possible for the system is 1. So one has to multiply the integral by a factor of $1/N!$

$$\frac{1}{N!} \int f(q,p) \frac{d^{3N}q d^{3N}p}{h^{3N}} \rightarrow \textbf{For N same particle arrangements} \tag{2.53}$$

In a nutshell, the total number of ways we can arrange N different particles are $N!$. So we can say that

$$\boxed{\textbf{N! ways of arrangement} \rightarrow \int f(q,p)\frac{d^{3N}qd^{3N}p}{h^{3N}}} \tag{2.54}$$

$$\boxed{\textbf{1 way of arrangement} \rightarrow \frac{1}{N!}\int f(q,p)\frac{d^{3N}qd^{3N}p}{h^{3N}}} \tag{2.55}$$

This $N!$ comes in the statistical physics in the name of Gibb's correction factor.

2.2.2. (a) Example 1

Consider a system with three particles A_1, A_2, A_3 i.e., the sample space according to the integral is like

$$S = \{(A_1, A_2, A_3), (A_1, A_3, A_2), (A_2, A_1, A_3),$$
$$(A_2, A_3, A_1), (A_3, A_1, A_2), (A_3, A_2, A_1)\}$$

That is 3!. When the particles are of same kind, sample space reduces to $S = (A, A, A)$. So one has to multiply the corresponding integral by a factor 1/3!

2.2.3 N particles with two kinds

Consider three particles A, A, B, the number of ways we can arrange them is

$$S = \{(A, A, B), (A, B, A), (B, A, A)\} \tag{2.56}$$

i.e., $3!/(2!1!) = 3$ ways.
If the particles were A,A,B,B then the sample space will be

$$S = \{(AABB), (ABAB), (ABBA), (BABA), (BAAB), (BBAA)\} \tag{2.57}$$

i.e., $4!/(2!2!) = 6$ ways.
Consider N such particles in which number of A particle is n_A and number of B particle is n_B, and $n_A + n_B = N$, then

$$\textbf{The number of ways we can arrange them is } \frac{N!}{n_A!n_B!} \tag{2.58}$$

So in such a case where two different particles are involved the integral equations we use should also change accordingly

$$\textbf{N! ways of arrangement} \rightarrow \int f(q,p)\frac{d^{3N}qd^{3N}p}{h^{3N}} \tag{2.59}$$

$$\frac{\mathbf{N!}}{\mathbf{n_A!n_B!}}\ \textbf{ways of arrangement} \rightarrow \frac{\mathbf{1}}{\mathbf{n_A!n_B!}}\int f(q,p)\frac{d^{3N}qd^{3N}p}{h^{3N}} \tag{2.60}$$

2.2.4 N particles with m kinds

If there are a total of N particles with m kinds, with each of them are of $n_1.n_2, \dots n_m$ in numbers with $n_1 + n_2 + \cdots + n_m = N$ then

$$\textbf{The number of ways it can be arranged} \rightarrow \frac{\mathbf{N!}}{\mathbf{n}_1\mathbf{!n}_2\mathbf{!} \dots \mathbf{n}_m\mathbf{!}} \tag{2.61}$$

Now

$$\textbf{N! ways of arrangement} \rightarrow \int f(q,p) \frac{d^{3N}qd^{3N}p}{h^{3N}} \tag{2.62}$$

$$\frac{\mathbf{N!}}{\mathbf{n_1!n_2!\dots n_m!}} \textbf{ ways of arrangement} \rightarrow \frac{\mathbf{1}}{\mathbf{n_1!n_2!\dots n_m!}} \int f(q,p) \frac{d^{3N}qd^{3N}p}{h^{3N}}$$

So for a total number of N particles with m kinds one has to write the integral as

$$\frac{\mathbf{1}}{\mathbf{n_1!n_2!\dots n_m!}} \int f(q,p) \frac{d^{3N}qd^{3N}p}{h^{3N}} = \frac{\mathbf{1}}{\mathbf{n_1!n_2!\dots n_m!}} \int f(q,p) \frac{d^{3N}qd^{3N}p}{\hbar^{3N}(2\pi)^{3N}} \tag{2.63}$$

2.3 Particles of same kind (Indistinguishable)

In Section 2.1.1, we have pointed out that we considered the particles as distinguishable and removed the $N!$ (Gibb's factor) from the calculations. Now we are considering particles of same kind i.e., particles that are indistinguishable. So from Section 2.2, we have to consider the $N!$. We had

$$\mathfrak{S}^{\text{dist}} = \sum_{n=2}^{\infty} c^n S_n = c^2 ⓐ\!\!-\!\!-\!\!ⓑ + \sum_{n\geq 2} \frac{(n-1)!}{2}\; c^n \int \langle g \rangle^n \;\frac{\mathrm{d}^3 p}{(2\pi)^3} \tag{2.64}$$

Now let us make it indistinguishable by adding $1/n!$ to it.
i.e.,

$$\begin{aligned}\mathfrak{S}^{\text{indist.}} &= \sum_{n=2}^{\infty} c^n S_n = \frac{c^2}{2} ⓐ\!\!-\!\!-\!\!ⓑ + \sum_{n\geq 2} \frac{(n-1)!}{2}\frac{1}{n!}\; c^n \int \langle g \rangle^n \;\frac{\mathrm{d}^3 p}{(2\pi)^3} \\ &= \frac{c^2}{2} ⓐ\!\!-\!\!-\!\!ⓑ + \sum_{n\geq 2} \frac{c^n}{2n} \int \langle g \rangle^n \;\frac{\mathrm{d}^3 p}{(2\pi)^3} \\ &= \frac{c^2}{2} ⓐ\!\!-\!\!-\!\!ⓑ - \frac{1}{2}\Big(\, c\;\langle g \rangle + \ln\left[1 - c\;\langle g \rangle\right] \Big)\end{aligned} \tag{2.65}$$

Now using the relation

$$\frac{P}{k_B T} - c = \mathfrak{S} - c\frac{\partial \mathfrak{S}}{\partial c} \tag{2.66}$$

we get

$$\frac{P}{k_B T} - c = -\left(\frac{c^2}{2}\textcircled{a}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{b} + \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\ln(1 - c\langle g\rangle) + \frac{c\ \langle g\rangle}{1 - c\ \langle g\rangle}\right]\right) \tag{2.67}$$

Now if we look back we know that

$$\begin{aligned} g(r) &= -\beta U(r) \\ \langle g\rangle &= -\beta\langle U\rangle \qquad \text{(Corresponding Fourier transform)} \end{aligned} \tag{2.68}$$

Therefore we redefine

$$\textcircled{a}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{b} = \frac{1}{V}\int g(r)\,\mathrm{d}^3 r \to -\beta\frac{1}{V}\int U(r)\,\mathrm{d}^3 r = \ -\beta\ \textcircled{a}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{b} \tag{2.69}$$

Then let us re write

$$\boxed{\frac{P}{k_B T} - c = \frac{\beta c^2}{2}\ \textcircled{a}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{b} + \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{c\ \beta\langle U\rangle}{1 + c\beta\ \langle U\rangle} - \ln\left[1 + \beta c\ \langle U\rangle\right]\right]} \tag{2.70}$$

where

$$\textcircled{a}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{b} = \frac{1}{V}\int U(r)\,\mathrm{d}^3 r \tag{2.71}$$

$$\langle U\rangle = \int U(r) e^{i\vec{\mathbf{p}}.\vec{\mathbf{r}}}\,\mathrm{d}^3 r \tag{2.72}$$

2.4 Particles of Two kinds with opposite charges

Consider two particles with opposite charges, from the probability section if we consider the particles as indistinguishable then

$$\frac{P}{k_B T} - c_i - c_e = \mathfrak{S} - c_i\frac{\partial \mathfrak{S}}{\partial c_i} - c_e\frac{\partial \mathfrak{S}}{\partial c_e} \tag{2.73}$$

It is easy to verify the equation, on putting $c_i + c_e = c$ and $\dfrac{\partial c}{\partial c_i} = \dfrac{\partial c}{\partial c_e} = 1$ leads us back to Eq. (2.30). For a two particle system, from Eqs. (2.46) and (2.60), let us write

$$\begin{aligned}\mathfrak{S} = & \sum_{n_i+n_e=2} \frac{c_i^{n_i} c_e^{n_e}}{2n_e! n_i!} \textcircled{i}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{j} \\ & + \sum_{n_i+n_e \geq 2} \frac{(n_i+n_e-1)!}{2n_i! n_e!} \frac{c_i^{n_i} c_e^{n_e}}{V} \int g_{12} g_{23} \dots g_{n-1,n} g_{n1} \; d\tau_{1,2,3,\dots,n}\end{aligned} \tag{2.74}$$

If these particles are charge dependent,
i.e,

$$g_{ij} \to z_i z_j g'_{ij} = z_i z_j g'(|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j|) \tag{2.75}$$

with $z_i, z_j \in \{+1, -1\}$. Then,

$$\begin{aligned} g_{ij} g_{ji} &= z_i z_j \times z_j z_i \times g'_{ij} g'_{ji} = z_i^2 z_j^2 g'_{ij} g'_{ji} = g'_{ij} g'_{ji} \\ g_{ij} g_{jk} g_{ki} &= z_i z_j \times z_j z_k \times z_k z_i \times g'_{ij} g'_{jk} g'_{ki} = z_i^2 z_j^2 z_k^2 g'_{ij} g'_{jk} g'_{ki} = g'_{ij} g'_{jk} g'_{ki} \\ &\vdots \\ g_{12} g_{23} \dots g_{n1} &= z_1^2 z_2^2 z_3^2 \dots z_n^2 \times g'_{12} g'_{23} \dots g'_{n1} = g'_{12} g'_{23} \dots g'_{n1} \end{aligned} \tag{2.76}$$

If the system has only two kinds of particles having charges either $z_i(+1)$ or z_e (-1), with the numbers n_i and n_e, such that $n_i + n_e = n$ then

$$z_1^2 z_2^2 z_3^2 \dots z_n^2 = z_i^{2n_i} z_e^{2n_e} = 1 \tag{2.77}$$

So

$$\begin{aligned}\mathfrak{S} = & \sum_{n_i+n_e=2} \frac{c_i^{n_i} c_e^{n_e}}{2n_e! n_i!} \textcircled{i}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{j} \\ & + \sum_{n_i+n_e \geq 2} \frac{(n_i+n_e-1)!}{2n_i! n_e!} \frac{c_i^{n_i} c_e^{n_e} z_i^{2n_i} z_e^{2n_e}}{V} \int g'_{12} g'_{23} \dots g'_{n-1,n} g'_{n1} \; d\tau_{1,2,\dots,n}\end{aligned} \tag{2.78}$$

2.4.1 The neutrality condition

Assume that the medium is charge neutral. i.e., $c_i z_i + c_e z_e = 0$, if $|z_i| = |z_e|$ then $c_i = c_e = \frac{c}{2}$. Let us take the first term from $\mathfrak{S}$

$$\mathfrak{S}_1 = \sum_{n_i+n_e=2} \frac{c_i^{n_i} c_e^{n_e}}{2n_e! n_i!} \textcircled{i}\!\!-\!\!\!-\!\!\!-\!\!\textcircled{j} = \sum_{n_i+n_e=2} \frac{c_i^{n_i} c_e^{n_e}}{2n_e! n_i!} \frac{1}{V} \int z_i^{n_i} z_e^{n_e} g'_{ij} \, \mathrm{d}^3 r_i \, \mathrm{d}^3 r_j \tag{2.79}$$

The term $z_i^{n_i} z_i^{n_e}$ came from $g_{ij} = z_i^{n_i} z_e^{n_e} g'_{ij}$ with $n_i + n_e = 2$, the permutation is considered here, i.e., both the two particle clusters can be of the same charge

(- - or ++), or both can be different (- +, + -). If it was $g_{ij}g_{ji}$ then we would get $z_i^{2n_i} z_e^{2n_e}$. However we have the relation

$$\begin{aligned}(a+b)^n &= \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \\ &= \sum_{j=0}^{n} \frac{n!}{(n-j)!j!} a^j b^{n-j} \\ &= \sum_{j=0}^{n} \frac{[(n-j)+j]!}{(n-j)!j!} a^j b^{n-j}\end{aligned} \tag{2.80}$$

Putting $n-j=i$ we get

$$\begin{aligned}(a+b)^n &= \sum_{i+j=n} \frac{(i+j)!}{i!j!} a^j b^i \\ (a+b)^2 &= \sum_{i+j=2} \frac{(i+j)!}{i!j!} a^j b^i = \frac{2}{i!j!} a^j b^i\end{aligned} \tag{2.81}$$

Now let us write

$$\begin{aligned}\mathfrak{S}_1 &= \sum_{n_i+n_e=2} \frac{c_i^{n_i} c_e^{n_e}}{2n_e!n_i!} \frac{1}{V} \int z_i^{n_i} z_e^{n_e} g_{12}\, \mathrm{d}^3r_1\, \mathrm{d}^3r_2 \\ &= \frac{1}{4} \sum_{n_i+n_e=2} \frac{2}{n_e!n_i!} (c_i z_i)^{n_i} (c_e z_e)^{n_e} \int g_{12}\, \mathrm{d}^3r_{12} \\ &= \frac{1}{4} (c_i z_i + c_e z_e)^2 \int g_{12}\, \mathrm{d}^3r_{12} \\ &\text{Now, if we apply neutrality condition then } c_i z_i + c_e z_e = 0 \\ &= 0\end{aligned} \tag{2.82}$$

Now we are left with

$$\begin{aligned}\mathfrak{S} &= \sum_{n_i+n_e \geq 2} \frac{(n_i+n_e-1)!}{2n_i!n_e!} \frac{c_i^{n_i} c_e^{n_e} z_i^{2n_i} z_e^{2n_e}}{V} \int g'_{12} g'_{23} \dots g'_{n-1,n} g'_{n1}\; d\tau_{1,2,3,\dots,n} \\ &= \sum_{n_i+n_e \geq 2} \frac{1}{2(n_i+n_e)} \frac{(n_i+n_e)!}{n_i!n_e!} \left(c_i z_i^2\right)^{n_i} \left(c_e z_e^2\right)^{n_e} \int g'_{12} g'_{23} \dots g'_{n-1,n} g'_{n1}\; d\tau_{1,2,3,\dots,n} \\ &= \sum_{n_i+n_e \geq 2} \frac{1}{2(n_i+n_e)} \left(c_i z_i^2 + c_e z_e^2\right)^{n_i+n_e} \int g'_{12} \dots g'_{n-1,n} g'_{n1}\; d\tau_{1,2,3,\dots,n}\end{aligned} \tag{2.83}$$

$$\begin{aligned}
& z_i^2 = z_e^2 = 1,\ c_i + c_e = c,\ n_i + n_e = n \\
& = \sum_{n\geq 2} \frac{c^n}{2n} \int g'_{12} g'_{23} \ldots g'_{n-1,n} g'_{n1}\ d\tau_{1,2,3,\ldots,n} \\
& = \sum_{n\geq 2} \frac{c^n}{2n} \int \langle g' \rangle^n \frac{\mathrm{d}^3 p}{(2\pi)^3} \\
& = -\frac{1}{2} \left[\ln\left(1 - c\langle g' \rangle\right) + c\langle g' \rangle \right]
\end{aligned} \tag{2.84}$$

Now using the relation

$$\frac{P}{k_B T} - c = \mathfrak{S} - c\frac{\partial \mathfrak{S}}{\partial c} \tag{2.85}$$

we get

$$\frac{P}{k_B T} - c = -\frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\ln\left(1 - c\langle g' \rangle\right) + \frac{c\langle g' \rangle}{1 - c\langle g' \rangle} \right] \tag{2.86}$$

If we redefine $\langle g' \rangle = -\beta \langle \phi \rangle$, we get Eq. (2.91).
In a nutshell, we approximated

$$\begin{aligned}
\eta_{ij} &= \exp\left(-\beta U_{ij}\right) - 1 \\
&\approx -\beta U_{ij} = g_{ij} \text{ (First order)} \\
g_{ij} &= z_i z_j g'_{ij} = -z_i z_j \beta \phi(r_{ij}) \\
\langle g' \rangle &= -\beta \langle \phi \rangle \qquad \text{(Corresponding Fourier transform)}
\end{aligned} \tag{2.87}$$

i.e.,

$$U(r_{ij}) = z_i z_j \phi(r_{ij}) \tag{2.88}$$

Now combining Eqs. (2.86) and (2.88), we get

$$\boxed{\frac{P}{k_B T} - c = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\frac{c\ \beta \langle \phi \rangle}{1 + c\beta\ \langle \phi \rangle} - \ln\left[1 + \beta c\ \langle \phi \rangle\right] \right]} \tag{2.89}$$

with

$$\langle \phi \rangle = \int \phi(r) e^{\mathbf{i}\vec{\mathbf{p}}.\vec{\mathbf{r}}}\, \mathrm{d}^3 r \tag{2.90}$$

In natural units $k_B = 1$ and $\beta = 1/T$, so

$$\left.\frac{P}{T}\right|_{q\bar{q}} - c = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\frac{\beta c \langle \phi \rangle}{1 + \beta c \langle \phi \rangle} - \ln\left[1 + \beta c\ \langle \phi \rangle\right] \right] \tag{2.91}$$

If the concentration is a function of temperature

$$\begin{aligned}\left.\frac{\varepsilon(T)}{T}\right|_{q\bar{q}} &= \frac{\partial}{\partial \ln(T)}\left[\frac{P}{T}\right] \\ &= \frac{\partial c}{\partial \ln(T)} + \frac{1}{2}\left(1 - \frac{\partial \ln(c)}{\partial \ln(T)}\right)\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right)^2.\end{aligned} \tag{2.92}$$

These are the two important equations for the pressure and energy density of the system, which has an equal number of particles and anti-particles. i.e., the system is neutral in terms of the corresponding charges. These Eqs. (2.91) and (2.92) are the lowest-order Mayer's cluster expansion equations.

In Chapter 3, the above equations and procedures are used to derive the equation of state for QGP using a modified liquid drop potential. A comparison with Gamow's liquid drop model is also done. The modified liquid drop model is validated by fitting it with the the lattice data.

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Chapter 3

Modified Liquid Drop Model in QGP

In 1995, Bannur [1] introduced a new method by combining Mayer's cluster expansion with the QGP number density function to study the equations of state of quark-gluon plasma. In his work, only the quark contribution is considered, and the potential is dependent on the charge of the interating particles. The model also demanded a neutral system in which the number of antiparticles and particles should be the same. In that work, the Cornell potential of the form $a/r - br$ was examined. Later, it was extended by Udayanandan and VM Bannur [2] to include the gluon contribution. Prasanth and Bannur estimated the transport coefficient for modified Cornell potential in [3].

3.1 Study of QGP using MCE with an arbitrary central potential

The general steps for computing the EoS of QGP using MCE are as follows:

1. Find the appropriate central potential f(r).
2. Check whether the potential has a dependence on charge and find the appropriate model equation.
3. Find an appropriate converging function that goes to unity when the parameter goes to zero.
4. Three dimensional Fourier Transform the central potential with a converging factor for momentum representation.

5. Apply it to the pressure momentum relation.

6. Using standard statistical mechanics, find out various thermodynamic quantities.

3.1.1 Fourier transform of of polynomial function in radial coordinate with inverse radial term

Consider the three dimensional Fourier transform of central potential of the form $a_n r^n$ where $n \in \{-1, 1, 2, 3, \ldots, \infty\}$ and a_n is a constant. Let us choose the converging function with converging parameter h as $\underset{h\to 0}{\mathrm{Lt}} \exp(-hr) = 1$. As mentioned earlier, for a central potential function f(r),

$$\begin{aligned} \langle f \rangle &= \int f(r) \exp(\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}})\, \mathrm{d}^3 r = \int f(r) \exp(-\mathtt{i}\vec{\mathbf{p}}.\vec{\mathbf{r}})\, \mathrm{d}^3 r \\ &= 4\pi \int_0^\infty f(r) \frac{\sin(pr)}{pr} \times r^2 dr \end{aligned} \tag{3.1}$$

i.e., For central potential, Eq. (3.1) is an even function in p. In other words, the three dimensional Fourier transform from radial coordinate to momentum is independent of the sign of $|p|$.

The three dimensional Fourier transform for an arbitrary function of the form $a_n r^n$ with the converging parameter $\underset{h\to 0}{\mathrm{Lt}} \exp(-hr)$ is

$$\begin{aligned} \langle a_n r^n \rangle &= 4\pi \int_0^\infty a_n r^n \times \exp(-hr) \times \frac{\sin(pr)}{pr} \times r^2 dr \\ &= \frac{4\pi a_n}{p} \ \mathrm{Im} \left\{ \int_0^\infty r^{n+1} \exp\left(-r(h - \mathtt{i}p)\right) \right\} \\ &= \frac{4\pi a_n}{p} \times \Gamma\left(n+2\right) \times \mathrm{Im} \left\{ (h - ip)^{-(n+2)} \right\} \\ &= \frac{4\pi a_n}{p} \frac{\Gamma\left(n+2\right)}{(p^2+h^2)^{n+2}} \mathrm{Im}\left\{ (h + \mathtt{i}p)^{n+2} \right\} \\ &\approx \frac{4\pi a_n}{p} \frac{\Gamma\left(n+2\right)}{(p^2+h^2)^{\frac{n+2}{2}}} \sin\left[(n+2) \sin^{-1}\left(\frac{p}{\sqrt{h^2+p^2}} \right) \right] \end{aligned} \tag{3.2}$$

i.e., If we substitute $s = p^2 + h^2$ (for ease in notation),

$$\begin{aligned} &\left\langle \frac{a}{r} \right\rangle = 4\pi \frac{a}{s} \implies \underset{h\to 0}{\mathrm{Lt}} \left\langle \frac{a}{r} \right\rangle = \underset{h\to 0}{\mathrm{Lt}} \ 4\pi \frac{a}{s} \\ &\left\langle a_1 r \right\rangle = 4\pi \left[\frac{8h^2}{s^3} - \frac{2}{s^2} \right] a_1 \implies \underset{h\to 0}{\mathrm{Lt}} \left\langle a_1 r \right\rangle = -\frac{8\pi a_1}{s^2} \end{aligned} \tag{3.3}$$

$$\left\langle a_2 r^2 \right\rangle = 4\pi \left[\frac{48h^3}{s^4} - \frac{24h}{s^3}\right] a_2 \implies \underset{h\to 0}{\mathrm{Lt}} \left\langle a_2 r^2 \right\rangle = 0$$
$$\left\langle a_3 r^3 \right\rangle = 4\pi \left[\frac{384h^4}{s^5} - \frac{288h^2}{s^4} + \frac{24}{s^3}\right] a_3 \implies \underset{h\to 0}{\mathrm{Lt}} \left\langle a_3 r^3 \right\rangle = \frac{96\pi}{s^3} a_3 \tag{3.4}$$

One could easily find that for $n =$ even, $\langle a_n r^n \rangle = 0$. Similarly,

$$\left\langle a_{2k-1}\, r^{2k-1} \right\rangle = (-1)^k \times 4\pi \frac{\Gamma(2k+1)}{s^{k+1}} \times a_{2k-1} \tag{3.5}$$

with $s = \underset{h\to 0}{\mathrm{Lt}}\ p^2 + h^2$ and k being a whole number.

3.1.2 Pressure cluster relation

We have derived in Eqs. (2.91) and (2.92) that the pressure relation in natural units is

$$\frac{P}{T} = c + \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\frac{c\beta\langle\phi\rangle}{1 + c\beta\langle\phi\rangle} - \ln\left[1 + \beta c\langle\phi\rangle\right]\right] \tag{3.6}$$

where the potential energy is $U_{ij} = z_i z_j \phi_{ij}$, with $z_i, z_j \in \{-1, 1\}$. Consider a potential of the form $\beta c\langle\phi\rangle = \sum_{j=1}^{n} \frac{a_j}{s^j}$, then

$$\frac{\beta c\langle\phi\rangle}{1 + \beta c\langle\phi\rangle} = \frac{\sum_{j=1}^{n} \frac{a_j}{s^j}}{1 + \sum_{j=1}^{n} \frac{a_j}{s^j}} = \frac{\sum_{j=1}^{n} a_j s^{n-j}}{s^n + \sum_{j=1}^{n} a_j s^{n-j}} = \frac{\mathcal{N}(s)}{\mathcal{D}(s)} \tag{3.7}$$

Let $\mathcal{D}(s) = \prod_{j=1}^{N} (s - z_j)^{l_j}$ with $\sum_{j=1}^{N} l_j = n$. In other words, $\mathcal{D}(s)$ will have N distinct roots with different degeneracy factors. The j^{th} root is called z_j with degeneracy l_j. Since $\mathcal{D}$(s) is real, let there be ν pairs of complex roots (z_ν and z_ν^*) and $N - 2\nu$ pairs of real roots (x_τ). Therefore, using partial fraction decomposition,

$$\frac{\mathcal{N}(s)}{\mathcal{D}(s)} = \frac{\mathcal{D}(s) - s^n}{\mathcal{D}(s)} = \sum_{j=1}^{\nu} \sum_{i=1}^{l_j} \left[\frac{A_{ji}}{(s - z_j)^i} + \frac{A_{ji}^*}{\left(s - z_j^*\right)^i}\right] + \sum_{\tau=1}^{N-2\nu} \sum_{i=1}^{\bar{l}_\tau} \frac{\bar{A}_{\tau,i}}{(s - x_\tau)^i} \tag{3.8}$$

with $2\sum_{j=1}^{\nu} l_\nu + \sum_{j=1}^{N-2\nu} \bar{l}_j = n$, where $\bar{l}_\tau$ denotes the degeneracy of τ^{th} real root (x_τ) and l_ν is the ν^{th} complex pair root's (z_ν and z_ν^*) degeneracy. Similarly

$$\mathcal{D}(s) = \prod_{j=1}^{\nu}(s - z_j)^{l_j}(s - z_j^*)^{l_j} \times \prod_{\tau=1}^{N-2\nu}(s - x_\tau)^{\bar{l}_\tau} \tag{3.9}$$

$$\begin{aligned}\ln\left(1 + \beta c\langle\phi\rangle\right) &= \ln\left(1 + \sum_{j=1}^{n}\frac{a_j}{s^j}\right) \\ &= \ln\left[\mathcal{D}(s)\right] - n\ln(s) \\ &= \sum_{j=1}^{\nu} l_j\left[\ln\left(s - z_j\right) + \ln\left(s - z_j^*\right)\right] + \sum_{\tau=1}^{N-2\nu}\bar{l}_\tau\ln(s - x_\tau) - n\ln(s)\end{aligned} \tag{3.10}$$

We can evaluate the pressure by using a method called *dimensional regularization* which we will explain in the next section. The integral results are added in the following equation. The complete derivation can be found in Section 3.3. Now

$$\begin{aligned}\frac{P}{T} = c &+ \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{j=1}^{\nu}\sum_{i=1}^{l_j}\left[\frac{A_{ji}}{\left(s - z_j\right)^i} + \frac{A_{ji}^*}{\left(s - z_j^*\right)^i}\right] \\ &+ \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{\tau=1}^{N-2\nu}\sum_{i=1}^{\bar{l}_\tau}\frac{\bar{A}_{\tau,i}}{\left(s - x_\tau\right)^i} \\ &- \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{j=1}^{\nu} l_j\left[\ln\left(s - z_j\right) + \ln\left(s - z_j^*\right)\right] \\ &- \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_{\tau=1}^{N-2\nu}\bar{l}_\tau\ln(s - x_\tau) + \frac{n}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ln(s)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\frac{P}{T} = c &+ \frac{1}{2}\sum_{j=1}^{\nu}\sum_{i=1}^{l_j}\left[\mathrm{Re}\left\{A_{ji}\right\}\mathtt{Q}_i(z_j) - \mathrm{Im}\left\{A_{ji}\right\}\mathtt{R}_i(z_j)\right] \\ &+ \frac{1}{2}\sum_{\tau=1}^{N-2\nu}\sum_{i=1}^{\bar{l}_\tau}\bar{A}_{\tau,i}\left[\mathtt{J}_i(x_\tau) + \mathtt{Ji}_i(x_\tau)\right] - \frac{1}{2}\sum_{j=1}^{\nu} l_j\ \mathtt{Mn}(z_j) \\ &- \frac{1}{2}\sum_{\tau=1}^{N-2\nu}\bar{l}_\tau\ \left[\mathtt{Ln}(x_\tau) + \mathtt{Lm}(x_\tau)\right] + \frac{n}{2}\mathtt{Ln}(0)\end{aligned} \tag{3.12}$$

with

$$\begin{aligned}\mathtt{Q}_n(z) &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{2\pi^2}\,\mathrm{Re}\left\{(-z)^{\frac{3}{2}-n}\right\} \\ &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{2\pi^2}|z|^{\frac{3}{2}-n}\cos\left[\left(\frac{3}{2}-n\right)\cos^{-1}\left(-\frac{\mathrm{Re}\{z\}}{|z|}\right)\right]\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathtt{R}_n(z) &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{2\pi^2}\,\mathrm{Im}\left\{(-z)^{\frac{3}{2}-n}\right\} \\ &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{2\pi^2}|z|^{\frac{3}{2}-n}\sin\left[\left(\frac{3}{2}-n\right)\cos^{-1}\left(-\frac{\mathrm{Re}\{z\}}{|z|}\right)\right]\mathtt{sgn}\left(-\mathrm{Im}(z)\right)\end{aligned} \tag{3.14}$$

$$\begin{aligned}\mathtt{Mn}(z) &= \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{3\pi^2}\,\mathrm{Re}\left\{(-z)^{\frac{3}{2}}\right\} \\ &= -\frac{\sqrt{2}}{6\pi}\left[\lambda_R\Lambda - \frac{\lambda_I^2}{\Lambda}\right]\end{aligned} \tag{3.15}$$

where $\Lambda = \sqrt{\lambda_R + \xi}$, $\quad \lambda_R = -\mathrm{Re}(z)$, $\quad \lambda_I = -\mathrm{Im}(z) \quad \xi = \sqrt{\lambda_R^2 + \lambda_I^2}$

$$\begin{aligned}\mathtt{Ln}(x) &= \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{6\pi^2}(-x)^{\frac{3}{2}}\,\Theta(-x) \\ \mathtt{Lm}(x) &= \mathtt{i}\frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{6\pi^2}(x)^{\frac{3}{2}}\,\Theta(x)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\mathtt{J}_n(x) &= \frac{B\left(n-\frac{3}{2},\frac{3}{2}\right)}{4\pi^2}(-x)^{\frac{3}{2}-n}\,\Theta(-x) \\ \mathtt{Ji}_n(x) &= \mathtt{i}\ (-1)^n\ \frac{B\left(n-\frac{3}{2},\frac{3}{2}\right)}{4\pi^2}(x)^{\frac{3}{2}-n}\,\Theta(x) \\ \mathtt{I}_n(x) &= \mathtt{J}_n(x) + \mathtt{Ji}_n(x) \\ \overline{\mathtt{Ln}} &= \mathtt{Ln}(x) + \mathtt{Lm}(x)\end{aligned} \tag{3.17}$$

3.1.3 Dealing with divergences

We have

$$\frac{P}{T} = c + \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{c\beta\langle\phi\rangle}{1+c\beta\langle\phi\rangle} - \ln\left[1+\beta c\langle\phi\rangle\right]\right] \tag{3.18}$$

On adding and subtracting $\beta c \langle\phi\rangle$ from both the terms we get,

$$
\begin{aligned}
\frac{P}{T} &= c + \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{c\beta\langle\phi\rangle}{1+c\beta\langle\phi\rangle} - \beta c\langle\phi\rangle\right] \\
&\quad + \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta c\langle\phi\rangle - \ln\left(1+\beta c\langle\phi\rangle\right)\right] \\
&= c + \frac{1}{2}\left[\mathfrak{t} - \beta c\frac{\partial \mathfrak{t}}{\partial(\beta c)}\right] \\
\text{with } \mathfrak{t} &= \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta c\langle\phi\rangle - \ln\left(1+\beta c\langle\phi\rangle\right)\right]
\end{aligned}
\tag{3.19}
$$

As shown in Section 3.3.3, assume that a divergence k_1 appear in the integral $\frac{\partial \mathfrak{t}}{\partial(\beta c)}$ as a constant of integration, which is independent of βc
i.e.,

$$
\begin{aligned}
\frac{\partial \mathfrak{t}}{\partial(\beta c)} &= \frac{-1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{\langle\phi\rangle}{1+c\beta\langle\phi\rangle} - \langle\phi\rangle\right] \\
&= \left[\frac{\partial \mathfrak{t}}{\partial(\beta c)}\right]^{\text{finite}} + k_1
\end{aligned}
\tag{3.20}
$$

Therefore

$$
\begin{aligned}
\mathfrak{t} &= \int \mathrm{d}[\beta c]\left(\left[\frac{\partial \mathfrak{t}}{\partial(\beta c)}\right]^{\text{finite}} + k_1\right) \\
&= \mathfrak{t}^{\text{finite}} + \beta c \;\; k_1
\end{aligned}
\tag{3.21}
$$

Combining Eqs. (3.19) to (3.21) we get

$$
\begin{aligned}
\frac{P}{T} &= c + \frac{1}{2}\left[\mathfrak{t} - \beta c\frac{\partial \mathfrak{t}}{\partial(\beta c)}\right] \\
&= c + \frac{1}{2}\left[\mathfrak{t}^{\text{finite}} - \beta c\frac{\partial \mathfrak{t}^{\text{finite}}}{\partial(\beta c)}\right]
\end{aligned}
\tag{3.22}
$$

i.e., The divergence terms cancel themselves, and the integral remains finite. Therefore

$$
\frac{P}{T} = c + \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{c\beta\langle\phi\rangle}{1+c\beta\langle\phi\rangle} - \ln\left[1+\beta c\langle\phi\rangle\right]\right] = \text{Finite} \tag{3.23}
$$

3.2 Examples

3.2.1 Potential of the form $\beta c\langle\phi\rangle = \frac{a_1}{s}$

Consider potential of the form $U(r_{ij}) = z_i z_j b/r$ in spherical polar coordinate. The three dimensional Fourier transform is of the form $\beta c\langle\phi\rangle = a_1/s$ with

$a_1 = 4\pi\beta bc$. Here

$$\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle} - \ln\left(1+\beta c\langle\phi\rangle\right) = \frac{a_1}{s+a_1} - \ln(s+a_1) + \ln(s) \tag{3.24}$$

So

$$\begin{aligned}\frac{P}{T} &= c + \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle} - \ln\left(1+\beta c\langle\phi\rangle\right)\right] \\ &= c + \frac{a_1\mathtt{I}_1(-a_1) - \overline{\mathtt{Ln}}(-a_1) + \overline{\mathtt{Ln}}(0)}{2}\end{aligned} \tag{3.25}$$

But the effective pressure is the real part of the the pressure. So

$$\left.\frac{P}{T}\right|_{\text{eff.}} = \frac{\operatorname{Re}\{P\}}{T} = c + \frac{a_1\mathtt{J}_1(-a_1) - \mathtt{Ln}(-a_1) + \mathtt{Ln}(0)}{2} \tag{3.26}$$

But

$$\mathtt{J}_1(x) = \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{4\pi^2}\sqrt{-x} = -\frac{\sqrt{-x}}{4\pi} \implies \mathtt{J}_1(-a_1) = -\frac{\sqrt{a_1}}{4\pi} \tag{3.27}$$

$$\mathtt{Ln}(x) = \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{6\pi^2}(-x)^{\frac{3}{2}} = -\frac{(-x)^{\frac{3}{2}}}{6\pi} \implies \mathtt{Ln}(-a_1) = -\frac{a_1^{\frac{3}{2}}}{6\pi} \tag{3.28}$$

Thus

$$\left.\frac{P}{T}\right|_{\text{eff.}} = c - \frac{a_1^{\frac{3}{2}}}{24\pi} \tag{3.29}$$

When $c = c_i + c_e$, and $a_1 = 4\pi\beta bc = \kappa^2$, So

$$\left.\frac{P}{T}\right|_{\text{eff.}} = c_i + c_e - \frac{\kappa^3}{24\pi} \tag{3.30}$$

The result is the same as that of [4].

3.2.2 Potential of the form $\beta c\langle\phi\rangle = \frac{a_1}{s} + \frac{a_2}{s^2}$

Here $\mathcal{D}(s) = s^2 + a_1 s + a_2$ can have all real roots when $D_1^2 = a_1^2 - 4a_2 > 0$, or two imaginary roots which are complex conjugate to each other i.e., $D_2^2 = 4a_2 - a_1^2 > 0$ or same root with degeneracy $D_3^2 = 0 \implies 4a_2 = a_1^2$. In the result below we have accommodated the three possibilities as shown.

$$
\begin{aligned}
&\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle} - \ln\left(1+\beta c\langle\phi\rangle\right) = \frac{a_1 s + a_2}{s^2 + a_1 s + a_2} - \ln\left(s^2 + a_1 s + a_2\right) + 2\ln(s) \\
&= \left[\frac{\alpha_1}{s-x_1} + \frac{\alpha_2}{s-x_2} - \ln(s-x_1) - \ln(s-x_2) + 2\ln(s)\right]\Theta(D_1^2) \\
&+ \left[\frac{\beta_1}{s-z_1} + \frac{\beta_2}{s-z_2} - \ln(s-z_1) - \ln(s-z_2) + 2\ln(s)\right]\Theta(D_2^2) \\
&+ \left[\frac{\gamma_1}{s-y} + \frac{\gamma_2}{(s-y)^2} - 2\ln(s-y) + 2\ln(s)\right]\delta(0, D_3)
\end{aligned}
\tag{3.31}
$$

with

$$
\begin{aligned}
&D_1^2 = a_1^2 - 4a_2 > 0\ ,\ \alpha_1 = -\frac{x_1^2}{x_1 - x_2},\ \alpha_2 = -\frac{x_2^2}{x_2 - x_1},\ x_1 = \frac{1}{2}\left(D_1 - a_1\right), \\
&x_2 = -\frac{1}{2}(D_1 + a_1) \\
&D_2^2 = 4a_2 - a_1^2 > 0\ ,\ \beta_1 = -\frac{z_1^2}{z_1 - z_2},\ \beta_2 = -\frac{z_2^2}{z_2 - z_1},\ z_1 = \frac{1}{2}(\mathtt{i}D_2 - a_1), \\
&\quad z_2 = z_1^* \\
&D_3^2 = 4a_2 - a_1^2 = 0,\ \gamma_2 = -a_2 \text{ and } \gamma_1 = \pm 2\sqrt{a_2} \text{ the sign } \pm \text{ is decided} \\
&\quad \text{by the sign of } a_1 \text{i.e., } 2\sqrt{a_2} = a_1, y = \mp\sqrt{a_2}.
\end{aligned}
\tag{3.32}
$$

So

$$
\begin{aligned}
&\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle} - \ln\left(1+\beta c\langle\phi\rangle\right)\right] \\
&= \left[\alpha_1 \mathtt{I}_1(x_1) + \alpha_2 \mathtt{I}_1(x_2) - \overline{\mathtt{Ln}}(x_1) - \overline{\mathtt{Ln}}(x_2) + 2\overline{\mathtt{Ln}}(0)\right]\Theta(D_1^2) \\
&+ \left[\beta_R \mathtt{Q}_1(z_1) - \beta_I \mathtt{R}_1(z_1) - \mathtt{Mn}(z_1) + 2\overline{\mathtt{Ln}}(0)\right]\Theta(D_2^2) \\
&+ \left[\gamma_1 \mathtt{I}_1(y) + \gamma_2 \mathtt{I}_2(y) - 2\overline{\mathtt{Ln}}(y) + 2\overline{\mathtt{Ln}}(0)\right]\delta_{0,D_3}
\end{aligned}
\tag{3.33}
$$

with

$$
\begin{aligned}
\beta_R &= \frac{a_1}{2} \\
\beta_I &= \frac{a_1^2 - D_2^2}{4D_2} = \frac{a_1^2 - 2a_2}{2\sqrt{4a_2 - a_1^2}} \\
\mathtt{Q}_1(z_1) &= \frac{B\left(-\frac{1}{2}, \frac{3}{2}\right)}{4\pi^2}\sqrt{a_1 + 2\sqrt{a_2}} = -\frac{\sqrt{2\sqrt{a_2} + a_1}}{4\pi} \\
\mathtt{R}_1(z_1) &= -\frac{B\left(-\frac{1}{2}, \frac{3}{2}\right)}{4\pi^2}\sqrt{-a_1 + 2\sqrt{a_2}} = \frac{\sqrt{2\sqrt{a_2} - a_1}}{4\pi}
\end{aligned}
\tag{3.34}
$$

Now the effective pressure is

$$\left.\frac{P}{T}\right|_{\text{eff.}} = c + \frac{1}{2}\left[\alpha_1 \mathtt{J}_1(x_1) + \alpha_2 \mathtt{J}_1(x_2) - \mathtt{Ln}(x_1) - \mathtt{Ln}(x_2)\right]\Theta(D_1^2) + \frac{\left[\beta_R \mathtt{Q}_1(z_1) - \beta_I \mathtt{R}_1(z_1) - \mathtt{Mn}(z_1)\right]}{2}\Theta(D_2^2) + \frac{\left[\gamma_1 \mathtt{J}_1(y) + \gamma_2 \mathtt{J}_2(y) - 2\mathtt{Ln}(y)\right]}{2}\delta_{0,D_3} \tag{3.35}$$

For $D_2^2 = 4a_2 - a_1^2 < 0$, and changing symbols $c = \kappa^2 T$, $a_i \to \kappa^2 a_i$ and applying it to the above equation, gives us

$$\frac{P}{T} = \kappa^2 T - \frac{1}{24\pi}\frac{a_2\kappa^{\frac{3}{2}} + a_1\sqrt{a_2}\kappa^{\frac{5}{2}} + a_1^2\kappa^{\frac{7}{2}}}{\sqrt{2\sqrt{a_2} + a_1\kappa}} \;. \tag{3.36}$$

This result is same as that of Bannur [1].

3.2.3 Potential of the form $\beta c\langle\phi\rangle = \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}$

$$\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle} - \ln\left(1+\beta c\langle\phi\rangle\right) = \left[\sum_{j=1}^{3}\frac{\alpha_j}{s-x_j} - \sum_{j=1}^{3}\ln(s-x_j) + 3\ln(s)\right]\Theta(-D) + \left[\frac{\alpha}{s-x} + \frac{\beta_1}{s-z} + \frac{\beta_1^*}{s-z_1^*} - \ln(s-x) - \ln(s-z_1) - \ln(s-z_1^*) + 3\ln(s)\right]\Theta(D) + \left[\frac{\gamma_1}{s-x} + \frac{\gamma_2}{(s-x)^2} + \frac{\gamma_3}{(s-x)^3} - 3\ln(s-x) + 3\ln(s)\right]\delta_{D,0} \tag{3.37}$$

Similarly

$$\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right]^2 = \left[\sum_{j=1}^{3}\frac{\alpha_i}{s-x_i}\right]^2\Theta(-D) + \left(\frac{\alpha}{s-x} + \frac{\beta_1}{s-z} + \frac{\beta_1^*}{s-z^*}\right)^2\Theta(D) + \left(\frac{\gamma_3}{s-y} + \sum_{j=1}^{2}\frac{\gamma_j}{(s-x)^j}\right)\delta_{D,0} \tag{3.38}$$

with

$$Q = \frac{3a_2 - a_1^2}{9} \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}, \quad D = Q^3 + R^2, \quad S = \left(R + \sqrt{D}\right)^{\frac{1}{3}}, \quad T = -\frac{Q}{S} \tag{3.39}$$

$$
\begin{aligned}
&x_1 = -\frac{a_1}{3} + 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right); \quad x_2 = -\frac{a_1}{3} + 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right); \\
&x_3 = -\frac{a_1}{3} + 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
\alpha_1 &= -\frac{x_1^3}{(x_1 - x_2)(x_1 - x_3)}; \quad \alpha_2 = -\frac{x_2^3}{(x_2 - x_1)(x_2 - x_3)}; \\
\alpha_3 &= -\frac{x_3^3}{(x_3 - x_1)(x_3 - x_2)}
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
x &= S + T - \frac{a_1}{3}; \quad z = -\frac{1}{2}(S+T) - \frac{a_1}{3} + \frac{\mathtt{i}\sqrt{3}}{2}(S - T); \\
\alpha &= -\frac{x^3}{(x-z)(x-z^*)}; \; \beta_R = -\operatorname{Re}\left\{\frac{z^3}{(z-x)(z-z^*)}\right\}; \\
\beta_I &= -\operatorname{Im}\left\{\frac{z^3}{(z-x)(z-z^*)}\right\}, \; \beta_1 = \beta_R + \mathtt{i}\beta_I
\end{aligned}
\tag{3.42}
$$

3.2.3. (a) When all roots are real and unequal

$$
\begin{aligned}
\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right]^2 &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\sum_{i=1}^{3}\frac{\alpha_i^2}{(s-x_i)^2} + \sum_{i,j=1, i\neq j}^{3}\frac{\alpha_i\alpha_j}{x_i - x_j}\left(\frac{1}{s-x_i} - \frac{1}{s-x_j}\right)\right] \\
&= \sum_{i=1}^{3}\alpha_i^2 \mathtt{I}_2(x_i) + 2\sum_{i>j=1}^{3}\frac{\alpha_i\alpha_j}{x_i - x_j}\left(\mathtt{I}_1(x_i) - \mathtt{I}_1(x_j)\right)
\end{aligned}
\tag{3.43}
$$

3.2.3. (b) When two roots are complex conjugate, and the third one is real

$$
\begin{aligned}
\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right]^2 &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\frac{\alpha}{s-x} + \frac{\beta_1}{s-z} + \frac{\beta_1^*}{s-z^*}\right]^2 \\
&= \alpha^2 \mathtt{I}_2(x) + \operatorname{Re}\left\{\beta_1^2\right\}\mathtt{Q}_2(z) - \operatorname{Im}\left\{\beta_1^2\right\}\mathtt{R}_2(z) \\
&+ 2\alpha\left[\operatorname{Re}\left\{\frac{\beta_1}{z-x}\right\}\mathtt{Q}_1(z) - \operatorname{Im}\left\{\frac{\beta_1}{z-x}\right\}\mathtt{R}_1(z)\right] \\
&+ \frac{|\beta_1|^2}{\operatorname{Im}(z)}\mathtt{R}_1(z) + 4\alpha\operatorname{Re}\left\{\frac{\beta_1}{x-z}\right\}\mathtt{I}_1(x)
\end{aligned}
\tag{3.44}
$$

3.2.3. (c) When atleast two roots are real and equal third one is real and unequal

$$\begin{aligned}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right]^2 = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\gamma_1}{s-x}+\frac{\gamma_2}{(s-x)^2}+\frac{\gamma_3}{s-y}\right]^2 \\ =\gamma_1^2\mathtt{I}_2(x)+\gamma_2^2\mathtt{I}_4(x)+\gamma_3^2\mathtt{I}_2(y) \\ +2\gamma_1\gamma_2\mathtt{I}_3(x)+\frac{2\gamma_1\gamma_3}{x-y}\left(\mathtt{I}_1(x)-\mathtt{I}_1(y)\right) \\ +2\gamma_2\gamma_3\left[\frac{1}{xy}+\frac{1}{x(x-y)}-\frac{x}{y(x-y)^2}\right]\mathtt{I}_1(x) \\ +2\gamma_2\gamma_3\left[\frac{\mathtt{I}_2(x)}{x-y}+\frac{\mathtt{I}_1(y)}{(x-y)^2}\right]\end{aligned} \tag{3.45}$$

3.2.3. (d) When all roots are real and equal

$$\begin{aligned}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right]^2 = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\gamma_1}{s-x}+\frac{\gamma_2}{(s-x)^2}+\frac{\gamma_3}{(s-x)^3}\right]^2 \\ =\gamma_1^2\mathtt{I}_2(x)+\gamma_2^2\mathtt{I}_4(x)+\gamma_3\mathtt{I}_6(x) \\ +2\gamma_1\gamma_2\mathtt{I}_3(x)+2\gamma_1\gamma_3\mathtt{I}_4(x)+2\gamma_2\gamma_3\mathtt{I}_5(x)\end{aligned} \tag{3.46}$$

So The general formula from Eq. (2.91) is

$$\frac{P}{T}=c+\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}-\ln\left(1+\beta c\langle\phi\rangle\right)\right] \tag{3.47}$$

and from Eq. (2.92) for $c \propto T^3$

$$\frac{\varepsilon}{T}=3c-\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{\beta c\langle\phi\rangle}{1+\beta c\langle\phi\rangle}\right]^2 \tag{3.48}$$

After fitting these equations with lattice data, the Eq. (3.44) becomes the best fit with lattice data. However, in the next section, we can see that the fitted parameters caused x to be positive, so $\mathtt{Ln}(x)$ goes to zero.

$$\frac{P}{T}=c+\frac{1}{2}\left[\alpha\mathtt{I}_1(x)+\mathrm{Re}\left\{\beta_1\right\}\mathtt{Q}_1(z)-\mathrm{Im}\left\{\beta_1\right\}\mathtt{R}_1(z)\right]-\frac{1}{2}\left[\mathtt{Ln}(x)+\mathtt{Mn}(z)\right]$$

$$\begin{aligned}\frac{\varepsilon}{T}=3c-\left(\alpha^2\mathtt{I}_2(x)+\mathrm{Re}\left\{\beta_1^2\right\}\mathtt{Q}_2(z)-\mathrm{Im}\left\{\beta_1^2\right\}\mathtt{R}_2(z)\right) \\ -\left(2\alpha\left[\mathrm{Re}\left\{\frac{\beta_1}{z-x}\right\}\mathtt{Q}_1(z)-\mathrm{Im}\left\{\frac{\beta_1}{z-x}\right\}\mathtt{R}_1(z)\right]\right) \\ -\left(\frac{|\beta_1|^2}{\mathrm{Im}(z)}\mathtt{R}_1(z)+4\alpha\,\mathrm{Re}\left\{\frac{\beta_1}{x-z}\right\}\mathtt{I}_1(x)\right)\end{aligned} \tag{3.49}$$

with

$$
\begin{aligned}
Q &= \frac{3a_2 - a_1^2}{9} \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}, \quad D = Q^3 + R^2, \\
S &= \left(R + \sqrt{D}\right)^{\frac{1}{3}}, \quad T = -\frac{Q}{S} \\
a_1 &= -4\pi\beta r_0 a_c, \quad a_2 = 8\pi\beta r_0^{-1} a_l, \quad a_3 = -96\pi\beta r_0^{-3} a_v. \\
x &= S + T - \frac{a_1}{3}; \quad z = -\frac{1}{2}(S+T) - \frac{a_1}{3} + \frac{\mathtt{i}\sqrt{3}}{2}(S-T); \\
\alpha &= -\frac{x^3}{(x-z)(x-z^*)}; \ \operatorname{Re}\{\beta_1\} = -\operatorname{Re}\left\{\frac{z^3}{(z-x)(z-z^*)}\right\}; \\
\operatorname{Im}\{\beta_1\} &= -\operatorname{Im}\left\{\frac{z^3}{(z-x)(z-z^*)}\right\} \ ; \beta_1 = \operatorname{Re}\{\beta_1\} + \mathtt{i}\operatorname{Im}\{\beta_1\}
\end{aligned} \tag{3.50}
$$

Here $x \in \mathbb{R}^+$, so $\mathtt{I}_n(x)$ contains imagniary term, so taking the real term contribution we get

$$
\begin{aligned}
\frac{\operatorname{Re}\{\varepsilon\}}{T} = \frac{\varepsilon}{T} - \mathtt{i}\operatorname{Im}\left(\frac{\varepsilon}{T}\right) = 3c &+ \left[\operatorname{Im}^2\{\beta_1\} - \operatorname{Re}^2\{\beta_1\}\right]\mathtt{Q}_2(z) + 2\operatorname{Re}\{\beta_1\}\operatorname{Im}\{\beta_1\}\mathtt{R}_2(z) \\
&+ 2\alpha\left[\operatorname{Im}\left\{\frac{\beta_1}{z-x}\right\}\mathtt{R}_1(z) - \operatorname{Re}\left\{\frac{\beta_1}{z-x}\right\}\mathtt{Q}_1(z)\right] \\
&- \frac{|\beta_1|^2}{\operatorname{Im}\{z\}}\mathtt{R}_1(z)
\end{aligned} \tag{3.51}
$$

Similarly

$$
\frac{\operatorname{Re}(P)}{T} = c + \frac{1}{2}\left[\operatorname{Re}\{\beta_1\}\mathtt{Q}_1(z) - \operatorname{Im}\{\beta_1\}\mathtt{R}_1(z) - \mathtt{Mn}(z)\right] \tag{3.52}
$$

The imaginary part contribution can be written as

$$
\begin{aligned}
\frac{\operatorname{Im}\{\varepsilon\}}{T} &= -\alpha^2\operatorname{Im}\{\mathtt{I}_2(x)\} - 4\alpha\operatorname{Re}\left\{\frac{\beta_1}{x-z}\right\}\operatorname{Im}\{\mathtt{I}_1(x)\} \\
\frac{\operatorname{Im}\{P\}}{T} &= \frac{\alpha}{2}\operatorname{Im}\{\mathtt{I}_1(x)\} - \frac{1}{2}\operatorname{Im}\{\mathtt{Ln}(x)\}
\end{aligned} \tag{3.53}
$$

The integral result shown above can be derived in the section below.

3.3 Integral Results

3.3.1 Regularization via Contour integration method

Consider integrals of the form $\int_{-\infty}^{\infty} N(x)/D(x)\, dx$, where $N(x)$ and $D(x)$ are functions of polynomials with degree of $N(x)$ is less than $D(x)$ by two, and

$D(x) \neq 0 \; \forall x \in \mathbb{R}$.
Then one could evaluate the integral by defining $f(x) = N(x)/D(x)$, and evaluate the integral by

$$\int_c f(z)dz = \int_{-\Gamma}^{\Gamma} f(z)dz + \int_\epsilon f(z)dz \tag{3.54}$$

As

$$\lim_{\Gamma\to\infty} \int_\epsilon f(z)dz = 0 \tag{3.55}$$

So

$$\int_c f(z)dz = \int_{-\infty}^{\infty} f(x)dx \tag{3.56}$$

3.3.1. (a) Examples

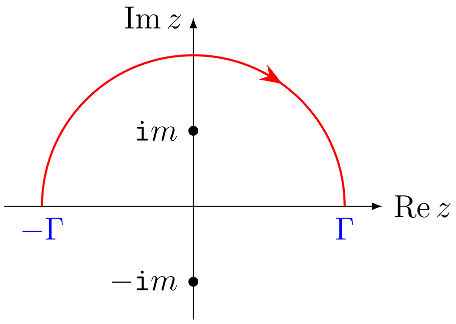
Consider the integral of the form

$$\begin{aligned}\mathfrak{I}_1(m^2) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2+m^2} \\ &= \int_0^\infty \frac{dp}{2\pi^2}\frac{p^2}{p^2+m^2}\end{aligned}$$

The numerator and denominator of above integral is not differ by a degree of two. In order to solve the integral one could take a differentiation w.r.t m^2.

$$\begin{aligned}\mathtt{I}_2(m^2) = -\frac{\partial}{\partial m^2}\mathfrak{I}_1(m^2) &= \int_0^\infty \frac{p^2}{(p^2+m^2)^2}\frac{dp}{2\pi^2} \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{p^2}{(p^2+m^2)^2}dp\end{aligned} \tag{3.57}$$

We are taking a semicircle with imaginary axis is taken along positive y axis.



Setting $f(p) = p^2/(p^2+m^2)^2$, with $N(p) = p^2$ and $D(p) = (p^2+m^2)^2$. Since

$N(p)$ is less than $D(p)$ by a degree of two, and $D(p) \neq 0 \forall p \in \mathbb{R}$, we could apply contour integral to find the solution

$$\begin{aligned}
\mathtt{I}_2(m^2) = -\frac{\partial}{\partial m^2}\mathtt{I}_1(m^2) &= \lim_{\Gamma\to\infty}\int_{-\Gamma}^{\Gamma}\frac{z^2}{(z^2+m^2)^2}\frac{dz}{4\pi^2} \\
&= 2\pi\mathtt{i} \times \text{Residue at}\left(\frac{z^2}{(z^2+m^2)^2}\right)\Bigg|_{z\to\mathtt{i}m}
\end{aligned}$$

We can take only the part in the upper half plane semicircle domain of the contour.

$$\begin{aligned}
\text{Residue at } z \to \mathtt{i}m &= \lim_{z\to\mathtt{i}m}\frac{d}{dz}\left\{(z-im)^2\frac{z^2}{(z^2+m^2)^2}\right\} \\
&= \lim_{z\to\mathtt{i}m}\frac{2\mathtt{i}mz}{(z+\mathtt{i}m)^3} = -\frac{\mathtt{i}}{4m}
\end{aligned}$$

$$\begin{aligned}
\text{So, } \mathtt{I}_2(m^2) = -\frac{\partial}{\partial m^2}\mathfrak{I}_1(m^2) &= 2\pi\mathtt{i} \times -\frac{\mathtt{i}}{4m} \times \frac{1}{4\pi^2} \\
-\frac{\partial}{\partial m^2}\mathfrak{I}_1(m^2) &= \frac{1}{8\pi m}
\end{aligned}$$

3.3.2 Regularization via Schwinger's proper time representation

$$\begin{aligned}
\mathtt{I}_2(m^2) = -\frac{\partial}{\partial m^2}\mathfrak{I}_1(m^2) &= \int_{-\infty}^{\infty}\frac{p^2}{(p^2+m^2)^2}\frac{dp}{4\pi^2} \\
&= \int_0^{\infty} d\tau\ \tau\int_{-\infty}^{\infty} p^2 e^{-\tau\left(p^2+m^2\right)}\frac{dp}{4\pi^2} \\
&= \frac{\Gamma\left(\frac{3}{2}\right)}{4\pi^2}\int_0^{\infty} d\tau\ \tau^{-\frac{1}{2}}e^{-\tau m^2} \\
&= \frac{\Gamma\left(\frac{3}{2}\right)}{4\pi^2}\frac{\Gamma\left(\frac{1}{2}\right)}{m} = \frac{1}{8\pi m}
\end{aligned} \tag{3.58}$$

3.3.3 Generalizing

Using the tricks of [5], the results can be obtained as,

$$\begin{aligned}
\mathfrak{I}_1(m^2) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(p^2+m^2)} = -\int_{\mathbf{a}}^{m^2}\mathrm{d}w\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(p^2+w)^2} + \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2+\mathbf{a}} \\
&= \underbrace{-\frac{1}{4\pi}\sqrt{m^2}}_{\mathtt{I}_1(m^2)} + \underbrace{\frac{1}{4\pi}\sqrt{\mathbf{a}} + \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2+\mathbf{a}}}_{k_1(\mathbf{a})} \\
&= \mathtt{I}_1(m^2) + k_1(\mathbf{a})
\end{aligned} \tag{3.59}$$

where $\mathbf{a}$ is an arbitrary lower limit of the integral.
We choose $a \to 0$, then

$$\mathfrak{I}_1(m^2) = \underbrace{-\frac{1}{4\pi}\sqrt{m^2}}_{\mathtt{I}_1(m^2)} + \underbrace{\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2}}_{k_1} \tag{3.60}$$

The terms that are independent of m^2, can be considered as constant of integration which contains divergence. i.e., k_1 is considered to be diverging. So in general,

$$\begin{aligned}
\mathtt{I}_1(m^2) &= -\int\frac{1}{8\pi\sqrt{m^2}}dm^2 = -\frac{1}{4\pi}\sqrt{m^2} \\
\mathfrak{I}_1(m^2) &= \int\frac{1}{p^2+m^2}\frac{\mathrm{d}^3p}{(2\pi)^3} = -\frac{m}{4\pi} + k_1
\end{aligned} \tag{3.61}$$

Consider the integral, with n being a natural number,

$$\mathtt{I}_n(m^2) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(p^2+m^2)^n} \tag{3.62}$$

The diverging constant in $\mathfrak{I}_1$ i.e., k_1 get's cancelled in Section 3.1.3. So in our calculation of pressure and energy density using MCE, the Eq. (3.62) definition at $n = 1$ causes no change in the result. Using the result

$$\frac{(-1)^{n-1}}{\Gamma(n)}\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{1}{x+p^2}\right) = \frac{1}{(x+p^2)^n} \tag{3.63}$$

Thus Eq. (3.62) can be expressed as

$$\begin{aligned}
\mathtt{I}_n(m^2) &= \frac{(-1)^{n-1}}{\Gamma(n)}\frac{\partial^{n-1}}{\partial(m^2)^{n-1}}\mathtt{I}_1(m^2) \\
&= -\frac{1}{4\pi}\frac{(-1)^{n-1}}{\Gamma(n)}\frac{\partial^{n-1}}{\partial(m^2)^{n-1}}\sqrt{m^2}
\end{aligned} \tag{3.64}$$

But

$$\frac{\partial^{n-1}}{\partial x^{n-1}}\sqrt{x} = \frac{1}{2}\frac{\Gamma\left(\frac{2n-3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{(-1)^n}{(x)^{\frac{2n-3}{2}}} \tag{3.65}$$

So Eq. (3.64) can be written as

$$\begin{aligned} \mathtt{I}_n(m^2) &= -\frac{1}{4\pi}\frac{(-1)^{n-1}}{\Gamma(n)}\frac{1}{2}\frac{\Gamma\left(n-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{(-1)^n}{m^{2n-3}} \\ &= \frac{1}{8\pi}\frac{\Gamma\left(n-\frac{3}{2}\right)}{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}\frac{1}{(m^2)^{n-\frac{3}{2}}} \\ &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{4\pi^2}\frac{1}{(m^2)^{n-\frac{3}{2}}} \end{aligned} \tag{3.66}$$

So in general

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(p^2+m^2)^n} = \frac{B(n-\frac{3}{2},\frac{3}{2})}{4\pi^2}\frac{1}{(m^2)^{n-\frac{3}{2}}}\Theta\left(m^2\right) \tag{3.67}$$

3.3.4 Circumventing The Poles

Dealing with plasma physics, the appearance of poles in the calculation of some integrals is not new; it corresponds to the plasma instability mechanism [6]. Plasma waves are typically slightly damped by collisions or amplified by some instability mechanism. Therefore, technically, the denominator of the integral never actually goes to zero. Consider integral of the form

$$\begin{aligned} J(m^2) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2-m^2} \\ \frac{\partial}{\partial m^2}J(m^2) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{(p^2-m^2)^2} \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{p^2}{(p^2-m^2)^2}dp \end{aligned} \tag{3.68}$$

The integrand is divergent in $p = \pm m$. So one cannot apply contour method because $Q(p^2) = 0$ at $p = \pm m \in \mathbb{R}$. If we try to circumvent the integral by deviating m^2 from $\mathbb{R} \to \mathbb{C}$ i.e., $m^2 \to m^2 + \mathtt{i}\epsilon$.

$$\begin{aligned} \frac{\partial}{\partial m^2}J(m^2+\mathtt{i}\epsilon) &= \frac{1}{4\pi^2}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{p^2}{(p^2-m^2-\mathtt{i}\epsilon)^2}dp \\ &= \frac{1}{4\pi^2}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\left(\frac{p}{p+\sqrt{m^2+i\epsilon}}\times\frac{1}{p-\sqrt{m^2+i\epsilon}}\right)^2 dp \end{aligned} \tag{3.69}$$

Residues are at $p = \pm z_1 = \pm\sqrt{m^2 + \mathtt{i}\epsilon}$

$$\pm z_1 = \pm\left(\sqrt{\frac{m^2 + \sqrt{m^4+\epsilon^2}}{2}} + \mathtt{i}\frac{\sqrt{2}\ \epsilon}{2\sqrt{m^4 + \sqrt{m^2+\epsilon^2}}}\right)$$

Now if pole is at $z = z_1 \in$ Upper Half plane,

$$\begin{aligned} \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= \frac{2\pi\mathtt{i}}{4\pi^2}\text{Residue at}\left(\frac{z^2}{(z^2 - z_1^2)^2}\right) \\ \text{Residue at } z \to z_1 &= \lim_{z\to z_1} \frac{d}{dz}\left\{(z - z_1)^2 \frac{z^2}{(z^2 - z_1^2)^2}\right\} \\ &= \lim_{z\to z_1} \frac{2z_1 z}{(z + z_1)^3} = \frac{1}{4z_1} \\ \text{So } \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= 2\pi\mathtt{i} \times \frac{1}{4z_1} \times \frac{1}{4\pi^2} \\ \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= \frac{\mathtt{i}}{8\pi\sqrt{m^2 + \mathtt{i}\epsilon}} \\ \frac{\partial}{\partial m^2} J(m^2) = \lim_{\epsilon\to 0} \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= \frac{\mathtt{i}}{8\pi m} \end{aligned} \tag{3.70}$$

If pole is in $z = -z_1 \in$ Lower Half plane using this circumventing pole technique one could derive

$$\begin{aligned} \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= \frac{2\pi\mathtt{i}}{4\pi^2}\text{Residue at}\left(\frac{z^2}{(z^2 - z_1^2)^2}\right) \\ \text{Residue at } z \to -z_1 &= \lim_{z\to -z_1} \frac{d}{dz}\left\{(z + z_1)^2 \frac{z^2}{(z^2 - z_1^2)^2}\right\} \\ &= \lim_{z\to -z_1} -\frac{2z_1 z}{(z - z_1)^3} = -\frac{1}{4z_1} \\ \text{So } \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= -2\pi\mathtt{i} \times \frac{-1}{4z_1} \times \frac{1}{4\pi^2} \\ \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= \frac{\mathtt{i}}{8\pi\sqrt{m^2 + \mathtt{i}\epsilon}} \\ \frac{\partial}{\partial m^2} J(m^2) = \lim_{\epsilon\to 0} \frac{\partial}{\partial m^2} J(m^2 + \mathtt{i}\epsilon) &= \frac{\mathtt{i}}{8\pi m} \end{aligned} \tag{3.71}$$

Using the idea described in Section 3.3.3 about the divergence,

$$\begin{aligned} J(m^2) &= \lim_{\epsilon\to 0} \int \frac{\mathtt{i}}{8\pi\sqrt{m^2 + \mathtt{i}\epsilon}} dm^2 = \frac{\mathtt{i}}{4\pi}\sqrt{m^2} + k_2 \\ J(m^2) &= \lim_{\epsilon\to 0} \int \frac{1}{p^2 - (m^2 \pm \mathtt{i}\epsilon)} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \mathtt{i}\frac{m}{4\pi} + k_2 \end{aligned} \tag{3.72}$$

where k_2 contains divergence. For $n > 1$,

$$\begin{aligned} J(m^2,n) &= \lim_{\epsilon\to 0}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\left(p^2-(m^2+\mathtt{i}\epsilon)\right)^n} \\ &= \frac{1}{\Gamma(n)}\frac{\partial^{n-1}}{\partial\left(m^2\right)^{n-1}}\lim_{\epsilon\to 0}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2-(m^2+\mathtt{i}\epsilon)} \\ &= \frac{\mathtt{i}}{4\pi}\frac{1}{\Gamma(n)}\frac{\partial^{n-1}}{\partial\left(m^2\right)^{n-1}}\sqrt{m^2}\;\Theta\left(m^2\right) \\ &= \mathtt{i}\frac{(-1)^n}{8\pi}\frac{\Gamma(n-3/2)}{\Gamma(n)\Gamma(1/2)}\frac{1}{\left(m^2\right)^{n-3/2}}\;\Theta\left(m^2\right) \end{aligned} \tag{3.73}$$

Therefore

$$J(m^2,n) = \mathtt{i}(-1)^n\frac{B\left(n-\frac{3}{2},\frac{3}{2}\right)}{4\pi^2}\left(m^2\right)^{\frac{3-2n}{2}}\;\Theta\left(m^2\right) \tag{3.74}$$

Combining the results, depending on the value of $m^2 - x$, the integral result $\mathtt{I}_n(m,x)$ for $n > 1$ can be redefined as

$$\begin{aligned} \mathtt{I}_n(m,x) &= \lim_{\epsilon\to 0}\int\frac{1}{\left(p^2+m^2-x\pm i\epsilon\right)^n}\frac{\mathrm{d}^3p}{(2\pi)^3} \\ &= \frac{B\left(n-\frac{3}{2},\frac{3}{2}\right)}{4\pi^2}\left[\frac{\Theta(m^2-x)}{(m^2-x)^{n-3/2}}+\mathtt{i}\frac{(-1)^n}{(x-m^2)^{n-3/2}}\Theta\left(x-m^2\right)\right] \end{aligned} \tag{3.75}$$

and

$$\begin{aligned} \mathfrak{I}_1(m,x) =& \lim_{\epsilon\to 0}\int\frac{1}{(p^2+m^2-x\pm i\epsilon)}\frac{\mathrm{d}^3p}{(2\pi)^3} \\ =& \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{4\pi^2}\left[\frac{\Theta(m^2-x)}{\sqrt{m^2-x}}-\mathtt{i}\frac{1}{\sqrt{x-m^2}}\Theta\left(x-m^2\right)\right] \\ &+ k_1(\mathbf{a})\Theta(m^2-x)+k_2\Theta(x-m^2) \\ =& \mathtt{I}_1(m,x)+k_1\Theta(m^2-x)+k_2\Theta(x-m^2) \end{aligned} \tag{3.76}$$

$$\begin{aligned} \mathtt{J_n}(x) &= \mathrm{Re}\left\{\mathtt{I}_n(0,x)\right\} \\ \mathtt{Ji_n}(x) &= \mathtt{i}\,\mathrm{Im}\left\{\mathtt{I}_n(0,x)\right\} \end{aligned} \tag{3.77}$$

Similarly, for $z \in \mathbb{C}$

$$\int\frac{1}{\left(p^2+m^2-z\right)^n}\frac{\mathrm{d}^3p}{(2\pi)^3} = \frac{B(n-\frac{3}{2},\frac{3}{2})}{4\pi^2}\frac{1}{\left(m^2-z\right)^{n-\frac{3}{2}}} \tag{3.78}$$

So, for $n \geq 2$,

$$\begin{aligned}
\mathtt{Q}_n(m,z) &= \int \mathrm{Re}\left\{\frac{1}{(p^2+m^2-z)^n} + \frac{1}{(p^2+m^2-z^*)^n}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3} \\
\mathtt{Q}_n(m,z) &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{2\pi^2}\,\mathrm{Re}\left\{\left(m^2-z\right)^{\frac{3}{2}-n}\right\} \\
&= \frac{B\left(\frac{3}{2},n-\frac{3}{2}\right)}{2\pi^2}\xi^{\frac{3}{2}-n}\cos\left[\left(\frac{3}{2}-n\right)\cos^{-1}\left(\frac{\lambda_R}{\xi}\right)\right]
\end{aligned} \tag{3.79}$$

for $n=1$, $\int \mathrm{Re}\left\{2/(p^2+m^2-z)\right\}\,\mathrm{d}^3p/(2\pi)^3 = \mathtt{Q}_1(m,z)+2k_1$ and for $n \geq 1$

$$\begin{aligned}
\mathtt{R}_n(m,z) &= \int \mathrm{Im}\left\{\frac{1}{(p^2+m^2-z)^n} - \frac{1}{(p^2+m^2-z^*)^n}\right\}\frac{\mathrm{d}^3p}{(2\pi)^3} \\
\mathtt{R}_n(m,z) &= \frac{B(n-\frac{3}{2},\frac{3}{2})}{2\pi^2}\,\mathrm{Im}\left\{\left(m^2-z\right)^{\frac{3}{2}-n}\right\} \\
&= \mathtt{sgn}(\lambda_I)\frac{B\left(\frac{3}{2},n-\frac{3}{2}\right)}{2\pi^2}\xi^{\frac{3}{2}-n}\sin\left[\left(\frac{3}{2}-n\right)\cos^{-1}\left(\frac{\lambda_R}{\xi}\right)\right] \\
\mathfrak{Mn}(m,z) &= \int \mathrm{Re}\left\{\ln\left(p^2+m^2-z\right)+\ln\left(p^2+m^2-z^*\right)\right\}\frac{\mathrm{d}^3p}{(2\pi)^3} \\
\mathfrak{Mn}(m,z) &= \mathtt{Mn}(m,z)+k_3
\end{aligned} \tag{3.80}$$

with

$$\mathtt{Mn}(m,z) = \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{3\pi^2}\,\mathrm{Re}\left\{\left(m^2-z\right)^{\frac{3}{2}}\right\} = -\frac{\sqrt{2}}{6\pi}\left[\lambda_R\Lambda - \frac{\lambda_I^2}{\Lambda}\right]$$

where $\Lambda = \sqrt{\lambda_R+\xi}$, $\lambda_R = m^2 - \mathrm{Re}(z)$, $\lambda_I = -\,\mathrm{Im}(z)$ $\xi = \sqrt{\lambda_R^2+\lambda_I^2}$ and k_3 is the constant of integration, that contains divergence.

$$\overline{\mathfrak{Ln}}(m,x) = \int \ln\left(s-x\right)\frac{\mathrm{d}^3p}{(2\pi)^3} \tag{3.81}$$

Solving Eq. (3.81) with [5] and Section 3.3.3,

$$\begin{aligned}
\overline{\mathfrak{Ln}}(m,x) =& \overline{\mathtt{Ln}}(m,x)+k_3 \\
\overline{\mathtt{Ln}}(m,x) =& \frac{B\left(-\frac{1}{2},\frac{3}{2}\right)}{6\pi^2}\left[\left(m^2-x\right)^{\frac{3}{2}}\Theta(m^2-x)+\mathtt{i}\left(x-m^2\right)^{\frac{3}{2}}\Theta\left(x-m^2\right)\right]
\end{aligned} \tag{3.82}$$

where k_3 diverges. In Sections 3.1.2, 3.2, 3.4 and 3.4.1. (a), we have defined

$$\begin{aligned}
\mathtt{Ln}(x) &= \mathrm{Re}\left\{\overline{\mathtt{Ln}}(0,x)\right\} \\
\mathtt{Lm}(x) &= \mathrm{Im}\left\{\overline{\mathtt{Ln}}(0,x)\right\} \\
\mathtt{Mn}(z) &= \mathtt{Mn}(0,z) \\
\mathtt{R}_n(z) &= \mathtt{R}_n(0,z) \\
\mathtt{Q}_n(z) &= \mathtt{Q}_n(0,z)
\end{aligned} \tag{3.83}$$

3.3.5 Some complex number summation results

The sum of certain complex numbers are involved in the final steps of integral calculation shown above. The formula we employed can be found as follows: Let $\lambda_R, \lambda_I \in \mathbb{R}$, then $\lambda_R + \mathtt{i}\lambda_I$ is a complex number for $\lambda_I \neq 0$.

$$(\lambda_R + \mathtt{i}\lambda_I)^n + (\lambda_R - \mathtt{i}\lambda_I)^n = 2|\lambda|^n \cos\left[n \cos^{-1}\left(\frac{\lambda_R}{\sqrt{\lambda_R^2 + \lambda_{\mathtt{I}}^2}}\right)\right] \tag{3.84}$$

Similarly

$$(\lambda_R + \mathtt{i}\lambda_I)^n - (\lambda_R - \mathtt{i}\lambda_I)^n = 2\mathtt{i}|\lambda|^n \sin\left[n \cos^{-1}\left(\frac{\lambda_R}{\sqrt{\lambda_R^2 + \lambda_{\mathtt{I}}^2}}\right)\right] \operatorname{sgn}(\lambda_I) \tag{3.85}$$

$$\sqrt{\lambda_R + i\lambda_I} + \sqrt{\lambda_R - i\lambda_I} = \sqrt{2}\Lambda \tag{3.86}$$

$$\sqrt{\lambda_R + i\lambda_I} - \sqrt{\lambda_R + i\lambda_I} = i\frac{\sqrt{2}\lambda_I}{\Lambda} \tag{3.87}$$

$$(\lambda_R + i\lambda_I)^{\frac{3}{2}} + (\lambda_R - i\lambda_I)^{\frac{3}{2}} = \sqrt{2}\left(\lambda_R\Lambda - \frac{\lambda_{\mathtt{I}}^2}{\Lambda}\right) \tag{3.88}$$

$$(\lambda_R + i\lambda_I)^{\frac{3}{2}} - (\lambda_R - i\lambda_I)^{\frac{3}{2}} = 2i\lambda_I\left(\sqrt{2}\Lambda - \frac{|\lambda|}{\sqrt{2}\Lambda}\right) \tag{3.89}$$

$$(\lambda_R + i\lambda_I)^{\frac{5}{2}} + (\lambda_R - i\lambda_I)^{\frac{5}{2}} = \sqrt{2}\left[\left(\lambda_R^2 - \lambda_{\mathtt{I}}^2\right)\Lambda - \frac{2\lambda_{\mathtt{I}}^2\lambda_R}{\Lambda}\right] \tag{3.90}$$

$$(\lambda_R + i\lambda_I)^{\frac{5}{2}} - (\lambda_R - i\lambda_I)^{\frac{5}{2}} = i\sqrt{2}\lambda_I\left[\frac{(\lambda_R^2 - \lambda_{\mathtt{I}}^2)}{\Lambda} + 2\lambda_R\Lambda\right] \tag{3.91}$$

$$(\lambda_R + i\lambda_I)^{\frac{7}{2}} + (\lambda_R - i\lambda_I)^{\frac{7}{2}} = \sqrt{2}\left[\Lambda\left(\lambda_R^3 - 3\lambda_R\lambda_{\mathtt{I}}^2\right) + \frac{\lambda_{\mathtt{I}}^4 - 3\lambda_{\mathtt{I}}^2\lambda_R^2}{\Lambda}\right] \tag{3.92}$$

$$(\lambda_R + i\lambda_I)^{\frac{7}{2}} - (\lambda_R - i\lambda_I)^{\frac{7}{2}} = i\sqrt{2}\lambda_I\left[\Lambda\left(3\lambda_R^2 - \lambda_{\mathtt{I}}^2\right) + \frac{\lambda_R^3 - 3\lambda_R\lambda_{\mathtt{I}}^2}{\Lambda}\right] \tag{3.93}$$

$$\frac{1}{\sqrt{\lambda_R + i\lambda_I}} + \frac{1}{\sqrt{\lambda_R - i\lambda_I}} = \frac{\sqrt{2}\Lambda}{|\lambda|} \tag{3.94}$$

$$\frac{1}{\sqrt{\lambda_R + i\lambda_I}} - \frac{1}{\sqrt{\lambda_R - i\lambda_I}} = -i\frac{\sqrt{2}\lambda_I}{|\lambda|\Lambda} \tag{3.95}$$

$$\frac{1}{(\lambda_R + i\lambda_I)^{\frac{3}{2}}} + \frac{1}{(\lambda_R - i\lambda_I)^{\frac{3}{2}}} = \frac{\sqrt{2}}{|\lambda|^3}\left[\lambda_R\Lambda - \frac{\lambda_{\mathtt{I}}^2}{\Lambda}\right] \tag{3.96}$$

$$\frac{1}{(\lambda_R + i\lambda_I)^{\frac{3}{2}}} - \frac{1}{(\lambda_R - i\lambda_I)^{\frac{3}{2}}} = -i\frac{2\lambda_I}{|\lambda|^3}\left[\sqrt{2}\Lambda - \frac{|\lambda|}{\sqrt{2}\Lambda}\right] \tag{3.97}$$

$$\frac{1}{(\lambda_R + i\lambda_I)^{\frac{5}{2}}} + \frac{1}{(\lambda_R - i\lambda_I)^{\frac{5}{2}}} = \frac{\sqrt{2}}{|\lambda|^5}\left[\left(\lambda_R^2 - \lambda_{\mathtt{I}}^2\right)\Lambda - \frac{2\lambda_{\mathtt{I}}^2\lambda_R}{\Lambda}\right] \tag{3.98}$$

$$\frac{1}{(\lambda_R + i\lambda_I)^{\frac{5}{2}}} - \frac{1}{(\lambda_R - i\lambda_I)^{\frac{5}{2}}} = -i\frac{\sqrt{2}}{|\lambda|^5}\lambda_I\left[\frac{(\lambda_R^2 - \lambda_{\mathtt{I}}^2)}{\Lambda} + 2\lambda_R\Lambda\right] \tag{3.99}$$

$$\frac{1}{(\lambda_R + i\lambda_I)^{\frac{7}{2}}} + \frac{1}{(\lambda_R - i\lambda_I)^{\frac{7}{2}}} = \frac{\sqrt{2}}{|\lambda|^7}\left[\Lambda\left(\lambda_R^3 - 3\lambda_R\lambda_{\mathtt{I}}^2\right) + \frac{\lambda_{\mathtt{I}}^4 - 3\lambda_{\mathtt{I}}^2\lambda_R^2}{\Lambda}\right] \tag{3.100}$$

$$\frac{1}{(\lambda_R + i\lambda_I)^{\frac{7}{2}}} - \frac{1}{(\lambda_R - i\lambda_I)^{\frac{7}{2}}} = -i\frac{\sqrt{2}}{|\lambda|^7}\lambda_I\left[\Lambda\left(3\lambda_R^2 - \lambda_{\mathtt{I}}^2\right) + \frac{\lambda_R^3 - 3\lambda_R\lambda_{\mathtt{I}}^2}{\Lambda}\right] \tag{3.101}$$

with $\Lambda = \sqrt{\lambda_R + \sqrt{\lambda_R^2 + \lambda_{\mathtt{I}}^2}}$, and $|\lambda| = \sqrt{\lambda_R^2 + \lambda_{\mathtt{I}}^2}$

3.4 Semi-empirical mass formula and Modified liquid drop model

The famous semi-empirical mass formula (SEMF) [7, 8, 9] describes the binding energy of the nucleus as a function of mass number A and charge/proton number Z. The model is based on the idea of a liquid drop, in which the binding energy is defined as

$$U_{B.E} = \underbrace{a_v A}_{\substack{\text{Volume}\\ \text{term}}} - \underbrace{a_s A^{\frac{2}{3}}}_{\substack{\text{Surface}\\ \text{term}}} - \underbrace{Z(Z-1)a_c A^{-\frac{1}{3}}}_{\substack{\text{Coulomb}\\ \text{Term}}} - \underbrace{a_A (N-Z)^2 A^{-1}}_{\substack{\text{Asymmetry}\\ \text{term}}} + \underbrace{\delta(N,Z)}_{\substack{\text{Pairing}\\ \text{term}}} \tag{3.102}$$

The names of these terms can be made clear when one rewrites the above equation with the idea of constant nuclear density. i.e.,

$$\rho_{\text{nuclear}} = \frac{A}{\frac{4}{3}\pi r^3} \implies r = r_0 A^{\frac{1}{3}} \tag{3.103}$$

Combining Eq. (3.103) with Eq. (3.102) we get

$$U_{B.E} = \underbrace{a_v \frac{r^3}{r_0^3}}_{\substack{\text{Volume}\\ \text{term}}} - \underbrace{a_s \frac{r^2}{r_0^2}}_{\substack{\text{Surface}\\ \text{term}}} - \underbrace{Z(Z-1)a_c \frac{r_0}{r}}_{\substack{\text{Coulomb}\\ \text{Term}}} - \underbrace{a_A (N-Z)^2 \frac{r_0^3}{r^3}}_{\substack{\text{Asymmetry}\\ \text{term}}} + \underbrace{\delta(N,Z)}_{\substack{\text{Pairing}\\ \text{term}}} \tag{3.104}$$

The first two terms, volume and surface, represent the strong force contribution, while the third term, Coulomb, represents the electro-static contribution. Fourth term coming from Pauli's exclusion principle. The fourth term is proportional to the difference between the neutron and proton numbers. The asymmetry term is also known as the Pauli term. The fifth term, known as the pairing term, is used to balance the spin coupling effects. The values of the terms in this equation are $a_v = 15.76$; $a_S = 17.81$; $a_c = 0.711$; $a_A = 23.702$ $a_p = 34$ [9], the units are in MeV.

However, in our work, we concentrate mainly on the first three terms. There is no alternative to the asymmetric term in QGP because the system is in a deconfined state. So the last two terms of the Eq. (3.104) make no sense in the context of QGP.

3.4.1 Modified liquid drop model

1. Nucleons are confined forms of quarks and gluons. But in our model, the quarks and gluons are in deconfined form. So we assume the strong force a_v, which acts towards the confinement of the nucleons. In our case we work on the quarks size range and in the high energy regime, in which the quarks and gluons have asymptotic freedom. So we change this confining a_v to $-a_v$.

2. The nuclear density is constant for nucleons within the nucleus. In QGP, the density is a function of temperature. $\rho_{\text{deconf.}} = \rho(T)$, which can be

obtained by integrating the Fermi-Dirac distribution of free particles over momentum space.

3. The charge dependence in SEMF is through the Coulomb term ($Z(Z-1)/r$). In the modified liquid drop model (MLDM), all terms are charge dependent. i.e., $U(r_{ij}) = z_i z_j \phi(r_{ij})$, with $\phi(r)$ being a central potential.

4. In MLDM, we are adding an additional linear term $a_l r$ but we haven't added the last two asymmetric term and pairing term as compared to SEMF.

5. The nucleons obey Fermi-Dirac distribution; MLDM assumes that the quark gluon plasma medium has collective behaviour dependent on the potential $U(r_{ij})$ followed by all particles in the medium.

The mathematical expression of MLDM is

$$\begin{aligned} U(r_{ij}) &= z_i z_j \phi(r_{ij}) \\ \phi(r) &= -a_v \left(\frac{r}{r_0}\right)^3 - a_s \left(\frac{r}{r_0}\right)^2 - a_c \left(\frac{r_0}{r}\right) - a_l \frac{r}{r_0} \\ \phi(r) &\approx \phi(r)\Big|_{\text{eff.}} = -a_v \left(\frac{r}{r_0}\right)^3 - a_c \left(\frac{r_0}{r}\right) - a_l \frac{r}{r_0} \end{aligned} \tag{3.105}$$

The approximation is possible because the three dimensional Fourier transform with the converging factor as mentioned in Eq. (3.3) $\underset{h\to 0}{\text{Lt}} \langle r^2 \rangle = 0$.

So the potential can be written as

$$U(r) = z_i z_j \Bigg(-a_v \left(\frac{r}{r_0}\right)^3 - a_c \left(\frac{r_0}{r}\right) - a_l \frac{r}{r_0} \Bigg) \tag{3.106}$$

So from Eq. (3.105), the three dimensional Fourier transform of ϕ is

$$\langle \phi \rangle = \underset{h\to 0}{\text{Lt}} - 4\pi \left[\frac{a_c r_0}{s} - \frac{2a_l}{r_0 s^2} + \frac{24 a_v}{s^3 r_0^3} \right] \tag{3.107}$$

with $s = p^2 + h^2$.

3.4.1. (a) Contribution of quarks and gluons to pressure

Now the total pressure in QGP is

$$\frac{P}{T}\Bigg|_{\text{eff}}^{\text{QGP}} = \frac{P}{T}\Bigg|_{\text{eff}}^{q\bar{q}} + \frac{P}{T}\Bigg|_{\text{eff}}^{g} \tag{3.108}$$

with

$$
\begin{aligned}
\left.\frac{P}{T}\right|_{\text{eff}}^{q\bar{q}} &= \left.\frac{P}{T}\right|^{q\bar{q}} - \mathtt{i}\,\mathrm{Im}\left(\left.\frac{P}{T}\right|^{q\bar{q}}\right) \\
&= C_{q\bar{q}} + \frac{1}{2}\left[\beta_{\text{qR}}\mathtt{Q}_1(z_q) - \beta_{\text{qI}}\mathtt{R}_1(z_q) - \mathtt{Mn}(z_q)\right]
\end{aligned} \tag{3.109}
$$

Similarly

$$
\left.\frac{P}{T}\right|_{\text{eff}}^{g} = C_g + \frac{1}{2}\left[\beta_{\text{gR}}\mathtt{Q}_1(z_g) - \beta_{\text{gI}}\mathtt{R}_1(z_g) - \mathtt{Mn}(z_g)\right] \tag{3.110}
$$

$$
C_{q\bar{q}} = C_q + C_{\bar{q}} = g_{q\bar{q}}\frac{\eta(3)}{\pi^2}T^3; \qquad C_g = g_g\frac{\eta(3)}{\pi^2}T^3 \tag{3.111}
$$

Thus the energy density of QGP can be written from Eq. (3.51) as

$$
\begin{aligned}
\left.\frac{\varepsilon}{T}\right|_{\text{eff}}^{\text{QGP}} =& \frac{\partial}{\partial\ln(T)}\left(\left.\frac{P}{T}\right|_{\text{eff}}^{\text{QGP}}\right) \\
=& 3C_{\text{q}\bar{\text{q}}} - \left[\left(\beta_{\text{qR}}^2 - \beta_{\text{qI}}^2\right)\mathtt{Q}_2(z_{\text{q}}) - 2\beta_{\text{qR}}\beta_{\text{qI}}\mathtt{R}_2(z_{\text{q}})\right] \\
&+ 2\alpha_{\text{q}}\left(\mathrm{Re}\left\{\frac{\beta_{1\text{q}}}{x_{\text{q}} - z_{\text{q}}}\right\}\mathtt{Q}_1(z_{\text{q}}) - \mathrm{Im}\left\{\frac{\beta_{1\text{q}}}{x_{\text{q}} - z_{\text{q}}}\right\}\mathtt{R}_1(z_{\text{q}})\right) \\
&+ 2\,\mathrm{Im}\left\{\frac{|\beta_{1\text{q}}|^2}{z_{\text{q}} - {z_{\text{q}}}^*}\right\}\mathtt{R}_1(z_{\text{q}}) \\
&+ 3C_{\text{g}} - \left[\left(\beta_{\text{gR}}^2 - \beta_{\text{gI}}^2\right)\mathtt{Q}_2(z_{\text{g}}) - 2\beta_{\text{gR}}\beta_{\text{gI}}\mathtt{R}_2(z_{\text{g}})\right] \\
&+ 2\alpha_g\left(\mathrm{Re}\left\{\frac{\beta_{1\text{g}}}{x_{\text{g}} - z_{\text{g}}}\right\}\mathtt{Q}_1(z_{\text{g}}) - \mathrm{Im}\left\{\frac{\beta_{1\text{g}}}{x_{\text{g}} - z_{\text{g}}}\right\}\mathtt{R}_1(z_{\text{g}})\right) \\
&+ 2\,\mathrm{Im}\left\{\frac{|\beta_{1\text{g}}|^2}{z_{\text{g}} - z_{\text{g}}^*}\right\}\mathtt{R}_1(z_{\text{g}})
\end{aligned} \tag{3.112}
$$

with

$$
\begin{aligned}
z_{\text{q/g}} &= -\frac{1}{2}\left(S_{\text{q/g}} + T_{\text{q/g}}\right) - \frac{\mathtt{a}_{\text{q/g}}}{3} + \frac{\mathtt{i}\sqrt{3}}{2}(S_{\text{q/g}} - T_{\text{q/g}}), \\
x_{\text{q/g}} &= S_{\text{q/g}} + T_{\text{q/g}} - \frac{\mathtt{a}_{\text{q/g}}}{3}; \quad \beta_{\text{q/g I}} = \mathrm{Im}\left\{\beta_{1\text{q/g}}\right\} \\
\beta_{1\text{q/g}} &= -\frac{z_{\text{q/g}}^3}{\left(z_{\text{q/g}} - x_{\text{q/g}}\right)\left(z_{\text{q/g}} - z_{\text{q/g}}^*\right)}, \ \beta_{\text{q/g R}} = \mathrm{Re}\left\{\beta_{1\text{q/g}}\right\}, \\
\alpha_{\text{q/g}} &= -\frac{x_{\text{q/g}}^3}{(x_{\text{q/g}} - z_{\text{q/g}})(x_{\text{q/g}} - z_{\text{q/g}}^*)}, \quad Q_{\text{q/g}} = \frac{3\mathtt{b}_{\text{q/g}} - \mathtt{a}_{\text{q/g}}^2}{9}, \\
R_{\text{q/g}} &= \frac{9\mathtt{a}_{\text{q/g}}\mathtt{b}_{\text{q/g}} - 27\mathtt{c}_{\text{q/g}} - 2\mathtt{a}_{\text{q/g}}^3}{54}, \ D_{\text{q/g}} = Q_{\text{q/g}}^3 + R_{\text{q/g}}^2
\end{aligned} \tag{3.113}
$$

$$S_{q/g} = \left(R_{q/g} + \sqrt{D_{q/g}}\right)^{\frac{1}{3}}, \quad T = -\frac{Q_{q/g}}{S_{q/g}},$$
$$\mathsf{a}_{q/g} = -4\pi\beta\ C_{qq/g}\ r_{qq/g}\ a_c; \ \mathsf{b}_{q/g} = 8\pi\beta\ C_{q\bar{q}/g}\ \frac{a_l}{r_{q\bar{q}/g}};$$
$$\mathsf{c}_{q/g} = -96\pi\beta\ C_{q\bar{q}/g}\ \frac{a_v}{r^3_{q\bar{q}/g}}, \ a_c = 0.711, \ a_v = 15.76$$
$$C_{q\bar{q}/g} = g_{q\bar{q}/g}\frac{\eta(3)}{\pi^2}T^3; \quad g_g = 16; \quad g_{q\bar{q}} = \frac{32}{3}n_f$$

When we fit the Eqs. (3.108), (3.112) and (4.29) of the proposed model with the lattice data in [10, 11] we get fitting parameter values as shown in Table 3.1.

n_f	$g_{q\bar{q}}$ $\frac{32}{3}n_f$	g_g (16)	a_l	$r_{q\bar{q}}$	r_g
3	32	16	9.052	1.374	0.061
2+1	28	16	8.638	1.435	0.043
2	64/3	16	9.389	1.565	0.075
0	-	16	8.853	-	1.682

Table 3.1: The parameters used in this table, when applied to Eqs. (3.108) and (3.112) through Eq. (4.29), give us Figs. 3.1 to 3.3 .

The coefficients of the Volume term (a_v) and Coulomb term (a_c) have values of 15.76 T_c and 0.711 T_c, respectively. The unit of a_l is also T_c. Since the available lattice data is in the units of T_c, for numerical calculations, we approximate $T_c \approx 1$.

3.5 Results and Conclusions

In Fig. 3.1, energy density ε/T^4 vs T/T_c is plotted and compared with the lattice data [10, 11]. The parameters used to plot the energy density are given in Table 3.1. The number density of particles for quarks and gluons are $\frac{32}{3}n_f$

and 16 in the units of $\frac{\eta(3)}{\pi^2}T^3$. The effective value for $2+1$ flavor is considered as 2.625. The lattice data points range from $T > 0.72T_c$ to $T \approx 3.5T_c$. The number of quarks and anti quarks are equal. i.e, The number density of quarks and anti-quarks are $C_{q\bar{q}}/2$. The lattice data points and the fitted equation of Eq. (3.112) fall within the errorbar. The pure gluon lattice data is taken from [11] and is without errorbar.

The pressure lattice data [10, 11] is without errorbars. The scaled pressure in Eq. (3.108) is in agreement with the lattice data as shown in Fig. 3.2.

The interaction measure (also known as trace anomaly) is defined as

$$\frac{I}{T^4} = \frac{\varepsilon - 3P}{T^4} \tag{3.114}$$

For an ideal gas of free particles obeying the Fermi-Dirac distribution, the energy density at zero chemical potential (fugacity = 1) is

$$\begin{aligned}\varepsilon &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\epsilon_p}{\exp(\beta\epsilon_p)+1} \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{|p|}{\exp(\beta|p|)+1} \\ &= 3\frac{\eta(4)}{\pi^2}T^4.\end{aligned} \tag{3.115}$$

From standard statistical mechanics

$$\begin{aligned}\frac{P}{T} - \frac{P_0}{T_0} &= \int_{T_0}^{T} \frac{\varepsilon}{T^2}dT \\ &= \frac{\eta(4)}{\pi^2}\left[T^3 - T_0^3\right]\end{aligned} \tag{3.116}$$

So

$$\frac{\varepsilon - 3P}{T} = \frac{3\eta(4)}{\pi^2}T_0^3 - 3\frac{P_0}{T_0} = \text{Constant} \tag{3.117}$$

Thus, for an ideal free particle gas, the theoretical value of the interaction measure should be such that $I/T^4 \propto T^{-3}$.

The interaction measure Fig. 3.3 shows that QGP is different from the ideal gas. The lattice data shows a gradual rise between $T/T_c \in [0.72, 1.25]$. After $T/T_c \approx 1.25$, the interaction measure I/T^4 goes down similar to T^{-3}. The Eqs. (3.108) and (3.112) are in good agreement with the interaction measure lattice data [11].

In Chapter 4, the modified liquid drop model will be extended to the magnetic field regime. The idea of a quantum harmonic oscillator result is coupled with the relativistic equations of a free particle having a charge in the presence of a magnetic field. The integral equations and the results are modified and redefined. The results are then compared with the lattice data. The pros, cons, and refinements needed in the model are also discussed in Chapter 4.

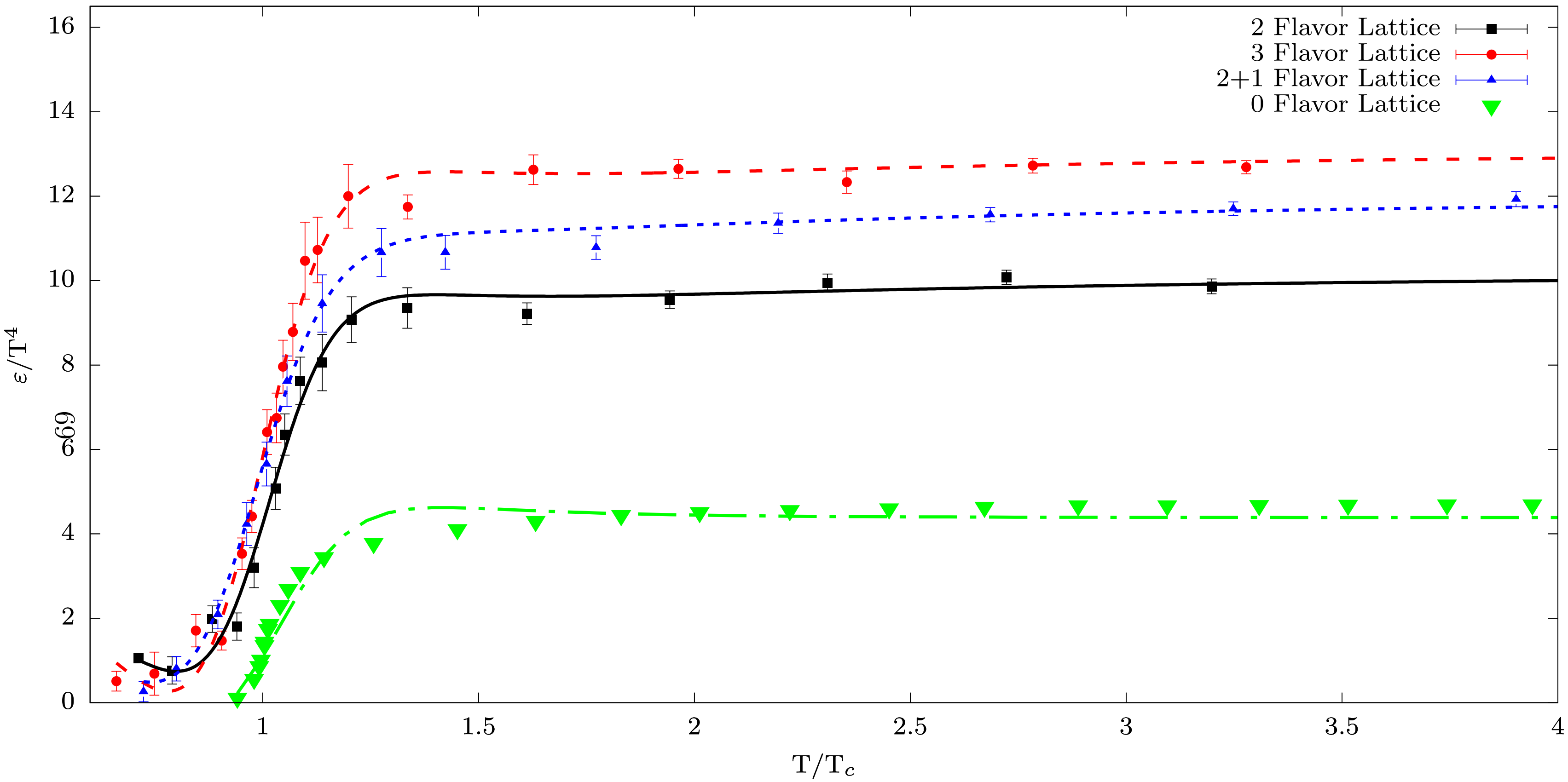


Figure 3.1: Energy density for 0, 2, 2+1, and 3 flavours is plotted. Lattice data is taken from [10, 11]. The model parameters used to fit the lattice data are given in Table 3.1.

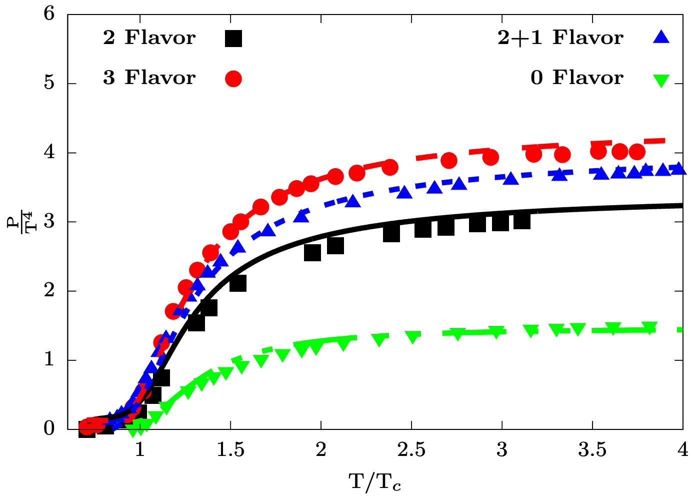


Figure 3.2: Pressure scaled by T^4, for $n_f = 2,\ 2+1, 3$ flavor QGP. Lattice data is taken from [10, 11]

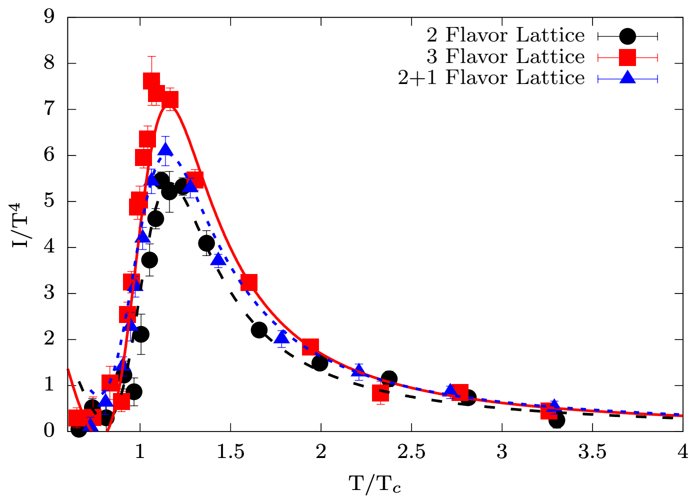


Figure 3.3: Interaction measure $I/T^4 = (\varepsilon - 3P)/T^4$, for $n_f = 2,\ 2+1, 3$ flavor QGP. Lattice data is taken from [10, 11]

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Chapter 4

Equation of state under magnetic field for relativistic particle

4.1 Introduction

Consider a quantum mechanical system whose square of the Hamiltonian in the absence of a magnetic field is

$$\hat{H}^2_{B=0} = c^2(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \hat{\mathbb{I}} m^2 c^4 \tag{4.1}$$

In the presence of a magnetic field with a magnetic vector potential,

$$\begin{aligned} \vec{\hat{A}} &= [0, x, 0]\, B \\ \implies \boldsymbol{\nabla} \times \vec{\hat{A}} &= B\hat{\mathbb{I}}\hat{z} \end{aligned} \tag{4.2}$$

Hamiltonian becomes

$$\begin{aligned} \hat{H}^2_{B\neq 0} &= c^2 \left(\hat{p}_x^2 + \left[\hat{p}_y - q\hat{A} \right]^2 + \hat{p}_z^2 \right) + m^2 c^4 \hat{\mathbb{I}} \\ &= c^2 \left(\hat{p}_x^2 + q^2 B^2 \left[\hat{x} - \frac{\hat{p}_y}{qB} \right]^2 + \hat{p}_z^2 \right) + m^2 c^4 \hat{\mathbb{I}} \end{aligned} \tag{4.3}$$

For a harmonic oscillator with Hamiltonian of the form

$$\begin{aligned} \hat{H}_{\text{HO}} &= \alpha^2(\hat{x} - \hat{x}_0)^2 + \beta^2(\hat{p}_x - \hat{p}_{x0})^2 \\ &\text{with properties} \\ [\hat{x}, \hat{p}_x] &= i\hbar \\ [\hat{x}, \hat{x}_0] &= [\hat{x}, \hat{p}_{x0}] = 0 \\ [\hat{x}_0, \hat{p}_{x0}] &= [\hat{p}_x, \hat{p}_{x0}] = 0 \\ &\text{and } \alpha, \beta \in \mathbb{R} \end{aligned} \tag{4.4}$$

can be reduced to the form

$$\begin{aligned}
&\hat{H}_{\text{HO}} = \left(2\hat{N} + 1\right) \hbar\alpha\beta \\
&\text{with annihilation operator} \\
&\hat{a} = \sqrt{\frac{\alpha}{2\hbar\beta}} \left(\hat{x} - \hat{x}_0\right) + i\sqrt{\frac{\beta}{2\hbar\alpha}} \left(\hat{p}_x - \hat{p}_{x0}\right) \\
&\text{and } \hat{N} = \hat{a}^\dagger \hat{a}
\end{aligned} \tag{4.5}$$

Combining Eq. (4.3) with Eq. (4.4), the Hamiltonian get reduced to Eq. (4.3) as

$$\hat{H}^2_{B\neq 0} = \left(2\hat{N} + 1\right) q\hbar B c^2 + \hat{p}_z^2 c^2 + m^2 c^4 \hat{\mathbb{I}} \tag{4.6}$$

Thus spin effect included Hamiltonian in terms of natural units is

$$\begin{aligned}
\hat{H}^2_{\sigma,B} &= \left(2\hat{N} + [\sigma + 1]\, \hat{\mathbb{I}}\right) qB + \hat{p}_z^2 + m^2 \hat{\mathbb{I}} \\
\langle \hat{H}^2_{\sigma,B} \rangle &= (2n + \sigma + 1)\, qB + p_z^2 + m^2
\end{aligned} \tag{4.7}$$

defining

$$\nu_\sigma = n + \frac{1 + \sigma}{2} \tag{4.8}$$

where σ can be ± 1 w.r.t spin up and down.

$$\begin{aligned}
\forall\, n \in \{0, 1, 2, \ldots, \infty\}\,, \exists\;\; \nu_{\sigma+} &= \{1, 2, \ldots, \infty\} \;\text{ and} \\
\nu_{\sigma-} &= \{0, 1, 2, \ldots, \infty\} \\
\therefore v_{\sigma+} + v_{\sigma-} &= 2\,\{0, 1, 2, \ldots, \infty\} - \{0\} \\
&= \{\mathbb{W}\}\,(2 - \delta_{w,0})
\end{aligned} \tag{4.9}$$

where $\{\mathbb{W}\} = \{0, 1, 2, 3, \ldots\}$ For ease in calculation we write $q = q_f e$, then Hamiltonian can be approximated as

$$\langle \hat{H}^2_{\sigma,B} \rangle = 2 q_f e B n_\sigma + p_z^2 + m^2 \tag{4.10}$$

with degeneracy $2 - \delta_{n_\sigma,0}$.

This is also known as relativistic Landau quantization. In this semi-classical approximation, the value of $\hat{p}_y$ only affects the wavefunction, not the energy ($\hat{x}_0 = \frac{\hat{p}_y}{qB}$). Corresponding change in phase space calculation [1, 2] becomes

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \to \frac{q_f|eB|}{2\pi} \sum_{n_\sigma=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{2\pi} \left(2 - \delta_{n_\sigma,0}\right) \tag{4.11}$$

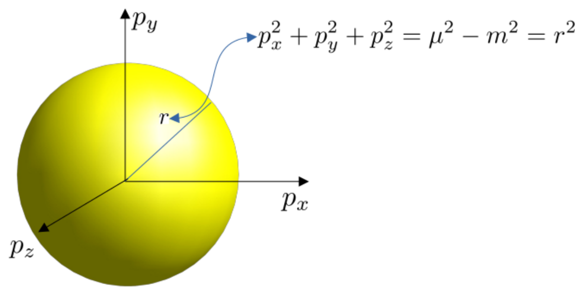

Figure 4.1: In the absence of magnetic field, the momentum space (with finite upper bound) is spherically symmetric.

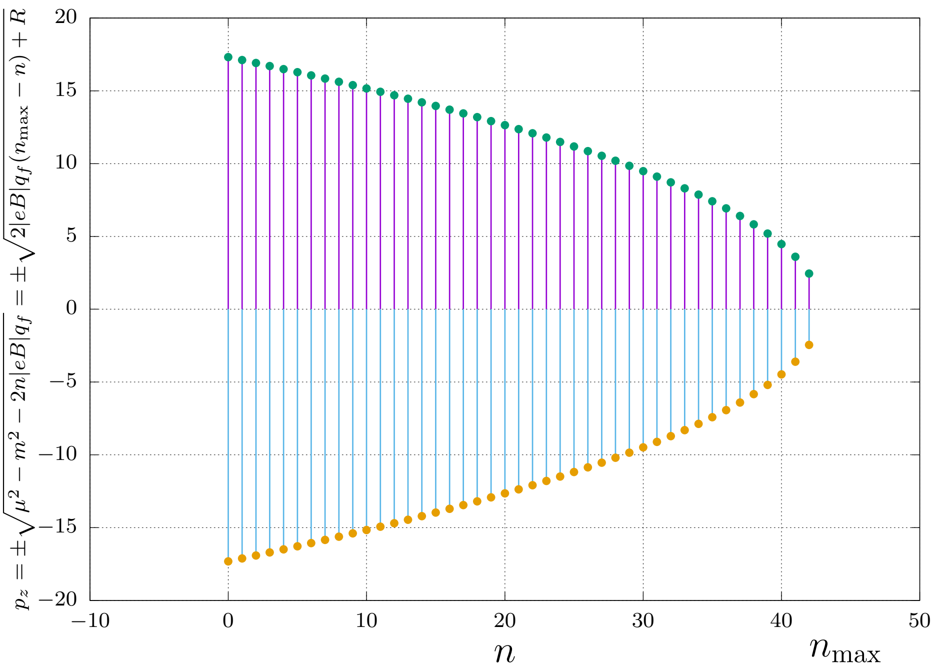

Figure 4.2: Quantized energy states in the presence of magnetic field and corresponding p_z values under fermi distribution (with finite upper bound). The equation involved here is $p_z^2 + 2nBq_fe = \mu^2 - m^2 = 2n_{\max}Bq_fe + R$ where $R < 2Bq_fe$, n is the level number and is an integer, $n \in [0, n_{\max}]$. The values are thus discrete

4.2 Ideal free particle gas

Let us define

$$\begin{aligned}
\widehat{I}_{\mathrm{ND}} &= \int \frac{d^N p}{(2\pi)^N} \\
\widehat{I}_{zB} &= \sum_{n=0}^{\infty} \int \frac{dp_z}{2\pi} \left(2 - \delta_{n,0}\right)
\end{aligned} \tag{4.12}$$

The free particle energy density can be written as

$$\begin{aligned}
\varepsilon &= \widehat{I}\left(\frac{|p|}{\exp\left(\beta|p|\right)+1}\right) = \widehat{I}\left(\sum_{n=1}^{\infty}(-1)^{n-1}|p|\exp\left(-n\beta|p|\right)\right) \\
\beta\varepsilon &= \widehat{I}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\beta|p|\exp\left(-n\beta|p|\right)\right)
\end{aligned} \tag{4.13}$$

From standard statistical mechanics

$$\begin{aligned}
\beta P &= \beta_0 P_0 - \int_{\beta_0}^{\beta} \varepsilon(\beta)\ d\beta \\
&= \beta_0 P_0 + \widehat{I}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{-n\beta|p|}}{n}\right]_{\beta_0}^{\beta} \\
&= \beta_0 P_0 + \widehat{I}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{e^{-n\beta|p|}}{n} - \frac{e^{-n\beta_0|p|}}{n}\right)\right]
\end{aligned} \tag{4.14}$$

Therefore

$$\begin{aligned}
\beta\left(\varepsilon - 3P\right) = -3\beta_0 P_0 &+ \widehat{I}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\left(\beta|p| - \frac{3}{n}\right)e^{-n\beta|p|}\right] \\
&+ \widehat{I}\left[\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{3}{n}\right)e^{-n\beta_0|p|}\right]
\end{aligned} \tag{4.15}$$

For an N dimensional integral

$$\begin{aligned}
\beta\left(\varepsilon - 3P\right) = -3\beta_0 P_0 &+ \frac{2}{\beta^N}\frac{\Gamma\left(N+1\right)}{(4\pi)^{\frac{N}{2}}\,\Gamma\left(\frac{N}{2}\right)}\eta(N+1)\left[1 - \frac{3}{N}\right] \\
&+ \frac{6}{\beta_0^N}\frac{\Gamma(N)}{(4\pi)^{\frac{N}{2}}}\frac{\eta(N+1)}{\Gamma\left(N/2\right)}
\end{aligned} \tag{4.16}$$

It is evident from the above equation that for three dimensions, $\beta\left(\varepsilon - 3P\right)$ is a constant. For all other dimensions, $\varepsilon - 3P$ varies with respect to temperature.

4.3 Modified liquid drop model under magnetic field

The magnetic field can affect the distribution of quarks in the presence of magnetic field. From now on we use H to denote the magnetic field instead of B, because we employ B to represent the beta function. Consider Fermi-Dirac distribution under a magnetic field for mass-less particles; the number density of particles under a magnetic field is

$$\begin{aligned} c_{q\bar{q}}^{H} &= \frac{|q_f eH|}{2\pi} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \frac{2-\delta_{j,0}}{\exp\left(\beta\sqrt{p_z^2+2j|q_f eH|}\right)+1} \frac{dp_z}{2\pi} \\ &= \frac{q_f|eH|T}{2\pi^2}\left[\eta(1)+2\sum_{\nu=1}^{\infty}\mathcal{S}_1\left(\beta\sqrt{2\nu q_f|eH|}\right)\right] \end{aligned} \tag{4.17}$$

with

$$\mathcal{S}_1(x) = \sum_{n=1}^{\infty}(-1)^{n-1} x K_1(nx) \tag{4.18}$$

$$\mathcal{S}_0(x) = -x\frac{\partial}{\partial x}\mathcal{S}_1(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\left[nx^2 K_0(nx)\right] \tag{4.19}$$

$$\bar{\mathcal{S}}_1(x) = \sum_{n=1}^{\infty}(-1)^{n-1} n^2 x^3 K_1(nx) \tag{4.20}$$

$$y\frac{\partial}{\partial y}\mathcal{S}_1(x) = \sum_{n=1}^{\infty}(-1)^{n}\left[nxK_0(nx)\right] y\frac{\partial x}{\partial y} \tag{4.21}$$

As one extends the modified liquid drop model into the magnetic field regime, one can get the result by replacing the three dimensional integral with the magnetic field integral, as shown in Eq. (4.11); the pressure relation is as straight forward as

$$\frac{P_{q\bar{q}}^{H}}{T} = \left.\frac{P}{T}\right|_{\text{eff}}^{H} = c_{q\bar{q}}^{H} + \frac{1}{2}\left[\beta_R \mathtt{Q}_1^H(z) - \beta_I \mathtt{R}_1^H(z) - \mathtt{Mn}^H(z)\right] \tag{4.22}$$

$$
\begin{aligned}
\frac{\varepsilon^H_{q\bar{q}}}{T} =& \frac{\varepsilon}{T}\bigg|^H_{\text{eff}} = \frac{\partial}{\partial \ln(T)}\left(\frac{P}{T}\bigg|_{\text{eff}}\right) \\
=& \frac{\partial c_{q\bar{q}}}{\partial \ln(T)} - \left[\left(\beta_R^2 - \beta_I^2\right) \mathtt{Q}_2^H(z) - 2\beta_R\beta_I \mathtt{R}_2^H(z)\right] \\
&+ 2\alpha\left(\mathrm{Re}\left\{\frac{\beta_1}{x-z}\right\}\mathtt{Q}_1^H(z) - \mathrm{Im}\left\{\frac{\beta_1}{x-z}\right\}\mathtt{R}_1^H(z)\right) \\
&+ 2\,\mathrm{Im}\left\{\frac{|\beta_1|^2}{z-z^*}\right\}\mathtt{R}_1^H(z)
\end{aligned} \tag{4.23}
$$

From Eqs. (3.114), (3.116) and (3.117), the interaction measure can be written as

$$
\frac{I_{q\bar{q}}}{T^4} = \frac{\varepsilon_{q\bar{q}} - 3P_{q\bar{q}}}{T^4} \tag{4.24}
$$

with

$$
\mathtt{Q}_n^H(z) = \frac{2G_n(q_f eH)}{(2|q_f eH|)^{n-\frac{1}{2}}}\,\mathrm{Re}\left\{\zeta\left(n-\frac{1}{2}, \frac{-z}{2|q_f eH|}\right)\right\} - 2|q_f eH| n \mathtt{Q}_{n+1}(z) \tag{4.25}
$$

$$
\mathtt{R}_n^H(z) = \frac{2G_n(q_f eH)}{(2|q_f eH|)^{n-\frac{1}{2}}}\,\mathrm{Im}\left\{\zeta\left(n-\frac{1}{2}, \frac{-z}{2|q_f eH|}\right)\right\} - 2|q_f eH| n \mathtt{R}_{n+1}(z) \tag{4.26}
$$

$$
G_n(q_f eH) = \frac{|q_f eH|}{2\pi^2}\mathrm{B}\left(\frac{1}{2}, n-\frac{1}{2}\right) \tag{4.27}
$$

$$
\begin{aligned}
\mathtt{Mn}^H(z) =& \frac{|2q_f eH|^{\frac{3}{2}}}{\pi}\left[\mathrm{Re}\left\{\zeta\left(-\frac{1}{2}, -\frac{z}{|2q_f eH|}\right)\right\} - \zeta(-1/2)\right] \\
&- \frac{\sqrt{2}|q_f eH|}{2\pi}\sqrt{|z| - \mathrm{Re}\{z\}} \\
|z| =& \sqrt{\mathrm{Re}^2\{z\} + \mathrm{Im}^2\{z\}}
\end{aligned} \tag{4.28}
$$

with

$$
\begin{aligned}
z &= -\frac{1}{2}(S+T) - \frac{\mathtt{a}}{3} + \frac{\mathtt{i}\sqrt{3}}{2}(S-T), \\
x &= S+T-\frac{\mathtt{a}}{3}; \quad \beta_{\,\mathrm{I}} = \mathrm{Im}\{\beta_1\} \\
\beta_1 &= -\frac{z^3}{(z-x)(z-z^*)}, \; \beta_{\,\mathrm{R}} = \mathrm{Re}\{\beta_1\}, \\
\alpha &= -\frac{x^3}{(x-z)(x-z^*)}, \quad Q = \frac{3\mathtt{b}-\mathtt{a}^2}{9}, \\
R &= \frac{9\mathtt{ab}-27\mathtt{c}-2\mathtt{a}^3}{54}, \; D = Q^3+R^2
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
S &= \left(R+\sqrt{D}\right)^{\frac{1}{3}}, \quad T = -\frac{Q}{S}, \\
\mathtt{a} &= -4\pi\beta\, c^H_{\mathrm{qq}}\, r_{\mathrm{qq}}\, a_c; \; \mathtt{b} = 8\pi\beta\, c^H_{\mathrm{q\bar{q}}}\, \frac{a_l}{r_{\mathrm{q\bar{q}}}}; \\
\mathtt{c} &= -96\pi\beta\, c^H_{\mathrm{q\bar{q}}}\, \frac{a_v}{r^3_{\mathrm{q\bar{q}}}}, \; a_c = 0.711, \; a_v = 15.76
\end{aligned}
$$

The expected result is that as the magnetic field increases, the energy density, pressure due to quarks increase.

4.4 Integral table

4.4.1 Integrals in presence of Magnetic field

Consider the integral of the form

$$
\begin{aligned}
\mathtt{I}^H_n(m^2) &= \frac{|q_f eH|}{2\pi} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{1}{\left(p_z^2+m_j^2\right)^n} (2-\delta_{j,0}) \\
&= \frac{|q_f eH|}{2\pi^2} \sum_{j=0}^{\infty} J(m_j^2) - \frac{|q_f eH|}{4\pi^2} J(m^2)
\end{aligned} \tag{4.30}
$$

with $m_j^2 = m^2 + 2j|q_f eH|$. So

$$
J(m^2) = \int_{-\infty}^{\infty} \frac{dp_z}{(p_z^2+m^2)^n} = \frac{B(n-\frac{1}{2},\frac{1}{2})}{(m^2)^{n-\frac{1}{2}}} \tag{4.31}
$$

So

$$\mathtt{I}_n^H(m^2) = \frac{|q_feH|}{2\pi^2} B\left(n-\frac{1}{2},\frac{1}{2}\right)\left[\sum_{j=0}^{\infty}\frac{1}{(m^2+2j|q_feH|)^{n-\frac{1}{2}}} - \frac{1}{2m^{2n-1}}\right]$$

$$= \frac{|q_feH|}{2\pi^2} B\left(n-\frac{1}{2},\frac{1}{2}\right)\left[\frac{\zeta\left(n-\frac{1}{2},\frac{m^2}{2|q_feH|}\right)}{(2|q_feH|)^{n-\frac{1}{2}}} - \frac{1}{2\left(m^2\right)^{n-\frac{1}{2}}}\right]$$

$$\mathtt{I}_n^H(m^2) = \left[\frac{|q_feH|}{2\pi^2}\frac{B\left(n-\frac{1}{2},\frac{1}{2}\right)}{(2|q_feH|)^{n-\frac{1}{2}}}\zeta\left(n-\frac{1}{2},\frac{m^2}{2|q_feH|}\right) - 2n|q_feH|I(m^2,n+1)\right]\Theta\left(m^2\right) \tag{4.32}$$

$$\boxed{\mathtt{I}_n^H(m^2) = \left[\frac{B\left(n-\frac{1}{2},\frac{1}{2}\right)}{(2|q_feH|)^{n-\frac{3}{2}}}\frac{\zeta\left(n-\frac{1}{2},\frac{m^2}{2|q_feH|}\right)}{4\pi^2} - 2n|q_feH|I(m^2,n+1)\right]\Theta\left(m^2\right)} \tag{4.33}$$

Similarly

$$\mathtt{Q}_n^H(m,z) = \frac{|q_feH|}{2\pi}\sum_{j=0}^{\infty}\int \mathrm{Re}\left\{\frac{1}{\left(p_z^2+m_j^2-z\right)^n} + \frac{1}{\left(p_z^2+m_j^2-z^*\right)^n}\right\}(2-\delta_{j,0})\,\frac{dp_z}{2\pi}$$

$$\mathtt{Q}_n^H(m,z) = \left[\frac{B\left(n-\frac{1}{2},\frac{1}{2}\right)}{(2|q_feH|)^{n-\frac{3}{2}}}\frac{\mathrm{Re}\left\{\zeta\left(n-\frac{1}{2},\frac{m^2-z}{2|q_feH|}\right)\right\}}{2\pi^2} - 2n|q_feH|\mathtt{Q}_{n+1}(m,z)\right] \tag{4.34}$$

repeating

$$\mathtt{R}_n^H(m,z) = \frac{|q_feH|}{2\pi}\sum_{j=0}^{\infty}\int \mathrm{Im}\left\{\frac{1}{\left(p_z^2+m_j^2-z\right)^n} - \frac{1}{\left(p_z^2+m_j^2-z^*\right)^n}\right\}(2-\delta_{j,0})\,\frac{dp_z}{2\pi}$$

$$\mathtt{R}_n^H(m,z) = \left[\frac{B\left(n-\frac{1}{2},\frac{1}{2}\right)}{(2|q_feH|)^{n-\frac{3}{2}}}\frac{\mathrm{Im}\left\{\zeta\left(n-\frac{1}{2},\frac{m^2-z}{2|q_feH|}\right)\right\}}{2\pi^2} - 2n|q_feH|\mathtt{R}_{n+1}(m,z)\right] \tag{4.35}$$

4.4.2 Special Case

Consider the integral

$$\begin{aligned}
\mathtt{I}_n^H(m^2,x) &= \frac{|q_f eH|}{2\pi}\sum_{j=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\frac{1}{\left(p_z^2+m_j^2-x\right)^n}\left(2-\delta_{j,0}\right) \\
&= \frac{|q_f eH|}{2\pi^2}\sum_{j=0}^{\infty}J(m_j^2)-\frac{|q_f eH|}{4\pi^2}J(m^2-x)
\end{aligned} \tag{4.36}$$

with $m_{jx}^2 = m^2+2j|q_f eH|-x$.

4.4.2. (a) Case 1: $m^2-x>0$

From Eq. (4.33), the result is

$$\begin{aligned}
\mathtt{I}_n^{H+}(m^2,x) &= \left[\frac{B\left(n-\frac{1}{2},\frac{1}{2}\right)}{\left(2|q_f eH|\right)^{n-\frac{3}{2}}}\frac{\zeta\left(n-\frac{1}{2},\frac{m^2-x}{2|q_f eH|}\right)}{4\pi^2}\right]\Theta\left(m^2-x\right) \\
&\quad -2n|q_f eH|\mathtt{I}_{n+1}(m^2,x)\Theta\left(m^2-x\right)
\end{aligned} \tag{4.37}$$

4.4.2. (b) Case 2: $m^2-x<0$

Then the summation can be divided into two with $2j|q_f eH|+m^2-x<0$ and $2j|q_f eH|+m^2-x>0$, so the summation term

$$\begin{aligned}
\sum_{j=0}^{\infty}f(j) &= \sum_{j=0}^{j_{\min}}f(j)\Theta\left(x-m^2-2j|q_f eH|\right)+\sum_{j_{\min}+1}^{\infty}f(j)\Theta\left(2j|q_f eH|+m^2-x\right) \\
&= \sum_{j=0}^{j_{\min}}f(j)\Theta\left(\frac{x-m^2}{2|q_f eH|}-j\right)+\sum_{j_{\min}+1}^{\infty}f(j)\Theta\left(j+\frac{m^2-x}{2|q_f eH|}\right) \\
&= \sum_{j=0}^{j_{\min}}f(j)^-+\sum_{j_{\min}+1}^{\infty}f(j)^+ \\
&= \sum_{j=0}^{j_{\min}}f(j)^-+\sum_{j=0}^{\infty}f(j+j_{\min}+1)^+
\end{aligned} \tag{4.38}$$

with $j_{\min} = \left\lfloor \frac{x-m^2}{2|q_f eH|} \right\rfloor$ Now

$$
\begin{aligned}
\mathtt{I}_n^{H-}(m^2,x) =& \frac{|q_f eH|}{2\pi} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{1}{\left(p_z^2 + m_j^2 - x\right)^n} \left(2 - \delta_{j,0}\right) \\
=& \frac{|q_f eH|}{2\pi^2} \left[\sum_{j=0}^{j_{\min}} J^-(2j|q_f eH| + m^2 - x) \right] - \frac{|q_f eH|}{4\pi^2} J^-(m^2 - x) \\
&+ \frac{|q_f eH|}{2\pi^2} \sum_{j=0}^{\infty} \left[J \left(2|q_f eH| \left[j + \frac{m^2 - x}{2|q_f eH|} + \left\lfloor \frac{x - m^2}{2|q_f eH|} \right\rfloor + 1 \right] \right) \right]
\end{aligned}
\tag{4.39}
$$

using the result

$$
\begin{aligned}
J(m^2) =& \int_{-\infty}^{\infty} \frac{dp_z}{(p_z^2 + m^2)^n} \Theta(m^2) = \frac{B\left(\frac{1}{2}, n - \frac{1}{2}\right)}{(m^2)^{n-\frac{1}{2}}} \Theta(m^2) \\
J^-(m^2) =& \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{dp_z}{(p_z^2 + m^2 - i\epsilon)^n} \Theta(-m^2) \\
=& \, \mathtt{i} B \left(\frac{1}{2}, n - \frac{1}{2} \right) \frac{(-1)^{n+1}}{(-m^2)^{n-\frac{1}{2}}} \Theta(-m^2)
\end{aligned}
\tag{4.40}
$$

So

$$
J^-(m^2 - x) = \mathtt{i} B \left(\frac{1}{2}, n - \frac{1}{2} \right) \frac{(-1)^{n+1}}{(x - m^2)^{n-\frac{1}{2}}} \Theta\left(x - m^2\right) \tag{4.41}
$$

Combining

$$
\begin{aligned}
\mathtt{I}_n^{H-}(m^2,x) =& \mathtt{i}(-1)^{n+1} \frac{|q_f eH|}{4\pi^2} B \left(\frac{1}{2}, n - \frac{1}{2} \right) \left[\sum_{j=0}^{j_{\min}} \frac{2}{(x - m^2 - 2j|q_f eH|)^{n-\frac{1}{2}}} - \frac{1}{(x - m^2)^{n-\frac{1}{2}}} \right] \\
&+ \frac{|q_f eH|}{2\pi^2} \frac{1}{(2|q_f eH|)^{n-\frac{1}{2}}} \, \zeta \left(n - \frac{1}{2}, \left\lfloor \frac{x - m^2}{2|q_f eH|} \right\rfloor + 1 - \frac{x - m^2}{2|q_f eH|} \right)
\end{aligned}
\tag{4.42}
$$

Thus

$$
\mathtt{I}_n^H(m^2,x) = \mathtt{I}_n^{H+} \Theta\left(m^2 - x\right) + \mathtt{I}_n^{H-} \Theta\left(x - m^2\right) \tag{4.43}
$$

$$
\begin{aligned}
\mathtt{R}_n^H(m,z) =& 2\,\mathrm{Im}\left\{\sum_{j=0}^{\infty}\int\frac{dp_z}{(2\pi)^2}\left[\frac{|q_f eH|\,(2-\delta_{j,0})}{(p_z^2+2j|q_f eH|+m^2-z)^n}\right]\right\} \\
=& \frac{B\left(\frac{1}{2},n-\frac{1}{2}\right)}{2\pi^2}\left(2|q_f eH|\right)^{\frac{3}{2}-n}\mathrm{Im}\left\{\zeta\left(n-1/2,\frac{m^2-z}{2|q_f eH|}\right)\right\} \\
& -2n|q_f eH|\mathtt{R}_{n+1}(m,z)
\end{aligned}
\tag{4.44}
$$

$$
\begin{aligned}
\mathtt{Q}_n^H(m,z) =& 2\,\mathrm{Re}\left\{\sum_{j=0}^{\infty}\int\frac{dp_z}{(2\pi)^2}\left[\frac{|q_f eH|\,(2-\delta_{j,0})}{(p_z^2+2j|q_f eH|+m^2-z)^n}\right]\right\} \\
=& \frac{B\left(\frac{1}{2},n-\frac{1}{2}\right)}{2\pi^2}\left(2|q_f eH|\right)^{\frac{3}{2}-n}\mathrm{Re}\left\{\zeta\left(n-1/2,\frac{m^2-z}{2|q_f eH|}\right)\right\} \\
& -2n|q_f eH|\mathtt{Q}_{n+1}(m,z)
\end{aligned}
\tag{4.45}
$$

Using the idea from Sections 3.1.3 and 3.3.3

$$
\begin{aligned}
\mathtt{Mn}^H(m,z) =& 2\,\mathrm{Re}\left\{\frac{|q_f eH|}{2\pi}\sum_{n=0}^{\infty}\int\frac{dp_z}{2\pi}\left[\ln\left(\frac{p_z^2+m_n^2-z}{p_z^2+m_n^2}\right)(2-\delta_{n,0})\right]\right\} \\
=& \frac{|2q_f eH|^{\frac{3}{2}}}{\pi}\left[\mathrm{Re}\left\{\zeta\left(-\frac{1}{2},\frac{m^2-z}{|2q_f eH|}\right)\right\}-\zeta\left(-\frac{1}{2},\frac{m^2}{2|q_f eH|}\right)\right] \\
& -\frac{\sqrt{2}|q_f eH|}{2\pi}\Lambda
\end{aligned}
\tag{4.46}
$$

where $\Lambda = \sqrt{\lambda_R+\xi}$, $\quad \lambda_R = m^2-\mathrm{Re}(z)$, $\quad \lambda_I = -\,\mathrm{Im}(z) \quad \xi = \sqrt{\lambda_R^2+\lambda_I^2}$

4.4.3 Renaming the formulae

Now let us take some cases where the parameter $s = p^2+m^2 \to p^2$ at $m = 0$

$$
\begin{aligned}
\mathtt{I}_n(0,x_1) &= \mathtt{I}_n(x_1) \\
\mathtt{Q}_n(0,x_1) &= \mathtt{Q}_n(x_1) \\
\mathtt{R}_n(0,x_1) &= \mathtt{R}_n(x_1) \\
\mathtt{I}_n^H(0,x_1) &= \mathtt{I}_n^H(x_1) \\
\mathtt{Q}_n^H(0,x_1) &= \mathtt{Q}_n^H(x_1) \\
\mathtt{R}_n^H(0,x_1) &= \mathtt{R}_n^H(x_1)
\end{aligned}
\tag{4.47}
$$

The above notations are used in Section 4.3.

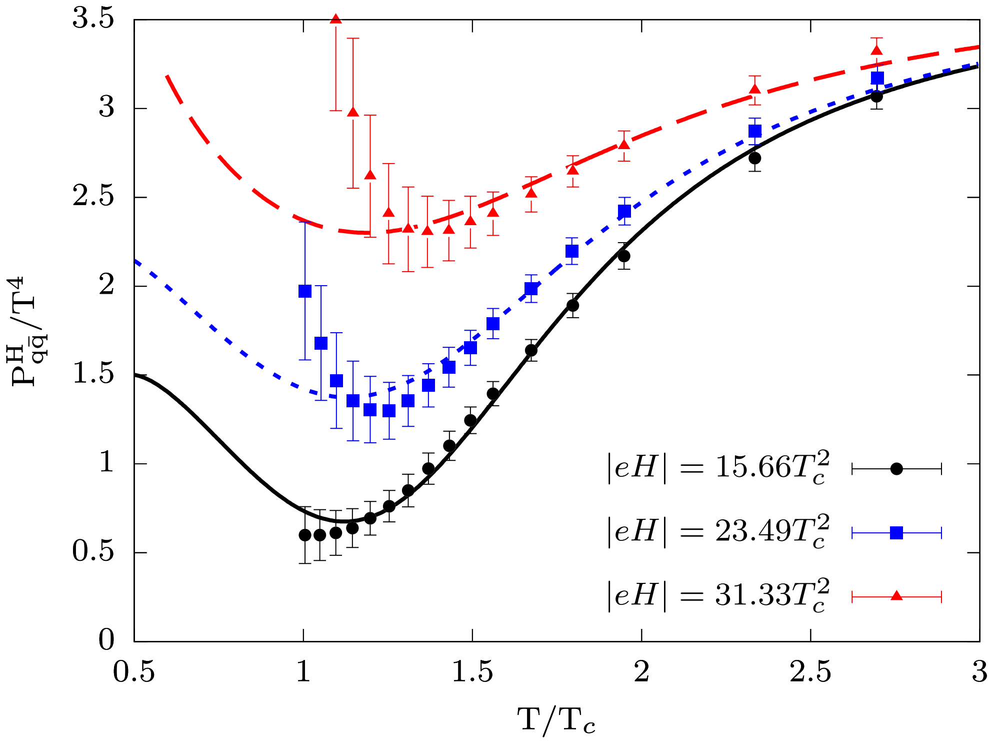

Figure 4.3: Pressure for Lattice data by [3], Here $T_c \approx 113$ MeV, $\mathrm{g}^{\mathrm{eff}} \approx 20.1$, $q_{\mathrm{f}}^{\mathrm{eff}} \approx \frac{1}{23}$, The radius factors $r_{q\bar{q}}$ are 0.986, 0.955 and 0.94 from bottom to top. The a_l factors are 4.735, 3.098 and 0.31 from bottom to top. The values of a_c and a_v are 0.711 (T_c) and 15.76 (T_c), respectively.

4.5 Results and conclusion

We have extended the modified liquid drop model in the presence of a magnetic field and derived the equation of state for magnetized quark matter. This is achieved by changing the integral equations of MLDM in Chapter 3 to a magnetic field dependent integral. The magnetic field dependent integral is formed on the basis of relativistic Landau level quantization.

In Figs. 4.4 to 4.7, the pressure, energy density, and entropy are plotted for qualitative purposes. The magnetic fields are described in terms of critical temperature ranging from $|qeH| = T_c^2$ to $40T_c^2$. The parameters used in the above mentioned plots are the same as those of two flavor data in Table 3.1, i.e.,$g_{q\bar{q}} = 64/3$, $a_l = 9.389$ and $r_{q\bar{q}} = 1.565$. The results match with the expected behaviour of [3].

The magnetic field can cause the production of new quarks [4, 5, 6, 7], which can cause a change in degeneracy. The magnetic field can also affect the charge neutrality of the plasma medium. This can be compensated by assuming an effective charge flavor q_f. The quantitative comparison is done in Fig. 4.3 with the lattice data [3].

The magnetic fields are $|eH| \approx 0.2$ GeV^2, 0.3 GeV^2, 0.4 GeV^2. In Fig. 4.3 the values are represented in terms of critical temperature. Fig. 4.3 is fitted with $\mathrm{g}^{\mathrm{eff}} = 20.1$, $\mathrm{q}_f^{\mathrm{eff}} \approx 1/22.8$, the radius factors are $r_{q\bar{q}} \approx 0.986, 0.955$ and 0.94 from bottom to top. The a_l factors are $4.735, 3.098$ and 0.31. The results are in quantitative agreement at the lower magnetic field.

The fluctuation between the fitted curve and the lattice data in $0.3\mathrm{GeV}^2 \approx 23.49T_c^2$ and $0.4\mathrm{GeV}^2 \approx 31.33T_c^2$ could be minimised if the degeneracy factor, which is considered a constant with respect to temperature, is modified by considering quark production happening in accordance with the magnetic field.

In Chapter 5, we derive the equation of state of deconfined quark matter in the presence of a magnetic field. Chapter 5 is dependent on the quasiparticle model of VM Bannur described in [8]. The integral equations in [8] are changed to magnetic field dependent integrals as shown in Eq. (4.11).

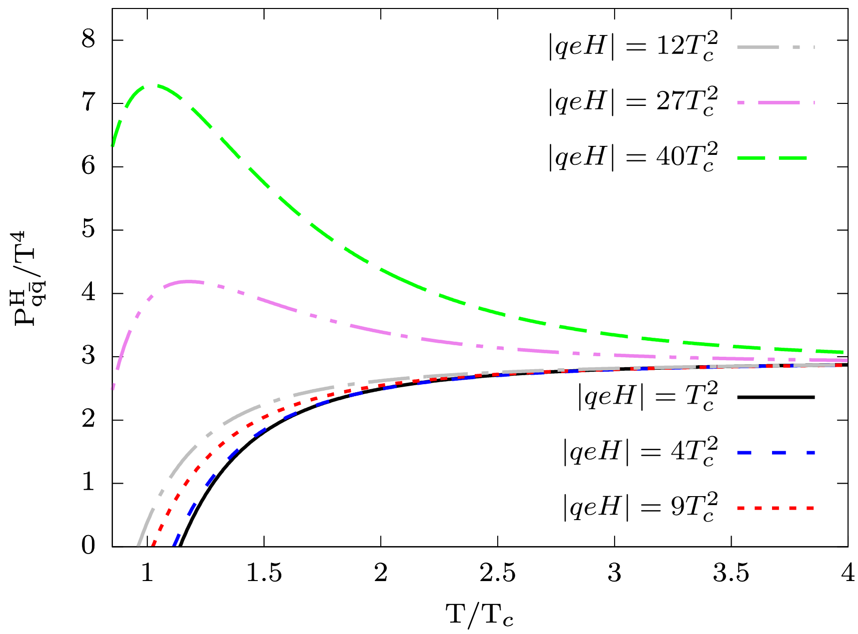

Figure 4.4: Quark contribution of pressure with $g_{q\bar{q}} \approx 64/3$, $a_l \approx 9.389$ and $r_{q\bar{q}} \approx 1.565$.

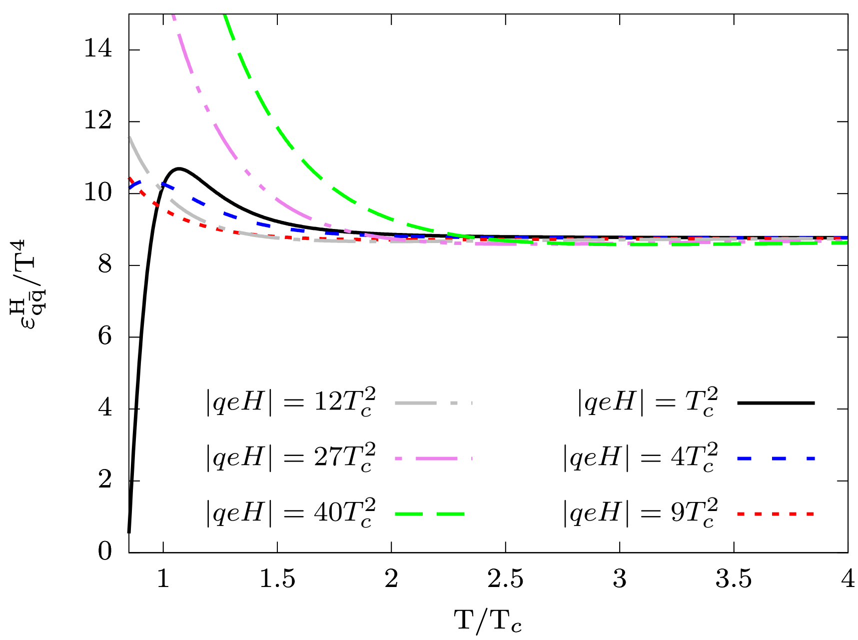

Figure 4.5: Quark contribution of energy density.

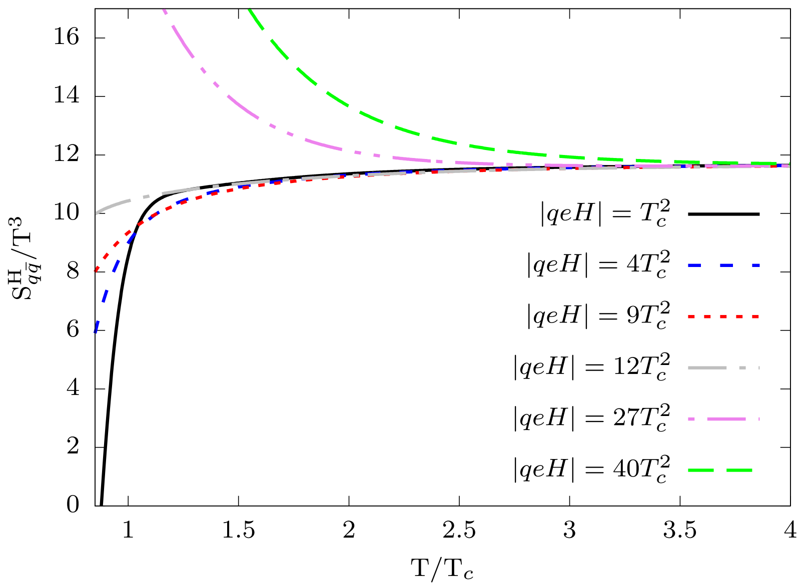

Figure 4.6: Quark contribution of entropy.

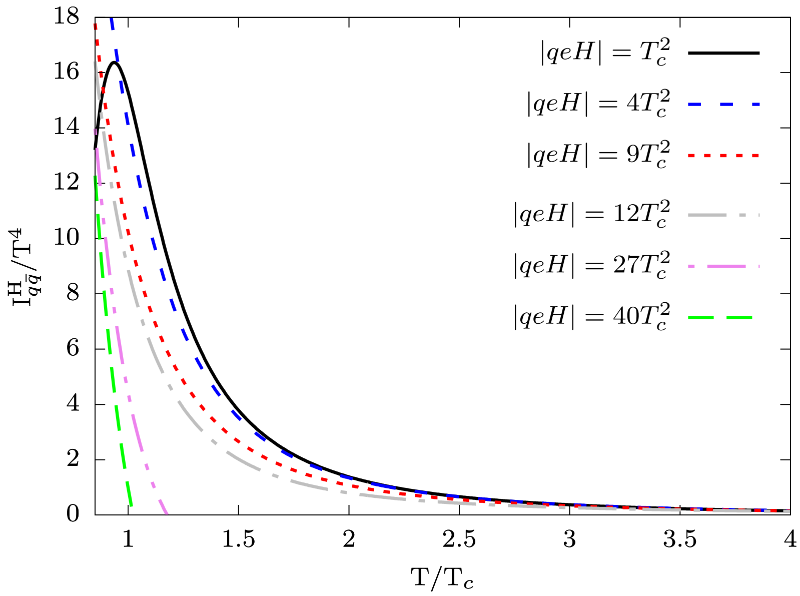

Figure 4.7: Quark contribution of Interaction Measure . Figs. 4.5 to 4.7 are plotted for various values of $|qeH|$ with $g_{q\bar{q}} \approx 64/3$, $a_l \approx 9.389$ and $r_{q\bar{q}} \approx 1.565$.

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Chapter 5

De-confined quark matter under magnetic field

When a massive supergiant star collapses, if the core has a mass about 1.4 times that of the sun, it forms a neutron star. The order of magnitude of the radius of a neutron star is about 10km [1, 2]. When the pressure inside a neutron star is so intense that the density of matter becomes several times greater than the nuclear density ($\rho_{\text{nuclear}} = 0.16\ \text{fm}^{-3}$), neutrons are transformed into quarks.

As the temperature reaches to zero, the distribution of quarks, which are fermions, gets bound by the chemical potential. The mathematical relations connecting the momentum, mass of the quasiparticle and chemical potential are given in Eqs. (5.10) to (5.12). The quasiparticle model of Bannur [3] is applied in the magnetic field regime. The magnetic field independent coupling constant and integrals are replaced by corresponding magnetic field dependent coupling constant from [4] and integral Eq. (5.16).

Using these approximations, the magnetic field contribution to pressure, energy density, and number density for various chemical potentials are derived.

Consider a quark star having the charge neutrality condition

$$\sum_i q_i n_i = 0 \implies \frac{2}{3}n_{\text{up}} - \frac{1}{3}n_{\text{down}} - \frac{1}{3}n_{\text{strange}} - n_e = 0 \tag{5.1}$$

where $\text{n}_{\text{up}}, \text{n}_{\text{down}}$ and $\text{n}_{\text{strange}}$, are the number density of up, down and strange quark. n_e is the electron number density. The chemical equilibrium attained through weak interaction can be written as

$$\begin{aligned} d \leftrightarrow u + e^- + \bar{\nu}_{e^-} &\implies \mu_d = \mu_u + \mu_e \\ s \leftrightarrow u + e^- + \bar{\nu}_{e^-} &\implies \mu_s = \mu_u + \mu_e \end{aligned} \tag{5.2}$$

At equilibrium

$$s + u \leftrightarrow d + u \implies \mu_d = \mu_u \tag{5.3}$$

assuming that there is no neutrino participation in the quark star's thermodynamic properties and approximating the rest mass of quarks in the medium to be zero leads to the approximation

$$\begin{aligned} &\mu_u = \mu_d = \mu_s \\ &\mu_e = 0 \end{aligned} \tag{5.4}$$

for the medium.

5.1 Quasi Particle Model and Fermi Dirac Distribution

Consider quark matter that has a medium-dependent mass. At absolute zero temperature, let the mass be a function of the chemical potential and rest mass of the constituent quark matter. i.e.,

$$m^2 = m_0^2 + \sqrt{2} m_\mu m_0 + m_\mu^2 \tag{5.5}$$

According to the quasiparticle model put forward by Bannur [3], at T = 0, the medium-dependent effective quark mass can be approximated as

$$m^2(\mu) = \frac{g^2(\sqrt{a}\mu)}{18\pi^2} \mu^2 \; n_f \tag{5.6}$$

with $a = \left(\frac{1.91}{2.91}\right)^2$.
For one loop approximation, when $B = 0$

$$g^2(\mu, \Lambda) = \frac{48\pi^2}{(33 - 2n_f) \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \tag{5.7}$$

For one loop approximation [4], when $B \neq 0$

$$g^2(\mu, \Lambda, |eB|) = g^2(\mu, \Lambda) \left[1 + \frac{\ln\left(\frac{\mu^2}{\mu^2 + |eB|}\right)}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)} \right]^{-1} \tag{5.8}$$

In this chapter, we use B to represent the magnetic field unlike H that was used in Chapter 4.

We use the same procedure as in [3] and take the same approximation

$$
\begin{aligned}
m_{\text{th}}^2(T,\mu) &\approx \frac{g^2T^2}{18} n_f\left(1+\frac{\mu^2}{\pi^2T^2}\right)\\
m_{\text{th}}^2(0,\mu)\Big|_{n_f=3} &\approx \frac{g^2\mu^2}{6\pi^2}
\end{aligned}
\tag{5.9}
$$

The general equation for average energy and number density is

$$
\langle\epsilon\rangle = \sum_i \sum_{\sigma=\pm1} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{g_i\ \omega_i}{1+\exp\left(\beta(\omega_i-\sigma\mu_i)\right)}\right] \tag{5.10}
$$

$$
\langle n\rangle = \sum_i \sum_{\sigma=\pm1} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{g_i\sigma}{1+\exp\left(\beta(\omega_i-\sigma\mu_i)\right)}\right] \tag{5.11}
$$

with g_i being the degeneracy factor of the i^{th} species. Since we are dealing with matter involving particles (not antiparticles), it is crucial to note that only the case where $\sigma = +1$ in Eqs. (5.10) and (5.11) will be considered for the subsequent calculations in this chapter. Now as $\beta \to \infty$ the systems goes to Fermi distribution. As a result, in phase space, a surface with the highest energy value is generated, which limits the range of energy values. When B=0, the equation becomes

$$
\begin{aligned}
&\left\langle H^2\right\rangle_{B=0} = p_x^2+p_y^2+p_z^2+m^2 = \mu^2\\
&\implies p_x^2+p_y^2+p_z^2 = \mu^2-m^2
\end{aligned}
\tag{5.12}
$$

and thus phase space has spherical symmetry bounded by surface with radius $\sqrt{\mu^2-m^2}$. So number density is

$$
\begin{aligned}
\langle n\rangle &= g_i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ \Theta\left(\sqrt{\mu^2-m^2}-|p|\right)\\
&= \frac{g_i}{2\pi^2}\int_0^{\infty} p^2\ \Theta\left(\sqrt{\mu^2-m^2}-|p|\right)\ \mathrm{d}p\\
&= \frac{g_i}{2\pi^2}\int_0^{\sqrt{\mu^2-m^2}} p^2\mathrm{d}p\\
&= \frac{g_i}{6\pi^2}\left(\sqrt{\mu^2-m^2}\right)^3
\end{aligned}
\tag{5.13}
$$

Similarly energy density is

$$
\begin{aligned}
\langle \epsilon \rangle =& g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{p^2+m^2}\ \Theta\left(\sqrt{\mu^2+m^2} - |p|\right) \\
=& g_i \int_0^{\sqrt{\mu^2-m^2}} p^2\ \sqrt{p^2+m^2} \frac{\mathrm{d}p}{2\pi^2} \\
& \text{Putting } p = m\sinh(x) \text{, then} \\
=& \frac{g_i}{2\pi^2} \int_0^{\sinh^{-1}\left(\sqrt{\frac{\mu^2-m^2}{m^2}}\right)} m^4 \cosh^2(x)\ \sinh^2(x) \mathrm{d}x \\
=& \frac{g_i}{2\pi^2} \int_0^{\sinh^{-1}\left(\sqrt{\frac{\mu^2-m^2}{m^2}}\right)} \frac{m^4}{8} \left[\cosh(4x) - 1\right] \mathrm{d}x \\
=& \frac{g_i}{16\pi^2} \left[\left(\mu\sqrt{\mu^2-m^2}\left[2\mu^2-m^2\right]\right) - \left(m^4 \sinh^{-1}\left(\frac{\sqrt{\mu^2-m^2}}{m}\right)\right) \right]
\end{aligned}
\tag{5.14}
$$

5.2 Quark matter under magnetic field

As discussed in Chapter 4, if we consider a magnetic field with magnetic vector potential

$$
\begin{aligned}
\hat{\vec{A}} &= Bx\ \hat{y} \\
\vec{B} &= \boldsymbol{\nabla} \times Bx\ \hat{y} = B\ \hat{z}
\end{aligned}
\tag{5.15}
$$

with B being the magnitude of magnetic field. The three dimensional integrals changes accordingly i.e.,

$$
\int f(p^2)\ \frac{\mathrm{d}^3 p}{(2\pi)^3} \to \frac{|qeB|}{2\pi} \sum_{n=0}^{\infty} \int f(p_z^2 + 2n|qeB|)\ (2-\delta_{n,0})\ \frac{dp_z}{2\pi} \tag{5.16}
$$

5.3 The integral table for deconfined matter under magnetic field

Consider the integral,

$$\begin{aligned}
\mathrm{I}_1 = F(a,x) &= \int \frac{\sqrt{x^2-a^2}}{x}\mathrm{d}x \text{ , Putting } u^2 = x^2 - a^2 \\
&= \int \frac{u^2}{u^2+a^2}\mathrm{d}u \\
&= \int \left(1 - \frac{a^2}{u^2+a^2}\right)\mathrm{d}u \\
&= \left(u - \int \left[\frac{a^2}{u^2+a^2}\right]\mathrm{d}u\right) \text{ , Putting } u = a\tan\theta \\
&= \left(u - a\theta\right) \\
&= \left[\sqrt{x^2-a^2} - a\tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right)\right]
\end{aligned} \tag{5.17}$$

If one puts $u = a\cot\theta$, then

$$\begin{aligned}
\mathrm{I}_1 &= \left(u - \int \left[\frac{a^2}{u^2+a^2}\right]\mathrm{d}u\right) \\
&= u + a\theta \\
&= \sqrt{x^2-a^2} + a\cot^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right) \\
&= \sqrt{x^2-a^2} + a\tan^{-1}\left(\frac{a}{\sqrt{x^2-a^2}}\right)
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\mathrm{I}_2 &= \int \frac{\ln\left(\mu + \sqrt{\mu^2-m^2}\right)}{\mu^2}\mathrm{d}\mu \\
&\text{Putting } \mu = m\cosh x \\
\mathrm{I}_2 &= \int \tanh(x)\,\mathrm{sech}(x)\left[\frac{x+\ln(m)}{m}\right]\mathrm{d}x
\end{aligned} \tag{5.19}$$

The integral with integrand $\tanh(x)\,\mathrm{sech}(x)$ can be solved as

$$
\begin{aligned}
\mathrm{I}_{21} &= \int x \tanh(x)\,\mathrm{sech}(x)\mathrm{d}x \\
&= x \int \tanh(x)\,\mathrm{sech}(x)\mathrm{d}x - \int\int \tanh(x)\,\mathrm{sech}(x)\mathrm{d}x \\
&= x \int -\mathrm{d}\left(\mathrm{sech}(x)\right) + \int\int \mathrm{d}\left(\mathrm{sech}(x)\right) \\
&= -x\,\mathrm{sech}(x) + \int \mathrm{sech}(x)\mathrm{d}x \\
&= -x\,\mathrm{sech}(x) + \int \frac{2e^x}{e^{2x}+1}\mathrm{d}x \\
&\text{putting } u = e^x \\
&= -x\,\mathrm{sech}(x) + 2\int \frac{1}{u^2+1}\mathrm{d}u \\
&= -x\,\mathrm{sech}(x) + 2\tan^{-1}(e^x)
\end{aligned}
\tag{5.20}
$$

On continuing

$$
\begin{aligned}
\mathrm{I}_2 &= \int \tanh(x)\,\mathrm{sech}(x)\left[\frac{x+\ln(m)}{m}\right]\mathrm{d}x \\
&= \frac{1}{m}\left[2\tan^{-1}(e^x) - x\,\mathrm{sech}(x) - \ln(m)\,\mathrm{sech}(x)\right] \\
&\text{putting } \mathrm{sech}(x) = m/\mu \\
&= \frac{1}{m}\left[2\tan^{-1}\left(\frac{\mu \pm \sqrt{\mu^2-m^2}}{m}\right) - \frac{m}{\mu}\ln\left(\frac{\mu \pm \sqrt{\mu^2-m^2}}{m}\right) - \frac{m}{\mu}\ln(m)\right] \\
&= \frac{1}{m}\left[2\tan^{-1}\left(\frac{\mu \pm \sqrt{\mu^2-m^2}}{m}\right) - \frac{m}{\mu}\ln\left(\mu \pm \sqrt{\mu^2-m^2}\right)\right]
\end{aligned}
\tag{5.21}
$$

So

$$
\begin{aligned}
G\left(m,\mu\right) &= \int \frac{\ln\left(\mu+\sqrt{\mu^2-m^2}\right)}{\mu^2}\mathrm{d}\mu \\
&= \frac{1}{m}\left[2\tan^{-1}\left(\frac{\mu + \sqrt{\mu^2-m^2}}{m}\right) - \frac{m}{\mu}\ln\left(\mu + \sqrt{\mu^2-m^2}\right)\right]
\end{aligned}
\tag{5.22}
$$

The other way of working out the integral is

$$\begin{aligned}
G(m,\mu) &= \int \frac{\ln\left(\mu+\sqrt{\mu^2-m^2}\right)}{\mu^2}\mathrm{d}\mu \\
&= \ln\left(\mu+\sqrt{\mu^2-m^2}\right)\int\frac{1}{\mu^2}\mathrm{d}\mu - \int\frac{1}{\sqrt{\mu^2-m^2}}\times -\frac{1}{\mu}\mathrm{d}\mu \\
&= -\frac{\ln\left(\mu+\sqrt{\mu^2-m^2}\right)}{\mu} + \int\frac{1}{\mu\sqrt{\mu^2-m^2}}\mathrm{d}\mu \qquad (5.23)\\
&\text{Putting } \mu = a\sec\theta \\
&= -\frac{\ln\left(\mu+\sqrt{\mu^2-m^2}\right)}{\mu} + \frac{1}{m}\sec^{-1}\left(\frac{\mu}{m}\right) \\
&= \frac{1}{m}\tan^{-1}\left(\frac{\sqrt{\mu^2-m^2}}{m}\right) - \frac{\ln\left(\mu+\sqrt{\mu^2-m^2}\right)}{\mu}
\end{aligned}$$

5.4 Number density in presence of magnetic field

The limit relation of momentum in z direction, chemical potential and mass can be developed as

$$\begin{aligned}
&2q_f|eB|n_\sigma + p_z^2 + m^2 = \mu^2 \qquad (5.24)\\
&\implies 2q_f|eB|n_\sigma + p_z^2 = \mu^2 - m^2
\end{aligned}$$

Here we have to give special consideration to the quantized state of the system. We introduced the limiting condition with the involvement of theta function.

$$\langle n\rangle_{\mathrm{q}} = \frac{q_f|eB|}{2\pi}\sum_{n_\sigma=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}(2-\delta_{n_\sigma,0})\Theta(\sqrt{\mu^2-m^2-2q_f|eB|n_\sigma}-|p_z|) \qquad (5.25)$$

In quasiparticle model mass is a function of chemical potential. For free particle approximation, the mass m =0 which leads to a change in relativistic energy $\sqrt{p^2+m^2} \to |p|$.

5.4.1 Number density in presence of magnetic field for QPM Model

$$
\begin{aligned}
\langle n\rangle_{\mathrm{q}} &= \frac{q_f|eB|}{2\pi}\sum_{n_\sigma=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}(2-\delta_{n_\sigma,0})\Theta(\sqrt{\mu^2-m^2-2q_f|eB|n_\sigma}-|p_z|) \\
&= \frac{q_f|eB|}{\pi^2}\sum_{n_\sigma=0}^{n_{\max}}\left[\sqrt{\mu^2-m^2-2q_f|eB|n_\sigma}\right]-\frac{q_f|eB|}{2\pi^2}\sqrt{\mu^2-m^2} \\
&= \frac{\sqrt{2}\,(|q_feB|)^{\frac{3}{2}}}{\pi^2}\sum_{n_\sigma=0}^{n_{\max}}\sqrt{n_\sigma+\left\{\frac{\mu^2-m^2}{2|q_feB|}\right\}}-\frac{q_f|eB|}{2\pi^2}\sqrt{\mu^2-m^2} \\
&= \frac{\sqrt{2}\,(|q_feB|)^{\frac{3}{2}}}{\pi^2}\psi\left(\frac{1}{2},\frac{\mu^2-m^2}{2|q_feB|}\right)-\frac{q_f|eB|}{2\pi^2}\sqrt{\mu^2-m^2} \qquad (5.26)
\end{aligned}
$$

with

$$
\begin{aligned}
&\{x\} = x-\lfloor x\rfloor \geq 0 \\
&\psi(s,x) = \sum_{n=0}^{\lfloor x\rfloor}(x-n)^s = \zeta\left(-s,\{x\}\right)-\zeta\left(-s,x+1\right), \quad s>0 \\
&n_{\max} = \left\lfloor\frac{\mu^2-m^2}{2q_f|eB|}\right\rfloor
\end{aligned}
$$

We used the mathematical operator floor $\lfloor\rfloor$ and ceil$\lceil\rceil$, ($\lfloor a\rfloor$ meaning the largest integer less than or equal to a. i.e $\left\lfloor\frac{3}{2}\right\rfloor = 1$, $\lceil a\rceil$ the smallest integer greater than or equal to a. $\left\lceil\frac{3}{2}\right\rceil = 2$).

The relation between mass, chemical potential and coupling constant for one loop approximation is

$$
\begin{aligned}
m^2(\mu) &= \frac{n_f\mu^2}{18\pi^2}\times\frac{48\pi^2}{(33-2n_f)\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\times\left[1+\frac{\ln\left(\frac{\mu^2}{\mu^2+|eB|}\right)}{\ln\left(\frac{a\mu^2}{\Lambda^2}\right)}\right]^{-1} \\
m^2(\mu)\Big|_{n_f=3} &= \left(\frac{2}{3}\right)^3\frac{\mu^2}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\left[1+\frac{\ln\left(\frac{\mu^2}{\mu^2+|eB|}\right)}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\right]^{-1}
\end{aligned} \qquad (5.27)
$$

with $m^2 = m_0^2+\sqrt{2}m_0m_\mu+m_\mu^2$. The calculation can be made simple if we consider the magnitude of $\Lambda = 1$ and rewrite both μ, μ_0 and $|eB|$ in the units of Λ. Thus

$$
\mu^2-m^2(\mu) = \mu^2\left[1-\left(\frac{2}{3}\right)^3\frac{1}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)+\ln\left(\frac{\mu^2}{\mu^2+|eB|}\right)}\right] \qquad (5.28)
$$

5.4.2 Number density in presence of magnetic field for free-particles

As we have mentioned, for free particle the mass $m = 0$ ($\sqrt{p^2+m^2} \to |p|$),

$$\therefore 2q_f|eB|n_\sigma + p_z^2 = \mu^2 \qquad (5.29)$$
$$\implies 2q_f|eB|n_\sigma + p_z^2 = \mu^2$$

The number density of free particle is

$$\begin{aligned}
\langle n\rangle_q^{\text{free}} &= \frac{q_f|eB|}{2\pi}\sum_{n_\sigma=0}^{\infty}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}(2-\delta_{n_\sigma,0})\Theta(\sqrt{\mu^2-2q_f|eB|n_\sigma}-|p_z|)\\
&= \frac{q_f|eB|}{\pi^2}\sum_{n_\sigma=0}^{n_{\max}}\left[\sqrt{\mu^2-2q_f|eB|n_\sigma}\right]-\frac{q_f|eB|}{2\pi^2}\sqrt{\mu^2}\\
&= \frac{\sqrt{2}\,(|q_feB|)^{\frac{3}{2}}}{\pi^2}\sum_{n_\sigma=0}^{n_{\max}}\sqrt{n_\sigma+\left\{\frac{\mu^2}{2|q_feB|}\right\}}-\frac{q_f|eB|}{2\pi^2}\sqrt{\mu^2} \qquad (5.30)\\
&= \frac{\sqrt{2}\,(|q_feB|)^{\frac{3}{2}}}{\pi^2}\psi\left(\frac{1}{2},\frac{\mu^2}{2|q_feB|}\right)-\frac{q_f|eB|}{2\pi^2}\sqrt{\mu^2}
\end{aligned}$$

with

$$n_{\max} = \left\lfloor\frac{\mu^2}{2q_f|eB|}\right\rfloor$$

Let us represent the total number density of quark in the qpm model as $\sum n_{q\bar{q}}^B = \langle n\rangle_{\text{up}} + \langle n\rangle_{\text{down}} + \langle n\rangle_{\text{strange}}$ and that of the free particle as $\sum n_{\text{free}}^B = \langle n\rangle_{\text{up}}^{\text{free}} + \langle n\rangle_{\text{down}}^{\text{free}} + \langle n\rangle_{\text{strange}}^{\text{free}}$. The ratio of $\sum n_{q\bar{q}}^B$ to $\sum n_{\text{free}}^B$ is plotted in Figs. 5.1 and 5.2.

In Fig. 5.1, x axis is the chemical potential in units of Λ ranging from 15 to 1000 . The Y axis is the ratio of the number density of quasi particles to that of free particles. It can be observed that the curve follows a similar trend as in non-magnetic regime [3]. But when the same plot is enlarged as shown in Fig. 5.2, it can be observed that when the magnetic field contribution $|eB|$ is near to the chemical potential, large oscillations can be observed. This is due to the quantization effect where the energy is quantized with a maximum value of quantization frequency $\left\lfloor\frac{\mu^2-m^2}{2|q_feB|}\right\rfloor$. For the charge neutrality to be fulfilled and to balance this charged particle oscillation, a corresponding positron-electron oscillation will also occur. But in our quasiparticle approximation we have given the chemical potential of electron as zero, thus it doesn't affect the charge neutrality.

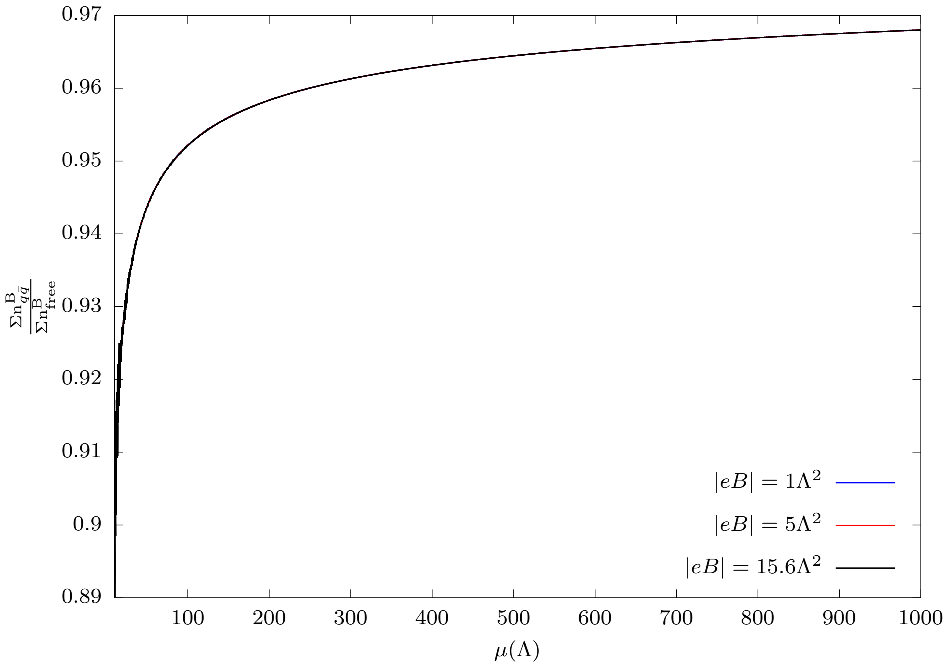

Figure 5.1: The scaled number density is plotted for higher chemical potential range.

5.5 Energy density in presence of magnetic field

In presence of magnetic field the Eq. (5.10) changes to

$$\langle \epsilon \rangle_q = \frac{q_f|eB|}{2\pi} \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \sqrt{p_z^2 + m^2 + 2j|eB|q_f}\ (2 - \delta_{j,0})\Theta(A)\ \frac{\mathrm{d}p_z}{2\pi} \tag{5.31}$$

where $A = \sqrt{\mu^2 - m^2 - 2q_f|eB|j} - |p_z|$.

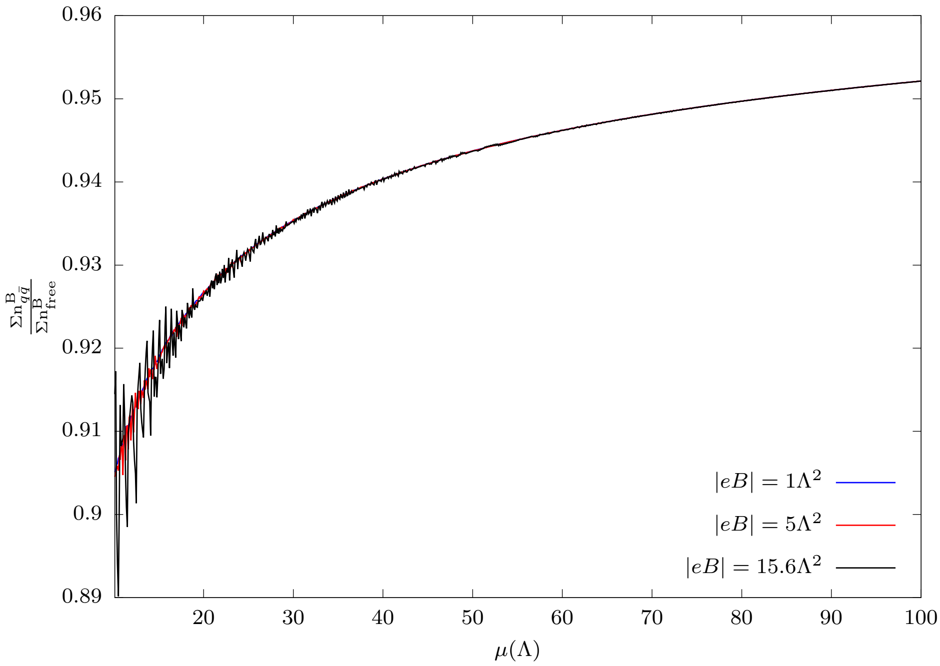

Figure 5.2: The number density of quasi particles scaled with that of free particles. The range of chemical potential is limited to 100 Λ to show the effect of the magnetic field

5.5.1 Energy density in presence of magnetic field for QPM Model

The energy density in presence of magnetic field is

$$\begin{aligned}\langle\epsilon\rangle_q &= \frac{q_f|eB|}{2\pi}\sum_{j=0}^{\infty}\int_{-\infty}^{\infty}\sqrt{p_z^2+{m_j}^2}\;(2-\delta_{j,0})\Theta(\sqrt{\mu^2-{m_j}^2}-|p_z|)\;\frac{\mathrm{d}p_z}{2\pi}\\ &= \frac{q_f|eB|}{2\pi^2}\left[2\sum_{j=0}^{n_{\max}}\int_0^{\sqrt{\mu^2-m_j^2}}\sqrt{p_z^2+m_j^2}\;\mathrm{d}p_z-\int_0^{\sqrt{\mu^2-m^2}}\sqrt{p_z^2+m^2}\;\mathrm{d}p_z\right]\end{aligned} \tag{5.32}$$

with $m_j^2 = m^2 + 2j|qeB|$. Eq. (5.32) can be solved by putting $p = m\sinh(x)$, then

$$\int_0^{\sqrt{\mu^2-m^2}}\sqrt{p^2+m^2}\mathrm{d}p = \frac{1}{2}\left(\mu\sqrt{\mu^2-m^2}+m^2\sinh^{-1}\left(\frac{\sqrt{\mu^2-m^2}}{m}\right)\right) \tag{5.33}$$

and $\sinh^{-1}(x) = \ln\left(x \pm \sqrt{x^2+1}\right)$, in case of real $\sinh^{-1}$ value, we can approximate $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$. Therefore

$$\int_0^{\sqrt{\mu^2-m^2}} \sqrt{p^2+m^2}\mathrm{d}p = \frac{\mu\sqrt{\mu^2-m^2}}{2} + \frac{m^2}{2}\ln\left[\frac{\mu+\sqrt{\mu^2-m^2}}{m}\right] \tag{5.34}$$

Thus the energy density is

$$\begin{aligned}\langle\varepsilon\rangle_q =& \frac{q_f|eB|}{2\pi^2}\sum_{n_\sigma=0}^{n_{\max}}\mu\sqrt{\mu^2-m^2-2|eB|q_f n_\sigma} \\ &+ \frac{|eB|q_f}{2\pi^2}\sum_{n=0}^{n_{\max}}(m^2+2n_\sigma|eB|q_f)\ln\left(\frac{\sqrt{\mu^2-m^2-2q_f|eB|n_\sigma}+\mu}{\sqrt{m^2+2n_\sigma|eB|q_f}}\right) \\ &- \frac{q_f|eB|}{4\pi^2}\left[\mu\sqrt{\mu^2-m^2}+m^2\ln\left(\frac{\sqrt{\mu^2-m^2}+\mu}{m}\right)\right]\end{aligned} \tag{5.35}$$

But as per Eq. (5.27), the square of the mass is a function of the coupling constant. i.e.,

$$m^2(\mu) \propto g^2(\mu, |q_f eB|)\mu^2 \tag{5.36}$$

One could re-write the equation as

$$\begin{aligned}\langle\varepsilon\rangle_q =& \frac{(q_f|eB|)^2}{2\pi^2}\left[\frac{\partial}{\partial s} - \ln(2q_f|eB|)\right]\phi\left(s, \frac{m^2}{2q_f|eB|}, \left\lfloor\frac{\mu^2-m^2}{2q_f|eB|}\right\rfloor\right)\Bigg|_{s=-1} \\ &+ \frac{q_f|eB|}{2\pi^2}\mu\sqrt{2q_f|eB|}\psi\left(\frac{1}{2}, \frac{\mu^2-m^2}{2q_f|eB|}\right) \\ &- \frac{q_f|eB|}{4\pi^2}\left(\mu\sqrt{\mu^2-m^2}+m^2\ln\left(\frac{\mu+\sqrt{\mu^2-m^2}}{m}\right)\right) \\ &+ \frac{q_f|eB|}{2\pi^2}\sum_{j=0}^{n_{\max}}\left(m^2+2j|qeB|\right)\ln\left(\mu+\sqrt{\mu^2-m^2-2jq_f|eB|}\right)\end{aligned} \tag{5.37}$$

with

$$\begin{aligned}\zeta(s,x) &= \sum_{n=0}^{\infty}\frac{1}{(n+x)^s}, \quad \frac{\partial\zeta(s,x)}{\partial s} = -\sum_{n=0}^{\infty}\frac{\ln(n+x)}{(n+x)^s} \\ \phi(s,x,n_{\max}) &= \sum_{n=0}^{n_{\max}}\frac{1}{(x+n)^s} = \zeta(s,x) - \zeta(s, x+n_{\max}+1)\end{aligned} \tag{5.38}$$

So

$$\sum_{j=0}^{n_{\max}} \frac{\mu\sqrt{\mu^2 - m_j^2}}{2} = \frac{\mu\sqrt{2q_f|eB|}}{2}\ \psi\left(\frac{1}{2}, \frac{\mu^2 - m^2}{2q_f|eB|}\right)$$

and

$$\begin{aligned}\sum_{j=0}^{n_{\max}} \frac{m_j^2}{2}\ln\left(\frac{1}{m_j}\right) &= -\sum_{j=0}^{n_{\max}} \frac{m^2 + 2j|q_f eB|}{4}\ln\left(m^2 + 2j|q_f eB|\right) \\ &= -\frac{q_f|eB|}{2}\sum_{j=0}^{n_{\max}}\left(\frac{m^2}{2q_f|eB|} + j\right)\left[\ln\left(2q_f|eB|\right) + \ln\left(\frac{m^2}{2q_f|eB|} + j\right)\right] \\ &= \frac{q_f|eB|}{2}\left[\frac{\partial}{\partial s} - \ln\left(2q_f|eB|\right)\right]\phi\left(s, \frac{m^2}{2q_f|eB|}, \left\lfloor\frac{\mu^2 - m^2}{2q_f|eB|}\right\rfloor\right)\Bigg|_{s=-1}\end{aligned} \tag{5.39}$$

The total energy density can be written as

$$\varepsilon_{\text{total}} = \varepsilon_{\text{up}} + \varepsilon_{\text{down}} + \varepsilon_{\text{strange}} \tag{5.40}$$

5.5.2 Free particle energy density

The free particle approximation of quarks in our model assumes the mass of the quark to be negligible. i.e., $m_q \approx 0$. Thus, the calculation is straight-forward and we get,

$$\begin{aligned}\langle\varepsilon\rangle^{\text{free}} =& \frac{q_f|eB|}{2\pi^2}\sum_{n_\sigma=0}^{n_{\max}} \mu\sqrt{\mu^2 - 2|eB|q_f n_\sigma} - \frac{q_f|eB|}{4\pi^2}\mu^2 \\ &+ \frac{|eB|q_f}{2\pi^2}\sum_{n=0}^{n_{\max}}(2n_\sigma|eB|q_f)\ln\left(\frac{\sqrt{\mu^2 - 2q_f|eB|n_\sigma} + \mu}{\sqrt{2n_\sigma|eB|q_f}}\right)\end{aligned} \tag{5.41}$$

with $n_{\max} = \left\lfloor \frac{\mu^2}{2q_f|eB|} \right\rfloor$

The total energy density of quark in the qpm model is represented as $\sum \varepsilon^B_{q\bar{q}} = \langle\varepsilon\rangle_{\text{up}} + \langle\varepsilon\rangle_{\text{down}} + \langle\varepsilon\rangle_{\text{strange}}$ and that of the free particle as $\sum \varepsilon^B_{\text{free}} = \langle\varepsilon\rangle^{\text{free}}_{\text{up}} + \langle\varepsilon\rangle^{\text{free}}_{\text{down}} + \langle\varepsilon\rangle^{\text{free}}_{\text{strange}}$.

The ratio of $\sum \varepsilon^B_{q\bar{q}}$ to $\sum \varepsilon^B_{\text{free}}$ is plotted in Figs. 5.3 and 5.4. The X axis of Fig. 5.3 is the chemical potential ranging from 15 to 1000 in the units of Λ. Comparing Fig. 5.3 with the enlarged version Fig. 5.4, very high oscillation can be observed for $|eB| = 15.6\Lambda^2$, when the chemical potential is comparable. As we have mentioned in the number density section, the oscillation of charged particles is balanced by a corresponding quasielectron-quasipositron oscillation. Since the chemical potential of this quasielectron and quasipositron is zero. The contribution to energy density is negligible.

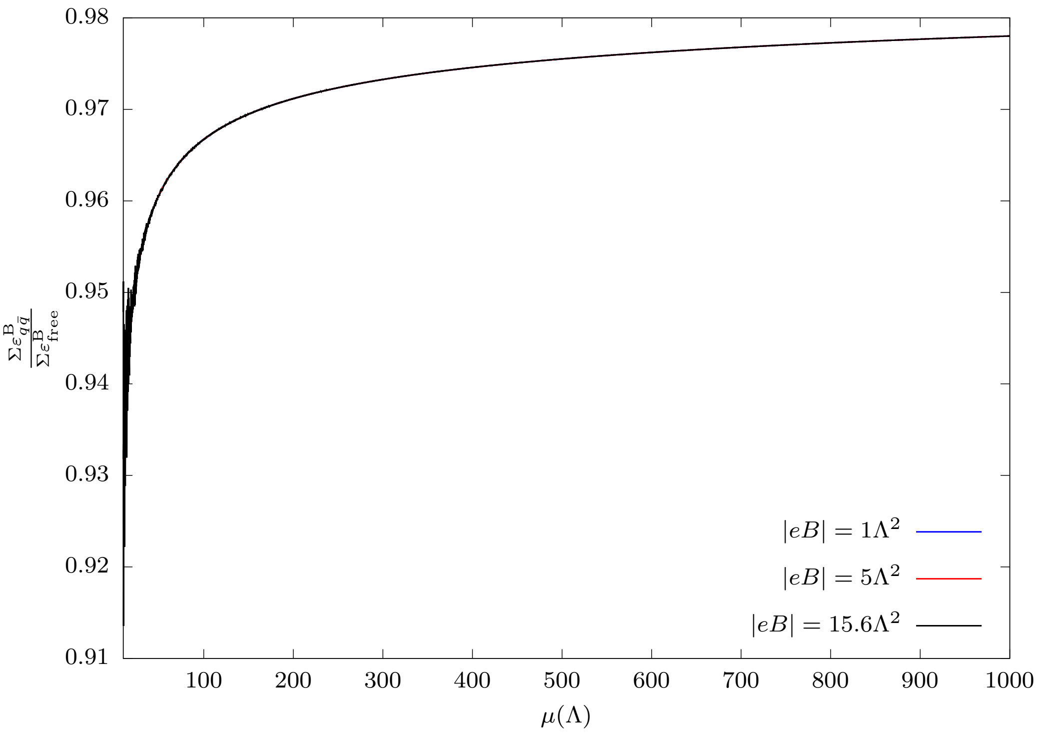

Figure 5.3: The scaled energy density is plotted for higher chemical potential range.

5.6 Pressure in presence of magnetic field

The pressure can be derived using the relation [3]

$$\frac{\varepsilon}{\mu} = \frac{\partial}{\partial \ln(\mu)} \left[\frac{P}{\mu}\right] \tag{5.42}$$

where ε is the energy density in terms of μ. P is the pressure and μ is the chemical potential. The corresponding integral expression is

$$\frac{P}{\mu} = \frac{P_0}{\mu_0} + \int_{\mu_0}^{\mu} \frac{\langle \varepsilon(\mu) \rangle}{\mu^2} \mathrm{d}\mu \tag{5.43}$$

.

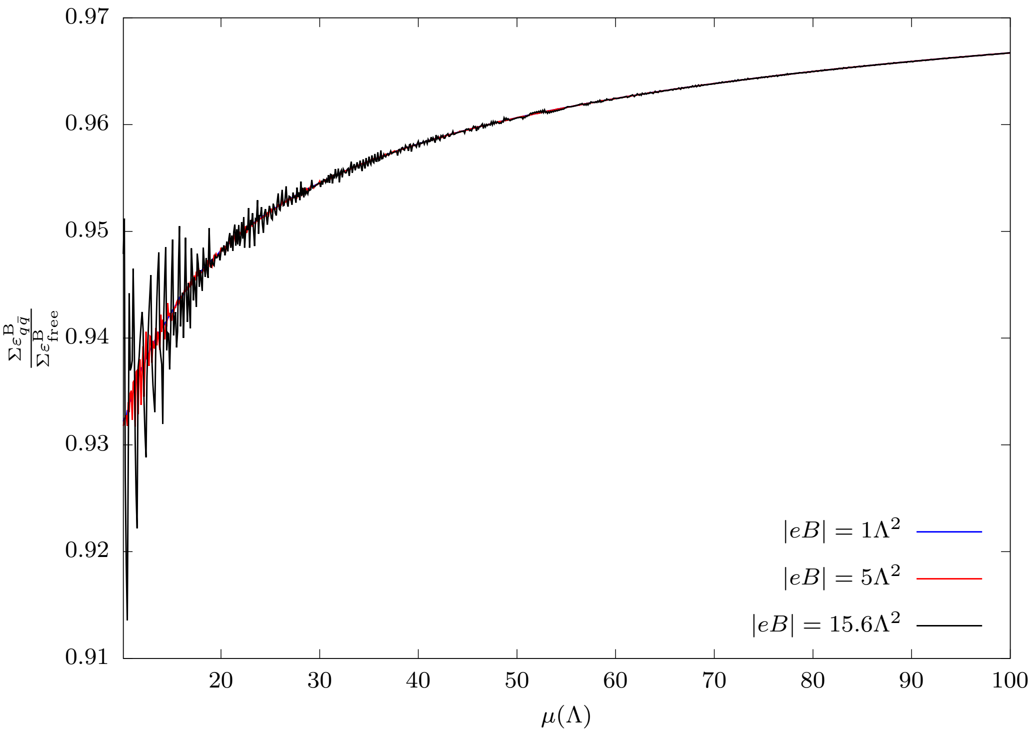

Figure 5.4: The energy density of quasi particles scaled with that of free particles. The range of chemical potential is limited to 100 Λ to show the effect of the magnetic field

5.6.1 Free particle pressure

The pressure can be derived using Section 5.5.2 and Eqs. (5.17) to (5.19) and (5.21),

$$
\begin{aligned}
\frac{P^{\text{free}}}{\mu} &= \frac{P_0}{\mu_0} + \int_{\mu_0}^{\mu} \frac{\langle \varepsilon(\mu) \rangle^{\text{free}}}{\mu^2} \mathrm{d}\mu \\
&= \frac{q_f|eB|}{2\pi^2} \left[\sum_{n_\sigma=0}^{\left\lfloor \frac{\mu^2}{2|eB|q_f} \right\rfloor} F\left(2|eB|q_f n_\sigma, \mu\right) \right]_{\mu_0}^{\mu} - \frac{q_f|eB|}{4\pi^2} \left[\mu\right]_{\mu_0}^{\mu} \\
&+ \frac{q_f|eB|}{2\pi^2} \left[\sum_{n_\sigma=0}^{\left\lfloor \frac{\mu^2}{2|eB|q_f} \right\rfloor} 2|eB|q_f n_\sigma \left(G(2n_\sigma|eB|q_f) + \frac{\ln\left(2n_\sigma|eB|q_f\right)}{2\mu} \right) \right]_{\mu_0}^{\mu} \\
&= \frac{q_f|eB|}{2\pi^2} \left[\sum_{n_\sigma=0}^{\left\lfloor \frac{\mu^2}{2q_f|eB|} \right\rfloor} \sqrt{\mu^2 - 2|eB|q_f n_\sigma} - \sqrt{2|eB|q_f n_\sigma} \tan^{-1}\left(\frac{\sqrt{\mu^2 - 2|eB|q_f n_\sigma}}{\sqrt{2|eB|q_f n_\sigma}} \right) \right]_{\mu_0}^{\mu}
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
&- \frac{q_f|eB|}{4\pi^2} \mu \Big|_{\mu_0}^{\mu} + \frac{q_f|eB|}{4\pi^2} \left[\sum_{n_\sigma=0}^{\left\lfloor \frac{\mu^2}{2q_f|eB|} \right\rfloor} \left(2q_f|eB|n_\sigma\right) \frac{\ln\left(2q_f|eB|n_\sigma\right)}{\mu} \right]_{\mu_0}^{\mu} \\
&+ \frac{q_f|eB|}{2\pi^2} \left[\sum_{n_\sigma=0}^{\left\lfloor \frac{\mu^2}{2q_f|eB|} \right\rfloor} \sqrt{2n_\sigma|eB|q_f} \sec^{-1}\left(\frac{\mu}{\sqrt{2n|eB|q_f}} \right) \right]_{\mu_0}^{\mu} \\
&- \frac{q_f|eB|}{2\pi^2} \left[\sum_{n_\sigma=0}^{\left\lfloor \frac{\mu^2}{2q_f|eB|} \right\rfloor} \left(2n|eB|q_f\right) \ln\left(\frac{\mu + \sqrt{\mu^2 - 2n_\sigma|eB|q_f}}{\mu} \right) \right]_{\mu_0}^{\mu}
\end{aligned}
$$

5.6.2 Pressure in presence of magnetic field for QPM model

The pressure can be derived using

$$
\frac{P_q}{\mu} = \frac{P_0}{\mu_0} + \int_{\mu_0}^{\mu} \frac{\langle \varepsilon_q(\mu) \rangle}{\mu^2} \mathrm{d}\mu \tag{5.45}
$$

with
$m_j^2 = m^2 + 2j|q_f eB|$

The analytical derivation is complex because of the involvement of mass, which is a function of the coupling constant. But one can solve it numerically. Let us represent the total pressure of quark in the qpm model as $\sum P_{q\bar{q}}^{B} = P_{\rm up} + P_{\rm down} + P_{\rm strange}$ and that of the free particle as $\sum P_{\rm free}^{B} = P_{\rm up}^{\rm free} + P_{\rm down}^{\rm free} + P_{\rm strange}^{\rm free}$.

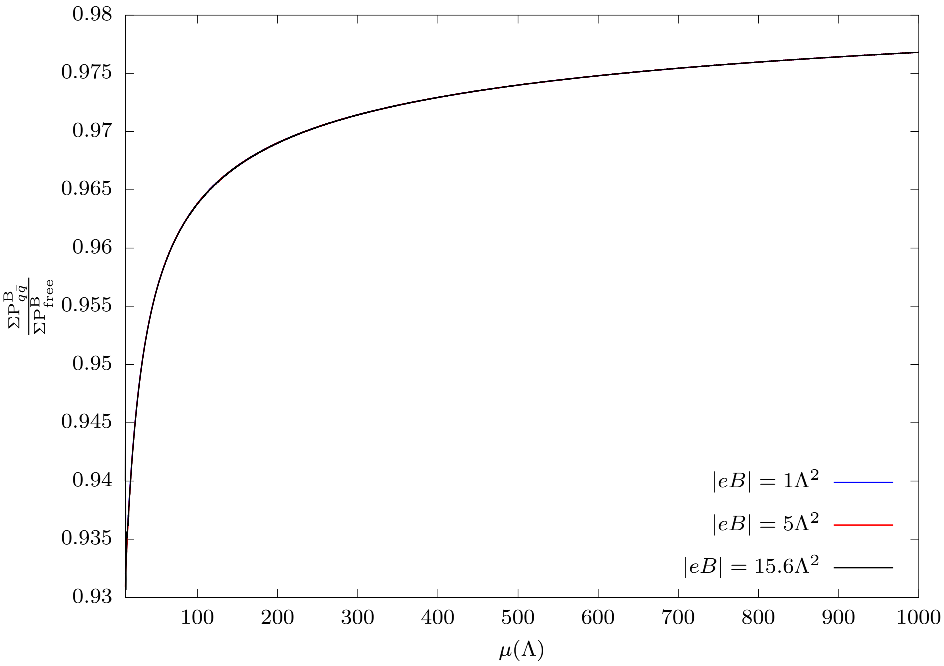

Figure 5.5: The scaled pressure density is plotted for higher chemical potential range.

The ratio of $\sum P_{q\bar{q}}^{B}$ to $\sum P_{\rm free}^{B}$ is plotted in Figs. 5.5 and 5.6. Due to quantization with a maximum frequency of $\left\lfloor \frac{\mu^2 - m^2}{2|q_f eB|} \right\rfloor$, oscillation can be observed in the enlarged plot Fig. 5.6. The ratio comparing the pressure of quasiparticles with that of free particles gives us insight into the ratio of non-ideal and ideal-behaviour of quasiparticles. Since the contribution of electron and positron pressure is negligible in this approximation, such pressure values are not considered. In addition to that the pressure of the magnetic field $P = \frac{B^2}{2}$ is not included in the plot.

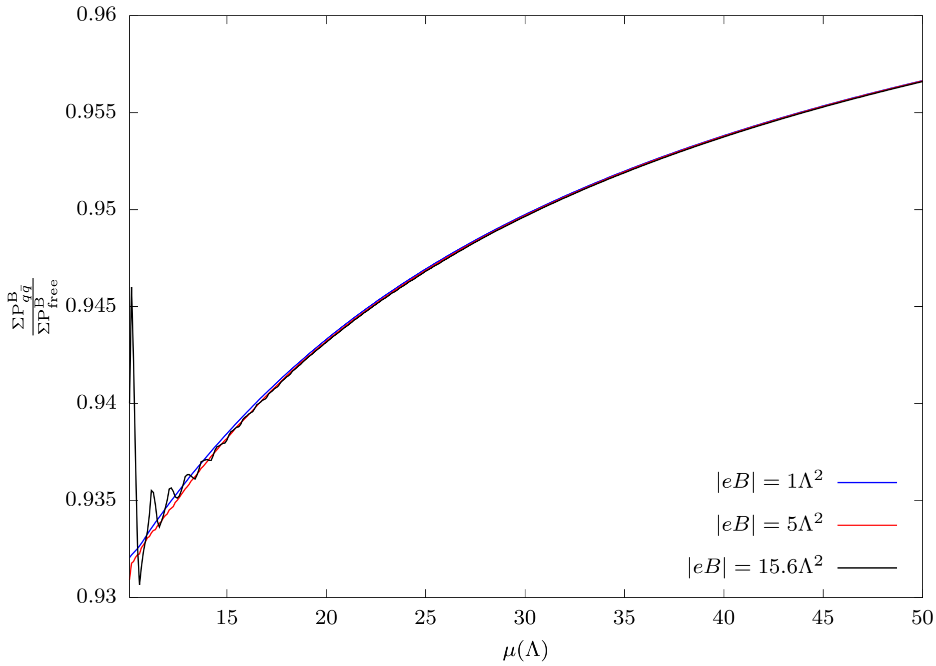

Figure 5.6: The pressure of quasi particles scaled with that of free particles. The range of chemical potential is limited to 50 Λ to show the effect of the magnetic field

5.7 Results and Discussion

For a finite chemical potential, we have taken into account the pressure, energy density, and number density of the magnetized quark at zero Kelvin. The formulas for number density, energy density, and pressure are obtained by combining the relative particle result with the quantized harmonic oscillator result, as demonstrated in Chapter 4.

We have compared the result with that of free particles. In Figs. 5.1 and 5.2, we have compared the relative number density of quasiparticle quarks with that of free particle gas at finite chemical potential and absolute zero temperature. When the chemical potential is comparable with the magnetic field, a larger oscillation can be seen. The plots for various magnetic fields can also be obtained. The reason for the oscillation is the quantization of phase space due to the applied magnetic field. This introduces a maximum frequency of $\left\lfloor \left[\frac{\mu^2 - m^2}{2|q_f eB|} \right] \right\rfloor$. The mass of a quasiparticle is a function of coupling constant which is a function of chemical potential. It also gives rise to the oscillation. In order to compensate the charge neutrality, quasielectron-quasipositron oscillation is assumed.

Comparing Figs. 5.3 and 5.4, the same kind of oscillation can be observed. In the Figs. 5.3 and 5.4, only the energy density due to the quasiparticle is taken into account. The total energy density requires the contribution caused by the pure magnetic field ie., $B^2/2$

Since the chemical potential of quasielectron and quasipositron is considered as zero, the contribution to energy density is negligible. The pressure is calculated for $\mu_0 = 10$, and $P_0 = 0$. Comparing this with Figs. 5.5 and 5.6, one can observe that when the magnetic field is greater than the chemical potential, a larger oscillation can be observed. Later, energy density vs. pressure are compared. The total pressure will also have a magnetic field contribution value of $B^2/2$.

In a nutshell, in this chapter, we have derived the equation of state for quasi particles at zero temperature at finite chemical potential. For symmetrical pressure, energy density and baryonic density, the Tolman–Oppenheimer–Volkoff (TOV) equation can be used to derive the radius and mass dependence of a star. But in our case because of this asymmetric magnetic field, we cannot apply our model in TOV. So we might need to transform TOV equations to the magnetic field regime first and then derive the mass-radius relation. We expect to do some research in that regard in the future.

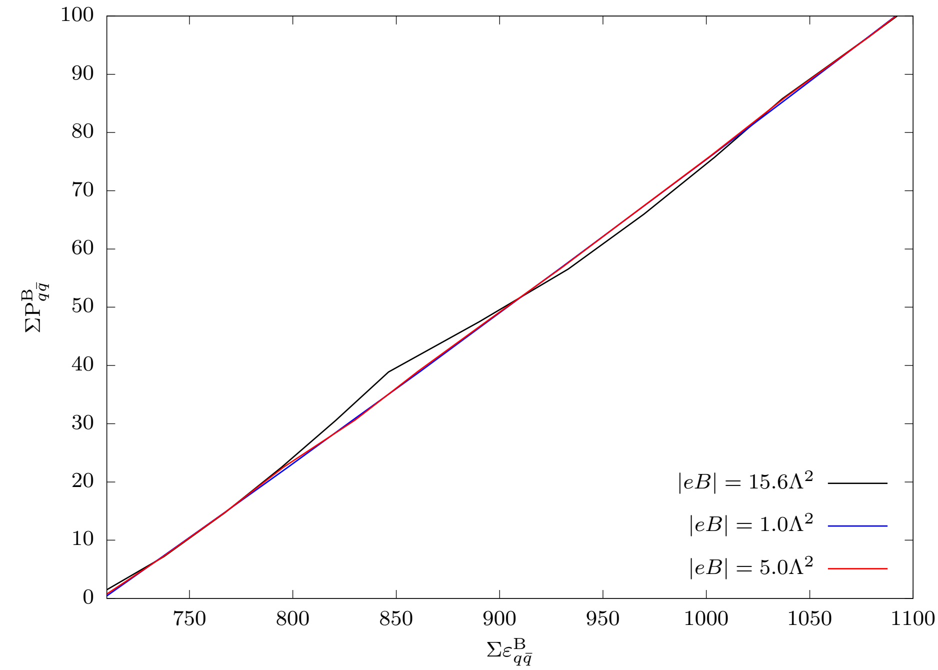

Figure 5.7: The pressure vs Energy of quasi particles is plotted. The range of chemical potential is limited to 708-1100 Λ to show the effect of the magnetic field. Units of pressure is Λ^4. One can observe a kink in this figure, and there are two reasons for this kink. Firstly, it is due to the initial condition we imposed to derive the pressure, i.e., $\mu_0 = 10$. The kink's position shifts as we change this initial value, μ_0. Secondly, the kink occurs when the magnetic field and chemical potential values are in the same order of magnitude. This is attributed to Landau quantization applied here, making the ratio $\left\lfloor \frac{\mu^2 - m^2(\mu)}{2|q_f eB|} \right\rfloor$ behave as a step function. When this ratio surpasses a specific value, the entire quantity increases significantly, leading to the observed kink.

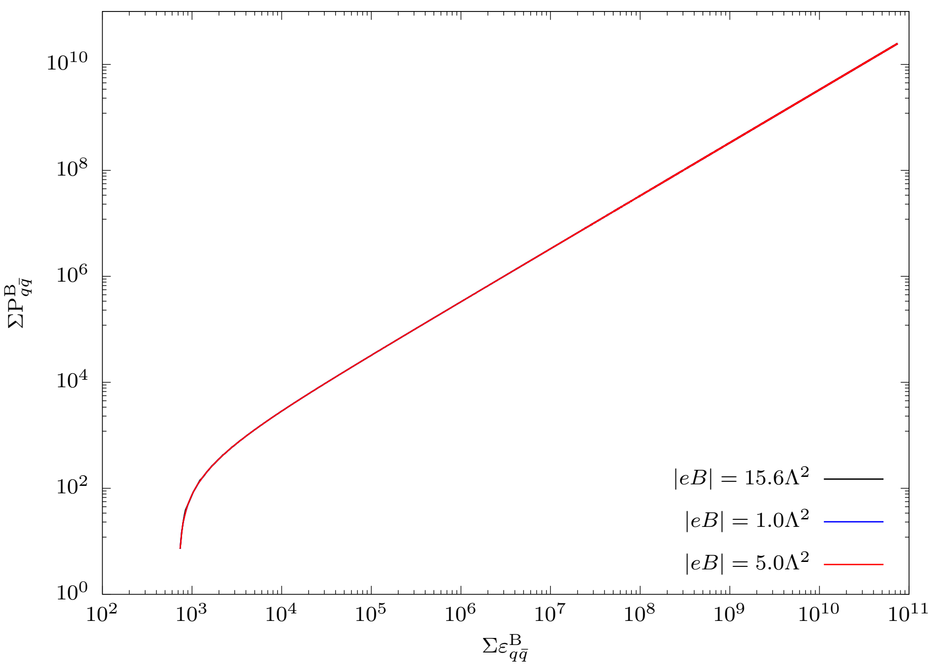

Figure 5.8: Pressure vs Energy of quasi particles is plotted for higher chemical potential range.

Bibliography

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Chapter 6

Quasiparticle model in thermal ϕ^4 theory having coupling constant up to two loop order

6.1 Introduction

6.1.1 Creation and Annihilation operator in quantum mechanics

Consider the quantum mechanical Harmonic Oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + \frac{1}{2}m\omega^2\hat{\vec{x}}^2 \tag{6.1}$$

The eigenvalue of the n^{th} energy state for the above Hamiltonian is

$$E_n = \left\langle n \middle| \left(\hat{N} + \frac{1}{2}\right)\hbar\omega \middle| n \right\rangle = \left(n + \frac{1}{2}\right)\hbar\omega \tag{6.2}$$

with $n \in \mathbb{W}$, where $\mathbb{W}$ is the set of all whole numbers. The $\hat{N}$ is the state operator, which can be written as

$$\hat{N} = \hat{a}^\dagger\hat{a} \tag{6.3}$$

where $\hat{a}$ is the annihilation operator, which when acted on, changes a state $|n\rangle$ to $|n-1\rangle$. And the creation operator $\hat{a}^\dagger$ does the opposite. i.e.,

$$\begin{aligned} \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}\,|0\rangle &= 0 \end{aligned} \tag{6.4}$$

The creation and annihilation operators can be expressed in terms of momentum and coordinates operator as

$$\begin{aligned}\hat{a} &= \frac{m\omega\hat{x} + \mathtt{i}\hat{p}}{\sqrt{2m\hbar\omega}} \\ \hat{a}^{\dagger} &= \frac{m\omega\hat{x} - \mathtt{i}\hat{p}}{\sqrt{2m\hbar\omega}} \\ \left[\hat{a}, \hat{a}^{\dagger}\right] &= \frac{[\hat{x}, \hat{p}]}{\mathtt{i}\hbar} = 1\end{aligned} \tag{6.5}$$

6.1.2 General description of QFT

We can represent every particle and wave in the universe as an excitation of a corresponding quantum field. As we have mentioned in the previous section, in quantum mechanics $\hat{a}_p$ annihilates a particle or changes the state to a lower state. i.e.

$$\hat{a_p} |1\rangle = |0\rangle \tag{6.6}$$

Similarly, $\hat{a_p}^{\dagger}$ creates a particle with momentum $\vec{\mathbf{p}}$ (or increases the state). i.e.

$$\hat{a_p}^{\dagger} |0\rangle = |1\rangle \tag{6.7}$$

where $|0\rangle$ is the lowest-order state, also known as the ground state. $|1\rangle$ is the next higher level of state.

At the same time in quantum field theory,

$$\hat{\phi}^{\dagger}(x) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\vec{p}} \hat{a_p}^{\dagger} e^{-i\vec{p}.\vec{x}} \tag{6.8}$$

creates a particle at $\vec{x}$ with momentum $\vec{p}$.
Similarly

$$\hat{\phi}(x) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\vec{p}} \hat{a}_p e^{i\vec{p}.\vec{x}} \tag{6.9}$$

annihilates a particle at $\vec{x}$.
Corresponding to the Lagrangian in classical and quantum mechanics, there exists a corresponding quantum field theory function known as Lagrangian density, which, when integrated over four dimensions, gives the Lagrangian.

$$\mathrm{L} = \int \mathcal{L}\ \mathrm{d}^4x \tag{6.10}$$

There are different kinds of Lagrangian equations; the prominent ones are the Klein-Gordon scalar field equation and the charged Klein-Gordon field equation.
For Klein-Gordon scalar field

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right) \tag{6.11}$$

solving using Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\right) = 0 \tag{6.12}$$

gives

$$\hat{\phi}(x) = \int \left(\hat{a_p}(t)e^{i(\vec{p}.\vec{x}-\omega t)} + \hat{a_p}^\dagger(t)e^{-i(\vec{p}.\vec{x}-\omega t)}\right)\frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2\omega}} \tag{6.13}$$

For charged Klein-Gordon field

$$\begin{aligned}
&\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi\right)\\
&\implies \hat{\phi}(x) = \int \left(\hat{b_p}(t)e^{i(\vec{p}.\vec{x}-\omega t)} + \hat{a_p}^\dagger(t)e^{-i(\vec{p}.\vec{x}-\omega t)}\right)\frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2\omega}}
\end{aligned} \tag{6.14}$$

Two types of particles involved here, i.e., a and b.

In quantum mechanics we have Schrodinger picture, and Heisenberg picture. In Schrodinger picture the state is time dependent and the operator is independent of time. The state of a particle or physical system at a time $\boldsymbol{t}$ can be obtained in the Schrodinger picture by multiplying the initial state with the time evolution operator $\hat{\mathbf{U}}(t)$
i.e.,

$$\psi(t) = \hat{\mathbb{U}}(t, t_0)\psi(t_0). \tag{6.15}$$

But in the Heisenberg picture, the state is time-independent, but the opertor is time-dependent. In both pictures, the expectation value of an operator should be the same, so one could relate both via

$$\begin{aligned}
&\langle\psi(t)|\hat{A}_s|\psi(t)\rangle = \langle\psi_H|\hat{A}_H(t)|\psi_H\rangle\\
\implies &\langle\psi(t)\underbrace{\hat{\mathbb{U}}^{-1}\hat{\mathbb{U}}}_{\mathbb{1}}|\hat{A}_s|\underbrace{\hat{\mathbb{U}}\hat{\mathbb{U}}^{-1}}_{\mathbb{1}}\psi(t)\rangle = \langle\psi_H|\hat{A}_H(t)|\psi_H\rangle\\
\implies &\langle\psi(t)\hat{\mathbb{U}}^{-1}|\hat{\mathbb{U}}^{-1}\hat{A}_s\hat{\mathbb{U}}|\hat{\mathbb{U}}^{-1}\psi(t)\rangle = \langle\psi_H|\hat{A}_H(t)|\psi_H\rangle
\end{aligned} \tag{6.16}$$

On one to one comparison of LHS with RHS,

$$\begin{aligned}
\psi_H &= \hat{\mathbb{U}}^{-1}\psi(t)\\
\hat{A}_H &= \hat{\mathbb{U}}^{-1}\hat{A}_s\hat{\mathbb{U}}
\end{aligned} \tag{6.17}$$

6.1.3 Heisenberg picture and Action priciple

In Heisenberg's picture, we have

$$|\phi\rangle_H = \hat{\mathbb{U}}^{-1} |\phi\rangle \tag{6.18}$$

where $\hat{\mathbb{U}}$ is the time evolution operator, and

$$\hat{A}_H(t) = \mathbb{U}^{-1}\hat{A}\mathbb{U} \tag{6.19}$$

where $\hat{A}_H(t)$ is the opertor in the Heisenberg picture and $\hat{A}(t)$ is the operator in the Schrodinger picture. Consider the propagation of a particle from

$$(x, t_1) \to (y, t_2) \tag{6.20}$$

the corresponding expectation value is

$$\mathrm{I} = \langle y, t_2|x, t_1\rangle_H = \langle y|\hat{\mathbb{U}}(t_2)\hat{\mathbb{U}}^{-1}(t_1)|x\rangle\,. \tag{6.21}$$

But $\hat{\mathbb{U}} = \exp\left(-i\hat{H}t\right)$

$$\implies \langle y, t_2|x, t_1\rangle_H = \langle y|\exp\left(-i\hat{H}\tau\right)|x\rangle \tag{6.22}$$

where $\tau = \mathbf{t}_2 - \mathbf{t}_1$. We can divide the long interval into $\boldsymbol{n}$ equal intervals of small size.

$$\begin{aligned}[t_1, t_2] =& [t_1, t_1 + \Delta\tau] + (t_1 + \Delta\tau, t_1 + 2\Delta\tau] + ... \\ &+ (t_1 + n\Delta\tau, t_1 + (n+1)\Delta\tau]\end{aligned} \tag{6.23}$$

with $t_2 = t_1 + (n+1)\Delta\tau$.

$\mathrm{I} = \langle y|\exp\left(-i\hat{H}\tau\right)|x\rangle = \langle y|\exp\left(-i\hat{H}(n+1)\Delta\tau\right)|x\rangle$

Now using the completeness relation

$$\int |x_n\rangle\langle x_n|\,\mathrm{d}x_n = \mathbb{1}$$

$$\mathrm{I} = \int \langle y|\exp\left(-i\hat{H}\Delta\tau\right)|x_n\rangle\langle x_n|\exp\left(-i\hat{H}\Delta\tau\right)|x_{n-1}\rangle ... \langle x_1|\exp\left(-i\hat{H}\Delta\tau\right)|x\rangle\,\mathrm{d}x_1..\mathrm{d}x_n$$

combining completeness relations in momentum coordinates

$$\langle x_n|\exp\left(-i\hat{H}\Delta\tau\right)|x_{n-1}\rangle = \int \langle x_n|p\rangle\exp\left(-iH_p\Delta\tau\right)\langle p|x_{n-1}\rangle\,\mathrm{d}p$$

$$= \int \exp\left(i\Delta\tau\left[p\frac{x_n - x_{n-1}}{\Delta\tau} - H_p\right]\right)\mathrm{d}p$$

$$= \int \exp\left(i\mathrm{L}\Delta\tau\right)\mathrm{d}p$$

$$(6.24)$$

In the continuum limit for each x_n, if we express it like this, then

$$\mathrm{I} = \langle y, t_2 | x, t_1 \rangle = \int \exp\left(i \int L\mathrm{d}t \right) \mathrm{D}x\mathrm{D}p \tag{6.25}$$

6.1.4 Green's Function

If we do the same as above in functional theory, then we get an exponential term containing

$$S = \int \mathrm{Ld}t = \int \mathcal{L}\mathrm{d}^4x \tag{6.26}$$

In case of Klein-Gordon equation

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right) \tag{6.27}$$

If we solve it for action S then

$$S = \int \frac{\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right)}{2}\mathrm{d}^4x = -\int \frac{\phi\left(\partial_\mu\partial^\mu\phi + m^2\phi\right)}{2}\mathrm{d}^4x \tag{6.28}$$

The solution to the equation $-\left(\partial_\mu\partial^\mu\phi + m^2\phi\right) = \delta^4(x - x')$, for ϕ is known as the Feynman propagator and is denoted as $\Delta_F(x - x')$.

6.1.4. (a) The idea behind Green's function

Consider an equation

$$\hat{A}_x\phi(x) = g(x) \tag{6.29}$$

Let there exist a Green's function such that

$$\hat{A}_xG(x, y) = \delta(x - y) \tag{6.30}$$

Eq. (6.29) and Eq. (6.30), when combined, become

$$
\begin{aligned}
g(x) &= \int g(y)\delta(x-y)\mathrm{d}y \\
&= \int g(y)\hat{A}_x G(x,y)\mathrm{d}y \\
&= \hat{A}_x \int g(y)G(x,y)\mathrm{d}y \\
&\implies g(x) = \hat{A}_x\phi(x) = \hat{A}_x \int g(y)G(x,y)\mathrm{d}y \\
&\implies \phi(x) = \int g(y)G(x,y)\mathrm{d}y
\end{aligned}
\tag{6.31}
$$

The beauty of Green's function is that once $G(x,y)$ is found, it can be readily applied to RHS to find the solution for any RHS function.

6.1.4. (b) Propagator in QFT

The solution to

$$
-\left(\partial_\mu\partial^\mu + m^2\right)G(x,x') = \delta^n(x-x') \tag{6.32}
$$

is known as the Feynmann propagator. The solution can be derived by defining $G(x,x')$ such that

$$
\begin{aligned}
&\implies -\left(\partial_\mu\partial^\mu + m^2\right)G(x,x') = \delta^n(x-x') \\
&\implies -\left(\partial_\mu\partial^\mu + m^2\right)\int G(p)e^{ip^\mu(x_\mu - x'_\mu)}\frac{\mathrm{d}^n p}{(2\pi)^n} = \int e^{ip^\mu(x_\mu - x'_\mu)}\frac{\mathrm{d}^n p}{(2\pi)^n} \\
&\implies -\int\left(-p_\mu p^\mu + m^2\right)G(p)e^{ip^\mu(x_\mu - x'_\mu)}\frac{\mathrm{d}^n p}{(2\pi)^n} = \int e^{ip^\mu(x_\mu - x'_\mu)}\frac{\mathrm{d}^n p}{(2\pi)^n} \\
&\implies -\left(-p_\mu p^\mu + m^2\right)G(p) = 1 \\
G(p) &= \frac{1}{(p_\mu p^\mu - m^2)} \\
G(x,x') &= \int \frac{e^{ip^\mu(x_\mu - x'_\mu)}}{(p_\mu p^\mu - m^2)}\frac{\mathrm{d}^n p}{(2\pi)^n} = \Delta_F(x-x') = \text{Feynman Propagator}
\end{aligned}
\tag{6.33}
$$

6.1.5 Quartic interaction

Consider a quartic self-interacting theory $\lambda\phi^4$, where the action can be defined as

$$
\begin{aligned}
A[\phi] &= \int\left(\frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right] - \frac{\lambda}{4!}\phi^4\right)\mathrm{d}^4x \\
&\text{From Eq. (6.28)} \\
A[\phi] &= S[\phi] - \frac{\lambda}{4!}\int\phi^4\mathrm{d}^4x
\end{aligned}
\tag{6.34}
$$

Thus a generating function $Z_\lambda(J)$, can be defined as

$$\begin{aligned}
Z_\lambda(J) &= \int \exp\left(\mathtt{i}S[\phi] + \mathtt{i}\int \mathrm{d}^4x \left[-\frac{\lambda}{4!}\phi^4 + J\phi\right]\right)\mathrm{D}\phi \\
Z_0(J) &= \int \exp\left(\mathtt{i}S[\phi] + \mathtt{i}\int \mathrm{d}^4x\,[J\phi]\right)\mathrm{D}\phi \\
Z_0(0) &= \int \exp\left(\mathtt{i}S[\phi]\right)\mathrm{D}\phi \\
Z_\lambda(0) &= \int \exp\left(\mathtt{i}S[\phi] + \mathtt{i}\int \mathrm{d}^4x \left[-\frac{\lambda}{4!}\phi^4\right]\right)\mathrm{D}\phi = \int \mathrm{D}\phi \exp\left(\mathtt{i}A[\phi]\right)
\end{aligned} \tag{6.35}$$

Differentiating with respect to J, we can connect the equations shown above as

$$\int \phi^4 \exp\left(\mathtt{i}S[\phi] + \mathtt{i}\int \mathrm{d}^4x\,[J\phi]\right)\mathrm{D}\phi = \int \frac{\delta^4}{\delta J^4}\exp\left(\mathtt{i}S[\phi] + \mathtt{i}\int \mathrm{d}^4x\,[J\phi]\right)\mathrm{D}\phi \tag{6.36}$$

Therefore

$$\begin{aligned}
Z_\lambda(J) &= Z_0(J)\sum_{n=0}^{\infty}\frac{1}{n!}(-\mathtt{i}\lambda/4!)^n\left\{\int \mathrm{d}^4x\phi^4(x)\right\}^n \\
&= Z_0(J)\sum_{n=0}^{\infty}\frac{1}{n!}(-\mathtt{i}\lambda/4!)^n\left[\prod_{l=0}^{n}\int \mathrm{d}^4x_l\ \phi^4(x_l)\right] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(-\mathtt{i}\lambda/4!)^n\left[\prod_{l=0}^{n}\int \mathrm{d}^4x_l\ \frac{\delta^4}{\delta J(x_l)^4}\right]Z_0(J)
\end{aligned} \tag{6.37}$$

and

$$Z_0(J) = Z_0(0)\sum_{n=0}^{\infty}\frac{(\mathtt{i}^n)}{n!}\left\{\prod_{l=0}^{n}\int \mathrm{d}^4x_l\ J(x_l)\ \phi(x_l)\right\} \tag{6.38}$$

One can correlate Green's function with $Z_\lambda(J)$ as shown below

$$Z_\lambda(J) = Z_0(0)\sum_{n=0}^{\infty}\frac{\mathtt{i}^n}{n!}\left\{\int G^{(n)}(x_1, x_2, \ldots, x_n)\prod_{l=0}^{n}J(x_l)\ \mathrm{d}^4x_l\right\} \tag{6.39}$$

with

$$Z_\lambda(J) = \int \mathrm{D}\phi\sum_{n=0}^{\infty}\frac{\mathtt{i}^{\mathtt{n}}}{n!}\left\{\prod_{l=0}^{n}\int \mathrm{d}^4x_l\ J(x_l)\ \phi(x_l)\right\}\exp\left(\mathtt{i}A[\phi]\right) \tag{6.40}$$

$G^{(n)}(x_1, x_2, \ldots x_n)$ is called n-point Green's function. Dividing Eq. (6.40) by Eq. (6.39) we get

$$1 = \frac{\int \mathrm{D}\phi \sum_{n=0}^{\infty}\frac{\mathtt{i}^n}{n!}\left\{\prod_{l=0}^{n}\int \mathrm{d}^4x_l\ J(x_l)\ \phi(x_l)\right\}\exp\left(\mathtt{i}A[\phi]\right)}{Z_0(0)\sum_{n=0}^{\infty}\frac{\mathtt{i}^n}{n!}\left\{\int G^{(n)}(x_1, x_2, \ldots, x_n)\prod_{l=0}^{n}\mathrm{d}^4x_l\ J(x_l)\right\}} \tag{6.41}$$

So for two-point functions,

$$1 = \frac{\int \mathrm{D}\phi \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 J(x_1)J(x_2)\phi(x_1)\phi(x_2) \exp\left(\mathtt{i}A[\phi]\right)}{Z_0(0) \int G(x_1, x_2)\ J(x_1)\ J(x_2)\ \mathrm{d}^4x_1\mathrm{d}^4x_2} \tag{6.42}$$

comparing the denominator and numerator,

$$G(x_1, x_2) = \frac{1}{Z_0(0)} \int \mathrm{D}\phi\ \phi(x_1)\phi(x_2) \exp\Big(\mathtt{i}A[\phi]\Big) \tag{6.43}$$

Similarly, four-point Green function can be written as

$$\begin{aligned} G(x_1, x_2, x_3, x_4) &= \frac{1}{Z_0(0)} \int \mathrm{D}\phi\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \exp\Big(\mathtt{i}A[\phi]\Big) \\ &= \frac{1}{Z_0(0)} \int \mathrm{D}\phi \exp\Big(\mathtt{i}A[\phi]\Big) \prod_{l=1}^{4} \phi(x_l) \end{aligned} \tag{6.44}$$

i.e., As shown in Eq. (6.39), $Z_\lambda(J)$ can be expanded in terms of Green's function.

The expansion of Green's function, can be rewritten in terms of momentum representation, as shown in the Section 6.1.4. Similarly, the expansion can be simplified using Feynmann diagrams. We are not going into detail on deriving the Feymann diagram's rule, but we simply state the rules while we interchange the integrals from Euclidean to Minkowski space.

1. Draw the diagram in the momentum space associating a momentum label, with each of the lines. We label each momentum k_i and associate them with a factor $\left(k_i^2 + m^2\right)^{-1}$
2. Assume that momentum is conserved at each vertex (Sum of incoming momentum = Sum of outgoing momentum)
   i.e., associate it with $(2\pi)^N \delta^N[k_1 + k_2 + \cdots + k_n]$
3. Momentum associated with internal lines are integrated over with measure $\frac{\mathrm{d}^N k}{(2\pi)^N}$
4. Associate a factor $-\lambda/4!$ with each vertex
5. Associate the correct symmetry factor and weight factor with diagram.

6.1.6 Thermal field theory : Imaginary Time Formalism

Here, the continuation of Heisenberg picture and Action principle in Section 6.1.3 is considered in the context of a canonical ensemble. In order to differentiate the formalism in ITF, we use ϕ and π instead of x and p used in Section 6.1.3. Similarly the expectation value of an operator $\hat{B}$ is considered as

$$\left\langle \hat{B} \right\rangle = \frac{\mathrm{Tr}\Big\{\hat{B} \exp\Big(-\beta\hat{H}\Big)\Big\}}{\mathrm{Tr}\Big\{\exp\Big(-\beta\hat{H}\Big)\Big\}} \tag{6.45}$$

The partition function is defined as [1]

$$Z = \sum_{\mathrm{n}} \int \mathrm{d}\phi_n \left\langle \phi_n \right| \exp\Big[-\beta(\hat{H} - \mu_i \hat{N}_i)\Big] \left| \phi_n \right\rangle \tag{6.46}$$

As the variable changes to imaginary, i.e., moving from real to complex plane, $\tau \to \mathtt{i}t$, a periodicity arises to conserve the charge density $\mathcal{N}$.
i.e., the Hamiltonian density changes from $\mathcal{H} \to \mathcal{H} - \mu\mathcal{N}$. Therefore, we skip to the basic formula [1]

$$Z = \int \mathrm{d}\pi \int_{\text{periodic}} \mathrm{d}\phi \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left(\mathtt{i}\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H} + \mu\mathcal{N}\right)\right]. \tag{6.47}$$

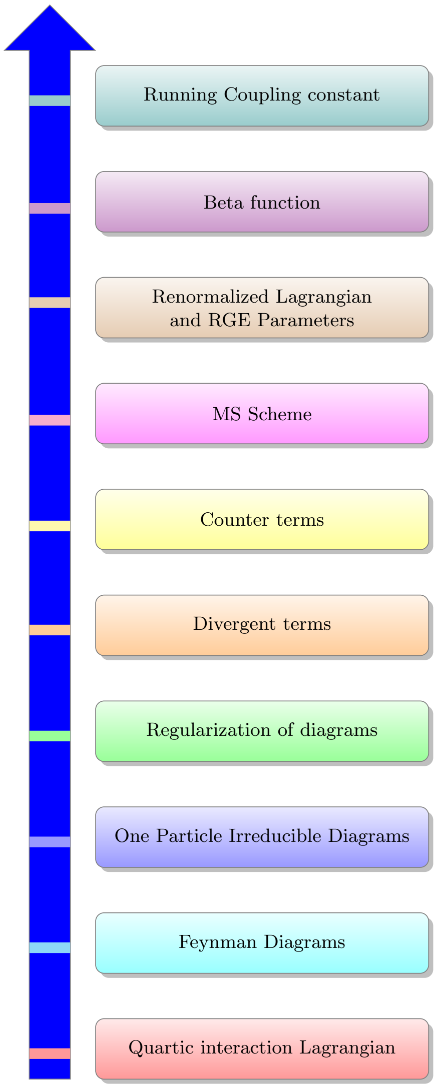
Here $\phi_a = \phi(x, 0) = \phi(x, \beta)$. Thus the Feynmann diagram in thermal ϕ^4 theory can be defined as

1. Draw all diagrams that are connected
2. Find the weight factor for each diagram
3. For each line give the factor $T \sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + p^2 + m^2}$, where $\omega_n = 2\pi nT$
4. For each vertex associate a factor $-\lambda$
5. For each vertex associate a factor $(2\pi)^3 \beta\delta(p_{\text{incoming}} - p_{\text{outgoing}})\delta_{\omega_{\text{in}},\omega_{\text{out}}}$. This delta function usage is to preserve energy-momentum conservation. At the end, a factor $(2\pi)^3\beta\delta(0) = \beta V$ will remain.

For comparison of thermal and non-thermal QFT, the difference can be written as

Characteristics	QFT ϕ^4	ITF ϕ^4
Weight factor	Same/Identical	Same/Identical
Vertex factor	$-\lambda$	$-\lambda$
Line factor	$\int \frac{\mathrm{d}^4P}{(2\pi)^4}\frac{1}{P^2+m^2}$	$T\sum_{n=-\infty}^{\infty}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\omega_n^2+p^2+m^2}$
Conservation	$(2\pi)^4\delta^4(P_{\text{incoming}}-P_{\text{outgoing}})$	$(2\pi)^3\beta\delta^3(p_{\text{incoming}}-p_{\text{outgoing}})\delta_{\omega_{\text{in}},\omega_{\text{out}}}$

6.2 Running coupling constant in ϕ^4 theory



The coupling constant in the ϕ^4 theory affects the quasiparticle model we employ in the work. Therefore, a temperature-dependent coupling constant must be derived in order to complete the model.

The typical procedure for determining the coupling constant involves defining the Lagrangian, selecting the appropriate order of approximation, writing down irreducible diagrams, and regularizing such diagrams and noting the divergences. Later inorder to remove the divergence and makes the theory in order, counter terms are derived which cancels the diverging terms of the Feynman diagrams. Renormalization group equations (RGE) are utilised to renormalize the Lagrangian. Several

RGE parameters were derived for it. Beta function is one of them. The coupling constant can be obtained by solving the beta function.

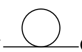
6.2.1 Dimensional Regularization of Feynmann diagram's in non thermal ϕ^4 theory

We use the dimensional regularization method [2, 3, 4], which successfully regularizes the non-Abelian gauge theory while maintaining symmetries. The minimal subtraction scheme (MS scheme) follows after regularization and in which it corresponds to the cancellation of pole terms ($\epsilon^{-n},\ n \geq 1$) using the counter-term method [5]. The vertex function, which is formed by the Feynmann diagrams, has divergences. When appropriate counter terms are added, the divergence terms get removed, and it becomes the proper vertex function [6, 7, 8, 9]. In the next stage, the corresponding renormalized group equation (RGE) is applied to the finite proper vertex function of the imaginary time formalism. Those diagrams which have a subscript QFT corresponds to non-themal ϕ^4 QFT diagram. The diagram which has an ITF subscript corresponds with the diagrams of the thermal ϕ^4 theory of the imaginary time formalism.

6.2.2 Examples of Dimensional Regularization

6.2.2. (a) One loop two-point function diagram

Consider the simple tadpole diagram and the integral expression

$\Big[\{-\lambda\}$  $_{\mathrm{QFT}}\Big]$

$$= -\lambda \int \left(\frac{1}{P^2 + m^2} \right) \frac{\mathrm{d}^4 P}{(2\pi)^4} \tag{6.48}$$

It is very clear from the equation itself that the integral is diverging. d^4P is proportional to P^3, thus the integral in Eq. (6.48) is divergent itself. Because the integrand has an effective power of $3-2=1$. But in order to pinpoint the diverging terms, the method of regularization can be used. For that, certain re-arrangements are done. The dimension of integral equation is changed from 4 to N

$$\frac{\mathrm{d}^4 P}{(2\pi)^4} \to \underset{\mathrm{N}\to 4-\epsilon}{\mathrm{Lt}} \frac{\mathrm{d}^{\mathrm{N}} P}{(2\pi)^{\mathrm{N}}} \tag{6.49}$$

The coupling constant changed with a coefficient of mass scale, i.e.,

$$\lambda \to g\mu^{\epsilon} \tag{6.50}$$

So

$$-g\mu^{\epsilon}\int\left(\frac{1}{P^2+m^2}\right)\frac{\mathrm{d}^{\mathrm{N}}P}{(2\pi)^{\mathrm{N}}}=-g\mu^{\epsilon}\int\left(\frac{1}{P^2+m^2}\right)\frac{1}{(2\pi)^{\mathrm{N}}}\,\mathrm{d}\left(\underbrace{\frac{\pi^{\mathrm{N}/2}P^{\mathrm{N}}}{\Gamma\left(\frac{\mathrm{N}}{2}+1\right)}}_{\substack{\text{N dimensional}\\ \text{volume}}}\right) \tag{6.51}$$

$$=\frac{-g\mu^{\epsilon}}{(4\pi)^{\frac{\mathrm{N}}{2}}\,\Gamma\left(\frac{\mathrm{N}}{2}\right)}\int_0^{\infty}\frac{2P^{\mathrm{N}-1}}{P^2+m^2}\mathrm{d}P$$

Using $\frac{1}{x}=\int_0^{\infty}\exp(-tx)\mathrm{d}t$

$$-g\mu^{\epsilon}\int\left(\frac{1}{P^2+m^2}\right)\frac{\mathrm{d}^{\mathrm{N}}P}{(2\pi)^{\mathrm{N}}}=\frac{-g\mu^{\epsilon}}{(4\pi)^{\frac{\mathrm{N}}{2}}\,\Gamma\left(\frac{\mathrm{N}}{2}\right)}\int_0^{\infty}\int_0^{\infty}\left[2P^{\mathrm{N}-1}\exp\left(-t(P^2+m^2)\right)\right]\mathrm{d}t\ \mathrm{d}P \tag{6.52}$$

Integrating with respect to P, we get

$$-g\mu^{\epsilon}\int\left(\frac{1}{P^2+m^2}\right)\frac{\mathrm{d}^{\mathrm{N}}P}{(2\pi)^{\mathrm{N}}}=\frac{-g\mu^{\epsilon}}{(4\pi)^{\frac{\mathrm{N}}{2}}\,\Gamma\left(\frac{\mathrm{N}}{2}\right)}\times\int_0^{\infty}\frac{\Gamma\left(\frac{\mathrm{N}}{2}\right)}{t^{\frac{\mathrm{N}}{2}}}\exp\left(-tm^2\right)\mathrm{d}t$$

$$=\frac{-g\mu^{\epsilon}}{(4\pi)^{\frac{\mathrm{N}}{2}}\,\Gamma\left(\frac{\mathrm{N}}{2}\right)}\times\frac{\Gamma\left(\frac{\mathrm{N}}{2}\right)\Gamma\left(1-\frac{\mathrm{N}}{2}\right)}{(m^2)^{1-\frac{\mathrm{N}}{2}}} \tag{6.53}$$

$$=\frac{-g\mu^{\epsilon}}{(4\pi)^{\frac{\mathrm{N}}{2}}}\frac{\Gamma\left(1-\frac{\mathrm{N}}{2}\right)}{(m^2)^{1-\frac{\mathrm{N}}{2}}}$$

Now putting back $\mathrm{N}\to 4-\epsilon$, we get

$$\underset{\mathrm{N}\to 4-\epsilon}{\mathrm{Lt}}-g\mu^{\epsilon}\int\left(\frac{1}{P^2+m^2}\right)\frac{\mathrm{d}^{\mathrm{N}}P}{(2\pi)^{\mathrm{N}}}=\underset{\mathrm{N}\to 4-\epsilon}{\mathrm{Lt}}\frac{-g\mu^{\epsilon}}{(4\pi)^{\frac{\mathrm{N}}{2}}}\frac{\Gamma\left(1-\frac{\mathrm{N}}{2}\right)}{(m^2)^{1-\frac{\mathrm{N}}{2}}}$$

$$=\frac{-gm^2}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}-1\right) \tag{6.54}$$

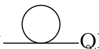
The expansion of $\Gamma\left(-n+\epsilon\right)=\dfrac{(-1)^n}{(n!)^2\,\epsilon}\Gamma\left(n+1+\epsilon\right)\left\{1+\epsilon^2\left[\dfrac{\pi^2}{6}-\psi^{'}(n+1)\right]+\mathcal{O}(\epsilon^4)\right\}$ with $\psi(n)=-\gamma+\displaystyle\sum_{l=1}^{n-1}\frac{1}{l}=\psi(1)+\sum_{l=1}^{n-1}\frac{1}{l}$. Thus

$$\Gamma\left(-1+\epsilon\right)=-\left\{\frac{1}{\epsilon}+\psi(2)+\epsilon\left[1+\frac{\psi^{'}(2)}{2}+\frac{\psi(2)^2}{2}\right]\right\}+\mathcal{O}(\epsilon^2) \tag{6.55}$$

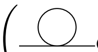
Therefore Eq. (6.54) becomes

$$\begin{aligned}\underset{N\to 4-\epsilon}{\text{Lt}} - g\mu^{\epsilon}\int\left(\frac{1}{P^2+m^2}\right)\frac{\mathrm{d}^{\mathrm{N}}P}{(2\pi)^{\mathrm{N}}} &= \frac{-gm^2}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}-1\right)\\ &= \frac{gm^2}{16\pi^2}\left[\frac{2}{\epsilon}+\psi(2)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+\mathcal{O}(\epsilon)\right]\end{aligned} \tag{6.56}$$

Therefore

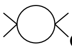
$$\left[\{-\lambda\}\right.$$  $$\left.{}_{\mathrm{QFT}}\right] = \frac{gm^2}{16\pi^2}\left[\frac{2}{\epsilon}+\psi(2)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+\mathcal{O}(\epsilon)\right] \tag{6.57}$$

We define a pole picking operator $\mathcal{K}$ which can pick out the diverging terms. i.e.,

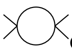
$$\mathcal{K}\left(\right.$$  $$\left.{}_{\mathrm{QFT}}\right) = \frac{gm^2}{8\pi^2}\frac{1}{\epsilon}. \tag{6.58}$$

6.2.2. (b) One loop four-point function diagram

The simple four-point function diagram [10] of first order is

$$\mathrm{I}_2 = \{\lambda^2\}$$  $$_{\mathrm{QFT}} = \lambda^2\int\frac{\mathrm{d}^4P}{(2\pi)^4}\left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{(P+K)^2+m^2}\right) \tag{6.59}$$

Now, following the dimensional regularization procedure, the integral can be re-written as

$$\mathrm{I}_2 = \{\lambda^2\}$$  $$_{\mathrm{QFT}} = \underset{N\to 4-\epsilon}{\text{Lt}} g^2\mu^{2\epsilon}\int\frac{\mathrm{d}^{\mathrm{N}}P}{(2\pi)^{\mathrm{N}}}\left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{(P+K)^2+m^2}\right) \tag{6.60}$$

Using gamma function properties

$$\begin{aligned}\frac{1}{XY} &= \int_0^{\infty}\int_0^{\infty}\exp\left(-tX-uY\right)\,\mathrm{d}t\,\mathrm{d}u\\ 1 &= \int_0^{\infty}\delta(q-t-u)\mathrm{d}q\\ \therefore\frac{1}{XY} &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\exp\left(-tX-uY\right)\delta(q-t-u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}q\\ &\text{Putting } t=qa \ \&\ u=qb\\ \frac{1}{XY} &= \int\int\int_0^{\infty}q^2\exp\left(-q\left[aX+bY\right]\right)\frac{\delta(1-a-b)}{q}\,\mathrm{d}a\,\mathrm{d}b\\ &= \int\int\frac{\delta(1-a-b)}{[aX+bY]^2}\,\mathrm{d}a\,\mathrm{d}b\\ &= \int_0^1\frac{\mathrm{d}a}{[aX+(1-a)Y]^2}\end{aligned} \tag{6.61}$$

Combining Eq. (6.61) with Eq. (6.60) becomes

$$\begin{aligned}
\mathrm{I}_2 &= g^2\mu^\epsilon \int \frac{\mathrm{d^N}P}{(2\pi)^{\mathrm{N}}} \left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{(P+K)^2+m^2}\right) \\
&= \frac{g^2\mu^\epsilon}{(2\pi)^N}\int_0^1 \mathrm{d}a \int \frac{1}{[m^2+[aP^2+(1-a)(P+K)^2]^2}\mathrm{d^N}P \\
&= \frac{g^2\mu^\epsilon}{(2\pi)^N}\int_0^1 \mathrm{d}a \int t\exp\left(-t(m^2+[aP^2+(1-a)(P+K)^2])\right)\ \mathrm{d^N}P\ \mathrm{d}t
\end{aligned} \tag{6.62}$$

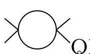
The N dimensional integral can be converted into N rectangular coordinates, So

$$\begin{aligned}
\mathrm{I}_2 = &\frac{g^2\mu^\epsilon}{(2\pi)^N}\int_0^1 \mathrm{d}a\int_0^\infty \mathrm{d}t\ t\exp\left(-tm^2\right)\prod_{\nu=1}^{N}\int_{-\infty}^{\infty}\exp\left(-t\left[{P_\nu}^2+2P_\nu K_\nu(1-a)+{K_\nu}^2(1-a)\right]\right) \\
= &\frac{g^2\mu^\epsilon}{(2\pi)^N}\int_0^1 \mathrm{d}a\int_0^\infty \mathrm{d}t\ t\exp\left(-tm^2\right)\prod_{\nu=1}^{N}\int_{-\infty}^{\infty}\exp\left(-t\left[(P_\nu+K_\nu(1-a))^2+{K_\nu}^2a(1-a)\right]\right)\mathrm{d}P_\mu \\
= &\frac{g^2\mu^\epsilon}{(2\pi)^N}\int_0^1 \mathrm{d}a\int_0^\infty \mathrm{d}t\ t\exp\left(-t(m^2+K^2a(1-a))\right)\prod_{\nu=1}^{N}\int_{-\infty}^{\infty}\exp\left(-tP_\mu^2\right)\mathrm{d}P_\mu \\
= &\frac{g^2\mu^\epsilon}{(2\pi)^N}\int_0^1 \mathrm{d}a\int_0^\infty \mathrm{d}t\ t\exp\left(-t(m^2+K^2a(1-a))\right)\left[\frac{\pi^{\mathrm{N}/2}}{t^{\mathrm{N}/2}}\right] \\
= &g^2\mu^\epsilon\frac{1}{(4\pi)^{\frac{\mathrm{N}}{2}}}\int_0^1 \mathrm{d}a\ \frac{\Gamma\left(2-\frac{\mathrm{N}}{2}\right)}{[m^2+K^2a(1-a)]^{2-\frac{\mathrm{N}}{2}}}
\end{aligned} \tag{6.63}$$

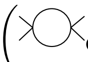
Therefore

$$\underset{\mathrm{N}\to 4-\epsilon}{\mathrm{Lt}}\ \mathrm{I}_2 = \frac{g^2}{16\pi^2}\int_0^1 \frac{(4\pi\mu^2)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)}{[m^2+K^2a(1-a)]^{\frac{\epsilon}{2}}}\mathrm{d}a \tag{6.64}$$

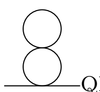
Now as shown in Example 1, using Digamma function the result can be written as

$\{\lambda^2\}$  $_{\mathrm{QFT}} = \dfrac{g^2\mu^\epsilon}{16\pi^2}\left\{\dfrac{2}{\epsilon}+\psi(1)+\displaystyle\int_0^1 \mathrm{d}a\ln\left[\dfrac{4\pi\mu^2}{K^2a(1-a)+m^2}\right]+\mathcal{O}(\epsilon)\right\}$ (6.65)

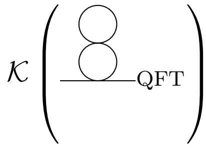
The diverging term of the above diagram is

$\mathcal{K}\Big($  $_{\mathrm{QFT}}\Big) = \dfrac{g^2}{8\pi^2}\dfrac{1}{\epsilon}$ (6.66)

The results of diagrams of Non-thermal ϕ^4 theory are well known [10]

 $_{\mathrm{QFT}} = \dfrac{-m^2g^2}{(4\pi)^4}\left[\dfrac{4}{\epsilon^2}+2\dfrac{\psi(1)+\psi(2)}{\epsilon}-\dfrac{4}{\epsilon}\ln\left(\dfrac{m^2}{4\pi\mu^2}\right)+\mathcal{O}(\epsilon^0)\right]$ (6.67)

So



$$= \frac{-m^2 g^2}{(4\pi)^4}\left[\frac{4}{\epsilon^2} + 2\frac{\psi(1)+\psi(2)}{\epsilon} - \frac{4}{\epsilon}\ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] \tag{6.68}$$

Further diagrams results can be found in standard textbooks [10]. We are not going to reproduce the same results, but in the upcoming section we will try to find the corresponding results for thermal ϕ^4 theory.
For calculation easiness proper vertex function is used in some calculations. The proper vertex function is defined as

$$\bar{\Gamma}^{(n)}(\vec{\mathbf{k}}_1 \ldots \vec{\mathbf{k}}_n) = -\int \frac{d^n l_1}{(2\pi)^N} \cdots \frac{d^n l_L}{(2\pi)^N} \; G_0\left(\vec{\mathbf{q}}_1(\vec{\mathbf{l}}, \vec{\mathbf{k}})\right) \; \ldots \; G_0\left(\vec{\mathbf{q}}_I(\vec{\mathbf{l}}, \vec{\mathbf{k}})\right) \tag{6.69}$$

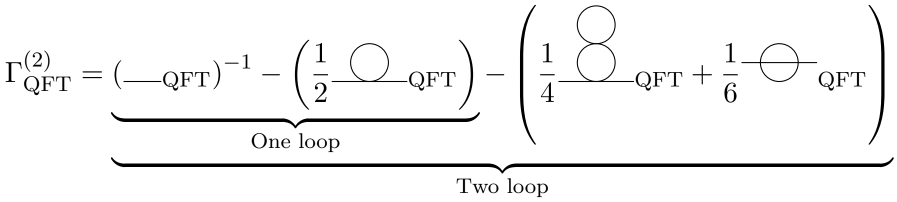
. The examples of two- and four-point vertex functions can be found in the coming sections. The procedure for deriving the coupling constant in ϕ^4 theory for Lagrangians can be written down as shown below [10].

6.2.3 Procedures for deriving coupling constant upto two loop order in non-thermal ϕ^4 theory

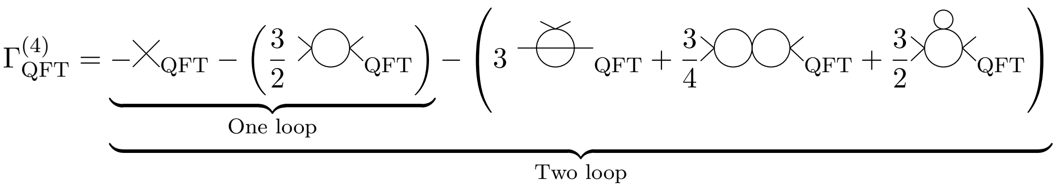
1. Define the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] - \frac{\lambda}{4!}\phi^4 \tag{6.70}$$

2. Write down the Feynman diagrams for two and four-point functions up to two loop order



(6.71)



(6.72)

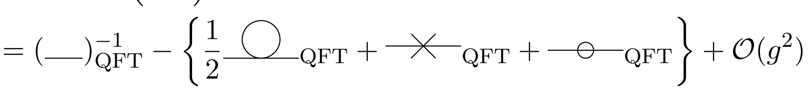
3. Use the regularization method to pick out the diverging terms and write down the counter-terms. Make the proper vertex function finite by adding the counter terms. The operator $\mathcal{K}$ can be used as a pole finding operator. The results for the non-thermal ϕ^4 theory can be taken from Kleinert [10], as shown below

Two-point function

One loop Counter terms

The finite two-point proper vertex function can be written as

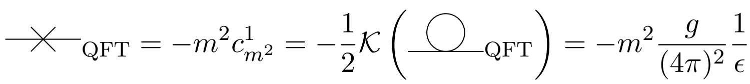
$$\widetilde{\Gamma}^{(2)}_{\text{QFT}} = \Gamma^2 - \mathcal{K}\left(\Gamma^{(2)}\right)$$



(6.73)

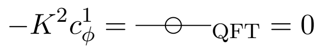
where the counter terms are

(a)



(6.74)

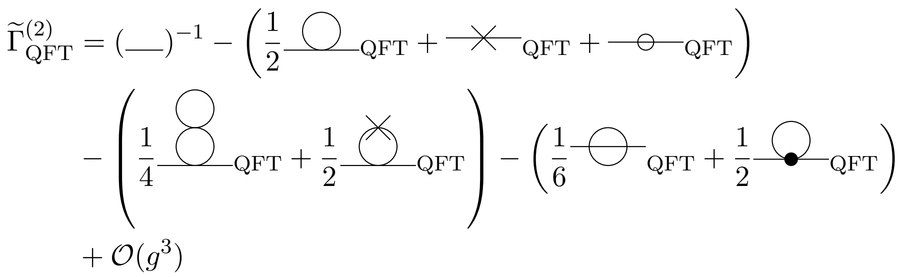
(b)



(6.75)

6.2.3. (a) Two loop Counter terms

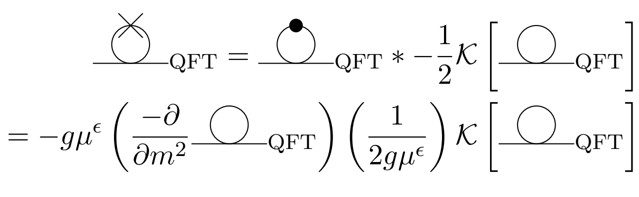
The finite proper vertex function for two-point function at two loop approximation is



(6.76)

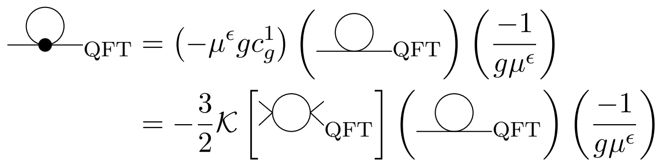
with $*$ being an operator that substitutes the appropriate counter term $-m^2 c_{m^2}$ or $-\mu^{\epsilon} g c_g$, the new counter terms in the two loop approximation being

(a)



(6.77)

(b)

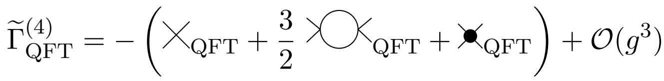


(6.78)

Four-point function

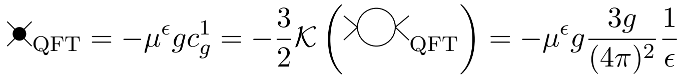
6.2.3. (b) One loop Counter terms

The finite four-point proper vertex function up to one loop order can be written as



(6.79)

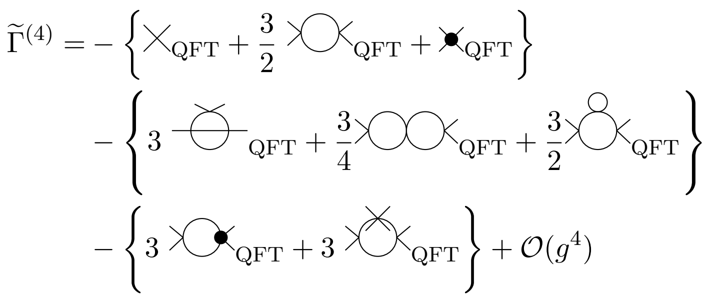
with the counter term



(6.80)

6.2.3. (c) Two loop Counter terms

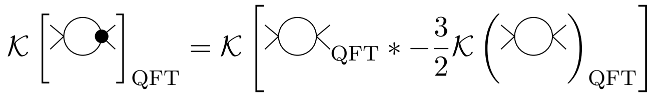
The finite four-point proper vertex function up to two loop order is



(6.81)

The counter terms are

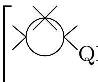
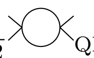
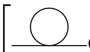
(a)

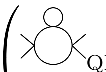


(6.82)

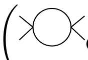
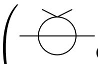
Similarly

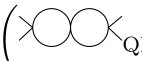
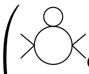
(b)

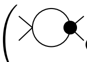
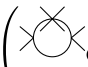
$\mathcal{K}\Big[$  $_{\text{QFT}}\Big] = \mathcal{K}\Big[-g\mu^{\epsilon}\Big(-\frac{1}{2}\frac{\partial}{\partial m^2}$  $_{\text{QFT}}\Big)\Big(\frac{-1}{\mu^{\epsilon}g}\Big)\Big(-\frac{1}{2}\mathcal{K}\Big[$  $_{\text{QFT}}\Big]\Big)\Big]$

$= -\frac{1}{2}\mathcal{K}\Big($  $_{\text{QFT},k_0=\omega_{n_k}}\Big)$ (6.83)

4. Find the renormalization constants via diagrammatic expansion as shown

$Z_g(g,\epsilon^{-1}) = 1 + \frac{1}{g\mu^{\epsilon}}\Big\{\frac{3}{2}\mathcal{K}\Big($  $_{\text{QFT}}\Big) + 3\mathcal{K}\Big($  $_{\text{QFT}}\Big)\Big\}$

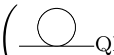
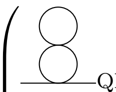
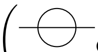
$+ \frac{1}{g\mu^{\epsilon}}\Big\{\frac{3}{4}\mathcal{K}\Big($  $_{\text{QFT}}\Big) + \frac{3}{2}\mathcal{K}\Big($  $_{\text{QFT}}\Big)\Big\}$ (6.84)

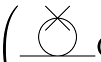
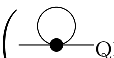
$+ \frac{1}{g\mu^{\epsilon}}\Big\{3\mathcal{K}\Big($  $_{\text{QFT}}\Big) + 3\mathcal{K}\Big($  $_{\text{QFT}}\Big)\Big\}$

So from standard textbook result of QFT [10] the analytical value of the calculation is,

$$Z_g(g,\epsilon^{-1}) = 1 + \frac{g}{(4\pi)^2}\frac{3}{\epsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{9}{\epsilon^2} - \frac{3}{\epsilon}\right) \tag{6.85}$$

and

$Z_{m^2} = 1 + \frac{1}{m^2}\Big\{\frac{1}{2}\mathcal{K}\Big($  $_{\text{QFT}}\Big) + \frac{1}{4}\mathcal{K}\Big($  $_{\text{QFT}}\Big) + \frac{1}{6}\mathcal{K}\Big($  $_{\text{QFT},K^2=0}\Big)\Big\}$

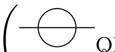
$+ \frac{1}{m^2}\Big\{\frac{1}{2}\mathcal{K}\Big($  $\text{QFT}\Big) + \frac{1}{2}\mathcal{K}\Big($  $_{\text{QFT}}\Big)\Big\}$

(6.86)

the analytical solution of Eq. (6.86) from [10] is

$$Z_{m^2}(g,\epsilon^{-1}) = 1 + \frac{g}{(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon}\right) \tag{6.87}$$

Finally

$Z_{\phi} = 1 + \frac{1}{K^2}\frac{1}{6}\mathcal{K}\Big($  $_{\text{QFT}}\Big)|_{m^2=0,k_0=\omega_{n_k}}$

$$= 1 + c_{\phi} \tag{6.88}$$

$$= 1 - \frac{g^2}{(4\pi)^4}\frac{1}{12\epsilon}$$

5. From the relation between Renormalization Group Equation (RGE) and proper vertex function

$$\begin{aligned}
&\frac{d}{d(\ln\mu)}\widetilde{\Gamma}^{(n)}\left(m,g,T,\mu\right)=0\\
&\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)+\gamma_m m\frac{\partial}{\partial m}\right]\widetilde{\Gamma}^{(n)}\left(m,g,T,\mu\right)\approx_{TLA}0
\end{aligned} \tag{6.89}$$

where

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}\ln(\mu)}&=\beta(g)\\
\frac{\mathrm{d}\ln(m)}{\mathrm{d}\ln(\mu)}&=\gamma_m(g)
\end{aligned} \tag{6.90}$$

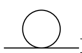
The renormalization constants can be derived [10] as

$$\begin{aligned}
\gamma(g)&=-Z_{\phi,1}=-\epsilon c_\phi\\
\gamma_m(g)&=\frac{1}{2}\frac{g}{(4\pi)^2}-\frac{1}{2}\frac{g^2}{(4\pi)^4}+\gamma(g)\\
\beta(g)&=-\epsilon g+\frac{3g^2}{(4\pi)^2}-\frac{6g^3}{(4\pi)^4}+4g\gamma(g)
\end{aligned} \tag{6.91}$$

6.3 Dimensional Regularization in Thermal ϕ^4 Theory

6.3.1 One loop two-point function: The Tadpole Diagram

The Tadpole diagram in Imaginary Time Formalism (ITF) is defined as



$$\Big|_{\mathrm{ITF}}=-\lambda\,T\sum_n\int\frac{1}{p^2+m^2+\omega_n^2}\frac{\mathrm{d}^3p}{(2\pi)^3}=-\lambda\,T\sum_n\int\frac{1}{\varepsilon_p^2+\omega_n^2}\frac{\mathrm{d}^3p}{(2\pi)^3} \tag{6.92}$$

with $\omega_n=2\pi nT$.

The summation

The summation can be done using the formula [11]

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}f(n)&=-\sum_k\mathrm{Res}\left[\pi f(z)\cot(\pi z)\right]\Big|_{z=z_k}\\
\sum_{n=-\infty}^{\infty}(-1)^n f(n)&=-\sum_k\mathrm{Res}\left[\pi f(z)\csc(\pi z)\right]\Big|_{z=z_k}
\end{aligned} \tag{6.93}$$

where z_k is the pole values of $f(z)$.
In order to solve Eq. (6.92) we have to do the summation

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \sum_{n=-\infty}^{\infty} f(n) \tag{6.94}$$

Here

$$f(z) = \frac{1}{z^2 + a^2} = \frac{1}{2\mathrm{i}a}\left[\frac{1}{z - \mathrm{i}a} - \frac{1}{z + \mathrm{i}a}\right] \tag{6.95}$$

So simple poles are at $z = \pm\mathrm{i}a$. Thus the summation can be evaluated as

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} &= -\sum_k \mathrm{Res}\left[\pi f(z)\cot(\pi z)\right] \\
&= -\mathrm{Res}\left[\pi \frac{1}{2\mathrm{i}a}\left[\frac{1}{z - \mathrm{i}a} - \frac{1}{z + \mathrm{i}a}\right]\cot(\pi z)\right] \\
&= -\frac{\pi}{2\mathrm{i}a}\left[\cot(\pi z)\Big|_{z=\mathrm{i}a} - \cot(\pi z)\Big|_{z=-\mathrm{i}a}\right] \\
&= \frac{\pi}{a}\coth(\pi a)
\end{aligned} \tag{6.96}$$

Defining $n_B(x) = \dfrac{1}{\exp(x) - 1}$ and using Eq. (6.96)

$$T\sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \varepsilon_p^2} = \frac{1}{2\varepsilon_p} + \frac{n_B(\beta\varepsilon_p)}{\varepsilon_p} \tag{6.97}$$

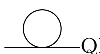
Here we can re write

$$\frac{1}{2\varepsilon_p} = \int_{-\infty}^{\infty} \frac{1}{p_0^2 + \varepsilon_p^2}\frac{\mathrm{d}p_0}{2\pi} \tag{6.98}$$

where $\varepsilon_p^2 = p^2 + m^2$, with $p = [p_x, p_y, p_z]$, So as pointed out by [12], these two diagrams can be connected like (combining Eq. (6.98) and Eq. (6.97))

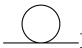
$$\begin{aligned}
\int T\sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \varepsilon_p^2}\frac{\mathrm{d}^3p}{(2\pi)^3} &= \int \frac{1}{p_0^2 + p^2 + m^2}\frac{d^3p}{(2\pi)^3}\frac{dp_0}{(2\pi)} + \int \frac{n_B(\beta\varepsilon_p)}{\varepsilon_p}\frac{\mathrm{d}^3p}{(2\pi)} \\
&= \int \frac{1}{P^2 + m^2}\frac{d^4P}{(2\pi)^4} + \int \frac{n_B(\beta\varepsilon_p)}{\varepsilon_p}\frac{\mathrm{d}^3p}{(2\pi)^3}
\end{aligned} \tag{6.99}$$

where $P = [p_0, p_x, p_y, p_z]$
On comparing the tadpole diagram in both QFTand ITFwe have

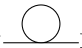
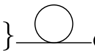


$$_{\mathrm{QFT}} = -\lambda \int \left(\frac{1}{P^2 + m^2}\right)\frac{\mathrm{d}^4P}{(2\pi)^4} \tag{6.100}$$

and

 $$_{\text{ITF}} = -\lambda\, T \sum_{n_l} \int \left(\frac{1}{\omega_{n_l}^2 + \varepsilon_l^2} \right) \frac{\mathrm{d}^3 l}{(2\pi)^3} \tag{6.101}$$

combining Eqs. (6.99) to (6.101)

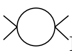
$\Big[\{-\lambda\}$  $_{\text{ITF}}\Big] = \Big[\{-\lambda\}$  $_{\text{QFT}}\Big] - \lambda S_1(m,T)$ (6.102)

where

$$S_1(m,T) = \int \frac{n_B(\beta\varepsilon_p)}{\varepsilon_p} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{m}{2\pi n\beta} \right) K_1(n\beta m) \tag{6.103}$$

Where $K(n,x)$ is the Modified Bessel function of second kind. The braces $\{\lambda\}$ used to denote the power of the integral in terms of λ and doesn't mean the multiplication purpose.

6.3.2 Four-point function at one loop order

$\mathrm{I}_{\text{ITF}}^{(4,1)} =$  $_{\text{ITF}} =$

$$\begin{aligned} &= \lambda^2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, T \sum_{n_p=-\infty}^{\infty} \left(\frac{1}{\omega_{n_p}^2 + \varepsilon_p^2} \right) \left(\frac{1}{\omega_{n_p-n_r}^2 + \varepsilon_{p-r}^2} \right) \\ &= \lambda^2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, T \sum_{n_p=-\infty}^{\infty} \left(\frac{1}{\omega_{n_p}^2 + \varepsilon_p^2} \right) \left(\frac{1}{\omega_{n_p+n_r}^2 + \varepsilon_{p-r}^2} \right) \end{aligned} \tag{6.104}$$

The summation of the integrand in Eq. (6.104) can be derived using Eq. (6.93) as

$$T \sum_{n_p=-\infty}^{\infty} \left(\frac{1}{\omega_{n_p}^2 + \varepsilon_p^2} \right) \left(\frac{1}{\omega_{n_p-n_r}^2 + \varepsilon_q^2} \right) = [t_1(p,q,n_r) + t_2(p,q,n_r) + t_2(q,p,n_r)] \tag{6.105}$$

with

$$t_1(p,q,n_r) = \sum_{\sigma=\pm 1} \left(\frac{1}{4\varepsilon_p \varepsilon_q} \right) \left(\left(\frac{1}{\varepsilon_p + \varepsilon_q + \mathtt{i}\sigma\omega_{n_r}} \right) \right) \tag{6.106}$$

$$t_2(p,q,n_r) = \sum_{\sigma,\sigma_1=\pm 1} \left(\frac{1}{4\varepsilon_p \varepsilon_q} \right) \left(\frac{1}{\sigma_1 \varepsilon_p + \varepsilon_q + \mathtt{i}\sigma\omega_{n_r}} \right) n_B(\beta\varepsilon_p) \tag{6.107}$$

and Eq. (6.104) for $\lambda = 1$, becomes

$$\mathrm{I}_{\text{ITF}}^{(4,1)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3}\ [t_1(p,q,n_q) + t_2(p,q,n_q) + t_2(q,p,n_q)]\ (2\pi)^3 \delta^3(p+q+r)$$

$$\tag{6.108}$$

Using the fact that $p \leftrightarrow q$ doesn't alter the integral in Eq. (6.108) gives

$$\mathrm{I}_{\mathrm{ITF}}^{(4,1)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3} \; [t_1(p,q,n_r) + 2t_2(p,q,n_r)] \; (2\pi)^3 \delta^3(p+q+r) \tag{6.109}$$

One can use the result that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{1}{q_0^2+\varepsilon_q^2}\right) \delta(p_0+q_0+\omega_{n_r}) \frac{\mathrm{d}q_0}{2\pi}\frac{\mathrm{d}p_0}{2\pi} \\
&= \frac{-1}{4\varepsilon_p\varepsilon_q} \int_{-\infty}^{\infty} \left[\left(\frac{1}{p_0-\mathtt{i}\varepsilon_p}\right) - \left(\frac{1}{p_0+\mathtt{i}\varepsilon_p}\right)\right]\left[\left(\frac{1}{q_0-\mathtt{i}\varepsilon_q}\right) - \left(\frac{1}{q_0+\mathtt{i}\varepsilon_q}\right)\right] \; \delta(p_0+q_0+\omega_{n_r}) \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}q_0}{2\pi} \\
&= \frac{-1}{4\varepsilon_p\varepsilon_q} \int_{-\infty}^{\infty} \left[\left(\frac{1}{p_0-\mathtt{i}\varepsilon_p}\right) - \left(\frac{1}{p_0+\mathtt{i}\varepsilon_p}\right)\right]\left[\left(\frac{1}{q_0-\mathtt{i}\varepsilon_q}\right) - \left(\frac{1}{q_0+\mathtt{i}\varepsilon_q}\right)\right] \; e^{i(p_0+q_0+\omega_{n_r})l} \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}q_0}{2\pi}\frac{\mathrm{d}l}{2\pi} \\
&= \frac{-1}{4\varepsilon_p\varepsilon_q} \int_{-\infty}^{\infty} \left[2\pi\mathtt{i}e^{-\varepsilon_p l}\theta(l) + 2\pi\mathtt{i}e^{\varepsilon_p l}\theta(-l)\right]\left[2\pi\mathtt{i}e^{-\varepsilon_q l}\theta(l) + 2\pi\mathtt{i}e^{\varepsilon_q l}\theta(-l)\right] \; e^{i\omega_{n_r} l} \frac{\mathrm{d}l}{(2\pi)^3} \\
&= \left(\frac{1}{4\varepsilon_p\varepsilon_q}\right) \int_{-\infty}^{\infty} \left[e^{-(\varepsilon_p+\varepsilon_q)l}\theta(l) + e^{(\varepsilon_p+\varepsilon_q)l}\theta(-l)\right] e^{i\omega_{n_r} l} \; \frac{\mathrm{d}l}{2\pi} \\
&= \left(\frac{1}{4\varepsilon_p\varepsilon_q}\right) \sum_{\sigma=\pm 1} \int_0^{\infty} \left[e^{-l(\varepsilon_p+\varepsilon_q+\mathtt{i}\sigma\omega_{n_r})}\right] \frac{\mathrm{d}l}{2\pi} \\
&= \left(\frac{1}{2\pi}\right)\left(\frac{1}{4\varepsilon_p\varepsilon_q}\right) \sum_{\sigma=\pm 1} \left(\frac{1}{\varepsilon_p+\varepsilon_q+\mathtt{i}\sigma\omega_{n_r}}\right)
\end{aligned} \tag{6.110}$$

Thus, we can write

$$t_1(p,q,n_r) = \int_{-\infty}^{\infty} \left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{1}{q_0^2+\varepsilon_q^2}\right) \; 2\pi\delta(p_0+q_0+\omega_{n_r}) \frac{\mathrm{d}q_0}{2\pi}\frac{\mathrm{d}p_0}{2\pi} \tag{6.111}$$

Now

$$\begin{aligned}
&\int \; t_1(p,q,n_r) \; (2\pi)^3\delta^3(p+q+r) \; \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\mathrm{d}^3 q}{(2\pi)^3} \\
&= \int \left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{Q^2+m^2}\right) \; (2\pi)^4\delta^4(P+Q+R)\frac{\mathrm{d}^4 P}{(2\pi)^4}\frac{\mathrm{d}^4 Q}{(2\pi)^4}
\end{aligned} \tag{6.112}$$

with

$$\begin{aligned}
P &= [p_0, p] \\
Q &= [q_0, q] \\
R &= [\omega_r, r]
\end{aligned} \tag{6.113}$$

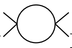
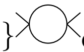
So

$$\begin{aligned}
\mathrm{I}_{\mathrm{ITF}}^{(4,1)} =& \int \left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{Q^2+m^2}\right)\ (2\pi)^4\delta^4(P+Q+R)\frac{\mathrm{d}^4P}{(2\pi)^4}\frac{\mathrm{d}^4Q}{(2\pi)^4} \\
&+\sum_{\sigma,\sigma_1=\pm1}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\left(\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p\varepsilon_q}\right)\left(\frac{(2\pi)^3\delta^3(p+q+r)}{\sigma_1\varepsilon_p+\varepsilon_q+\mathtt{i}\sigma\omega_{n_r}}\right) \qquad (6.114)\\
=&\mathrm{I}_{\mathrm{QFT}}^{(4,1)}+\sum_{\sigma,\sigma_1=\pm1}\int\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p\varepsilon_{p+r}}\left(\frac{1}{\sigma_1\varepsilon_p+\varepsilon_{p+r}+\mathtt{i}\sigma\omega_{n_r}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}
\end{aligned}$$

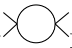
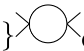
i.e.,

$$\begin{aligned}
\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,T\sum_{n_p=-\infty}^{\infty}\left(\frac{1}{\omega_{n_p}^2+\varepsilon_p^2}\right)\left(\frac{1}{\omega_{n_p-n_r}^2+\varepsilon_{p-r}^2}\right)=&\int\left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{(P-R)^2+m^2}\right)\frac{\mathrm{d}^4P}{(2\pi)^4}\\
&+\sum_{\sigma,\sigma_1=\pm1}\int\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p\varepsilon_{p+r}}\left(\frac{1}{\sigma_1\varepsilon_p+\varepsilon_{p+r}+\mathtt{i}\sigma\omega_{n_r}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3}
\end{aligned}$$

In terms of diagrammatic representation, the equation can be expressed as

$\{\lambda^2\}$  $_{\mathrm{ITF}} = \{\lambda^2\}$  $_{\mathrm{QFT},R_0=\omega_{n_r}}$

$$+\lambda^2\sum_{\sigma,\sigma_1=\pm1}\int\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p\varepsilon_{p+r}}\left(\frac{1}{\sigma_1\varepsilon_p+\varepsilon_{p+r}+\mathtt{i}\sigma\omega_{n_r}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3} \qquad (6.115)$$

$\{\lambda^2\}$  $_{\mathrm{ITF}} = \{\lambda^2\}$  $_{\mathrm{QFT},R_0=\omega_{n_r}} + \lambda^2 W(r,n_r)$

Now let us look at the nature of integral

$$W(r,n_r)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\frac{2n_B(\beta\varepsilon_p)}{\varepsilon_p}\right)\frac{\left[r^2+2pr\cos\theta+\omega_{n_r}^2\right]}{\left[(r^2+2pr\cos\theta+\omega_{n_r}^2)^2+4\varepsilon_p^2\omega_{n_r}^2\right]} \qquad (6.116)$$

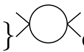
when we apply the limit $R^2=\omega_{n_r}^2+\vec{r}^2=0$ Then integral becomes

$$W(r,n_r)_{R^2=0}=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\frac{2n_B(\beta\varepsilon_p)}{\varepsilon_p}\right)\left(\frac{2pr\cos\theta}{(2pr\cos\theta)^2+4\varepsilon_p^2\omega_{n_r}^2}\right) \qquad (6.117)$$

Taking integration of angular coordinate we get

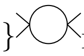
$$W(r,n_r)_{R^2=0}=\int\frac{p^2dp}{4\pi^2}\frac{2n_B(\beta\varepsilon_p)}{\varepsilon_p}\int_{-1}^{1}\frac{2pr\cos\theta}{\left[(2pr\cos\theta)^2+4\varepsilon_p^2\omega_{n_r}^2\right]}\,d\cos\theta \qquad (6.118)$$

Since $\frac{2pr\cos\theta}{(2pr\cos\theta)^2+4\varepsilon_p^2\omega_{n_r}^2}$ is an odd function w.r.t $\cos\theta$, the integral becomes zero. From result from standard textbook [10]

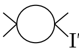
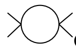
$\{g^2\mu^\epsilon\}$  $_{\mathrm{QFT}} = \frac{g^2\mu^\epsilon}{(4\pi)^2}\left(\frac{2}{\epsilon}+\psi(1)+\int_0^1 dx\log\left[\frac{4\pi\mu^2}{R^2x(1-x)+m^2}\right]+\mathcal{O}(\epsilon)\right)$

(6.119)

Thus

$\{g^2\mu^\epsilon\}$  $_{\rm ITF} = \frac{g^2\mu^\epsilon}{(4\pi)^2}\left(\frac{2}{\epsilon}+\psi(1)+\int_0^1 dx \log\left[\frac{4\pi\mu^2}{R^2 x(1-x)+m^2}\right]\right)|_{R_0=\omega_{n_r}} + g^2 W(r,n_r)$ (6.120)

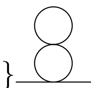
The result can be thus obtained as

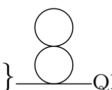
 $_{\rm ITF} =$  $_{{\rm QFT},Q_0=\omega_{n_q}} + (g\mu^\epsilon)^2 W(q,n_q)$ (6.121)

with

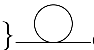
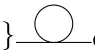
$$W(r,n_r) = \int \frac{2n_B(\beta\varepsilon_p)\left(r^2+2pr\cos\theta+\omega_{n_r}^2\right)}{\varepsilon_p\left[\left(r^2+2pr\cos\theta+\omega_{n_r}^2\right)^2+4\varepsilon_p^2\omega_{n_r}^2\right]}\frac{d^3p}{(2\pi)^3} \tag{6.122}$$

6.3.3 Two-point two loop order diagrams

The integral expression of $\{\lambda^2\}$  in QFT is

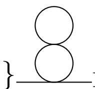
$\{\lambda^2\}$  $_{\rm QFT} = \lambda^2\int\left(\frac{1}{P_1^2+m^2}\right)\left[\left(\frac{1}{P_2^2+m^2}\right)\right]^2\frac{{\rm d}^4P_1}{(2\pi)^4}\frac{{\rm d}^4P_2}{(2\pi)^4}$

$= \left(-\lambda\int\frac{1}{P_1^2+m^2}\frac{{\rm d}^4P_1}{(2\pi)^4}\right)\left(-\frac{\partial}{\partial m^2}\left[-\lambda\int\frac{1}{P_2^2+m^2}\frac{{\rm d}^4P_2}{(2\pi)^4}\right]\right)$

$= \Big(\{-\lambda\}$  $_{\rm QFT}\Big)\Big(-\frac{\partial}{\partial m^2}\{-\lambda\}$  $_{\rm QFT}\Big)$

(6.123)

The same kind of arrangements of diagrams is also applicable in ITF thus the corresponding diagram in ITF is

$\{\lambda^2\}$  $_{\rm ITF} = \int\lambda^2T^2\sum_{n_{p1}=-\infty}^{\infty}\sum_{n_{p2}=-\infty}^{\infty}\left(\frac{1}{\omega_{n_{p1}}^2+\varepsilon_{p1}^2}\right)\left[\left(\frac{1}{\omega_{n_{p2}}^2+\varepsilon_{p2}^2}\right)\right]^2\frac{{\rm d}^3p_1}{(2\pi)^3}\frac{{\rm d}^3p_2}{(2\pi)^3}$

$$= \left(-\lambda T\int\sum_{n_{p1}=-\infty}^{\infty}\frac{1}{\omega_{n_{p1}}^2+\varepsilon_{p1}^2}\frac{{\rm d}^3p_1}{(2\pi)^3}\right)\left(-\frac{\partial}{\partial m^2}\left[-\lambda T\int\sum_{n_{p1}=-\infty}^{\infty}\frac{1}{\omega_{n_{p1}}^2+\varepsilon_{p1}^2}\frac{{\rm d}^3p_1}{(2\pi)^3}\right]\right)$$

(6.124)

Now

$$
\begin{aligned}
\{\lambda^2\}\,\underset{}{\overset{\bigcirc}{\bigcirc}}_{\text{ITF}} &= \left(\{-\lambda\}\,\bigcirc_{\text{ITF}}\right)\left(-\frac{\partial}{\partial m^2}\{-\lambda\}\,\bigcirc_{\text{ITF}}\right) \\
&= \left(\bigcirc_{\text{QFT}} - \lambda S_1(m,T)\right)\left(-\frac{\partial}{\partial m^2}\left[\bigcirc_{\text{QFT}} - \lambda S_1(m,T)\right]\right) \\
&= \bigcirc_{\text{QFT}}\left(-\frac{\partial}{\partial m^2}\bigcirc_{\text{QFT}}\right) + \frac{\lambda^2}{4\pi}S_1(m,T)S_0(m,T) \\
&\quad -\frac{\lambda}{4\pi}S_0(m,T)\left[\bigcirc_{\text{QFT}}\right] + \lambda S_1(m,T)\frac{\partial}{\partial m^2}\bigcirc_{\text{QFT}}
\end{aligned}
\tag{6.125}
$$

So

$$
\begin{aligned}
\{\lambda^2\}\,\overset{\bigcirc}{\bigcirc}_{\text{ITF}} &= \{\lambda^2\}\,\overset{\bigcirc}{\bigcirc}_{\text{QFT}} + \frac{\lambda^2}{4\pi}S_1(m,T)S_0(m,T) \\
-\frac{\lambda}{4\pi}S_0(m,T)&\left[\{-\lambda\}\,\bigcirc_{\text{QFT}}\right] + \lambda S_1(m,T)\frac{\partial}{\partial m^2}\{-\lambda\}\,\bigcirc_{\text{QFT}}
\end{aligned}
\tag{6.126}
$$

where

$$
S_N(m,T) = \left(\frac{1}{\pi}\right)\sum_{n=1}^{\infty}\left(\frac{m}{2\pi n\beta}\right)^N K_N(nm\beta) \tag{6.127}
$$

Using the results from [10] and from Section 6.3.1 we can write

$$
\begin{aligned}
\overset{\bigcirc}{\bigcirc}_{\text{ITF}} = \overset{\bigcirc}{\bigcirc}_{\text{QFT}} &- \frac{g}{4\pi}S_0(m,T)\left[\bigcirc_{\text{QFT}}\right] + gS_1(m,T)\frac{\partial}{\partial m^2}\bigcirc_{\text{QFT}} \\
&+ \frac{g^2}{4\pi}S_1(m,T)S_0(m,T)
\end{aligned}
\tag{6.128}
$$

with

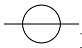
$$
\overset{\bigcirc}{\bigcirc}_{\text{QFT}} = -\frac{m^2g^2}{(4\pi)^4}\left[\frac{4}{\epsilon^2} + 2\left(\frac{\psi(1)+\psi(2)}{\epsilon}\right) - \frac{4}{\epsilon}\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^0)\right] \tag{6.129}
$$

$$
\left[\bigcirc_{\text{QFT}}\right] = \frac{m^2g}{(4\pi)^2}\left[\frac{2}{\epsilon} + \psi(2) + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon) \tag{6.130}
$$

$$
\frac{\partial}{\partial m^2}\left[\bigcirc_{\text{QFT}}\right] = \frac{g}{(4\pi)^2}\left[\frac{2}{\epsilon} + \psi(1) + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon) \tag{6.131}
$$

6.3.4 Sunset/Sunrise Diagram

The next important two-point two loop diagram can be written as

$\mathrm{I}^{(2,2)}_{\mathrm{ITF}} =$  $_{\mathrm{ITF}}$

$$
\begin{aligned}
&= \lambda^2 T^2 \int \sum_{n_p,n_q=-\infty}^{\infty} \left(\frac{1}{\omega_{n_p}^2+\varepsilon_p^2}\right)\left(\frac{1}{\omega_{n_q}^2+\varepsilon_q^2}\right)\left(\frac{1}{\omega_{n_p+n_q+n_r}^2+\varepsilon_{p+q+r}^2}\right)\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} \\
&= \lambda^2 T^2 \int \sum_{n_p,n_q=-\infty}^{\infty} \left(\frac{1}{\omega_{n_p}^2+\varepsilon_p^2}\right)\left(\frac{(2\pi)^3}{\omega_{n_q}^2+\varepsilon_q^2}\right)\left(\frac{\delta^3(p+q+r+s)}{\omega_{n_p+n_q+n_r}^2+\varepsilon_s^2}\right)\prod_{\Omega=p,q,s}\left[\frac{\mathrm{d}^3\Omega}{(2\pi)^3}\right]
\end{aligned}
\tag{6.132}
$$

with $\prod_{\Omega=p,q,s}\left[\frac{\mathrm{d}^3\Omega}{(2\pi)^3}\right] = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3s}{(2\pi)^3}$ and $\sum_{n_p,n_q=-\infty}^{\infty} = \sum_{n_p=-\infty}^{\infty}\sum_{n_q=-\infty}^{\infty}$.
The summation result using Eq. (6.93) is

$$
T^2 \sum_{n_p=-\infty}^{\infty}\sum_{n_q=-\infty}^{\infty} \left(\frac{1}{\omega_{n_p}^2+\varepsilon_p^2}\right)\left(\frac{1}{\omega_{n_q}^2+\varepsilon_q^2}\right)\left(\frac{1}{\omega_{n_p+n_q+n_r}^2+\varepsilon_r^2}\right) = S_1+S_2+S_3
\tag{6.133}
$$

with

$$
\begin{aligned}
S_1 &= \left(\frac{1}{8\varepsilon_p\varepsilon_q\varepsilon_r}\right)\left(\sum_{\sigma=\pm1}\left(\frac{1}{\varepsilon_p+\varepsilon_q+\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}\right)\right) \\
S_2 &= \left(\frac{1}{8\varepsilon_p\varepsilon_q\varepsilon_r}\right)\sum_{\sigma,\sigma_1=\pm1}\left(\frac{n_B(\beta\varepsilon_p)}{\sigma_1\varepsilon_p+\varepsilon_q+\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}+\frac{n_B(\beta\varepsilon_q)}{\varepsilon_p+\sigma_1\varepsilon_q+\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}\right) \\
&\quad + \left(\frac{1}{8\varepsilon_p\varepsilon_q\varepsilon_r}\right)\sum_{\sigma,\sigma_1=\pm1}\left(\frac{n_B(\beta\varepsilon_r)}{\varepsilon_p+\varepsilon_q+\sigma_1\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}\right) \\
S_3 &= \left(\frac{1}{8\varepsilon_p\varepsilon_q\varepsilon_r}\right)\sum_{\sigma,\sigma_1,\sigma_2=\pm1}\left(\frac{n_B(\beta\varepsilon_q)n_B(\beta\varepsilon_r)}{\varepsilon_p+\sigma_1\varepsilon_q+\sigma_2\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}+\frac{n_B(\beta\varepsilon_p)n_B(\beta\varepsilon_q)}{\sigma_1\varepsilon_p+\sigma_2\varepsilon_q+\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}\right) \\
&\quad + \left(\frac{1}{8\varepsilon_p\varepsilon_q\varepsilon_r}\right)\sum_{\sigma,\sigma_1,\sigma_2=\pm1}\left(\frac{n_B(\beta\varepsilon_p)n_B(\beta\varepsilon_r)}{\sigma_1\varepsilon_p+\varepsilon_q+\sigma_2\varepsilon_r+\mathtt{i}\sigma\omega_{n_r}}\right)
\end{aligned}
\tag{6.134}
$$

We can use the relation that

$$
\begin{aligned}
\mathrm{I} &= \int_{-\infty}^{\infty} \delta\left(\sum_{n=1}^{m} p_{0n} - b\right) \; \mathrm{d}p_{01}\ \mathrm{d}p_{02}\ \mathrm{d}p_{03}\ \ldots\ \mathrm{d}p_{0m} \prod_{n=1}^{m}\left[\left(\frac{1}{p_{0n}^2 + \varepsilon_{p_{0n}}^2}\right)\right] \\
&= \prod_{n=1}^{m} \int_{-\infty}^{\infty} \frac{\exp\left(\mathtt{i} l p_{0n}\right)}{p_{0n}^2 + \varepsilon_{p0n}^2} \; \mathrm{d}p_{0n} \; \exp(-\mathtt{i} l b) \frac{\mathrm{d}l}{2\pi} \\
&= \int_{-\infty}^{\infty} \prod_{n=1}^{m} \left[\frac{\pi}{\varepsilon_{p_{0n}}} \left[\exp(-l\varepsilon_{p_{0n}})\theta(l) + \exp(l\varepsilon_{p_{0n}})\theta(-l)\right]\right] \exp(\mathtt{i} b l) \frac{\mathrm{d}l}{2\pi} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}l}{2\pi} \exp(\mathtt{i} b l) \; \left(\frac{\pi^m}{\prod_{n=1}^{m} \varepsilon_{p_{0n}}} \left[\exp\left(-l \sum_{n=1}^{m} \varepsilon_{p_{0n}}\right)\theta(l) + \exp\left(l \sum_{n=1}^{m} \varepsilon_{p_{0n}}\right)\theta(-l)\right]\right) \\
&= \left(\frac{1}{2\pi}\right) \; \left(\frac{\pi^m}{\Pi_{n=1}^{m} \varepsilon_{p_{0n}}}\right) \sum_{\sigma=\pm 1} \left(\frac{1}{\sum_{n=1}^{m} \varepsilon_{p_{0n}} + \mathtt{i}\sigma b}\right)
\end{aligned}
\tag{6.135}
$$

Using the relation Eq. (6.135) the sum of certain terms can be written as an integral equation as

$$
\begin{aligned}
S_1 &= \left(\frac{1}{8\varepsilon_q \varepsilon_p \varepsilon_r}\right) \sum_{\sigma=\pm 1} \left(\frac{1}{\varepsilon_p + \varepsilon_q + \varepsilon_r + \mathtt{i}\sigma\omega_{n_k}}\right) \\
&= \int \left(\frac{1}{p_0^2 + \varepsilon_p^2}\right) \left(\frac{1}{q_0^2 + \varepsilon_q^2}\right) \left(\frac{1}{r_0^2 + \varepsilon_r^2}\right) (2\pi)\delta(p_0 + q_0 + r_0 + s_0)\Big|_{s_0=\omega_{n_k}} \frac{\mathrm{d}p_0}{2\pi} \frac{\mathrm{d}q_0}{2\pi} \frac{\mathrm{d}r_0}{2\pi}
\end{aligned}
\tag{6.136}
$$

$$
\begin{aligned}
S_2^{(p)} &= \frac{n_B(\beta\varepsilon_p)}{8\varepsilon_p \varepsilon_q \varepsilon_r} \sum_{\sigma,\sigma=\pm 1} \left(\frac{1}{\sigma_1\varepsilon_p + \varepsilon_q + \varepsilon_r + \mathtt{i}\sigma\omega_{n_k}}\right) \\
&= \sum_{\sigma_1=\pm 1} \frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p} \int \left(\frac{1}{q_0^2 + \varepsilon_q^2}\right) \left(\frac{1}{r_0^2 + \varepsilon_r^2}\right) 2\pi\delta(p_0 + q_0 + r_0 + s_0)\Big|_{\substack{s_0=\omega_{n_k},\\ p_0=\mathtt{i}\sigma_1\varepsilon_p}} \frac{\mathrm{d}p_0}{2\pi} \frac{\mathrm{d}r_0}{2\pi}
\end{aligned}
\tag{6.137}
$$

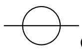
$S_2^{(p)}$ is the first term of S_2 in Eq. (6.134). Replacing p by q and p by r gives the second and third term of S_2 respectively.

$$
\begin{aligned}
S_3^{(p)} &= \frac{n_B(\beta\varepsilon_p) n_B(\beta\varepsilon_q)}{8\varepsilon_p \varepsilon_q \varepsilon_r} \sum_{\sigma,\sigma_1,\sigma_2=\pm 1} \left(\frac{1}{\sigma_1\varepsilon_p + \sigma_2\varepsilon_q + \varepsilon_r + \mathtt{i}\sigma\omega_{n_k}}\right) \\
&= \frac{n_B(\beta\varepsilon_p) n_B(\beta\varepsilon_q)}{4\varepsilon_p \varepsilon_q} \sum_{\sigma_1,\sigma_2=\pm 1} \int \frac{2\pi\delta(p_0 + q_0 + r_0 + s_0)}{q_0^2 + \varepsilon_q^2}\Big|_{\substack{p_0=\sigma_1\varepsilon_p,\\ r_0=\sigma_2\varepsilon_r,\\ s_0=\omega_{n_k}}} \frac{dq_0}{2\pi}
\end{aligned}
\tag{6.138}
$$

$S_3^{(p)}$ is the first term of S_3 in Eq. (6.134). Replacing p by q and p by r gives the second and third term of S_3 respectively.
One can evaluate the integral result by combining Eqs. (6.132) and (6.136) to (6.138) and including the symmetry in the Dirac delta function as

$$\mathrm{I}_{\mathrm{ITF}}^{(2,2)} = \mathrm{I}_{\mathrm{QFT}}^{(2,2)} + \mathrm{I}_2 + \mathrm{I}_3 \tag{6.139}$$

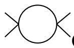
where

$\mathrm{I}_{\mathrm{QFT}}^{(2,2)} =$  $_{\mathrm{QFT}}$

$$= \lambda^2 \int \left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{Q^2+m^2}\right)\left(\frac{1}{R^2+m^2}\right)(2\pi)^4\delta^4(...)\frac{\mathrm{d}^4P}{(2\pi)^4}\frac{\mathrm{d}^4Q}{(2\pi)^4}\frac{\mathrm{d}^4R}{(2\pi)^4} \tag{6.140}$$

with $S = [\omega_{n_k}, \vec{s}]$ and $\delta^4(...) = \delta^4(P+Q+R+S)$.

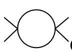
$$\mathrm{I}_2 = \lambda^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{3n_B(\beta\varepsilon_p)}{2\varepsilon_p}\sum_{\sigma_1=\pm1}\int (2\pi)^4\delta^4(...)\left(\frac{1}{Q^2+m^2}\right)\left(\frac{1}{R^2+m^2}\right)\frac{\mathrm{d}^4Q}{(2\pi)^4}\frac{\mathrm{d}^4R}{(2\pi)^4}$$

$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{3n_B(\beta\varepsilon_p)}{2\varepsilon_p}\sum_{\sigma_1=\pm1}$  $_{\mathrm{QFT}}\left((P+S)^2\right)$ (6.141)

with

$$\begin{aligned} P &= [i\sigma_1\varepsilon_p, \vec{p}] \\ S &= [\omega_{n_k}, \vec{s}] \end{aligned} \tag{6.142}$$

and

 $_{\mathrm{QFT}}(K^2) = \lambda^2 \int \frac{\mathrm{d}^4P}{(2\pi)^4}\left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{(P+K)^2+m^2}\right)$ (6.143)

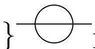
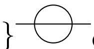
The third term of $\mathrm{I}_{\mathrm{ITF}}^{(2,2)}$ is

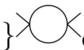
$$\begin{aligned} \mathrm{I}_3 &= \lambda^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{3n_B(\beta\varepsilon_p)n_B(\beta\varepsilon_q)}{4\varepsilon_p\varepsilon_q}\sum_{\sigma_1,\sigma_2=\pm1}\int \frac{(2\pi)^4\delta^4(...)}{R^2+m^2}\frac{\mathrm{d}^4R}{(2\pi)^4} \\ &= \lambda^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{3n_B(\beta\varepsilon_p)n_B(\beta\varepsilon_q)}{4\varepsilon_p\varepsilon_q}\sum_{\sigma_1,\sigma_2=\pm1}[\dots] \end{aligned} \tag{6.144}$$

with

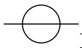
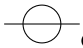
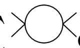
$$\begin{aligned} [\dots] &= \left(\frac{1}{(\mathtt{i}\sigma_1\varepsilon_p + \mathtt{i}\sigma_2\varepsilon_q + \omega_{n_k})^2 + (\vec{p}+\vec{q}+\vec{s})^2 + m^2}\right) \\ \delta^4(\dots) &= \delta^4(P+Q+R+S) \\ P &= [\mathtt{i}\sigma_1\varepsilon_p, \vec{p}], \quad Q = [\mathtt{i}\sigma_2\varepsilon_q, \vec{q}], \quad S = [\omega_{n_k}, \vec{s}] \end{aligned} \tag{6.145}$$

(However, when $\omega_{n_k} = 0$ and $s \neq 0$, integral I_3 is not analytical [13] for some external momenta s.)
We use $\{\}$, to denote the order of coupling factor. So

$\{\lambda^2\}$  $_{\mathrm{ITF}} = \{\lambda^2\}$  $_{\mathrm{QFT}}$

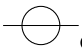
$$+ \int \frac{d^3p}{(2\pi)^3} \frac{3n_B(\beta\epsilon_p)}{2\epsilon_p} \sum_{\sigma_1=\pm1} \{\lambda^2\}$$  $$_{\mathrm{QFT}}\left((P+S)^2\right) + \mathrm{I}_3 \tag{6.146}$$

Now by taking the corresponding result from [10] and Eq. (6.59), when $\lambda \to g\mu^\epsilon$, we can write

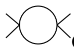
 $_{\mathrm{ITF}} = \underbrace{\text{[diagram]}_{\mathrm{QFT},K_0=\omega_{n_k}}}_{\mathrm{I}^{(2,2)}_{\mathrm{QFT}}}$  $+ \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{3n_B(\beta\epsilon_p)}{2\epsilon_p} \sum_{\sigma_1} \text{[diagram]}_{\mathrm{QFT}}(P+S)^2}_{\mathrm{I}_2}$ 

$$+ \mathrm{I}_3 \tag{6.147}$$

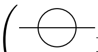
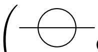
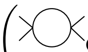
where

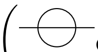
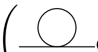
 $$_{\mathrm{QFT},k_0=\omega_{n_k}} = -g^2 \frac{m^2}{(4\pi)^4}\left(\frac{6}{\epsilon^2} + \frac{S^2}{2m^2\epsilon}\right) - g^2 \frac{m^2}{(4\pi)^4}\frac{6}{\epsilon}\left[\frac{3}{2} + \psi(1) + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon) \tag{6.148}$$

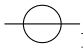
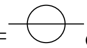
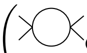
with $S^2 = \omega_{n_k}^2 + s^2$
Similarly

$\{g^2\mu^\epsilon\}\sum_{\sigma_1}$ 
$$_{\mathrm{QFT}}(P+S) = \frac{g^2\mu^\epsilon}{(4\pi)^2} \sum_{\sigma=\pm1} \left(\int_0^1 dx \log\left[\frac{4\pi\mu^2}{[(\mathtt{i}\sigma\epsilon_p + \omega_{n_k})^2 + (p+s)^2]x(1-x) + m^2}\right]\right) + \frac{2g^2\mu^\epsilon}{(4\pi)^2}\left(\frac{2}{\epsilon} + \psi(1)\right) + \mathcal{O}(\epsilon) \tag{6.149}$$

Now combining the result we can write it as in the case of pole term. i.e.,

$\mathcal{K}\Big($  $_{\mathrm{ITF}}\Big) = \mathcal{K}\Big($  $_{\mathrm{QFT},k_0=\omega_{n_k}}\Big) + 3S_1(m,T)\,\mathcal{K}\Big($  $_{\mathrm{QFT}}\Big)$

$= \mathcal{K}\Big($  $_{\mathrm{QFT},k_0=\omega_{n_k}}\Big) + 3S_1(m,T)\,\dfrac{\partial}{\partial m^2}\mathcal{K}\Big($  $_{\mathrm{QFT}}\Big)$ (6.150) when $\omega_n k = 0$ and $s \neq 0$ integral I_3 is not analytical for some external momenta s [13]. Similarly when external momentum $S = 0$ the whole result can be written as

 $_{\text{ITF},S=0} =$  $_{\text{QFT},S=0} + \underbrace{3S_1(m,T)\ \mathcal{K}\Big(}_{}$  $\underbrace{_{\text{QFT}}\Big)}_{\text{I}_2}$

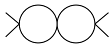
$$+ \underbrace{3S_1(m,T)\frac{g^2\mu^\epsilon}{(4\pi)^2}\left(\psi(1) - \int_0^1 \log\left(1 - x + x^2\right)dx + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right)}_{\text{I}_2}$$

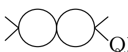
$$+ \underbrace{\frac{3g^2m^2}{32\pi^4}\int_0^\infty\int_0^\infty U(x)U(y)G(x,y)\ dx\ dy}_{\text{I}_3} \tag{6.151}$$

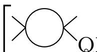
with

$$\begin{aligned} U(x) &= \frac{\sinh(x)}{\exp\left(\beta m \cosh(x)\right) - 1} \\ G(x,y) &= \ln\left(\frac{1 + 2\cosh(x-y)}{1 + 2\cosh(x+y)}\frac{1 - 2\cosh(x+y)}{1 - 2\cosh(x-y)}\right) \\ \int_0^1 \log\left(1 - x + x^2\right)dx &= \frac{\sqrt{3}\pi}{3} - 2 \end{aligned} \tag{6.152}$$

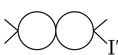
6.3.5 Four-point function at two loop order: Diagram 1

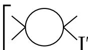
One of the two loop order four-point function is  . The expression of the diagram in non-thermal QFT in terms of integral and diagrammatic representation [10] is

 $$_{\text{QFT}} = -\left(\frac{1}{\lambda}\right)\left[\lambda^2\int\frac{\mathrm{d}^4P}{(2\pi)^4}\left(\frac{1}{(P-K)^2 + m^2}\right)\left(\frac{1}{P^2 + m^2}\right)\right]^2$$

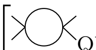
$$= -\left(\frac{1}{\lambda}\right)\Big[$$  $$_{\text{QFT}}\Big]^2 \tag{6.153}$$

The corresponding diagram's thermal form is

 $$_{\text{ITF}} = -\left(\frac{1}{\lambda}\right)\left[\lambda^2 T\sum_{n_p=-\infty}^{\infty}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\frac{1}{\epsilon_{p-k}^2 + \omega_{n_p-n_k}^2}\right)\left(\frac{1}{\epsilon_p^2 + \omega_{n_p}^2}\right)\right]^2$$

$$= -\left(\frac{1}{\lambda}\right)\Big[$$  $$_{\text{ITF}}\Big]^2$$

Using the results from Section 6.3.2,

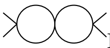
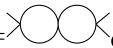
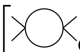
$$= -\left(\frac{1}{\lambda}\right)\Big[$$  $$_{\text{QFT}} + \lambda^2 W(r, n_r)\Big]^2$$

(6.154)

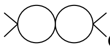
with

$$W(r,n_r)=\sum_{\sigma,\sigma_1=\pm1}\int\frac{n_B(\beta\epsilon_p)}{2\epsilon_p\epsilon_{p+r}}\left(\frac{1}{\sigma_1\epsilon_p+\epsilon_{p+r}+\mathrm{i}\sigma\omega_{n_r}}\right)\frac{\mathrm{d}^3p}{(2\pi)^3} \tag{6.155}$$

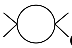
Then

 $_{\mathrm{ITF}} =$  $_{\mathrm{QFT},k_0=\omega_{n_r}} - 2gW(r,n_r)\Big[$  $_{\mathrm{QFT},k_0=\omega_{n_r}}\Big]$
$- g^3W^2(r,n_r)$ (6.156)

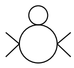
If we take the results from [10], we can write

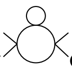
 $_{\mathrm{QFT},k_0=\omega_{n_r}} = -g\mu^\epsilon\frac{g^2}{(4\pi)^4}\left(\frac{4}{\epsilon^2}+\frac{4}{\epsilon}\psi(1)\right)$
$- g\mu^\epsilon\frac{g^2}{(4\pi)^4}\frac{4}{\epsilon}\int_0^1 dx\log\left[\frac{4\pi\mu^2}{K^2x(1-x)+m^2}\right]+\mathcal{O}(\epsilon^0)$ (6.157)

and $K^2=\omega_{n_r}^2+k^2$.
From Section 6.3.2,

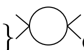
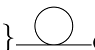
 $_{\mathrm{QFT}} = \frac{g^2\mu^\epsilon}{(4\pi)^2}\left(\frac{2}{\epsilon}+\psi(1)+\int_0^1 dx\log\left[\frac{4\pi\mu^2}{K^2x(1-x)+m^2}\right]+\mathcal{O}(\epsilon)\right)$ (6.158)

6.3.6 Four-point function at two loop order: Diagram 2

The next four-point, two loop order diagram  in non-thermal ϕ^4 theory is

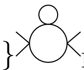
$\{-\lambda^3\}$  $_{\mathrm{QFT}} = -\lambda^3\int\frac{\mathrm{d}^4P}{(2\pi)^4}\left(\frac{1}{(P-K)^2+m^2}\right)\left(\frac{1}{(P^2+m^2)^2}\right)\int\frac{\mathrm{d}^4Q}{(2\pi)^3}\left(\frac{1}{Q^2+m^2}\right)$

$$=\left[-\left(\frac{1}{2}\right)\frac{\partial}{\partial m^2}\int\frac{\mathrm{d}^4P}{(2\pi)^4}\left(\frac{1}{(P-K)^2+m^2}\right)\frac{\lambda^2}{P^2+m^2}\right]\left[-\lambda\int\frac{\mathrm{d}^3Q}{(2\pi)^4}\left(\frac{1}{Q^2+m^2}\right)\right]$$

$=\left[-\frac{1}{2}\frac{\partial}{\partial m^2}\Big(\{\lambda^2\}\right.$  $\left._{\mathrm{QFT}}\Big)\right]\Big[\{-\lambda\}$  $_{\mathrm{QFT}}\Big]$

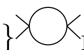
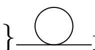
(6.159)

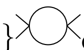
Similar diagram in ITF can be written as

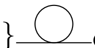
$\{-\lambda^3\}$  $_{\text{ITF}}$

$$= -\lambda^3 T^2 \sum_{n_p=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{\epsilon_{p-k}^2 + \omega_{n_p-n_k}^2} \right) \left(\frac{1}{(\epsilon_p^2 + \omega_{n_p}^2)^2} \right) \times \sum_{n_q=-\infty}^{\infty} \int \frac{d^3q}{(2\pi)^3} \left(\frac{1}{\epsilon_q^2 + \omega_{n_q}^2} \right)$$

$$= \left[-\left(\frac{1}{2}\right) \frac{\partial}{\partial m^2} \sum_{n_p=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{\epsilon_{p-k}^2 + \omega_{n_p-n_k}^2} \right) \frac{\lambda^2 T}{\epsilon_p^2 + \omega_{n_p}^2} \right] \times -\lambda T \sum_{n_q=-\infty}^{\infty} \int \frac{d^3q}{(2\pi)^3} \left(\frac{1}{\epsilon_q^2 + \omega_{n_q}^2} \right)$$

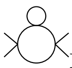
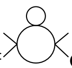
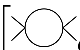
$= \left[-\left(\frac{1}{2}\right) \frac{\partial}{\partial m^2} \left(\{\lambda^2\} \right.\right.$  $\left.\left._{\text{ITF}} \right) \right] \left[\{-\lambda\} \right.$  $\left._{\text{ITF}} \right]$

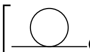
$= \left[-\left(\frac{1}{2}\right) \frac{\partial}{\partial m^2} \left(\{\lambda^2\} \right.\right.$  $\left.\left._{\text{QFT}} + \lambda^2 W(k, n_k) \right) \right]$

$\times \left[\{-\lambda\} \right.$  $\left._{\text{QFT}} - \lambda S_1(m, T) \right]$

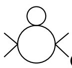
(6.160)

On solving, we get

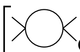
 $_{\text{ITF}} =$  $_{\text{QFT}, k_0 = \omega_{n_k}} + \frac{g S_1(m,T)}{2} \frac{\partial}{\partial m^2} \left[\right.$  $\left._{QFT} \right]$

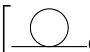
$- \frac{g^2}{2} \frac{\partial\ W(k, n_k)}{\partial m^2} \left[\right.$  $\left._{\text{QFT}} \right] + g^3 \frac{S_1(m,T)}{2} \frac{\partial}{\partial m^2} W(k, n_k)$ (6.161)

Using [10] and the results from Sections 6.3.1 and 6.3.2, we can write

 $_{\text{QFT}, k_0 = \omega_{n_k}} = -g\mu^\epsilon \frac{g^2}{(4\pi)^4} \frac{2}{\epsilon} \left[\int_0^1 dx \frac{m^2(1-x)}{(K^2 x(1-x) + m^2)} + \mathcal{O}(\epsilon) \right]$ (6.162)

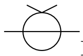
and $K^2 = \omega_{n_k}^2 + k^2$,

$\frac{\partial}{\partial m^2} \left[\right.$  $\left._{\text{QFT}} \right] = -\frac{g^2 \mu^\epsilon}{(4\pi)^2} \int_0^1 \left(\frac{1}{(K^2 x(1-x) + m^2)^2} \right) dx$ (6.163)

$\left[\right.$  $\left._{\text{QFT}} \right] = \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln\left(\frac{4\pi\mu^2}{m^2} \right) \right] + \mathcal{O}(\epsilon)$ (6.164)

6.3.7 Four-point function at two loop order: Diagram 3

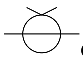
The next diagram on the list is



$$
{}_{\mathrm{ITF}} = \sum_{n,\theta=-\infty}^{\infty} \int \left(\frac{-\lambda^3}{\omega_n^2+\varepsilon_p^2}\right)\left(\frac{T^2}{\omega_\theta^2+\varepsilon_q^2}\right)\left(\frac{1}{\omega_{n-\alpha}^2+\varepsilon_r^2}\right)\left(\frac{(2\pi)^6\delta^6}{\omega_{n-\theta+\eta}^2+\varepsilon_s^2}\right)\prod_{\Omega=p,q,r,s}\left[\frac{\mathrm{d}^3\Omega}{(2\pi)^3}\right] \tag{6.165}
$$

where $\delta^6 = \delta^3(\vec{r}+\vec{p}-\vec{k}_1-\vec{k}_2)\ \delta^3(\vec{s}+\vec{q}-\vec{p}-\vec{k}_3)$, $\sum_{n,\theta=-\infty}^{\infty} = \sum_{n=-\infty}^{\infty}\sum_{\theta=-\infty}^{\infty}$ and $\prod_{\Omega=p,q,r,s}\left[\frac{\mathrm{d}^3\Omega}{(2\pi)^3}\right] = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3r}{(2\pi)^3}\frac{\mathrm{d}^3s}{(2\pi)^3}$.

The corresponding expression in non-thermal QFT is



$$
{}_{\mathrm{QFT}} = \int \left(\frac{1}{P^2+m^2}\right)\left(\frac{1}{Q^2+m^2}\right)\left(\frac{(2\pi)^8}{R^2+m^2}\right)\left(\frac{\delta^8(\dots)}{S^2+m^2}\right)\prod_{\Omega=P,Q,R,S}\left[\frac{\mathrm{d}^4\Omega}{(2\pi)^4}\right] \tag{6.166}
$$

with $\delta^8(\dots) = \delta^4(R+P-K_1-K_2)\delta^4(S+Q-P-K_3)$.
As we follow the summation method using the idea of residue, we get

$$
\begin{aligned}
&T^2\sum_{n=-\infty}^{\infty}\sum_{\theta=-\infty}^{\infty}\left(\frac{1}{\omega_n^2+\varepsilon_p^2}\right)\left(\frac{1}{\omega_\theta^2+\varepsilon_q^2}\right)\left(\frac{1}{\omega_{n-\alpha}^2+\varepsilon_r^2}\right)\left(\frac{1}{\omega_{n-\theta+\eta}^2+\varepsilon_s^2}\right)\\
&= T^2\sum_{n=-\infty}^{\infty}\sum_{\theta=-\infty}^{\infty}\left(\frac{1}{\omega_n^2+\varepsilon_p^2}\right)\left(\frac{1}{\omega_\theta^2+\varepsilon_s^2}\right)\left(\frac{1}{\omega_{n-\alpha}^2+\varepsilon_r^2}\right)\left(\frac{1}{\omega_{n-\theta+\eta}^2+\varepsilon_q^2}\right)\\
&= \frac{T_{s2}^1+T_{s2}^2+T_{s2}^3}{16\varepsilon_p\varepsilon_r\varepsilon_q\varepsilon_s}
\end{aligned} \tag{6.167}
$$

$$T_{s2}^1 = \sum_{\sigma=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_p - i\sigma\omega_\alpha} \right) \left(\frac{1}{\varepsilon_p + \varepsilon_q + \varepsilon_s + i\sigma\omega_\eta} \right)$$

$$+ \sum_{\sigma=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_p + i\sigma\omega_\alpha} \right) \left(\frac{1}{\varepsilon_r + \varepsilon_q + \varepsilon_s + i\sigma\omega_{\eta+\alpha}} \right)$$

$$+ \sum_{\sigma=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_q + \varepsilon_s + i\sigma\omega_{\alpha+\eta}} \right) \left(\frac{1}{\varepsilon_p + \varepsilon_q + \varepsilon_s + i\sigma\omega_\eta} \right)$$

$$T_{s2}^2 = t_{21} + t_{22} + t_{23} + t_{24} + t_{25} + t_{26} + t_{27} + t_{28}$$

$$T_{s2}{}^3 = t_{31} + t_{32} + t_{33} + t_{34} + t_{35}$$

$$t_{21} = \sum_{\sigma,\sigma_1,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_r + \sigma_1(\varepsilon_p - i\sigma\omega_\alpha)} \right) \left(\frac{1}{\varepsilon_q + \varepsilon_s + \sigma_2(\varepsilon_p + i\sigma\omega_\eta)} \right) n_B(\beta\varepsilon_p)$$

$$t_{22} = \sum_{\sigma,\sigma_1,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_p + \sigma_1(\varepsilon_r + i\sigma\omega_\alpha)} \right) \left(\frac{1}{\varepsilon_q + \varepsilon_s + \sigma_2(\varepsilon_r + i\sigma\omega_{\eta+\alpha})} \right) n_B(\beta\varepsilon_r)$$

$$t_{23} = \sum_{\sigma,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_s + \varepsilon_r + \sigma_2\varepsilon_q + i\sigma(\omega_{\eta+\alpha})} \right) \left(\frac{1}{\varepsilon_p + \varepsilon_s + \sigma_2\varepsilon_q + i\sigma\omega_\eta} \right) n_B(\beta\varepsilon_q)$$

$$t_{24} = \sum_{\sigma,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_p + i\sigma\omega_\alpha} \right) \left(\frac{1}{\varepsilon_p + \varepsilon_s + \sigma_2\varepsilon_q - i\sigma\omega_\eta} \right) n_B(\beta\varepsilon_q)$$

$$t_{25} = \sum_{\sigma,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_p + i\sigma\omega_\alpha} \right) \left(\frac{1}{\varepsilon_r + \varepsilon_s + \sigma_2\varepsilon_q + i\sigma\omega_{\eta+\alpha}} \right) n_B(\beta\varepsilon_q)$$

$$t_{26} = \sum_{\sigma,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_q + \varepsilon_r + \sigma_2\varepsilon_s + i\sigma\omega_{\eta+\alpha}} \right) \left(\frac{1}{\varepsilon_p + \varepsilon_q + \sigma_2\varepsilon_s + i\sigma\omega_\eta} \right) n_B(\beta\varepsilon_s)$$

$$t_{27} = \sum_{\sigma,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_p + i\sigma\omega_\alpha} \right) \left(\frac{1}{\varepsilon_p + \varepsilon_q + \sigma_2\varepsilon_s - i\sigma\omega_\eta} \right) n_B(\beta\varepsilon_s)$$

$$t_{28} = \sum_{\sigma,\sigma_2=\pm1} \left(\frac{1}{\varepsilon_r + \varepsilon_p + i\sigma\omega_\alpha} \right) \left(\frac{1}{\varepsilon_r + \varepsilon_q + \sigma_2\varepsilon_s + i\sigma\omega_{\eta+\alpha}} \right) n_B(\beta\varepsilon_s)$$

$$t_{31} = \sum_{\sigma,\sigma_1,\sigma_2,\sigma_3=\pm1} \left(\frac{n_B(\beta\varepsilon_p)}{\varepsilon_r + \sigma_1(\varepsilon_p - i\sigma\omega_\alpha)} \right) \left(\frac{n_B(\beta\varepsilon_q)}{\varepsilon_s + \sigma_2\varepsilon_q + \sigma_3(\varepsilon_p + i\sigma\omega_\eta)} \right)$$

$$t_{32} = \sum_{\sigma,\sigma_1,\sigma_2,\sigma_3=\pm1} \left(\frac{n_B(\beta\varepsilon_p)}{\varepsilon_r + \sigma_1(\varepsilon_p - i\sigma\omega_\alpha)} \right) \left(\frac{n_B(\beta\varepsilon_s)}{\varepsilon_q + \sigma_2\varepsilon_s + \sigma_3(\varepsilon_p + i\sigma\omega_\eta)} \right)$$

$$t_{33} = \sum_{\sigma,\sigma_1,\sigma_2,\sigma_3=\pm1} \left(\frac{n_B(\beta\varepsilon_r)}{\varepsilon_p + \sigma_1(\varepsilon_r + i\sigma\omega_\alpha)} \right) \left(\frac{n_B(\beta\varepsilon_s)}{\varepsilon_q + \sigma_2\varepsilon_s + \sigma_3(\varepsilon_r + i\sigma\omega_{\eta+\alpha})} \right)$$

$$t_{34} = \sum_{\sigma,\sigma_1,\sigma_2,\sigma_3=\pm1} \left(\frac{n_B(\beta\varepsilon_r)}{\varepsilon_p + \sigma_1(\varepsilon_r - i\sigma\omega_\alpha)} \right) \left(\frac{n_B(\beta\varepsilon_q)}{\varepsilon_s + \sigma_2\varepsilon_q + \sigma_3(\varepsilon_r - i\sigma\omega_{\eta+\alpha})} \right)$$

$$t_{35} = \sum_{\sigma,\sigma_1,\sigma_2,\sigma_3=\pm1} \left(\frac{n_B(\beta\varepsilon_s)}{\varepsilon_p + \sigma_2\varepsilon_s + \sigma_1\varepsilon_q + i\sigma\omega_\eta} \right) \left(\frac{n_B(\beta\varepsilon_q)}{\varepsilon_r + \sigma_3(\sigma_2\varepsilon_s + \sigma_1\varepsilon_q + i\sigma\omega_{\eta+\alpha})} \right)$$

with

$$n_B(x) = (e^x - 1)^{-1}$$

$$\beta = 1/T$$

One can rewrite the summation as integral as shown below

$$\frac{t_{11}}{16\varepsilon_p\varepsilon_q\varepsilon_q\varepsilon_r} = \left(\frac{1}{4\pi^2}\right)\int\left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{1}{r_0^2+\varepsilon_r^2}\right)\left(\frac{1}{q_0^2+\varepsilon_q^2}\right)\left(\frac{\delta^2}{s_0^2+\varepsilon_s^2}\right)\prod_{\Omega=p_0,r_0,q_0,s_0}\mathrm{d}\Omega \tag{6.168}$$

with $\delta^2 = \delta(r_0+p_0-\omega_\alpha)\,\delta(s_0+q_0-p_0-\omega_\eta)$ and $\prod_{\Omega=p_0,r_0,q_0,s_0}\mathrm{d}\Omega = \mathrm{d}p_0\;\mathrm{d}r_0\;\mathrm{d}q_0\;\mathrm{d}s_0$.

$$\frac{t_{21}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma=\pm1}\frac{n_B(\beta\varepsilon_p)}{4\pi\varepsilon_p}\int\left(\frac{1}{r_0^2+\varepsilon_r^2}\right)\left(\frac{1}{q_0^2+\varepsilon_q^2}\right)\left(\frac{\delta^2}{s_0^2+\varepsilon_s^2}\right)\,dr_0\;dq_0\;ds_0|_{p_0=-i\sigma\varepsilon_p} \tag{6.169}$$

$$\frac{t_{22}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma=\pm1}\frac{n_B(\beta\varepsilon_r)}{4\pi\varepsilon_r}\int\left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{1}{q_0^2+\varepsilon_q^2}\right)\left(\frac{\delta^2}{s_0^2+\varepsilon_s^2}\right)\,\mathrm{d}p_0\;\mathrm{d}q_0\;\mathrm{d}s_0\Big|_{r_0=-i\sigma\varepsilon_r} \tag{6.170}$$

$$\frac{t_{23}+t_{24}+t_{25}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma=\pm1}\frac{n_B(\beta\varepsilon_q)}{4\pi\varepsilon_q}\int\left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{1}{r_0^2+\varepsilon_r^2}\right)\left(\frac{\delta^2}{s_0^2+\varepsilon_s^2}\right)\,\mathrm{d}p_0\;\mathrm{d}r_0\;\mathrm{d}s_0\Big|_{q_0=i\sigma\varepsilon_q} \tag{6.171}$$

$$\frac{t_{26}+t_{27}+t_{28}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma=\pm1}\frac{n_B(\beta\varepsilon_s)}{4\pi\varepsilon_s}\int\left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{1}{r_0^2+\varepsilon_r^2}\right)\left(\frac{\delta^2}{q_0^2+\varepsilon_q^2}\right)\,\mathrm{d}p_0\;\mathrm{d}r_0\;\mathrm{d}q_0\Big|_{s_0=i\sigma\varepsilon_s} \tag{6.172}$$

$$\frac{t_{31}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma_1,\sigma_3=\pm1}\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p}\frac{n_B(\beta\varepsilon_q)}{2\varepsilon_q}\int\left(\frac{1}{r_0^2+\varepsilon_r^2}\right)\left(\frac{\delta^2}{s_0^2+\varepsilon_s^2}\right)\,\mathrm{d}r_0\;\mathrm{d}s_0\Big|_{\substack{p_0=-i\sigma_3\varepsilon_p,\\ q_0=-i\sigma_1\varepsilon_q}} \tag{6.173}$$

$$\frac{t_{32}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma_1,\sigma_3=\pm1}\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p}\frac{n_B(\beta\varepsilon_s)}{2\varepsilon_s}\int\left(\frac{1}{r_0^2+\varepsilon_r^2}\right)\left(\frac{\delta^2}{q_0^2+\varepsilon_q^2}\right)\,\mathrm{d}q_0\;\mathrm{d}r_0\Big|_{\substack{p_0=-i\sigma_3\varepsilon_p,\\ s_0=-i\sigma_1\varepsilon_s}} \tag{6.174}$$

$$\frac{t_{33}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma_1,\sigma_3=\pm1} \frac{n_B(\beta\varepsilon_r)}{2\varepsilon_p}\frac{n_B(\beta\varepsilon_s)}{2\varepsilon_s} \int \left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{\delta^2}{q_0^2+\varepsilon_q^2}\right) \, \mathrm{d}q_0 \, \mathrm{d}p_0\Big|_{\substack{r_0=-i\sigma_3\varepsilon_r,\\ s_0=-i\sigma_1\varepsilon_s}} \tag{6.175}$$

$$\frac{t_{34}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma_1,\sigma_3=\pm1} \frac{n_B(\beta\varepsilon_r)}{2\varepsilon_r}\frac{n_B(\beta\varepsilon_q)}{2\varepsilon_q} \int \left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{\delta^2}{s_0^2+\varepsilon_s^2}\right) \, \mathrm{d}s_0 \, \mathrm{d}p_0\Big|_{\substack{r_0=-i\sigma_3\varepsilon_r,\\ q_0=-i\sigma_1\varepsilon_q}} \tag{6.176}$$

$$\frac{t_{35}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s} = \sum_{\sigma_1,\sigma_3=\pm1} \frac{n_B(\beta\varepsilon_s)}{2\varepsilon_s}\frac{n_B(\beta\varepsilon_q)}{2\varepsilon_q} \int \left(\frac{1}{p_0^2+\varepsilon_p^2}\right)\left(\frac{\delta^2}{r_0^2+\varepsilon_r^2}\right) \, \mathrm{d}r_0 \, \mathrm{d}p_0\Big|_{\substack{s_0=-i\sigma_3\varepsilon_s,\\ q_0=-i\sigma_1\varepsilon_q}} \tag{6.177}$$

The summation result can be connected with the integral equations as shown below.

$$\mathrm{I}_{\rho\kappa} = \int \frac{t_{\rho\kappa}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s}(2\pi)^6\delta^6 \frac{\mathrm{d}^3r}{(2\pi)^3}\frac{\mathrm{d}^3s}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3} \tag{6.178}$$

By combining Eqs. (6.169) and (6.178) by omitting λ for a moment, one can express

$$\begin{aligned}\mathrm{I}_{21} &= \int \frac{t_{21}}{16\varepsilon_p\varepsilon_q\varepsilon_r\varepsilon_s}(2\pi)^6\delta^6 \frac{\mathrm{d}^3r}{(2\pi)^3}\frac{\mathrm{d}^3s}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p}\left(\sum_{\sigma=\pm1}\int\left(\frac{(2\pi)^8}{r_0^2+\varepsilon_r^2}\right)\left(\frac{\delta^2\delta^6}{q_0^2+\varepsilon_q^2}\right)\left(\frac{1}{s_0^2+\varepsilon_s^2}\right)\right)\prod_{\Omega=r,q,s}\left(\frac{\mathrm{d}^3\Omega\mathrm{d}\Omega_0}{(2\pi)^4}\right)\Big|_{p_0=-i\sigma\varepsilon_p}\end{aligned} \tag{6.179}$$

with $\prod_{\Omega=p,q,s}\left(\frac{\mathrm{d}^3\Omega\mathrm{d}\Omega_0}{(2\pi)^4}\right) = \frac{\mathrm{d}^3r}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3s}{(2\pi)^3}\frac{dr_0}{2\pi}\frac{dq_0}{2\pi}\frac{ds_0}{2\pi}$
i.e.,

$$\mathrm{I}_{21} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p}\left(\sum_{\sigma=\pm1}\int\left(\frac{(2\pi)^8}{r_0^2+\varepsilon_r^2}\right)\left(\frac{\delta^6}{s_0^2+\varepsilon_s^2}\right)\left(\frac{\delta^2|_{p_0=-i\sigma\varepsilon_p}}{q_0^2+\varepsilon_q^2}\right)\right)\prod_{\Omega=R,Q,S}\left[\frac{\mathrm{d}^4\Omega}{(2\pi)^4}\right] \tag{6.180}$$

with $\delta^6\delta^2 = \delta^8 = \delta^4(R + P - K_1 - K_2)\ \delta^4(S + Q - P - K_3)$

$$\begin{aligned} R &= [r_0\ ,\ \vec{r}] \\ P &= [p_0\ ,\ \vec{p}] \\ K_1 + K_2 &= \left[\omega_\alpha\ ,\ \vec{k}_1 + \vec{k}_2\right] \\ K_3 &= \left[\omega_\eta\ ,\ \vec{k}_3\right] \end{aligned} \tag{6.181}$$

Now Eq. (6.180) becomes

$$\mathrm{I}_{21} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p} \left(\sum_{\sigma=\pm 1} \int \frac{(2\pi)^8 \delta^8|_{p_0=-i\sigma\varepsilon_p}}{(R^2+m^2)(Q^2+m^2)(S^2+m^2)} \frac{\mathrm{d}^4R}{(2\pi)^4} \frac{\mathrm{d}^4Q}{(2\pi)^4} \frac{\mathrm{d}^4S}{(2\pi)^4} \right) \tag{6.182}$$

If we define

$$\begin{aligned} \mathrm{J}(K^2) &= \int \left(\frac{1}{P^2+m^2}\right) \left(\frac{1}{(P-K)^2+m^2}\right) \frac{\mathrm{d}^4P}{(2\pi)^4} \\ &= \int \left(\frac{1}{P^2+m^2}\right) \left(\frac{1}{(P+K)^2+m^2}\right) \frac{\mathrm{d}^4P}{(2\pi)^4} \end{aligned} \tag{6.183}$$

then integrating the variables R and Q will give

$$\begin{aligned} \mathrm{I}_{21} &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p} \sum_{\sigma=\pm 1} \mathrm{I}_{21}^{(1)}(P,K)\Big|_{p_0=-i\sigma\varepsilon_p} \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p} \sum_{\sigma=\pm 1} \left[\frac{\mathrm{J}\left[(P+K_3)^2\right]}{(P-K_1-K_2)^2+m^2} \right]_{p_0=-i\sigma\varepsilon_p} \end{aligned} \tag{6.184}$$

with

$$\mathrm{I}_{21}^{(1)}(P,K) = \left(\frac{1}{(P-K_1-K_2)^2+m^2}\right) \int \left[\left(\frac{1}{(S-P-K_3)^2+m^2}\right) \left(\frac{1}{S^2+m^2}\right) \right] \frac{\mathrm{d}^4S}{(2\pi)^4} \tag{6.185}$$

Similarly, combining Eqs. (6.170) and (6.178), one can express

$$\begin{aligned} \mathrm{I}_{22} &= \int \frac{\mathrm{d}^3r}{(2\pi)^3} \frac{n_B(\beta\epsilon_r)}{2\epsilon_r} \sum_{\sigma=\pm 1} \mathrm{I}_{22}^{(1)}(R,K)\Big|_{r_0=-i\sigma\epsilon_r} \\ &= \int \frac{\mathrm{d}^3r}{(2\pi)^3} \frac{n_B(\beta\epsilon_r)}{2\epsilon_r} \sum_{\sigma=\pm 1} \left[\frac{\mathrm{J}\left[(K_1+K_2+K_3-R)^2\right]}{(R-K_1-K_2)^2+m^2} \right] \end{aligned} \tag{6.186}$$

with

$$\mathrm{I}_{22}^{(1)} = \left(\frac{1}{(R-K_1-K_2)^2+m^2}\right) \int \left[\left(\frac{1}{(S+R-K_1-K_2-K_3)^2+m^2}\right) \left(\frac{1}{S^2+m^2}\right) \right] \frac{\mathrm{d}^4S}{(2\pi)^4}$$

$$ \tag{6.187} $$

Combining Eqs. (6.171) and (6.178), we get

$$ \mathrm{I}_{23}+\mathrm{I}_{24}+\mathrm{I}_{25}=\int\frac{d^3q}{(2\pi)^3}\frac{n_B(\beta\epsilon_q)}{2\epsilon_q}\sum_{\sigma=\pm1}\mathrm{I}_2^{(2)}(R,K)\Big|_{q_0=i\sigma\epsilon_q} \tag{6.188} $$

with

$$ \mathrm{I}_2^{(2)}=\int\left[\left(\frac{1}{(R-K_1-K_2)^2+m^2}\right)\left(\frac{1}{R^2+m^2}\right)\left(\frac{1}{(R+Q-K_1-K_2-K_3)^2+m^2}\right)\frac{d^4R}{(2\pi)^4}\right] \tag{6.189} $$

Similarly, combining Eqs. (6.172) and (6.178),

$$ \mathrm{I}_{26}+\mathrm{I}_{27}+\mathrm{I}_{28}=\int\frac{d^3s}{(2\pi)^3}\frac{n_B(\beta\epsilon_s)}{2\epsilon_s}\sum_{\sigma=\pm1}\mathrm{I}_2^{(3)}\Big|_{s_0=i\sigma\epsilon_s} \tag{6.190} $$

with

$$ \mathrm{I}_2^{(3)}=\int\left(\frac{1}{(R-K_1-K_2)^2+m^2}\right)\left(\frac{1}{R^2+m^2}\right)\left(\frac{1}{(R+S-K_1-K_2-K_3)^2+m^2}\right)\frac{d^4R}{(2\pi)^4} \tag{6.191} $$

Combining Eqs. (6.173) and (6.178),

$$ \mathrm{I}_{31}=\int\frac{d^3p}{(2\pi)^3}\frac{n_B(\beta\epsilon_p)}{2\epsilon_p}\frac{d^3q}{(2\pi)^3}\frac{n_B(\beta\epsilon_q)}{2\epsilon_q}\sum_{\sigma_1,\sigma_3=\pm1}\mathrm{I}_{31}^{(1)}(P,Q,K)\Big|_{\substack{p_0=i\sigma_1\varepsilon_p,\\ q_0=i\sigma_3\varepsilon_q}} \tag{6.192} $$

with

$$ \mathrm{I}_{31}^{(1)}(P,Q,K)=\left[\left(\frac{1}{(P-K_1-K_2)^2+m^2}\right)\left(\frac{1}{(Q-P-K_3)^2+m^2}\right)\right] \tag{6.193} $$

Also, Eqs. (6.174) and (6.178), give

$$ \mathrm{I}_{32}=\int\frac{d^3p}{(2\pi)^3}\frac{n_B(\beta\epsilon_p)}{2\epsilon_p}\frac{d^3s}{(2\pi)^3}\frac{n_B(\beta\epsilon_s)}{2\epsilon_s}\sum_{\sigma_1,\sigma_3=\pm1}\mathrm{I}_{31}^{(1)}(P,S,K)\Big|_{\substack{s_0=i\sigma_1\varepsilon_s,\\ p_0=i\sigma_3\varepsilon_p}} \tag{6.194} $$

Eqs. (6.175) and (6.178), give

$$ \mathrm{I}_{33}=\int\frac{d^3s}{(2\pi)^3}\frac{n_B(\beta\epsilon_s)}{2\epsilon_p}\frac{d^3r}{(2\pi)^3}\frac{n_B(\beta\epsilon_r)}{2\epsilon_s}\sum_{\sigma_1,\sigma_3=\pm1}\mathrm{I}_{33}{}^{(1)}(S,R,K)\Big|_{\substack{s_0=i\sigma_1\epsilon_s,\\ r_0=i\sigma_3\epsilon_r}} \tag{6.195} $$

with

$$ \mathrm{I}_{33}{}^{(1)}(S,R,K)=\left(\frac{1}{(R-K_1-K_2)^2+m^2}\right)\left(\frac{1}{(S+R-K_1-K_2-K_3)^2+m^2}\right) $$

(6.196)

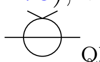
Eqs. (6.176) and (6.178) leads to

$$\mathrm{I}_{34} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{n_B(\beta\epsilon_r)}{2\epsilon_r}\frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{n_B(\beta\epsilon_q)}{2\epsilon_q}\sum_{\sigma_1,\sigma_3=\pm1}\mathrm{I}_{33}{}^{(1)}(Q,R,K)\Big|_{\substack{q_0=i\sigma_1\varepsilon_q,\\ r_0=i\sigma_3\varepsilon_r}} \tag{6.197}$$

Using Eqs. (6.177) and (6.178), I_{35} becomes

$$\mathrm{I}_{35} = \int \frac{\mathrm{d}^3 s}{(2\pi)^3}\frac{n_B(\beta\epsilon_s)}{2\epsilon_s}\frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{n_B(\beta\epsilon_q)}{2\epsilon_q}\sum_{\sigma_1,\sigma_3=\pm1}\mathrm{I}_{35}^{(1)}(Q,S,K)_{\substack{q_0=i\sigma_1\epsilon_q,\\ s_0=i\sigma_3\epsilon_s}} \tag{6.198}$$

$$\mathrm{I}_{35}^{(1)}(Q,S,K) = \left(\frac{1}{(S+Q-K_3)^2+m^2}\right)\left(\frac{1}{(Q+S+K_1+K_2-K_3)^2+m^2}\right) \tag{6.199}$$

Combining Eqs. (6.168) and (6.178), one can express the result as the same as a corresponding QFT diagram  $\underbrace{\quad{}_{\mathrm{QFT}}\Big|_{K_i}}_{\mathrm{I}_{11}}$ with

$$\begin{aligned} K_1+K_2 &= \left[\omega_\alpha\ ,\ \vec{k}_1+\vec{k}_2\right] \\ K_3 &= \left[\omega_\eta\ ,\ \vec{k}_3\right] \end{aligned} \tag{6.200}$$

Now if we look at the integral, we can find one thing: the first three terms of the integral $(\mathrm{I}_{11}, \mathrm{I}_{21}, \mathrm{I}_{22})$ diverge, and the rest becomes a finite one. The sum of the rest of the terms $(\mathrm{I}_{23}+\mathrm{I}_{24}+\mathrm{I}_{25}+\cdots+\mathrm{I}_{35})$ can be written as the sum of finite terms $(2\mathrm{I}_{F1}+2\mathrm{I}_{F2}+2\mathrm{I}_{F3}+\mathrm{I}_{F4})$ with

$$\mathrm{I}_{F1} = \int \frac{d^3 q}{(2\pi)^3}\frac{n_B(\beta\varepsilon_q)}{2\varepsilon_q}\sum_{\sigma=\pm1} L(R-K_1-K_2, R, R+Q-K_1-K_2-K_3)\frac{d^4 R}{(2\pi)^4}{}_{q_0=i\sigma\varepsilon_q} \tag{6.201}$$

$$\mathrm{I}_{F2} = \int \frac{d^3 p}{(2\pi)^3}\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p}\frac{d^3 q}{(2\pi)^3}\frac{n_B(\beta\varepsilon_q)}{2\varepsilon_q}\sum_{\sigma_1,\sigma_3=\pm1} G(P-K_1-K_2, Q-P-K_3)\big|^{q_0=i\sigma_3\varepsilon_q}_{p_0=i\sigma_1\varepsilon_p} \tag{6.202}$$

$$\mathrm{I}_{F3} = \int \frac{d^3s}{(2\pi)^3} \frac{d^3r}{(2\pi)^3} \frac{n_B(\beta\varepsilon_s)n_B(\beta\varepsilon_r)}{4\varepsilon_s\varepsilon_r} \sum_{\sigma_1,\sigma_3=\pm1} \mathrm{I}^{(1)}_{F3}(R,K)|^{r_0=i\sigma_3\varepsilon_r}_{s_0=i\sigma_1\varepsilon_s} \tag{6.203}$$

with

$$\mathrm{I}^{(1)}_{F3}(R,K) = G(R-K_1-K_2, S+R-K_1-K_2-K_3) \tag{6.204}$$

$$\mathrm{I}_{F4} = \int \frac{d^3q}{(2\pi)^3} \frac{d^3s}{(2\pi)^3} \frac{n_B(\beta\varepsilon_s)n_B(\beta\varepsilon_q)}{4\varepsilon_s\varepsilon_q} \sum_{\sigma_1,\sigma_3=\pm1} \mathrm{I}^{(1)}_{F4}(S,Q,K)|^{s_0=i\sigma_3\varepsilon_s}_{q_0=i\sigma_1\varepsilon_q} \tag{6.205}$$

with

$$\mathrm{I}^{(1)}_{F4}(S,Q,K) = G(S+Q-K_3, Q+S+K_1+K_2-K_3) \tag{6.206}$$

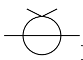
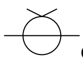
where

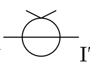
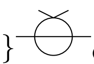
$$L(A,B,C) = \left(\frac{1}{A^2+m^2}\right)\left(\frac{1}{B^2+m^2}\right)\left(\frac{1}{C^2+m^2}\right) \tag{6.207}$$

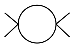
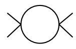
and

$$G(A,B) = \left(\frac{1}{A^2+m^2}\right)\left(\frac{1}{B^2+m^2}\right) \tag{6.208}$$

If we define the pole finding operator $\mathcal{K}$, then by the structure, we can write

 $_{\mathrm{ITF}} = \underbrace{\text{}}$  $_{\mathrm{QFT}}$ $_{\mathrm{I}_{11}}$ $+\mathrm{I}_{21}+\mathrm{I}_{22}+2(\mathrm{I}_{F1}+\mathrm{I}_{F2}+\mathrm{I}_{F3})+\mathrm{I}_{F4}$ (6.209)

$\mathcal{K}\Big[\{-\lambda^3\}$  $_{\mathrm{ITF}}\Big] = \mathcal{K}\Big[\{-\lambda^3\}$  $_{\mathrm{QFT},k_0=\omega_{n_k}}\Big]$

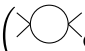
$-\Bigg(\lambda\int\frac{d^3p}{(2\pi)^3}\frac{n_B(\beta\varepsilon_p)}{2\varepsilon_p}\times\mathcal{K}\Bigg[\{\lambda^2\}\sum_{\sigma=\pm1}$  $(P+K_3)+$  $(-P+K_1+K_2+K_3)$ $\Big/\,\big((P-K_1-K_2)^2+m^2\big)\,|_{p_0=-i\sigma\varepsilon_p}\Bigg]\Bigg)$

We rewrite

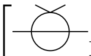
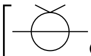
$$\begin{aligned}
\int_{-1}^{1} d\cos\theta \sum_{\sigma=\pm1}\left(\frac{1}{(P-K)^2+m^2}\right) &= \int_{-1}^{1} d\cos\theta \sum_{\sigma=\pm1}\left(\frac{1}{(p-k)^2+(i\sigma\varepsilon_p+\omega_\alpha)^2+m^2}\right) \\
&= \int_{-1}^{1} d\cos\theta\ \frac{2(k^2-2pk\cos\theta+\omega_\alpha^2)}{(k^2-2pk\cos\theta+\omega_\alpha^2)^2+4\varepsilon_p^2\omega_\alpha^2} \\
&= \int_{-1}^{1} d\cos\theta\ 2\frac{(k^2+2pk\cos\theta+\omega_\alpha^2)}{(k^2+2pk\cos\theta+\omega_\alpha^2)^2+4\varepsilon_p^2\omega_\alpha^2}
\end{aligned}$$

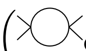
$$W(r, n_r) = \int \frac{d^3p}{(2\pi)^3} \frac{2n_B(\beta\varepsilon_p)}{\varepsilon_p} \frac{r^2 + 2pr\cos\theta + \omega_{n_r}^2}{(r^2 + 2pr\cos\theta + \omega_{n_r}^2)^2 + 4\varepsilon_p^2\omega_{n_r}^2} \tag{6.210}$$

We know that from Eq. (6.65)

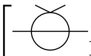
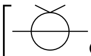
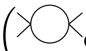
$\mathcal{K}\Big($  $_{\text{QFT}}\Big) = \frac{g^2\mu^\epsilon}{(4\pi)^2}\left(\frac{2}{\epsilon}\right)$ (6.211)

and

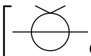
$\mathcal{K}\Big[$  $_{\text{ITF}}\Big] = \mathcal{K}\Big[$  $_{\text{QFT}, k_{0i}=\omega_{n_i}}\Big]$

$- g\mu^\epsilon \left(\int \frac{d^3p}{(2\pi)^3} \frac{2n_B(\beta\varepsilon_p)}{\varepsilon_p} \frac{(k^2 + 2pk\cos\theta + \omega_\alpha^2)}{(k^2 + 2pk\cos\theta + \omega_\alpha^2)^2 + 4\varepsilon_p^2\omega_\alpha^2} \mathcal{K}\Big($  $_{\text{QFT}}\Big)\right)$

So the pole term relation can be written as

$\mathcal{K}\Big[$  $_{\text{ITF}}\Big] = \mathcal{K}\Big[$  $_{\text{QFT}, k_{0i}=\omega_{n_i}}\Big] - g\mu^\epsilon W(k_i, n_{k_i})\mathcal{K}\Big($  $_{\text{QFT}}\Big)$ (6.212)

where

$\mathcal{K}\Big[$  $_{\text{QFT}}\Big] = g\mu^\epsilon \frac{g^2}{(4\pi)^4}\frac{2}{\epsilon^2}\left(1 + \frac{\epsilon}{2} + \epsilon\,\psi(1)\right)$

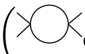
$- g\mu^\epsilon \frac{g^2}{(4\pi)^4}\frac{2}{\epsilon}\int_0^1 dx \ln\left[\frac{(K_1 + K_2)^2 x(1-x) + m^2}{4\pi\mu^2}\right]$ (6.213)

with $K_i = [\omega_{n_{k_i}}, \vec{k}_i]$.

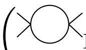
6.4 Counter terms in Thermal ϕ^4 theory

6.4.1 Counter term 1

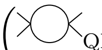
From [10], the counter term for divergence for the four-point function derived is

 $_{\text{QFT}} = -\mu^\epsilon g c_g^1 = -\frac{3}{2}\mathcal{K}\Big($  $_{\text{QFT}}\Big)$ (6.214)

The corresponding diagram in imaginary time formalism is

 $_{\text{ITF}} = -\mu^\epsilon g c_g^1 = -\frac{3}{2}\mathcal{K}\Big($  $_{\text{ITF}}\Big)$ (6.215)

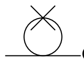
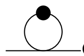
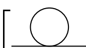
From the tadpole diagram result, one can find that the diverging term is the same for the diagram in imaginary time formalism and non-thermal QFT; thus, we can write

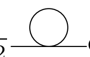
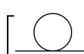
 $_{\mathrm{ITF}} = -\frac{3}{2}\mathcal{K}\Big($  $_{\mathrm{QFT}}\Big)$ (6.216)

$=$  $_{\mathrm{QFT}} = -\mu^{\epsilon} g \frac{3g}{(4\pi)^2}\frac{1}{\epsilon}$

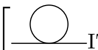
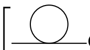
6.4.2 Counter term 2

From [10] Defining $*$ operator the substitution of the counter term $-m^2 c_{m^2}$ or $-\mu^{\epsilon} g c_g$, we can express counter terms as

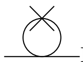
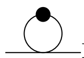
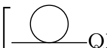
 $_{\mathrm{QFT}} =$  $_{\mathrm{QFT}} * -\frac{1}{2}\mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]$ (6.217)

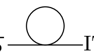
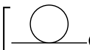
$= -g\mu^{\epsilon}\Big(\frac{-\partial}{\partial m^2}$  $_{\mathrm{QFT}}\Big)\Big(\frac{1}{2g\mu^{\epsilon}}\Big)\Big(\mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]\Big)$

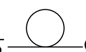
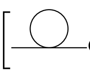
We have from tadpole diagram result the relation

$\mathcal{K}\Big[$  $_{\mathrm{ITF}}\Big] = \mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]$ (6.218)

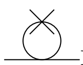
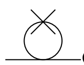
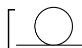
So, for ITF, the corresponding derivation is

 $_{\mathrm{ITF}} =$  $_{\mathrm{ITF}} * -\frac{1}{2}\mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]$ (6.219)

$= -g\mu^{\epsilon}\Big(\frac{-\partial}{\partial m^2}$  $_{\mathrm{ITF}}\Big)\Bigg(\frac{\mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]}{2g\mu^{\epsilon}}\Bigg)$

$= -g\mu^{\epsilon}\Big(\frac{-\partial}{\partial m^2}$  $_{\mathrm{QFT}} - \frac{gS_0(m,T)}{4\pi}\Big)\Big(\frac{1}{2g\mu^{\epsilon}}\Big)\Big(\mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]\Big)$

So

 $_{\mathrm{ITF}} =$  $_{\mathrm{QFT}} + \frac{g}{4\pi}\frac{S_0(m,T)}{2}\mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]$

$$= \frac{2m^2 g^2}{(4\pi)^4}\left[\frac{1}{\epsilon^2} + \frac{\psi(1)}{2\epsilon} - \frac{1}{2\epsilon}\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^0)\right] + \frac{g^2 m^2 S_0(m,T)}{(4\pi)^3}\frac{1}{\epsilon} \tag{6.220}$$

6.4.3 Counter term 3

From [10], the calculation proceeds as

$$
\begin{aligned}
\text{[diagram]}_{\text{QFT}} &= \left(\text{[diagram]}_{\text{QFT}}\right)\left(\frac{-1}{g\mu^{\epsilon}}\right)\left(-\mu^{\epsilon} g c_g^1\right) \\
&= \left(\text{[diagram]}_{\text{QFT}}\right)\left(\frac{-1}{g\mu^{\epsilon}}\right)\left(-\frac{3}{2}\mathcal{K}\left[\text{[diagram]}_{\text{QFT}}\right]\right) \qquad (6.221)
\end{aligned}
$$

the corresponding diagram made with results from Sections 6.3.1 and 6.3.2 is

$$
\begin{aligned}
\text{[diagram]}_{\text{ITF}} &= \text{[diagram]}_{\text{ITF}}\left(\frac{-1}{g\mu^{\epsilon}}\right)\left(-\frac{3}{2}\mathcal{K}\left[\text{[diagram]}_{\text{ITF}}\right]\right) \qquad (6.222) \\
&= \left(\text{[diagram]}_{\text{QFT}} - g\mu^{\epsilon} S_1(m,T)\right)\left(\frac{-1}{g\mu^{\epsilon}}\right)\left(-\frac{3}{2}\mathcal{K}\left[\text{[diagram]}_{\text{ITF}}\right]\right) \\
&= \text{[diagram]}_{\text{QFT}} - \frac{3}{2}S_1(m,T)\mathcal{K}\left(\text{[diagram]}_{\text{ITF}}\right) \\
&= \text{[diagram]}_{\text{QFT}} - \frac{3}{2}S_1(m,T)\mathcal{K}\left(\text{[diagram]}_{\text{QFT}}\right) \\
&= \frac{6m^2g^2}{(4\pi)^4}\left[\frac{1}{\epsilon^2} + \frac{\psi(2)}{2\epsilon} - \frac{1}{2\epsilon}\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^0)\right] \\
&- \frac{3\mu^{\epsilon}g^2}{(4\pi)^2}\frac{S_1(m,T)}{\epsilon}
\end{aligned}
$$

6.4.4 Counter term 4

From [10], the diagram evaluated is

$$
\mathcal{K}\left[\text{[diagram]}\right]_{\text{QFT}} = \mathcal{K}\left[\text{[diagram]}_{\text{QFT}} * -\frac{3}{2}\mathcal{K}\left(\text{[diagram]}\right)_{\text{QFT}}\right] \qquad (6.223)
$$

Using the results of Section 6.3.2 corresponding diagram in ITF can be written as

$$
\begin{aligned}
\mathcal{K}\left[\text{[diagram]}\right]_{\text{ITF}} &= \mathcal{K}\left[\text{[diagram]}_{\text{ITF}} * -\frac{3}{2}\mathcal{K}\left(\text{[diagram]}\right)_{\text{ITF}}\right] \qquad (6.224) \\
&= \mathcal{K}\left[\text{[diagram]}\right]_{QFT,k_0=\omega_{n_k}} + \frac{3gW(k,n_k)}{2}\mathcal{K}\left(\text{[diagram]}\right)_{\text{ITF}} \\
&= \mathcal{K}\left[\text{[diagram]}\right]_{QFT,k_0=\omega_{n_k}} + W(k,n_k)\frac{3g^3}{(4\pi)^2}\frac{1}{\epsilon}
\end{aligned}
$$

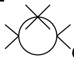
with

$$
\mathcal{K}\left[\text{[diagram]}\right]_{QFT,k_0=\omega_{n_k}} = \frac{\mu^{\epsilon}g^3}{(4\pi)^4}\left[\frac{6}{\epsilon^2} + \frac{3\psi(1)}{\epsilon} - \frac{3}{\epsilon}\int_0^1 \ln\left[\frac{m^2+K^2(x(1-x))}{4\pi\mu^2}\right]dx\right] \qquad (6.225)
$$

where $K^2 = k^2 + \omega_{n_k}^2$

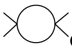
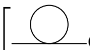
6.4.5 Counter term 5

From [10], one can derive the diagram as

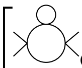
$\mathrm{I}_c^{\mathrm{QFT}} = \mathcal{K}\Big[$  $_{\mathrm{QFT}}\Big]$ (6.226)

$$= \mathcal{K}\left[-\lambda^3 \int \left(\frac{1}{(P^2+m^2)^2}\right)\left(\frac{1}{(P-K)^2+m^2}\right)\frac{d^4P}{(2\pi)^4}\left(-m^2 c^1_{m^2}\right)\left(\frac{-1}{g\mu^\epsilon}\right)\right]$$

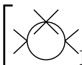
$$= \mathcal{K}\left[\frac{\lambda}{2}\frac{\partial}{\partial m^2}\lambda^2 \int \left(\frac{1}{(P^2+m^2)}\right)\left(\frac{1}{(P-K)^2+m^2}\right)\frac{d^4P}{(2\pi)^4}\left(-m^2 c^1_{m^2}\right)\left(\frac{-1}{g\mu^\epsilon}\right)\right]$$

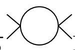
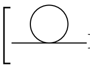
$= \mathcal{K}\Big[-g\mu^\epsilon\Big(-\left(\frac{1}{2}\right)\frac{\partial}{\partial m^2}$  $_{\mathrm{QFT}}\Big)\left(\frac{-1}{\mu^\epsilon g}\right)\Big(-\left(\frac{1}{2}\right)\mathcal{K}\big[$  $_{\mathrm{QFT}}\big]\Big)\Big]$

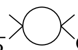
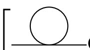
$$= \mathcal{K}\left[-\frac{g^2\mu^\epsilon}{(4\pi)^2}\left[\left(\frac{1}{2}\right)\int_0^1 \left(\frac{1}{K^2x(1-x)+m^2}\right)\,dx\right]\left(\frac{m^2 g}{(4\pi)^2}\left(\frac{1}{\epsilon}\right)\right)\right]$$

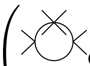
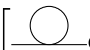
$= -\left(\frac{1}{2}\right)\mathcal{K}\Big[$  $_{\mathrm{QFT},k_0=\omega_{n_k}}\Big]$ (6.227)

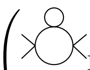
The corresponding counter term in imaginary time formalism can be written as

$\mathrm{I}_c^{\mathrm{ITF}} = \mathcal{K}\Big[$  $_{\mathrm{ITF}}\Big]$ (6.228)

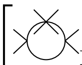
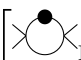
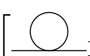
$= \mathcal{K}\Big[(-g\mu^\epsilon)\Big(-\left(\frac{1}{2}\right)\frac{\partial}{\partial m^2}$  $_{\mathrm{ITF}}\Big)\left(\frac{-1}{\mu^\epsilon g}\right)\Big(-\left(\frac{1}{2}\right)\mathcal{K}\big[$  $_{\mathrm{ITF}}\big]\Big)\Big]$ (6.229)

$= \mathcal{K}\Big[-g\mu^\epsilon\Big(-\left(\frac{1}{2}\right)\frac{\partial}{\partial m^2}$  $_{\mathrm{QFT}} - \frac{g^2}{2}\frac{\partial W(k,n_k)}{\partial m^2}\Big)\left(\frac{-1}{\mu^\epsilon g}\right)\Big(\frac{-1}{2}\mathcal{K}\big[$  $_{\mathrm{QFT}}\big]\Big)\Big]$

$= \mathcal{K}\Big($  $_{\mathrm{QFT},k_0=\omega_{n_k}}\Big) + \left(\frac{1}{4}\right)g^2\left(\frac{\partial W(k,n_k)}{\partial m^2}\right)\mathcal{K}\big[$  $_{\mathrm{QFT}}\big]$ (6.230)

$= -\left(\frac{1}{2}\right)\mathcal{K}\Big($  $_{\mathrm{ITF}}\Big)$

It can be also derived using * operation [10] i.e,

$\mathcal{K}\Big[$  $_{\mathrm{ITF}}\Big] = \mathcal{K}\Big[$  $_{\mathrm{ITF}} * -\left(\frac{1}{2}\right)\mathcal{K}\big[$  $_{\mathrm{ITF}}\big]\Big]$ (6.231)

6.5 Renormalization MS Scheme

6.5.1 One and Two loop Calculation

6.5.1. (a) Two-point function one loop calculation

We have to find the counter term for first order g, and if we follow the [10] as the reference text, then the finite proper vertex function is

$$\widetilde{\Gamma}^{(2)}(k) = (\underline{\qquad})^{-1}_{\text{ITF}} - \left(\frac{1}{2}\underline{\bigcirc}_{\text{ITF}} + -\!\!\times\!\!- + -\!\!\ominus\!\!- + \mathcal{O}(g^2)\right) \tag{6.232}$$

where $-\!\!\times\!\!-$ represents contribution of mass counter term, and $-\!\!\ominus\!\!-$ represents field contribution.

$$\begin{aligned} -\!\!\times\!\!- = -m^2 c^1_{m^2} &= -\frac{1}{2}\mathcal{K}\left(\underline{\bigcirc}_{\text{ITF}}\right) \\ &\quad \text{from Sec.(6.3.1)} \\ &= -\frac{1}{2}\mathcal{K}\left(\underline{\bigcirc}_{\text{QFT}}\right) \\ &= -m^2\frac{g}{(4\pi)^2}\frac{1}{\epsilon} \end{aligned} \tag{6.233}$$

The counter term that is proportional to K^2 in first order is zero, so

$$-K^2 c^1_{\phi} = -\!\!\ominus\!\!- = 0 \tag{6.234}$$

Thus the renormalized proper vertex function, which is finite at $\epsilon \to 0$,

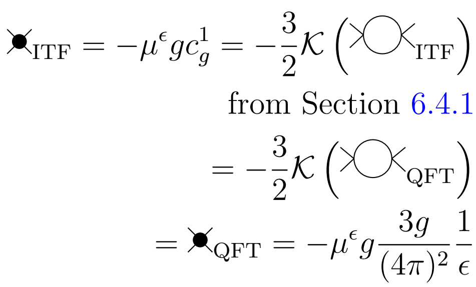
$$\begin{aligned} \widetilde{\Gamma}^{(2)}(k) &= (\underline{\qquad})^{-1}_{\text{ITF}} - \left(\frac{1}{2}\underline{\bigcirc}_{\text{ITF}} - \frac{1}{2}\mathcal{K}\left(\underline{\bigcirc}_{\text{ITF}}\right) + \mathcal{O}(g^2)\right) \\ &= (\underline{\qquad})^{-1}_{\text{ITF}} - \left(\frac{1}{2}\underline{\bigcirc}_{\text{QFT}} - \frac{1}{2}\mathcal{K}\left(\underline{\bigcirc}_{\text{QFT}}\right) - \frac{g}{2}S_1(m,T) + \mathcal{O}(g^2)\right) \end{aligned} \tag{6.235}$$

6.5.1. (b) Four-point function one loop

Similar to two-point function, the corresponding four-point finite proper vertex function can be derived as

$$\widetilde{\Gamma}^{(4)} = -\left(\times + \frac{3}{2}\times\!\!\bigcirc\!\!\times_{\text{ITF}} + \bullet\!\!\!\!\times_{\text{ITF}}\right) + \mathcal{O}(g^3) \tag{6.236}$$

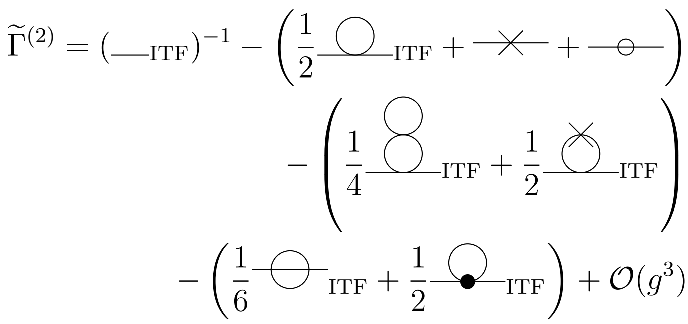
with



(6.237)

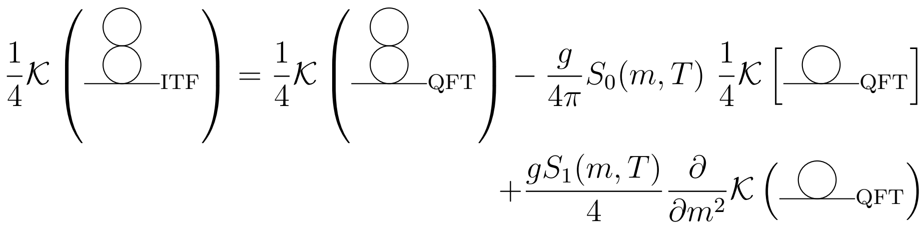
6.5.1. (c) Two loop calculation

Finite two-point function up to two loop order for ITF can be written as



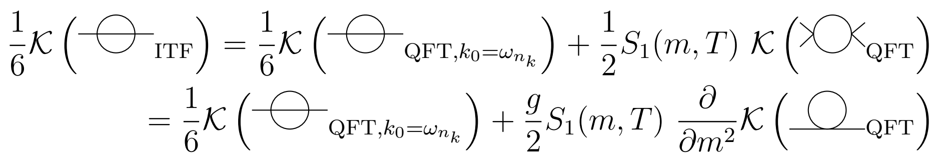
(6.238)

As per Section 6.3.3, the diagram in ITF can be expanded as



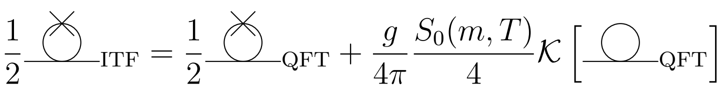
(6.239)

As per Section 6.3.4,



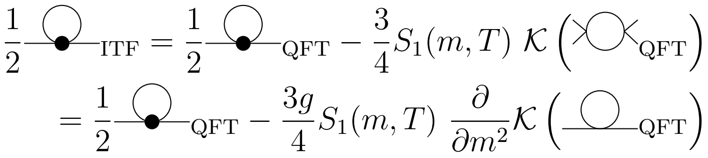
(6.240)

As per Section 6.4.2,



(6.241)

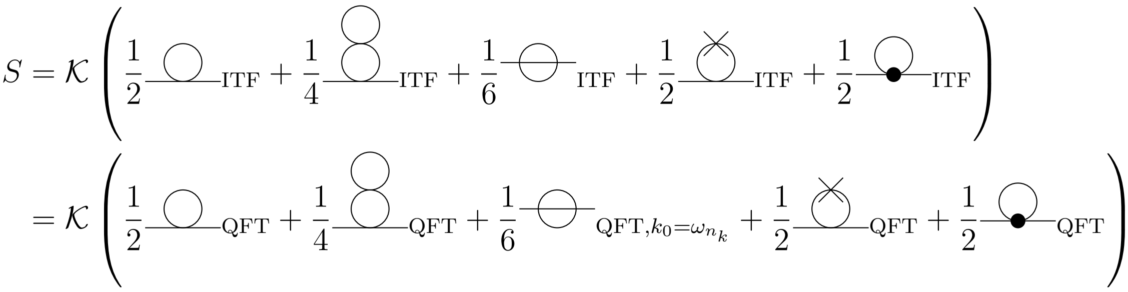
From Section 6.4.3,



(6.242)

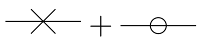
6.5.1. (d) Two loop renormalization two-point functions

It is interesting that the sum of pole terms in the above diagrams in ITF is the same as that of Non-thermal QFT up to two loop orders for two-point functions.
i.e.,



(6.243)

All other terms cancel with each other. So from [10]

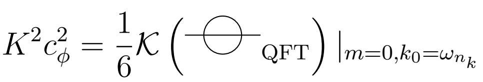


$$= -S = -\left[\frac{g}{(4\pi)^2}\frac{m^2}{\epsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2m^2}{\epsilon^2} - \frac{m^2}{2\epsilon} - \frac{K^2}{12\epsilon}\right)\right] \tag{6.244}$$

with $K^2 = \omega_{n_k}^2 + \vec{k}^2 = K^2$.
i.e., In ITF, if we follow the textbook procedure [10], then the counter terms are the same as those of QFT, with $k_0 = \pm\omega_{n_k}$. When $K^2 = 0$, both ITF and QFT will be in the same form. Now if we extract polynomials with coefficients m^2 and K^2, then one can write ([10])

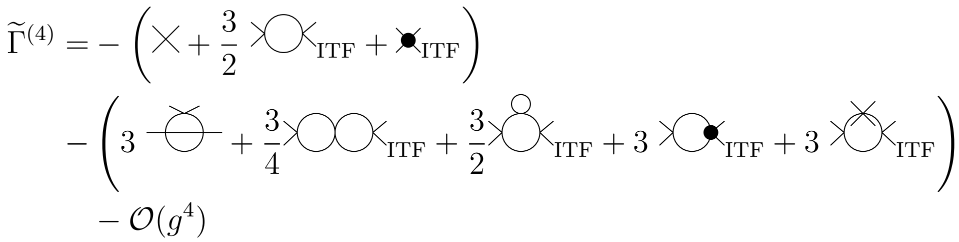
$$m^2(c^1_{m^2} + c^2_{m^2}) = m^2\left[\frac{g}{(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon}\right)\right] \tag{6.245}$$

For field renormalization, we have to consider the term proportional to K^2,



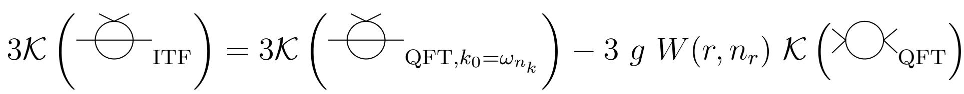
$$= -K^2\frac{g^2}{(4\pi)^4}\frac{1}{12\epsilon} \tag{6.246}$$

6.5.1. (e) Two loop Renormalization of four-point function



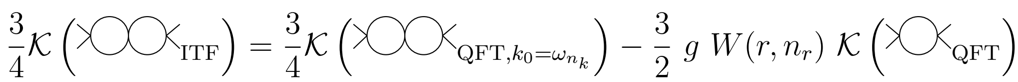
(6.247)

From counter terms section, one can verify the above results with complete derivation. Taking those results now we can write from Section 6.3.7



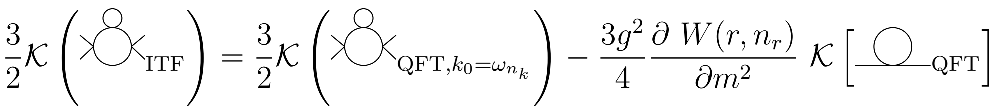
(6.248)

similarly Section 6.3.5 gives



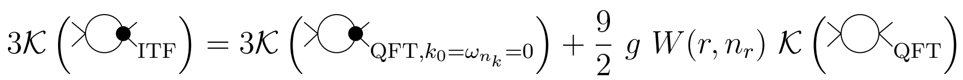
(6.249)

and Section 6.3.6 gives



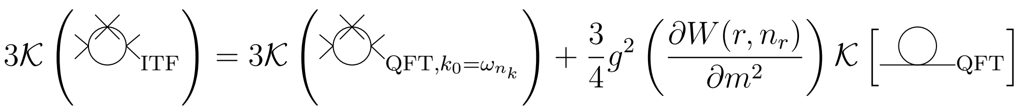
(6.250)

From Section 6.4.4,



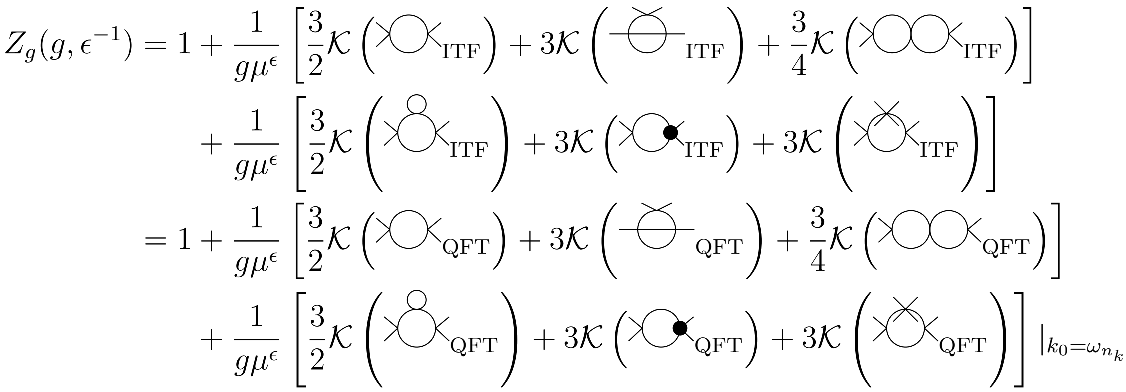
(6.251)

From Section 6.4.5,



(6.252)

Thus one can write

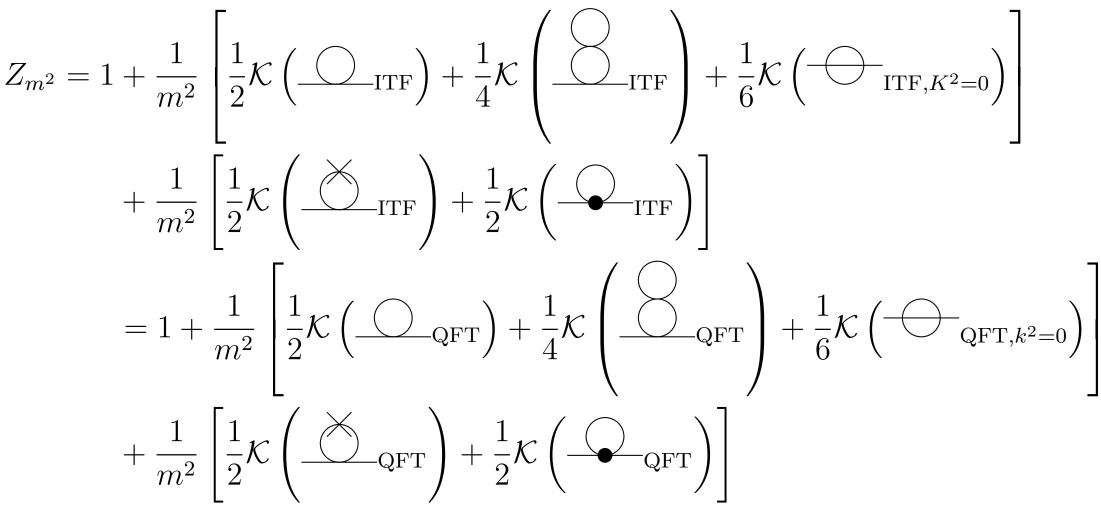


$$= 1 + c_g$$

Thus referring from [10],

$$= 1 + \frac{3g}{(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{9}{\epsilon^2} - \frac{3}{\epsilon}\right) \tag{6.253}$$

Similarly, from Eq. (6.243) and Sections 6.3.1, 6.5.1. (c) and 6.5.1. (d)



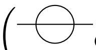
$$= 1 + c_{m^2}$$

From [10],

$$= 1 + \frac{g}{(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2}{(4\pi)^4}\left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon}\right) \tag{6.254}$$

From Section 6.5.1. (d) and Eq. (6.246),

$$Z_\phi = 1 + \frac{1}{K^2}\frac{1}{6}\mathcal{K}\Big($$



$$_{\text{QFT}}\Big)\,|_{m^2=0,k_0=\omega_{n_k}}$$

$$= 1 + c_\phi$$

Thus from [10]

$$= 1 - \frac{g^2}{(4\pi)^4}\frac{1}{12\epsilon} \tag{6.255}$$

6.6 Renormalization constants

We have from [10],

$$\gamma(g) = -Z_{\phi,1} = -\epsilon c_\phi \tag{6.256}$$

$$\gamma_m(g) = \frac{1}{2}\frac{g}{(4\pi)^2} - \frac{1}{2}\frac{g^2}{(4\pi)^4} + \gamma(g) \tag{6.257}$$

$$\beta(g) = -\epsilon g + \frac{3g^2}{(4\pi)^2} - \frac{6g^3}{(4\pi)^4} + 4g\gamma(g) \tag{6.258}$$

6.6.1 Case 1: $K \neq 0$

In this case, as per Eq. (6.88)

$$K^2 c_\phi = -\frac{g^2}{(4\pi)^4}\frac{K^2}{12}\frac{1}{\epsilon} \tag{6.259}$$

So, Eqs. (6.256) to (6.258) becomes

$$\gamma(g) = \frac{g^2}{(4\pi)^4}\frac{1}{12} \tag{6.260}$$

$$\gamma_m(g) = \frac{1}{2}\frac{g}{(4\pi)^2} - \frac{5}{12}\frac{g^2}{(4\pi)^4} \tag{6.261}$$

$$\beta(g) = -\epsilon g + \frac{3g^2}{(4\pi)^2} - \frac{17g^3}{3(4\pi)^4} \tag{6.262}$$

6.6.2 Case 2: $K = 0$

Now Eqs. (6.256) to (6.258) changes because of

$$K^2 c_\phi = 0 \tag{6.263}$$

So,

$$\gamma(g) = 0 \tag{6.264}$$

$$\gamma_m(g) = \frac{1}{2}\frac{g}{(4\pi)^2} - \frac{1}{2}\frac{g^2}{(4\pi)^4} \tag{6.265}$$

$$\beta(g) = -\epsilon g + \frac{3g^2}{(4\pi)^2} - \frac{6g^3}{(4\pi)^4} \tag{6.266}$$

6.6.3 Relation

We can relate them as

$$\gamma(g)_{k=0} = \gamma(g)_{k\neq 0} - \frac{g^2}{(4\pi)^4}\frac{1}{12} = 0 \tag{6.267}$$

$$\gamma_m(g)_{k=0} = \gamma_m(g)_{k\neq 0} - \frac{g^2}{(4\pi)^4}\frac{1}{12} \tag{6.268}$$

$$\beta(g)_{k=0} = \beta(g)_{k\neq 0} - \frac{1}{3}\frac{g^3}{(4\pi)^4} \tag{6.269}$$

6.7 Same Mass scale and Coupling (SMC) approximation

In the following section, we introduce a new scheme, in which the coupling constant and mass scale for thermal and non-thermal ϕ^4 theory considered to be equal and same. The one to one correspondence with the RGE parameters and renormalization constants has lead us to this scheme of approximation. The similarity between the underlying mathematical structure also supports such a scheme. The details can be found in the upcoming sections.

1. Up to two loop order, we have seen that any two-point, one-loop ITF diagram can be expressed as a combination of a QFT diagram with thermal factors.

2. The same is true for the two loop approximation, where one component of the external momenta of QFT is approximated with those of the thermal one.

3. Even though the same kind of diagram in two different formalisms (ITF and QFT at $k_0 = \omega_{n_k}$), is different, the renormalization constants $(Z_{m^2}, Z_g, ...c_{m^2}, c_g)$ are in the same form.

4. Thus, if we make the assumption that QFT and ITF at external momentum zero have the same coupling constant and mass scale, then one can write,

$$\widetilde{\Gamma}^{(2)\mathrm{ITF}}_{K=0} = \widetilde{\Gamma}^{(2)\mathrm{QFT}}_{K=0} + \Gamma^{\mathrm{diff}} \tag{6.270}$$

where $\widetilde{\Gamma}^{(2)} = \Gamma^{(2)} - \mathcal{K}\left(\Gamma^{(2)}\right)$

5. Thus any renormalization group equation which is true for both ITF, and QFT, will also be true for their differences ($\Gamma^{\mathrm{diff}} = \Gamma^{(2)\mathrm{ITF}} - \Gamma^{(2)\mathrm{QFT}}$).

6. Thus if we assume that the Callan Symanzik Equation is true for ITF and QFT for the same mass scale and coupling (since both have the same renormalization constants), then in that particular case

$$\begin{aligned} &\frac{d}{d(\ln\mu)}\widetilde{\Gamma}^{(n)}(m,g,T,\mu) = 0 \\ &\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + \gamma_m m\frac{\partial}{\partial m}\right]\widetilde{\Gamma}^{(n)}(m,g,T,\mu) \approx_{TLA} 0 \end{aligned} \tag{6.271}$$

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma(g) + \gamma_m m\frac{\partial}{\partial m}\right]\widetilde{\Gamma}^{(2)}_{\mathrm{QFT}}(m,g,T,\mu) \approx_{TLA} 0 \tag{6.272}$$

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma(g) + \gamma_m m\frac{\partial}{\partial m}\right]\widetilde{\Gamma}^{(2)}_{\mathrm{ITF}}(m,g,T,\mu) \approx_{TLA} 0 \tag{6.273}$$

Subtracting Eq. (6.272) from Eq. (6.273) we get

$$\boxed{\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma(g) + \gamma_m m\frac{\partial}{\partial m}\right]\Gamma^{\mathrm{diff}}(m,g,T,\mu) \approx_{TLA} 0} \tag{6.274}$$

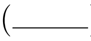
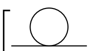
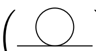
$$\begin{aligned} &\frac{dg}{d\ln(\mu)} = \beta(g) \approx \beta_2 g^2 + \beta_3 g^3 \\ &\frac{d\ln(m(\mu))}{d\ln(\mu)} = \gamma_m(g) \approx \gamma_{m_1} g + \gamma_{m_2} g^2 \\ &\gamma(g) \approx \gamma_2 g^2 \end{aligned} \tag{6.275}$$

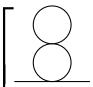
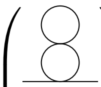
Subscript TLA means two loop approximation.

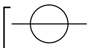
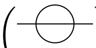
We have already shown that both ITF and QFT have the same structure renormalization constants, so we can approximate both having the same $\beta(g), \gamma(g)$ and $\gamma_m(g)$ under same mass scale and same coupling (SMC) assumption as given in Section 6.6.2

$$\begin{aligned} \gamma(g)_{k=0} &= \gamma(g)_{k\neq 0} - \frac{g^2}{(4\pi)^4}\frac{1}{12} = 0 \\ \gamma_m(g)_{k=0} &= \gamma_m(g)_{k\neq 0} - \frac{g^2}{(4\pi)^4}\frac{1}{12} = \frac{1}{2}\frac{g}{(4\pi)^2} - \frac{1}{2}\frac{g^2}{(4\pi)^4} \\ \beta(g)_{k=0} &= \beta(g)_{k\neq 0} - \frac{1}{3}\frac{g^3}{(4\pi)^4} = -\epsilon g + \frac{3g^2}{(4\pi)^2} - \frac{6g^3}{(4\pi)^4} \end{aligned} \tag{6.276}$$

with

$\widetilde{\Gamma}^{(2)} = ($  $)^{-1} - \frac{1}{2}\Big[$  $- \mathcal{K}\big($  $\big)\Big]$ (6.277)

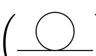
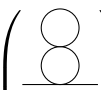
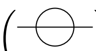
$- \frac{1}{4}\Big[$  $- \mathcal{K}\Big($  $\Big)\Big]$

$- \frac{1}{6}\Big[$  $- \mathcal{K}\big($  $\big)\Big]$

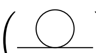
i.e., The two-point proper function is made finite by subtracting out the diverging terms, the mathematical compensation is done via renormalization constants and $\beta(g), \gamma(g), \gamma_m(g)$ functions. Let us define an operator

$$\Delta(\mathbb{A}) = \mathbb{A}_{ITF}|_{k,\omega_{n_k}=0} - \mathcal{K}\left(\mathbb{A}_{ITF}\right)|_{k,\omega_{n_k}=0} - \mathbb{A}_{QFT}|_{k_0,k=0} + \mathcal{K}\left(\mathbb{A}_{QFT}\right)|_{k_0,k=0} \tag{6.278}$$

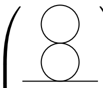
where $\mathbb{A}$ represents the appropriate diagram. Since we defined $\Gamma^{\text{diff}}_{n_k,\vec{k}=0} = \Gamma^{(2)\text{ITF}}|_{n_k,k=0} - \Gamma^{(2)\text{QFT}}|_{k_0,k=0}$ we get

$-\Gamma^{\text{diff}}_{n_k,\vec{k}=0} = \frac{1}{2}\Delta\big($  $\big) + \frac{1}{4}\Delta\Big($  $\Big) + \frac{1}{6}\Delta\big($  $\big)$ (6.279)

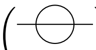
From Sec.(6.3.1),

$\frac{1}{2}\Delta\big($  $\big) = -\frac{g}{2}S_1(m, T)$ (6.280)

From Sec.(6.3.3), we get

$$\frac{1}{4}\Delta\left(\right.$$  $$\left. \right) = \frac{g^2}{16\pi}S_0(m,T)S_1(m,T) - \frac{g^2m^2}{4(4\pi)^3}S_0(m,T)\left[\psi(2)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \frac{g^2}{4(4\pi)^2}S_1(m,T)\left[\psi(1)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] \tag{6.281}$$

From Sec.(6.3.4), we get

$$\frac{1}{6}\Delta\left(\right.$$  $$\left. \right) = \frac{g^2}{2(4\pi)^2}S_1(m,T)\left(\psi(1)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+2-\frac{\sqrt{3}\pi}{3}\right) + \frac{g^2m^2}{64\pi^4}Y(m,T) \tag{6.282}$$

with

$$Y(m,T) = \int_0^\infty\int_0^\infty U(x)U(y)G(x,y)\;dx\;dy \tag{6.283}$$

$$U(x) = \frac{\sinh(x)}{\exp\left(\beta m\cosh(x)\right)-1} \tag{6.284}$$

$$G(x,y) = \ln\left(\frac{1+2\cosh(x-y)}{1+2\cosh(x+y)}\frac{1-2\cosh(x+y)}{1-2\cosh(x-y)}\right) \tag{6.285}$$

Therefore on combining above results, we get

$$\begin{aligned}\Gamma^{\text{diff}}_{n_k,\vec{k}=0} = \;& \frac{g}{2}\;S_1(m,T) - \frac{3g^2}{4}\frac{S_1(m,T)}{(4\pi)^2}\left[\psi(1)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] \\ & - \frac{g^2}{4(4\pi)}S_0(m,T)S_1(m,T) + \frac{g^2m^2}{4(4\pi)^3}S_0(m,T)\left[\psi(2)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] \\ & - \frac{g^2m^2}{64\pi^4}Y(m,T) - \frac{g^2}{32\pi^2}S_1(m,T)\left[2-\frac{\pi}{\sqrt{3}}\right]\end{aligned} \tag{6.286}$$

6.8 Coupling constant calculation

The coupling constant derivation using RGE equations and Γ^{diff} can be solved as shown below.

Re writing the function with coefficients of g,

$$\begin{aligned}
\Gamma^{\text{diff}}_{n_k,\vec{k}=0} &= gT_1 + g^2 T_2 \\
\frac{\partial}{\partial \ln \mu}\Gamma^{\text{diff}}_{n_k,\vec{k}=0} &= gT_{1,\ln\mu} + g^2 T_{2,\ln\mu} \\
\frac{\partial}{\partial \ln m}\Gamma^{\text{diff}}_{n_k,\vec{k}=0} &= gT_{1,\ln m} + g^2 T_{2,\ln m}
\end{aligned} \tag{6.287}$$

with

$$\begin{aligned}
S_N(m,T) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{m}{2\pi n\beta}\right)^N K_N(nm\beta) \\
T_1 &= \frac{1}{2}S_1(m,T) \\
T_2 &= V_1(m,T) + V_2(m,T)\ln(\mu)
\end{aligned} \tag{6.288}$$

$$T_{1,\ln\mu} = 0 \tag{6.289}$$

$$T_{1,\ln m} = \frac{-m^2}{4\pi}S_0(m,T) \tag{6.290}$$

$$T_{2,\ln\mu} = V_2(m,T) \tag{6.291}$$

$$T_{2,\ln m} = V_{1,\ln m} + V_{2,\ln m}\ln\mu \tag{6.292}$$

$$\begin{aligned}
V_1(m,T) &= \frac{m^2}{4(4\pi)^3}S_0(m,T)\left[\psi(2) + \ln\left(\frac{4\pi}{m^2}\right)\right] \\
&- \frac{3}{4}\frac{S_1(m,T)}{(4\pi)^2}\left[\psi(1) + \ln\left(\frac{4\pi}{m^2}\right)\right] \\
&- \frac{1}{4(4\pi)}S_0(m,T)S_1(m,T) - \frac{m^2}{64\pi^4}Y(m,T) \\
&- \frac{S_1(m,T)}{32\pi^2}\left[2 - \frac{\sqrt{3}\pi}{3}\right] \\
V_2(m,T) &= \left(\frac{m^2}{2(4\pi)^3}S_0(m,T) - \frac{3S_1(m,T)}{2(4\pi)^2}\right) \\
V_{2,\ln m} &= \frac{4m^2 S_0(m,T)}{(4\pi)^3} - \frac{m^4 S_{-1}(m,T)}{(4\pi)^4}
\end{aligned} \tag{6.293}$$

$$
\begin{aligned}
V_{1,\ln m} = & \frac{2m^2}{(4\pi)^3} S_0(m,T) \left[\psi(1) + \ln\left(\frac{4\pi}{m^2}\right) \right] \\
& - \frac{m^4}{2(4\pi)^4} S_{-1}(m,T) \left[\psi(2) + \ln\left(\frac{4\pi}{m^2}\right) \right] \\
& + \frac{3}{2} \frac{S_1(m,T)}{(4\pi)^2} + \frac{m^2 S_0^2(m,T)}{2(4\pi)^2} + \frac{m^2}{2(4\pi)^2} S_1(m,T) S_{-1}(m,T) \\
& + \frac{m^2 S_0(m,T)}{(4\pi)^3} \left[2 - \frac{\sqrt{3}\pi}{3} \right] - \frac{m^2}{32\pi^4} Y(m,T) - \frac{m^4}{32\pi^4} \frac{\partial Y(m,T)}{\partial m^2}
\end{aligned} \tag{6.294}
$$

Defining operator $\widehat{\mathrm{RGE}}$ as,

$$
\frac{\mathrm{d}}{\mathrm{d}\ln\mu} = \widehat{\mathrm{RGE}} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + \gamma_m m \frac{\partial}{\partial m} \tag{6.295}
$$

with

$$
\begin{aligned}
\beta(g) &= \beta_2 g^2 + \beta_3 g^3 \\
\gamma(g) &= \gamma_2 g^2 \\
\gamma_m(g) &= \gamma_{m1} g + \gamma_{m2} g^2
\end{aligned} \tag{6.296}
$$

In the two loop approximation,

$$
\widehat{\mathrm{RGE}}\, \Gamma^{\mathrm{diff}}(m,g,T,\mu) \approx_{TLA} 0. \tag{6.297}
$$

The results can be expressed in terms of polynomial in g.

$$
\begin{aligned}
\beta(g)\frac{\partial}{\partial g}\Gamma^{\mathrm{diff}}_{n_k,\vec{k}=0} =& g^4\{2\beta_3 T_2\} + g^3\{2\beta_2 T_2 + \beta_3 T_1\} \\
& + g^2\{\beta_2 T_1\}
\end{aligned} \tag{6.298}
$$

$$
\begin{aligned}
\gamma_m(g)\frac{\partial}{\partial \ln m}\Gamma^{\mathrm{diff}}_{n_k,\vec{k}=0} =& g^4\{\gamma_{m2} T_{2,\ln m}\} + g^3\{\gamma_{m1} T_{2,\ln m} + \gamma_{m2} T_{1,\ln m}\} \\
& + g^2\{\gamma_{m1} T_{1,\ln m}\}
\end{aligned} \tag{6.299}
$$

$$
-2\gamma(g)\Gamma^{\mathrm{diff}}_{n_k,\vec{k}=0} = g^4\{-2\gamma_2 T_2\} + g^3\{-2\gamma_2 T_1\} \tag{6.300}
$$

$$
\frac{\partial}{\partial \ln\mu}\Gamma^{\mathrm{diff}}_{n_k,\vec{k}=0} = g^2 T_{2,\ln\mu} + g T_{1,\ln\mu} \tag{6.301}
$$

$$\widehat{\mathrm{RGE}}\ \Gamma^{\mathrm{diff}}_{n_k,\vec{k}=0} = g^4\left\{2\left(\beta_3-\gamma_2\right)T_2+\gamma_{m2}T_{2,\ln m}\right\}$$
$$+\,g^3\left\{2\beta_2T_2+(\beta_3-2\gamma_2)T_1+\gamma_{m1}T_{2,\ln m}+\gamma_{m2}T_{1,\ln m}\right\}$$
$$+\,g^2\left\{\beta_2T_1+\gamma_{m1}T_{1,\ln m}+T_{2,\ln\mu}\right\}+gT_{1,\ln\mu}$$

Applying the Eqs. (6.288) to (6.291) to the above equation, it is clear that terms that have coefficient g, and g^2 are zero.

$$\widehat{\mathrm{RGE}}\ \Gamma^{\mathrm{diff}} = 0 \implies g = \frac{A(m,T)\ln\mu+B(m,T)}{C(m,T)\ln\mu+D(m,T)} \tag{6.302}$$

and

$$\begin{aligned} A &= \left[-\gamma_{m1}V_{2,\ln m}-2\beta_2V_2(m,T)\right]\\ C &= \left[2(\beta_3-\gamma_2)V_2+\gamma_{m2}V_{2,\ln m}\right]\\ B &= (2\gamma_2-\beta_3)T_1-2\beta_2V_1(m,T)-\gamma_{m1}V_{1,\ln m}-\gamma_{m2}T_{1,\ln m}\\ D &= 2(\beta_3-\gamma_2)V_1+\gamma_{m2}V_{1,\ln m} \end{aligned} \tag{6.303}$$

Combining beta coupling relation with mass scale as,

$$\frac{\mathrm{d}g(\mu)}{\mathrm{d}\ln(\mu)} = \beta_2g^2+\beta_3g^3 \tag{6.304}$$

give rise to the result,

$$\ln(\mu) = \int^g\frac{1}{\beta_2t^2+\beta_3t^3}dt = -\frac{1}{\beta_2\ g}+\frac{\beta_3}{\beta_2^2}\ln\left(\beta_3+\frac{\beta_2}{g}\right)+\ln\mu_0 \tag{6.305}$$

The corresponding running mass and coupling relation is

$$\frac{\mathrm{d}\ln(m)}{\mathrm{d}\ln(\mu)} = \gamma_m(g) \tag{6.306}$$

Combining with the above relation

$$\frac{\partial\ \ln(m)}{\partial g}\frac{\mathrm{d}g}{\mathrm{d}\ln(\mu)} = \gamma_m(g) \implies \frac{\partial\ \ln(m)}{\partial g} = \frac{\gamma_m(g)}{\beta(g)} \tag{6.307}$$

Solving by substituting

$$\frac{\partial\ \ln(m)}{\partial g} = \frac{\gamma_{m1}+\gamma_{m2}g}{\beta_2g+\beta_3g^2} \tag{6.308}$$

$$\ln\left(\frac{m}{m_0}\right) = \chi_2+\frac{\gamma_{m1}}{\beta_2}\ln(g)+\left(\frac{\gamma_{m2}}{\beta_3}-\frac{\gamma_{m1}}{\beta_2}\right)\ln(\beta_3\ g+\beta_2) \tag{6.309}$$

The integral constants are $\ln(\mu_0)$, m_0 and χ_2. We have three equations containing coupling constant g, running mass m, and mass scale μ. Therefore solving Eqs. (6.302), (6.305) and (6.309) simultaneously, we get temperature dependent running mass and coupling constant.

6.8.1 Coupling g Limit Case T $\to$ 0

Consider the limit case $\beta \to \infty$, i.e., $T \to 0$ when $m \neq 0$. In order to find the coupling nature of μ at $T \approx 0$, $\beta m \to \infty$, we have to find the rate of convergence of $A(m,T)$, $B(m,T)$, $C(m,T)$ and $D(m,T)$ as $T \to 0$, since

$$g = \frac{A(m,T)\ln\mu + B(m,T)}{C(m,T)\ln\mu + D(m,T)} \tag{6.310}$$

Here

$$\lim_{\beta m\to\infty} S_N(m,T) \to 0 \tag{6.311}$$

Because both the numerator and denominator of Eq. (6.310) contain $S_N(m,T)$ with varying N. The rate of convergence of the ratios is important. So

$$\lim_{\beta m\to\infty} \frac{S_{N+1}(m,T)}{S_N(m,T)} \to 0 \tag{6.312}$$

As N rises, the rate of $S_N(m,T)$ convergence also grows. The convergence of $Y(m,T)$ can be derived as

$$\begin{aligned} Y(m,T) &= \int_0^\infty \int_0^\infty U(x)U(y)G(x,y)\; dx\; dy \\ U(x) &= \frac{\sinh(x)}{[\exp\left(\beta m\cosh(x)\right) - 1]} \\ G(x,y) &= \ln\left(\frac{1+2\cosh(x-y)}{1+2\cosh(x+y)}\frac{2\cosh(x+y)-1}{2\cosh(x-y)-1}\right) \\ S_N(m,T) &= \frac{1}{\pi}\sum_{j=1}^{\infty}\left(\frac{m}{2\pi j\beta}\right)^N K_N(j\beta m) \end{aligned} \tag{6.313}$$

The limit of $G(x,y)$ can be found from its logarithmic expression, which is $G(x,y) < \ln(3)$. So,

$$\begin{aligned} Y(m,T) &< \ln(3)\left[\int_0^\infty \frac{\sinh(x)}{[\exp(\beta m\cosh(x)) - 1]}dx\right]^2 \\ &< \ln(3)\left[\frac{2\pi}{m}S_{\frac{1}{2}}(m,T)\right]^2 \end{aligned} \tag{6.314}$$

Therefore

$$\lim_{\beta m\to\infty} \frac{Y(m,T)}{S_{-1}(m,T)} \to 0 \tag{6.315}$$

Dividing both the numerator and denominator by $S_{-1}(m,T)$ in g gives

$$\lim_{\beta m\to\infty} g = \lim_{\beta m\to\infty} \frac{\frac{A(m,T)}{S_{-1}(m,T)}\ln\mu + \frac{B(m,T)}{S_{-1}(m,T)}}{\frac{C(m,T)}{S_{-1}(m,T)}\ln\mu + \frac{D(m,T)}{S_{-1}(m,T)}} \approx -\left(\frac{\gamma_{m1}}{\gamma_{m2}}\right) \tag{6.316}$$

where

$$\begin{aligned}
\lim_{\beta m\to\infty} \frac{A(m,T)}{S_{-1}(m,T)} &\approx -\gamma_{m1}\left(\frac{-m^4}{2(4\pi)^4}\left[\psi(2)+\ln\left(\frac{4\pi}{m^2}\right)\right]\right)\\
\lim_{\beta m\to\infty} \frac{B(m,T)}{S_{-1}(m,T)} &\approx -\gamma_{m1}\left(\frac{-m^4}{(4\pi)^4}\right)\\
\lim_{\beta m\to\infty} \frac{C(m,T)}{S_{-1}(m,T)} &\approx \gamma_{m2}\left(\frac{-m^4}{2(4\pi)^4}\left[\psi(2)+\ln\left(\frac{4\pi}{m^2}\right)\right]\right)\\
\lim_{\beta m\to\infty} \frac{D(m,T)}{S_{-1}(m,T)} &\approx \gamma_{m2}\left(\frac{-m^4}{(4\pi)^4}\right)
\end{aligned} \tag{6.317}$$

We have equations connecting μ and g as

$$\ln\left(\frac{\mu}{\mu_0}\right) = \frac{-1}{\beta_2 g} + \frac{\beta_3}{\beta_2^2}\ln\left(\beta_3 + \frac{\beta_2}{g}\right) \tag{6.318}$$

Applying the result of Eq. (6.316) to the above equations, we get

$$\lim_{\beta m\to\infty} \ln\left(\frac{\mu}{\mu_0}\right) = \frac{\gamma_{m2}}{\beta_2\gamma_{m1}} + \frac{\beta_3}{\beta_2^2}\ln\left(\beta_3 - \frac{\gamma_{m2}\beta_2}{\gamma_{m1}}\right) \tag{6.319}$$

The RHS of Eq. (6.319) changes to a complex number at the zero momentum limit in this approximation. One can still make $\mu(T\approx 0)$ a real number if we choose μ_0 in LHS appropriately (i.e., to a complex number or complex function approximation at $T\approx 0$).

The relation between running mass m and coupling relation g at $T\to 0$ is approximated as

$$\lim_{\beta m\to\infty} \ln\left(\frac{m}{m_0}\right) = \chi_2 + \frac{\gamma_{m1}}{\beta_2}\ln\left(\frac{-\gamma_{m1}}{\gamma_{m2}}\right) + \left(\frac{\gamma_{m2}}{\beta_3} - \frac{\gamma_{m1}}{\beta_2}\right)\ln\left(\beta_2 - \frac{\gamma_{m1}\beta_3}{\gamma_{m2}}\right) \tag{6.320}$$

At this approximation, one can choose the running mass $m(T\approx 0)$ as real or complex by intentionally choosing χ_2 and $\ln\mu_0$ accordingly.

6.8.2 Pressure P Limit Case T $\to$ 0

In Fig. 3, we have selected different values as T_0, P_0, χ_2, $\ln\mu$. All those results show a similar trend. i.e., $T \to \infty$, $P \to P_{\text{Ideal}}$, irrespective of the initial value.

The energy density of a quasiparticle with zero chemical potential, obeying the relativistic Bose-Einstein distribution, can be derived from standard statistical mechanics.

$$\langle \varepsilon \rangle = \int \frac{\sqrt{p^2+m^2}}{\left(\exp\left(\beta\sqrt{p^2+m^2}\right)-1\right)} \frac{\mathrm{d}^3p}{(2\pi)^3} \tag{6.321}$$

$$\begin{aligned} \langle \varepsilon \rangle &= \frac{1}{2\pi^2}\int_0^\infty \frac{p^2\sqrt{p^2+m^2}}{\exp\left(\beta\sqrt{p^2+m^2}\right)-1}\mathrm{d}p \\ &\text{Put } p = m\sinh x \\ &= \frac{m^4}{16\pi^2}\int_0^\infty \frac{[\cosh(4x)-1]}{[\exp(\beta m\cosh(x))-1]}\,\mathrm{d}x \end{aligned} \tag{6.322}$$

At $T \to 0$,
i.e., $\beta m \to \infty$, we have

$$\begin{aligned} \lim_{\beta m\to\infty}\langle \varepsilon \rangle &= \frac{m^4}{16\pi^2}\int_0^\infty (\cosh(4x)-1)\exp[-\beta m\cosh(x)]\mathrm{d}x \\ &= \frac{m^4}{16\pi^2}\left(K_4(\beta m)-K_0(\beta m)\right) \end{aligned} \tag{6.323}$$

We have

$$\lim_{x\to\infty} K(N,x) \to 0 \therefore \lim_{\beta m\to\infty}\langle \varepsilon \rangle = 0 \tag{6.324}$$

The equation connecting pressure with energy is

$$P(T) = \frac{T}{T_0}P_0 + T\int_{T_0}^{T}\frac{\varepsilon(T)}{T^2}dT \tag{6.325}$$

In Eq. (6.325) as $T \to T_0$, the integration part goes to zero. (Integral becomes a zero width integral). So at $T \to T_0$, $P \to P_0$. P_0 can have negative or positive or zero values depending upon the initial conditions we impose on it. But in the plot of pressure vs. temperature we have shown that irrespective of value of P_0, the pressure goes to the ideal limit.

In the case of the integrand $\varepsilon(T)/T^2$ at zero temperature limit, assume $m \neq 0$, and $T \to 0 \implies \beta m \to \infty$, according to Eq. (6.323),

$$\lim_{\beta m \to \infty} (\beta m)^2 K_N(\beta m) \to 0 \tag{6.326}$$

Because the energy density at the temperature limits of zero achieves values of zero. The value of pressure at a point can be made negative at some points if one chooses P_0 as negative at appropriate T_0. In our case, we found that it reaches the ideal value at a high-temperature limit, irrespective of the initial value of P_0.

6.9 Quasiparticle Model

Combining these results with the quasiparticle model of Bannur [14, 15, 16], we get an expression for energy density and pressure as

$$\begin{aligned}
\varepsilon(T) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\frac{\varepsilon_p}{\exp(\beta \varepsilon_p) - 1} \right] \\
&= g_f \frac{m^4}{2\pi^2} \sum_{n=1}^{\infty} \left[\frac{3K_2(\frac{nm}{T})}{(\frac{nm}{T})^2} + \frac{K_1(\frac{nm}{T})}{\frac{nm}{T}} \right] \\
&= g_f \frac{m^4}{16\pi^2} \sum_{n=1}^{\infty} \left[K_4\left(\frac{nm}{T}\right) - K_0\left(\frac{nm}{T}\right) \right]
\end{aligned} \tag{6.327}$$

with $\varepsilon_p = \sqrt{p^2 + m^2}$, and

$$\frac{P}{T} - \frac{P_0}{T_0} = \int_{T_0}^{T} \frac{\varepsilon(m(T), T)}{T^2} \, dT \tag{6.328}$$

6.10 Results and Discussion

In order to derive the equation of state for the quasiparticle model in thermal ϕ^4 theory, we need the running mass, which is a function of the coupling constant and mass scale. In thermal and non thermal ϕ^4 theory, the renormalization constants and coupling constant relations are not enough to derive the thermal dependent coupling constant and running mass. So we introduce a new scheme known as the same mass scale and coupling scheme, in which thermal and non-thermal theory are combined on the basis of the same mathematical structure. In this work, each thermal diagram is written in terms of

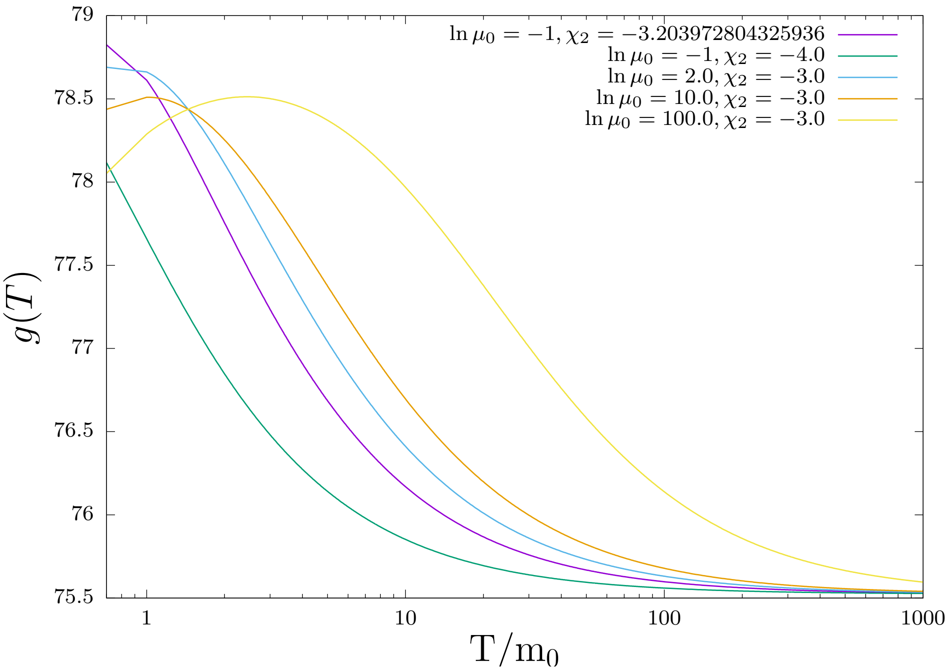


Figure 6.1: Two loop coupling constant results. g against T/m_0 plotted with varying values of integration constants $\ln \mu_0$ and χ_2 with $m_0 \approx 1$

its corresponding non-thermal diagram, which has coefficients that are temperature dependent.

Applying the Renormalization Group Equation simultaneously to the finite proper vertex function in both non-thermal and thermal ϕ^4 theory under SMC [17] has produced a new coupling constant equation, as shown in Eq. (6.302). This new equation, in addition to the already existing renormalization equations, is sufficient to produce the temperature dependent coupling constant and running mass. This is achieved in this work by solving Eqs. (6.302), (6.305) and (6.309) simultaneously.

We have plotted the results in Figs. 6.1 to 6.4 with different integration constants. The two loop coupling constant is plotted against the temperature in Fig. 6.1. It is qualitatively in agreement with the predicted behaviour. i.e., as the temperature goes to infinity, the coupling constant goes to zero. The running mass per temperature is plotted in Fig. 6.2, and it also goes to zero as the temperature tends to infinity. In Fig. 6.3, the scaled pressure is plotted, where the pressure is divided by an ideal pressure value of $\frac{\pi^2}{90}T^4$. The ideal

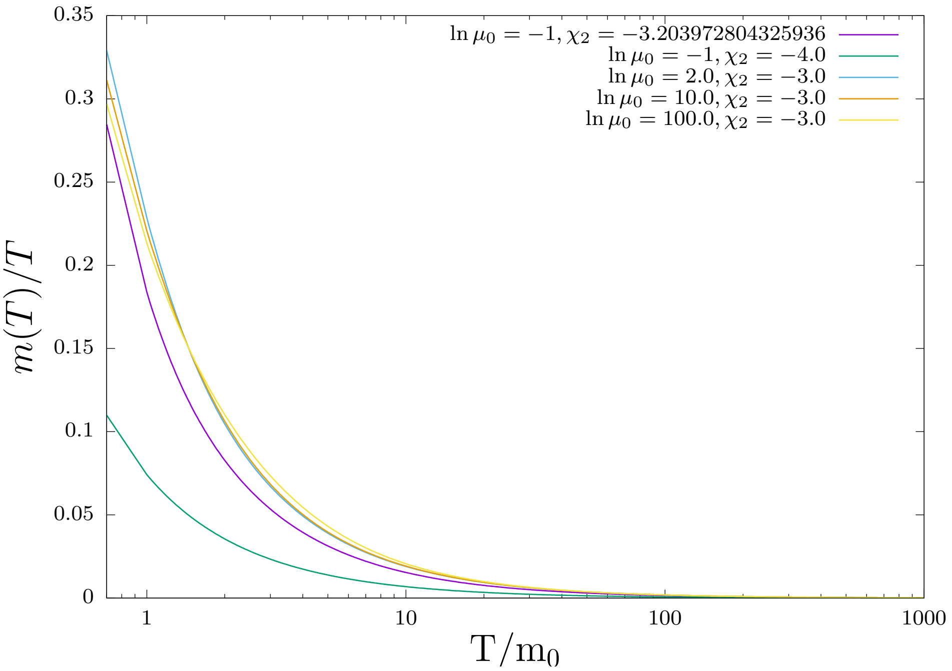


Figure 6.2: Two loop running mass results. The difference between the curves is due to the different integration constants, as shown in the figure.

pressure value is the pressure value that corresponds to a free particle under the Bose-Einstein distribution. It has been observed in Fig. 6.3 that, irrespective of the initial value, the pressure reaches its ideal behaviour as the temperature goes to infinity. The mass scale against the temperature is plotted in Fig. 6.4 for various integration constants.

When lattice data becomes available in the future, we hope to compare this work with those lattice data. We hope the extension of this SMC model from ϕ^4 to QCD might lead us to some new insights on new methods in the future.

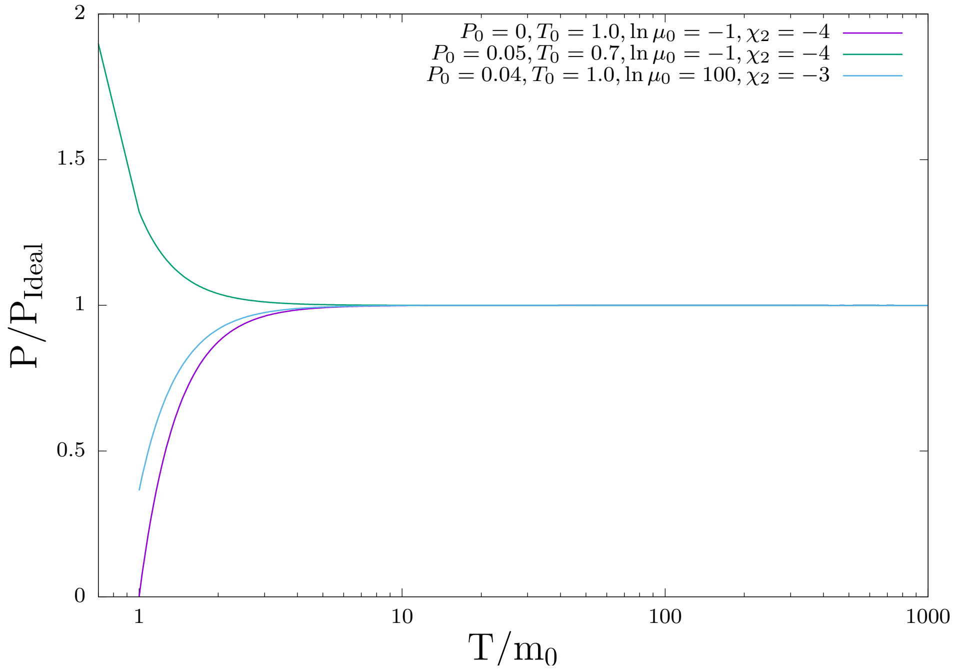


Figure 6.3: Pressure scaled by $\left[\frac{\pi^2}{90}T^4\right]^{-1}$ against T/m_0, with varying values of $T_0, P_0, \ln\mu_0, \chi_2$.

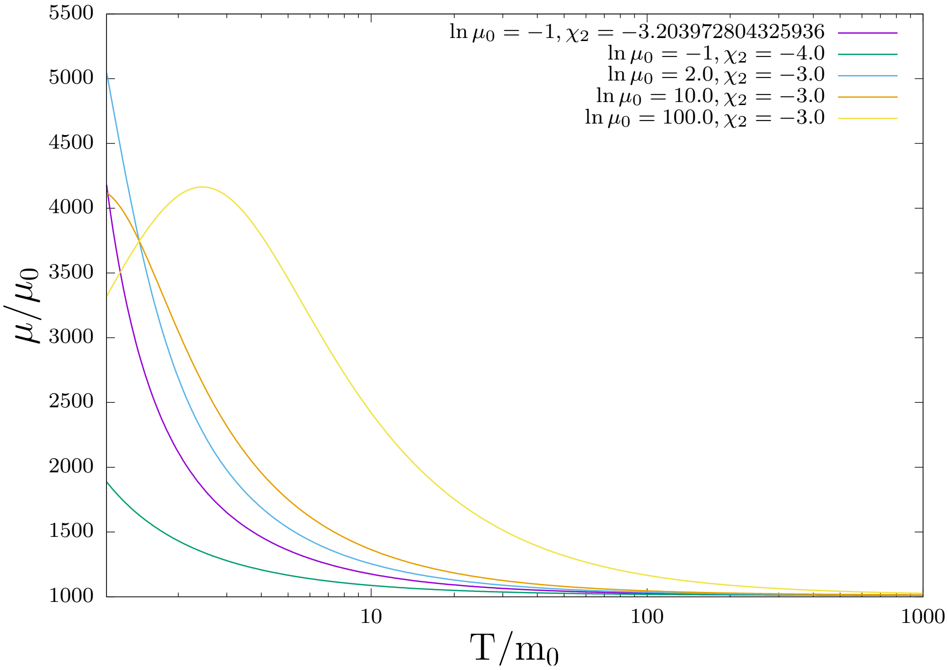


Figure 6.4: $\frac{\mu}{\mu_0}$ plotted against T/m_0 with varying integration constants $\ln(\mu_0)$ and χ_2 with $m_0 \approx 1$

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Chapter 7

Summary and Future Plans

In Chapter 1, we have given a general introduction to quark-gluon plasma and some phenemenological models. Phenemenological models are good tools for predicting the equation of state of quark-gluon plasma. The phenemenological models mentioned include the MIT Bag model, the relativistic harmonic oscillator model, and quasiparticle models. The quasiparticle models [1, 2, 3, 4, 5] approximate mass as a function of coupling constants, which are functions of temperature. Many of the coupling constant-dependent models predict the equation of state, which is in good agreement with the lattice data for the temperature range $T/T_c > 1$ [3, 4]. The relation between quasiparticle mass and coupling constant can be approximated as

$$m^2(T) = g^2(T)T^2 = 4\pi\alpha_s(T)T^2$$
$$\alpha_s(T) = \frac{6\pi}{(33-2n_f)\ln\left(T/\Lambda_T\right)}\left[1 - \frac{3(153-19n_f)}{(33-2n_f)^2}\frac{\ln\left(2\ln\left(T/\Lambda_T\right)\right)}{\ln\left(T/\Lambda_T\right)}\right] \tag{7.1}$$

where Λ_T is the QCD scale factor. In many published works, Λ_T is represented as a multiple of critical temperature T_c.

The coupling constants both in one loop and two loop order have factors of $\ln(T/\Lambda_T)$. Thus, when T goes to Λ_T or less than Λ_T, the mass goes complex, negative, or even towards divergence, depending on the value of T/Λ_T.

In the magnetic field regime, the coupling constant still depends on $\ln(T/\Lambda_T)$ and $\ln((T^2 + |eB|)/\Lambda_T^2)$. Thus, this makes the model deviate from the lattice data for low values of T. So the coupling constant independent pheneomenological model has an upper hand in the regime of $T < \Lambda_T$ or $T = \Lambda_T$.
In 1995, Vishnu Mayya Bannur combined the work of Balescu [6], which was based on Mayer's cluster expansion (MCE) for charged particles, with the quark number density [7]. Using Cornell potential, Bannur successfully fitted the derived equation of state with lattice data for $T > T_c$. The model extended by Udayanandan and Bannur [8] involves the gluon contribution to the EoS. The model used Cornell potential for both quarks and gluons.

In Chapter 2, we describe Mayer's cluster expansion for the lowest order step by step. We also described how the correction factor works between distinguishable particle integrals and indistinguishable particle integrals with different indistinguishable particle species. The equations of state corresponding to pressure and energy density were also described. We have studied the statistical mechanics and thermodynamics of interacting systems using the cluster expansion method in both the presence and absence of a magnetic field.

In Chapter 3, we introduced the modified liquid drop model in QGP. Fourier transforms of central potential functions (i.e., polynomials in radial coordinates) are studied. Previous published work on these topics involving MCE is covered. Various integrals involved in the formulation of our model are discussed. The poles in the integral equations are removed by introducing an infinitesimal imaginary term in the integrand. The contour integration method is used for this purpose. In the proposed modified liquid drop model, the effective terms in the potential are linear, volume, and Coulomb. The energy density, pressure, entropy and number density is calculated for zero magnetic field. It is found that the model does fit with the available lattice data for certain parameters.

In Chapter 4, the liquid drop model extended to magnetic field environment. The idea of a harmonic oscillator is used with a magnetic vector potential, giving rise to modifications in the integral equation. A new integral table with the modified integrating technique in the presence of a magnetic field is also derived. The model was concentrated heavily on quarks in Chapter 4, and we have shown that the data is in good agreement with the expected behavior in both quantitative and qualitative terms.

In Chapter 5, we have used quasi particle model of VM Bannur [9] in presence of magnetic field to study the behaviour of quarks with finite chemical potential at zero temperature. We have taken a neutron star case in which, due to high pressure at the core, the quarks became deconfined and resulted in a quark star. Chapter is entirely different from Chapters 3 and 4. The QPM equations [9] are changed to accommodate the magnetic field effects. The EoS results are compared with that of free particles.

In Chapter 6, the equation of state of the thermal and non-thermal ϕ^4 theories are discussed. We have developed a new method for resolving the beta function pole problem in the RGE equation. The method is known as the SMC method, where simultaneously the thermal and non-thermal quantum field theory proper vertex functions up to two loop orders are solved by the assumption of the same coupling and mass scale. This method works, and we get a coupling constant and running mass that are in qualitative agreement with the expected behaviour of the equation of state of ϕ^4 theory.

7.1 Future Plan

It is important to study the transport coefficients of QGP in the presence of a magnetic field to find the non-ideal behaviour of QGP in the presence of external forces. Chapters 3 and 4 can be extended in this way to find the transport coefficient by using Boltzmann transport equations for relativistic particles.

In Chapter 5, the longitudinal pressure of quarks causes an asymmetry of pressure in different directions. Thus, TOV cannot be applied to find the radius of a quark star under the present QPM model. So we have to use an asymmetric pressure equation for TOV to further explore the properties of QGP.

The method used in Chapter 3 and 4 works with a constant density at an infinite volume. The density is constant only with respect to the volume, but it can still be a function of temperature and magnetic field. If one uses the above concept in the context of cosmology, i.e., the Big Bang, space-time, and the idea of atoms of spacetime [10], one could possibly make the density of the space-time particles a function of some parameters. The quasiparticle model of Bannur [3] can also be rearranged to accommodate the idea of a spacetime atom.

We hope the rules of statistical mechanics for atoms in spacetime can be derived using these approaches. We will have to proceed towards accomplishing these goals in the future.

The SMC method we introduced in Chapter 6, is qualitatively in good agreement with the expected behaviour. Once lattice data is available, we would like to compare our model with it. We hope the extension of this SMC model from ϕ^4 to QCD might lead us to some new insights into QGP.

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Recommendations

The study of matter in its diverse forms has played a profoundly significant role throughout human history. Our comprehension of fundamental particles has provided us with invaluable insights into the nature of the universe. While it is impossible to journey back in time to gain direct knowledge about the behavior of matter during different phases of the universe's expansion, we can leverage high-energy experimental findings and theoretical models based on such experiments to investigate and simulate those specific periods.

In our study to examine the equation of state of quark gluon plasma, we have devised a model that agrees with the equation of state obtained from lattice data. The model, namely the modified liquid drop model developed in this study, enable us to make predictions about the equation of state of quark gluon plasma across a wide range of temperatures and magnetic fields. Furthermore, the same mass and coupling method we have developed can be extended for further applications across other domains of physics.