A STUDY ON SOME TOPOLOGICAL CONCEPTS IN IDEAL TOPOLOGICAL SPACES

Thesis submitted to the University of Calicut

for the award of the degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

under the Faculty of Science

by

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DECEMBER 2022

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I hereby certify that, this thesis entitled **"A STUDY ON SOME TOPOLOGICAL CONCEPTS IN IDEAL TOPOLOGICAL SPACES "** is a bonafide record of research work carried out by **Ms. Sangeetha M. V.**, Research Scholar, Department of Mathematics, St. Joseph's College (Autonomous), Devagiri, under my supervision and guidance for the award of the Degree of Doctor of Philosophy in Mathematics, of the University of Calicut. The work reported herein does not form part of any other thesis or dissertation submitted previously for the award of any degree or diploma of any other university or institution. Also certified that the contents of the thesis have been checked using anti-plagiarism data base and no unacceptable similarity was found through the software check.

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DECLARATION

I, Sangeetha M. V., hereby declare that this thesis entitled **"A STUDY ON SOME TOPOLOGICAL CONCEPTS IN IDEAL TOPOLOGICAL SPACES "** submitted to the University of Calicut for the award of the degree of Doctor of Philosophy in Mathematics is a bonafide record of the work done by me under the guidance and supervision of Dr. Baby Chacko, Department of Mathematics, St. Joseph's College, Devagiri. This thesis contains no material which has been accepted for the award of any degree or diploma of any university or institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference was made.

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ACKNOWLEDGEMENT

During the preparation of my thesis I have enjoyed the support and encouragement of numerous persons whom I need to thank. First and foremost, I am deeply indebted to my supervising teacher, Dr. Baby Chacko, Research guide, P. G. and research department of Mathematics, St. Joseph's College (Autonomous), Devagiri, for accepting me as his research student and for all the help and guidance.

No words can communicate my heartfelt gratitude to my beloved parents Venkiteswaran and Geetha, my husband Sumith and my children Harihar and Neerav, for their quiet but very meaningful support through all the ups and downs of my research work. They lightened my burden in numerous ways.

I thank profusely my fellow research scholar Ms. Muhsina V., for the generosity, concern and timely interventions she extended at every stage of my research work. I am much obliged to my colleague Dr. Aparna Aravindakshan M. for her unstinting support and wholehearted encouragement for the successful completion of my work.

I have to place on record my gratitude to Dr. Saji Mathew, Head of the Department, and teachers Dr. T. K. Muraleedharan, Mr. Jaison Joseph, Mr. Shinoj K. M., Dr. Anjaly Jose and Fr. Jinu B. George for their support.

The co-operation wholeheartedly rendered by the guest lecturers, research scholars and students of the Department of Mathematics, St.Joseph's College (Autonomous), Devagiri during the period of my research work deserves special mention.

I would like to express my sincere thanks to the management, Principal, library staff and office staff of St. Joseph's College (Autonomous), Devagiri for all the facilities provided to me.

Finally, I am thankful to all who were considerate as well extending my apologies that I could not mention everyone individually.

Everything fell in place and I could achieve this by the blessings of the Almighty. I bow before that Grace.

Sangeetha M. V.

17^{th} December 2022

Contents

1 Introduction

Ideal topological space has been a topic of study since 1960 through the paper by Vaidyanathaswamy. Kuratowski and Vaidyanathaswamy [39] studied the concept of ideals in topological spaces. Later in 1990 the contributions of Hamlett and Jankovic [6] in ideal topological spaces initiated the generalization of some important properties in general topology via topological ideals.

By a space (X, τ), we mean a topological space with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X, τ), the system of open neighborhoods of x is denoted by $N(x) = \{U \in \tau : x \in U\}$. For a given subset A of a space (X, τ), $Cl(A)$ and $Int(A)$ are used to denote the closure of A and interior of A respectively, with respect to the topology.

A non-empty collection of subsets of a set X is said to be an ideal $\mathcal{I}$ on X, if it satisfies the following two conditions: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal topological space (or ideal space) $(X, \tau, \mathcal{I})$ means a topological space (X, τ) with an ideal $\mathcal{I}$ defined on X. Let (X, τ) be a topological space with an ideal $\mathcal{I}$ defined on X. Then for any subset A of X, $A^*(\mathcal{I}, \tau) = \{x \in X / A \cap U \notin \mathcal{I} for\ every\ U \in N(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ. We say that a set $A \subset X$ has the property $\mathcal{I}$ locally at a point x, if there exists a neighborhood U_x of x, such that $U_x \cap A \in \mathcal{I}$. The set of points at which A does not have the property $\mathcal{I}$ locally is denoted by $A^*(\mathcal{I})$. Thus $x \in A^*(\mathcal{I})$ means that for every neighborhood U_x of x, $U_x \cap A$ does not have the property $\mathcal{I}$ or briefly is not in $\mathcal{I}$ [6, 37].

Let $P(X)$ be the class of all subsets of X. If $()^c : P(X) \to P(X)$ is a function satisfying (1) $\phi^c = \phi$, (2) $A \in P(X) \to A \subset A^c$, (3) $A \in P(X), B \in P(X) \to (A \cup B)^c = A^c \cup B^c$, and (4) $A \in P(X) \to (A^c)^c = A^c$ then $()^c$ is called a Kuratowski closure operator and $\{A \in P(X) : A = A^c\}$ is a collection of closed sets for a topology on X. Also if $d : P(X) \to P(X)$ is a function satisfying (1) $d(\phi) = \phi$, (2) $d(A \cup B) = d(A) \cup d(B)$, and (3) $d(d(A)) \subset d(A)$, then $()^c : P(X) \to P(X)$ defined by $A^c = A \cup d(A)$ is a Kuratowski closure operator on $P(X)$. Since $()^* : P(X) \to P(X)$ is a function satisfies all the required conditions for the function d, we have that $Cl^*(A) = A \cup A^*$ is a Kuratowski closure operator [6]. The topology generated by Cl^* is denoted by τ^* where τ is the original topology on X, i.e, $\tau^* = \{U \subset X : Cl^*(X - U) = X - U\}$. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$ finer than τ , generated by $\beta(\mathcal{I}, \tau) = \{U - I : U \in \tau \;\; and \;\; I \in \mathcal{I}\}$, but in general $\beta(\mathcal{I}, \tau)$ is not always a topology [6]. Further investigations in this field are done by (Arenas, Dontchev et al.2000; Jankovic and Hamlett 1990, 1992; Mukherjee et al. 2007; Nasef and Mahmoud 1992) [5, 6, 10, 27, 31, 45]

From ideal topological point of view, many generalizations in operators and open sets are currently in research. Within this framework certain concepts in ideal topological space are discussed here.

The idea of regular open sets was introduced by M. H. Stone in 1937 and is defned as $Int(Cl(A)) = A$. Similarly, regular closed sets as $Cl(Int(A)) = A$. Velicko (1968) also studied the same and is called as δ-open sets. The term δ-cluster point (resp. δ-interior) of A was defined by Velicko (1968) denoted as $Cl_{\delta}(A)$ (resp. $Int_{\delta}(A)$). Also it is proved that the collection of δ-open sets forms a topology denoted as τ^{δ}.

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $R-\mathcal{I}-$open if $Int(Cl^{*}(A)) = A$. We call a subset A of $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-$closed if its complement is $R-\mathcal{I}-$open [42].

Separation axioms always render a vital part in general topology. Many authors had defined different types of open sets and set forth several separation axioms with regard to those open sets. In literature, pre$-T_i(i = 0, 1, 2)$ axioms, semi$-T_i(i = 0, 1, 2)$ axioms were some of them. Separation axioms are also developed in ideal topological spaces.

The notion of R_0 topological space came to light in 1943 by Shanin [33]. Later in 1961, Davis [3] studied this weak separation axiom and some of its properties and also introduced the notion of R_1 topological space. This space is independent of T_0 and T_1 spaces. Also, it is weaker than T_2 space. S.A. Naimpally [40] in 1967 and Hall, Murphy and Rozycki [8] in 1971 also carried out further studies on R_0 topological spaces. K.K. Dube [22, 23] in 1974 and 1982 further studied the R_0 and R_1 spaces and obtained several characterisations.

The separation axioms weaker than R_0 was introduced by Jingcheng Tong [20] in 1983 and later by Guiseppe Di Maio [14] in 1985. They defined independently the separation axioms which were strictly weaker than R_0 axiom. D.N. Mishra and K.K. Dube [7] studied certain separation axioms weaker than R_0 axiom in 1973 but were not weaker than those defined by Jingcheng Tong.

A. S. Mashhour [4] in 1983 introduced supra open sets and studied supra topological space and the continuous maps in supra topological space. Later, many topologists introduced different kinds of supra open sets and continuous functions like supra b-open sets and supra b-continuity [35].

Another class of open (as well closed) sets which is a topic of interest is minimal open (as well minimal closed) sets and maximal (as well maximal closed) sets. This was introduced by F. Nakaoka and N. Oda [11, 12, 13] in 2001 and 2003 and had given many characterisations. Later, minimal and maximal continuous functions were under discussion.

Many topologists have introduced and several generalizations of continuous functions in general topology and as well in ideal topological space. In 1971, Gentry and Hoyle [25] introduced somewhat continuous functions. N. Levine [34] in 1961 introduced the decomposition of continuity in topological spaces. Many weaker form of continuous functions are in literature.

J. Dontchev [15] in 1996 introduced and investigated a concept of non-continuity called contra-continuous functions. Contra-continuous functions are extended in terms of several open sets by researchers. This notion is also extended in ideal topological space.

The concept of continuity has been extended to define multifunctions and later many generalizations have been made in terms of weak forms of open sets. V. Popa [50, 48, 49, 47] has investigated many layers of multi-function.

The objectives of this work are:

- To study certain topological concepts in terms of $R-\mathcal{I}-$open sets.
- To study $R-\mathcal{I}-T_i(i=0,1,2)$ and $R-\mathcal{I}-R_i(i=0,1)$ spaces and spaces weaker than $R-\mathcal{I}-R_i(i=0,1)$ spaces.
- To extend the notion of $R-\mathcal{I}-$open sets to introduce supra $R-\mathcal{I}-$open sets and minimal (maximal) $R-\mathcal{I}-$open sets.
- To extend the notion of $R-\mathcal{I}-$open sets to introduce somewhat $R-\mathcal{I}-$ continuous functions, contra $R-\mathcal{I}-$ continuous functions,

almost contra $R-\mathcal{I}-$ continuous functions, $R-\mathcal{I}-$ continuous multifunctions, almost $R-\mathcal{I}-$ continuous multifunctions and weakly $R-\mathcal{I}-$ continuous multifunctions.

Organisation of Thesis

The whole work is divided into eleven chapters. This chapter, the first one, includes the introduction and the historical growth.

The second chapter contains the literature reviews.

In chapter 3, the idea of $R-\mathcal{I}-$open sets is widened to advance a generalization of separation axioms. In this chapter we developed the separation axioms $R-\mathcal{I}-T_i$ $i=0,1,2$, $R-\mathcal{I}-$regular, $R-\mathcal{I}-$normal, completely $R-\mathcal{I}-$normal spaces and studied some properties of these axioms. Also the hereditary nature of those axioms are discussed. Moreover we initiated the concept of compactness in this context.

In chapter 4, we extended the notion of $R-\mathcal{I}-$open sets to develop weak separation axioms $R-\mathcal{I}-R_0$ axiom and $R-\mathcal{I}-R_1$ axiom. We studied certain characterisations of these axioms and analysed the relationship between them. Further, relations with the separation axioms discussed in chapter 3 are examined.

In chapter 5, we have discussed the weak separation axioms $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_1$. In this chapter, we concentrate on separation axioms which are weaker than $R-\mathcal{I}-R_0$. At first we will define $R-\mathcal{I}-$cluster point of a set and studied some of its properties. Also, the relation connecting the $R-\mathcal{I}-$cluster point and $R-\mathcal{I}-$regular space as well as the $R-\mathcal{I}-R_1$ space is examined. Afterwards, we will switch to the main purpose of the chapter: separation axioms namely $R-\mathcal{I}-R_S$, $R-\mathcal{I}-R_D$, $R-\mathcal{I}-R_T$, weakly $R-\mathcal{I}-R_0$ and weakly $R-\mathcal{I}-C_0$.

In chapter 6, we present a new class of sets and functions in a supra ideal topological space. We have brought in via ideal, supra $R-\mathcal{I}-$open sets. We have introduced and studied certain properties of supra $R-\mathcal{I}-$continuous functions, supra* $R-\mathcal{I}-$continuous functions, supra $R-\mathcal{I}-$irresolute functions, supra $R-\mathcal{I}-$open maps, supra $R-\mathcal{I}-$closed maps, supra $R-\mathcal{I}-$homeomorphism. Separation axioms in terms of supra $R-\mathcal{I}-$open sets are also looked into.

Till now, we have explored mainly on separation axioms in various modes and also on supra space. From this chapter onwards we will pertain to $R-\mathcal{I}-$open sets and $R-\mathcal{I}-$continuous functions and its generalizations. In chapter 7, we intend to propose a new class of sets and a new class of continuous functions in ideal topological space incorporating the idea of minimal open sets and $R-\mathcal{I}-$open sets. We brought in minimal $R-\mathcal{I}-$open set in the section 2. Further, we surveyed $R-\mathcal{I}-T_{min}$ and $R-\mathcal{I}-T_{max}$ spaces, in the section 3. In the section 4, we move on to the continuous functions namely minimal $R-\mathcal{I}-$continuous function and maximal $R-\mathcal{I}-$continuous function. The relation between the above defined continuous functions with certain other continuous functions is also investigated.

In chapter 8, we put forward the notions of somewhat $R-\mathcal{I}-$continuous functions and somewhat $R-\mathcal{I}-$open functions. In section 2, we will study about somewhat $R-\mathcal{I}-$continuous functions and its relationship with other classes of functions. Also some of its characterizations and properties are obtained besides giving examples and counter examples. In section 6, we will study about somewhat $R-\mathcal{I}-$open functions and get results which go parallel with the results of somewhat $R-\mathcal{I}-$continuous functions. In other sections new definitions related with somewhat $R-\mathcal{I}-$continuous functions are introduced and certain properties are analysed.

In chapter 9, we aim to set forth a class of functions using the concepts in ideal topological space called the contra $R-\mathcal{I}-$continuous functions.

We also introduce almost contra $R-\mathcal{I}-$continuous functions and investigate certain properties and several characterisations of such concepts. Further, we will deal with contra $R-\mathcal{I}-$closed graphs and strongly contra $R-\mathcal{I}-$closed graphs. In the section 2., using the notion of $R-\mathcal{I}-$open sets, the contra $R-\mathcal{I}-$continuous functions are presented and studied. In section 3., almost contra $R-\mathcal{I}-$continuous functions are investigated. Section 4. handles with $R-\mathcal{I}-$closed, contra $R-\mathcal{I}-$closed and strongly contra $R-\mathcal{I}-$closed graphs.

In chapter 10, we widen the concept of continuity to set multifunctions in ideal topological space by extending the class of $R-\mathcal{I}-$open sets. We introduce a class of continuous multifunctions namely upper and lower $R-\mathcal{I}-$continuous multifunctions and explored several characterisations of the same. This is done in section 2. We also present and examined two weaker forms of the upper and lower $R-\mathcal{I}-$continuous multifunctions. In section 3., the weaker form of $R-\mathcal{I}-$continuous multifunction, namely almost $R-\mathcal{I}-$continuous multifunction is developed. In section 4., another weaker form of $R-\mathcal{I}-$continuous multifunction, namely weakly $R-\mathcal{I}-$continuous multifunction is studied.

In the final chapter, chapter 11, the conclusion of the work is stated. This chapter also put forward certain proposals for further research.

2 Background

2.1 Preamble

We recollect concepts and results which are relevant in this study and which we have to do with in the succeeding chapters. As mentioned earlier, $(X, \tau, \mathcal{I})$ denotes a topological space (X, τ) with an ideal $\mathcal{I}$ defined on X where no separation axioms are assumed unless otherwise explicitly stated. Our area of discussion is mainly on $R-\mathcal{I}-$space. For our convenience, we mean $(X, \tau, \mathcal{I})$ as a $R-\mathcal{I}-$space.

2.2 Ideal Topogical space

Definition 2.2.1. *[24, 6] A non-empty collection of subsets of a set X is said to be an ideal $\mathcal{I}$ on X, if it satisfies the following two conditions:*
(i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$
(ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

Definition 2.2.2. *[24, 6]*
(1) The minimal ideal $\mathcal{I}$ is the empty set ϕ in any topological space (X, τ).
(2) The maximal ideal $\mathcal{I} = P(X)$ is the power set of X in any topological space (X, τ).

Definition 2.2.3. *[24, 6] For any subset A of X,*

$A^*(\mathcal{I},\tau) = \{x \in X / A \cap U \notin \mathcal{I} \;\; for \;\; every \;\; U \in N(x)\}$ *is called the local function of* A *with respect to* $\mathcal{I}$ *and* τ. *If there is no ambiguity, we will write* $A^*(\mathcal{I})$ *or simply* A^* *for* $A^*(\mathcal{I},\tau)$.

Definition 2.2.4. *[24, 6] Given a topological space* (X,τ) *and an ideal* $\mathcal{I}$ *on* X, $\mathcal{I}$ *is said to be compatible with* τ, *(*$\mathcal{I} \sim \tau$*), if the following condition is satisfied for each* $A \subset X$*: if for every* $a \in A$ *there exists a neighborhood* U *of* a *such that* $U \cap A \in \mathcal{I}$, *then* $A \in \mathcal{I}$.

Definition 2.2.5. *For any subset* A *of* X, $Cl^*(A) = A \cup A^*$ *defines a Kuratowski closure operator. The topology generated by* Cl^* *is denoted by* $\tau^*(\mathcal{I})$ *or simply* τ^* *which is finer than* τ, *the original topology on* X.

If $A \in \mathcal{I}$ and $U \cap A \in \mathcal{I}$ for any $U \in \tau$, then $A^* = \phi$. When $\mathcal{I} = \phi$, then $A^* = Cl(A)$. Hence in this case $Cl^*(A) = Cl(A)$ and $\tau = \tau^*$. If $\mathcal{I} = P(X)$, then $A^* = \phi$ for every $A \subset X$ and hence $\tau^*(\mathcal{I})$ is the discrete topology.

Lemma 2.2.1. *[24, 6] For any two sets* A *and* B,
$Cl^*(A \cup B) = Cl^*(A) \cup Cl^*(B)$.

Lemma 2.2.2. *[6] Let* (X,τ) *be a topological space,* $\mathcal{I}$ *and* $\mathcal{J}$ *be ideals on* X, *and let* A *and* B *be subsets of* X. *Then*

(1) $A \subset B \Rightarrow A^* \subset B^*$.

(2) If $\mathcal{I} \subset \mathcal{J}$, *then* $A^*(\mathcal{I}) \supset A^*(\mathcal{J})$.

(3) $A^* = Cl(A^*) \subset Cl(A)$ *(i.e,* A^* *is a closed subset of* $Cl(A)$*).*

(4) If $A \subset A^*$, *then* $A^* = Cl(A^*) = Cl(A) = Cl^*(A)$.

(5) $(A^*)^* \subset A^*$.

(6) $(A \cup B)^* = A^* \cup B^*$.

(7) If $U \in \tau$, *then* $U \cap A^* \subset (U \cap A)^*$.

Lemma 2.2.3. *[6] Let* (X,τ) *be a topological space with an ideal* $\mathcal{I}$ *on* X, *and* A *and* B *be subsets of* X. *Then*

(1) $A^* - B^* = ((A - B)^* - B^*) \subset (A - B)^*$.

(2) If $I \in \mathcal{I}$*, then* $(A \cup I)^* = A^* = (A - I)^*$.

Remark 2.2.1. *[24] For any subset* U *of* X*,* $Cl^*(U) \subset Cl(U)$ *since* $\tau \subset \tau^*$.

Remark 2.2.2. *[24]* $\tau^*(\mathcal{I}) = \tau$ *if and only if* $I \in \mathcal{I}$ *is closed in* (X, τ) *for every* $I \in \mathcal{I}$.

2.3 $R - \mathcal{I}-$open sets

Definition 2.3.1. *[42] A subset* A *of an ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be* $R - \mathcal{I}-$*open if* $Int(Cl^*(A)) = A$. *We call a subset* A *of* $(X, \tau, \mathcal{I})$ *is* $R - \mathcal{I}-$ *closed if its complement is* $R - \mathcal{I}-$*open.*

Definition 2.3.2. *A* $R - \mathcal{I}-$*space is an ideal topological space if it satisfies the following axioms:*

(i) Finite intersection of $R - \mathcal{I}-$*closed sets is* $R - \mathcal{I}-$*closed.*

(ii) Arbitrary union of $R - \mathcal{I}-$*open sets is* $R - \mathcal{I}-$*open.*

Definition 2.3.3. *The intersection of all* $R - \mathcal{I}-$*closed sets containing the set A is called the* $R - \mathcal{I}-$*closure of A and is denoted by* $R - \mathcal{I}-$*Cl(A). The* $R - \mathcal{I}-$*interior of A is defined as the union of all* $R - \mathcal{I}-$*open sets contained in A and is denoted by* $R - \mathcal{I} - Int(A)$. *The family of all* $R - \mathcal{I}-$*open (resp.* $R - \mathcal{I}-$*closed) sets of* $(X, \tau, \mathcal{I})$ *containing a point* $x \in X$ *is denoted by* $R\mathcal{I}O(X, x)$ *(resp.* $R\mathcal{I}C(X, x)$*).*

Lemma 2.3.1. *[42] Let* A *and* B *be subsets of an ideal topological space* $(X, \tau, \mathcal{I})$. *Then the following properties hold:*

(1) $Int(Cl^*(A))$ *is* $R - \mathcal{I}-$*open.*

(2) If A *and* B *are* $R - \mathcal{I}-$*open, then* $A \cap B$ *is* $R - \mathcal{I}-$*open.*

(3) If A *is regular open, then it is* $R - \mathcal{I}-$*open.*

Proof. (1) Let A be a subset of X and $V = Int(Cl^*(A))$. Then, we have $Int(Cl^*(V)) = Int(Cl^*(Int(Cl^*(A)))) \subset Int(Cl^*(Cl^*(A))) = Int(Cl^*(A)) = V$ and also $V = Int(V) \subset Int(Cl^*(V))$. Hence, we obtain $Int(Cl^*(V)) = V$.

(2) Let A and B be $R - \mathcal{I}-$open. Then, we have $A \cap B = Int(Cl^*(A)) \cap Int(Cl^*(B)) = Int(Cl^*(A) \cap Cl^*(B)) \supset Int(Cl^*(A \cap B)) \supset Int(A \cap B) = A \cap B$. Therefore, we obtain $A \cap B = Int(Cl^*(A \cap B))$. This shows that $A \cap B$ is $R - \mathcal{I}-$open.

(3) Let A be regular open. Since $\tau^* \supset \tau$, we have $A = Int(A) \subset Int(Cl^*(A)) \subset Int(Cl(A)) = A$ and hence $A = Int(Cl^*(A))$. Therefore, A is $R - \mathcal{I}-$open. □

2.4 Different forms of continuity

Definition 2.4.1. *[24] A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be **somewhat continuous** if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$, there exists an open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$.*

Definition 2.4.2. *[32] A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be **somewhat r-continuous** if for $U \in \sigma$ and $f^{-1}(U) \neq \phi$, there exists a regular open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$.*

Definition 2.4.3. *[18] A function $f : X \to Y$ is said to be **completely continuous** if $f^{-1}(V)$ is a regular open set in X, for every open set V in Y.*

Definition 2.4.4. *[18] A function $f : X \to Y$ is said to be **almost completely continuous** if $f^{-1}(V)$ is a regular open set in X for every regular open set V in Y.*

Definition 2.4.5. *[17] A function $f : X \to Y$ is said to be **perfectly continuous** if $f^{-1}(V)$ is clopen in X for every open set V in Y.*

Definition 2.4.6. *[17] A function $f : X \to Y$ is said to be* ***almost perfectly continuous*** *if $f^{-1}(V)$ is clopen for every regular open set V in Y.*

Definition 2.4.7. *[17] A function $f : X \to Y$ is said to be $cl-$**supercontinuous** if for each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.*

Definition 2.4.8. *[17] A function $f : X \to Y$ is said to be* ***almost*** *$cl-$**supercontinuous** if for each $x \in X$ and each regular open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.*

Definition 2.4.9. *[2] A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $R-\mathcal{I}-$**continuous** if for each $x \in X$ and for any open set $V \in \sigma$ containing $f(x)$, there exists a $R-\mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset V$.*

3

Separation axioms in terms of $R - \mathcal{I} -$open sets

3.1 Introduction

The idea of $R - \mathcal{I}-$open sets is widened to advance a generalization of separation axioms. What we developed in this chapter are the separation axioms $R - \mathcal{I} - T_i$ $i = 0, 1, 2$, $R - \mathcal{I}-$regular, $R - \mathcal{I}-$normal, completely $R - \mathcal{I}-$normal spaces. We study some properties of the above mentioned separation axioms. Also the hereditary nature of those axioms are discussed. Moreover we initiated the concept of compactness in this context.

Certain definitions are to be get acquainted with in prior to the main topic.

Definition 3.1.1. *A subset A of $(X, \tau, \mathcal{I})$ is called a $R - \mathcal{I}-$neighbourhood ($R - \mathcal{I}-$nbd) of $x \in X$ if there exists a $R - \mathcal{I}-$open set U such that $x \in U \subset A$.*

Theorem 3.1.1. *Let $x \in X$ and $V \subset X$. Then , if for any $R - \mathcal{I}-$open nbd U of x, $U \cap V \neq \phi$, then $x \in R - \mathcal{I} - Cl(V)$. The converse is also true.*

Definition 3.1.2. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is $R - \mathcal{I}-$open if the image of every $R - \mathcal{I}-$open set is $R - \mathcal{J}-$open.*

Definition 3.1.3. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is* $R^* - \mathcal{I}-$*continuous if the inverse image of every open set is* $R - \mathcal{I}-$*open.*

Definition 3.1.4. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is* $R^* - \mathcal{I}-$*irresolute if the inverse image of every* $R - \mathcal{J}-$*open set is* $R - \mathcal{I}-$*open.*

Definition 3.1.5. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is almost* $R - \mathcal{I}-$*continuous if* $f^{-1}(V)$ *is* $R - \mathcal{I}-$*open in* X *for every regular open set* V *in* Y.

3.2 $R - \mathcal{I}$ separation axioms

Definition 3.2.1. *A topological space* $(X, \tau, \mathcal{I})$ *is said to be:*

(i) $R - \mathcal{I} - T_0$ *if for each pair of distinct points* x *and* y *in* X, *there exists a* $R - \mathcal{I}-$*open set containing* x *but not* y.

(ii) $R - \mathcal{I} - T_1$ *if for each pair of distinct points* x *and* y *in* X, *there exist* $R - \mathcal{I}-$*open sets* U *and* V *in* X *such that* $x \in U, y \notin U$ *and* $y \in V, x \notin V$.

(iii) $R - \mathcal{I} - T_2$ *if for each pair of distinct points* x *and* y *in* X, *there exist disjoint* $R - \mathcal{I}-$*open sets* U *and* V *in* X *such that* $x \in U$ *and* $y \in V$.

Definition 3.2.2. *An ideal topological space* X *is* $R - \mathcal{I}-$*regular at a point* $x \in X$ *if for each pair consisting of a point* $x \in X$ *and a* $R - \mathcal{I}-$*closed set* B *not containing* x, *there exist disjoint* $R - \mathcal{I}-$*open sets containing* x *and* B *respectively.*

Definition 3.2.3. *An ideal topological space* X *is* $R - \mathcal{I}-$*normal if for each pair* A, B *of disjoint* $R - \mathcal{I}-$*closed sets of* X, *there exist disjoint* $R - \mathcal{I}-$*open sets containing* A *and* B *respectively.*

Definition 3.2.4. *An ideal topological space X is $R-\mathcal{I}-$Urysohn if for any two distinct points x and y, there exist $R-\mathcal{I}-$open sets U and V such that $x \in U,\ y \in V$ and $Cl^*(U) \cap Cl^*(V) = \phi$.*

Definition 3.2.5. *An ideal topological space X is completely $R-\mathcal{I}-$normal or hereditarily $R-\mathcal{I}-$normal if for any two sets A and B of X such that $Cl^*(A) \cap B = \phi = A \cap Cl^*(B)$, there exist disjoint $R-\mathcal{I}-$open sets U and V such that $A \subset U$ and $B \subset V$.*

Definition 3.2.6. *A $R-\mathcal{I}-$homeomorphism from a space X to a space Y is a bijection $f : (X, \tau, I) \to (Y, \sigma, \mathcal{J})$ such that f is $R^*-\mathcal{I}-$irresolute and $R-\mathcal{I}-$open.*

Theorem 3.2.1. *An ideal topological space $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-$regular if and only if for any $x \in X$ and any $R-\mathcal{I}-$open set G containing x there exists a $R-\mathcal{I}-$open set U with $x \in U$ and $R-\mathcal{I}-Cl(U) \subset G$.*

Proof. One part of the theorem is trivial from the definition of $R-\mathcal{I}-$regular space.

Now assume that X is $R-\mathcal{I}-$regular. Let $x \in X$ and G be a $R-\mathcal{I}-$open set containing x. Then by the regularity of X, there exists disjoint $R-\mathcal{I}-$open sets U and V such that $x \in U$ and $(X \setminus G) \subset V$. Then $R-\mathcal{I}-Cl(U) \subset (X \setminus V)$. But $(X \setminus V) \subset G$ gives $R-\mathcal{I}-Cl(U) \subset G$. □

Theorem 3.2.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-$normal if and only if for any $R-\mathcal{I}-$closed set C and any $R-\mathcal{I}-$open set G containing C there exists a $R-\mathcal{I}-$open set U such that $C \subset U$ and $R-\mathcal{I}-Cl(U) \subset G$.*

Proof. We are omitting the proof as it can be proved by the similar arguement as in the above theorem. □

Theorem 3.2.3. *Let Y be a $R-\mathcal{I}-T_2$ space. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ and $g : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ are $R-\mathcal{I}-$continuous functions, then $S = \{x \in X : f(x) = g(x)\}$ is $R-\mathcal{I}-$closed in X.*

Proof. Let $x \notin S$. That is $x \in (X \setminus S)$ and so $f(x) \neq g(x)$. Since Y is $R-\mathcal{I}-T_2$, there exist $R-\mathcal{I}-$open sets U and V of Y containing $f(x)$ and $g(x)$ respectively such that $U \cap V = \phi$. Since f and g are $R-\mathcal{I}-$continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are $R-\mathcal{I}-$open in X. Also, $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$. Name $f^{-1}(U) = P$ and $g^{-1}(V) = Q$ and let $R = P \cap Q$. Then R is a $R-\mathcal{I}-$open set in X containing x. Now, $f(R) \cap g(R) = f(P \cap Q) \cap g(P \cap Q) \subset f(P) \cap g(Q) = U \cap V = \phi$. Hence $S \cap R = \phi$ which implies $S \subset (X \setminus R)$. Thus $R-\mathcal{I}-Cl(S) \subset (X \setminus R)$ and so $x \notin R-\mathcal{I}-Cl(S)$. Thus $S = \{x \in X : f(x) = g(x)\}$ is $R-\mathcal{I}-$closed in X. □

Theorem 3.2.4. *An ideal topological space* $(X, \tau, \mathcal{I})$ *is* $R-\mathcal{I}-T_0$ *if and only if* $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ *for every distinct* x *and* y *in* X.

Proof. Let X be $R-\mathcal{I}-T_0$ and $x \neq y \in X$. Then there exists $R-\mathcal{I}-$open set U containing x but not y. Then $X-U$ is a $R-\mathcal{I}-$closed set containing y but not x. Hence $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$.

Conversely suppose $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ for $x \neq y \in X$. If X is not $R-\mathcal{I}-T_0$, then every $R-\mathcal{I}-$closed set containing x always contains y. But then $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{y\})$. This contradiction proves that X is $R-\mathcal{I}-T_0$. □

Theorem 3.2.5. *An ideal topological space* $(X, \tau, \mathcal{I})$ *is* $R-\mathcal{I}-T_1$ *if and only if every singleton is* $R-\mathcal{I}-$*closed.*

Proof. Let X be $R-\mathcal{I}-T_1$ and let $x \in X$. For any $y \in X$ with $y \neq x$, there exists a $R-\mathcal{I}-$open set V such that $y \in V$ and $x \notin V$. Then $\{x\} \subset V^c$. So for any $y \neq x \in X$, $y \notin R-\mathcal{I}-Cl(\{x\})$. Thus $R-\mathcal{I}-Cl(\{x\}) = \{x\}$ which is $R-\mathcal{I}-$closed.

Conversely, let $x, y \in X$ and let $\{x\}, \{y\}$ be $R-\mathcal{I}-$closed sets. Then $X-\{x\}$ and $X-\{y\}$ are $R-\mathcal{I}-$open sets. Hence X is $R-\mathcal{I}-T_1$. □

Theorem 3.2.6. *An ideal topological space $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-T_2$ if and only if for each $x \in X$, $\{x\} = \bigcap\{F_i :\ F_i\ \ is\ \ a\ \ R-\mathcal{I}-closed\ \ nbd\ \ of\ \ x\}$.*

Proof. Let X be $R-\mathcal{I}-T_2$. For $x \neq y \in X$, there exist two disjoint $R-\mathcal{I}-$open sets U_x and V_y such that $x \in U_x$ and $y \in V_y$. Then $U_x \subset X - V_y$ and so $R-\mathcal{I}-Cl(U_x) \subset X - V_y$. Denoting $R-\mathcal{I}-Cl(U_x)$ by F_x, $\{x\} = \bigcap\{F_x :\ F_x\ \ is\ \ a\ \ R-\mathcal{I}-closed\ \ nbd\ \ of\ \ x\}$.
For $x \in X$, assume the converse. Then for $x \neq y$, there exists a $R-\mathcal{I}-$closed nbd F such that $x \in F$ and $y \notin F$. Hence there exist disjoint $R-\mathcal{I}-$open sets U and V where $x \in U \subset F$ and $y \in V = F^c$ which will make X, a $R-\mathcal{I}-T_2$ space. □

Theorem 3.2.7. *If X is $R-\mathcal{I}-T_1$, then for each $x \in X$, $\{x\} = \bigcap\{U_i :\ U_i\ \ is\ \ a\ \ R-\mathcal{I}-open\ \ nbd\ \ of\ \ x\}$.*

Proof. Let X be $R-\mathcal{I}-T_1$. Fix $x \in X$. Then for each $y_i \in X$, there exist $R-\mathcal{I}-$open sets U_i and V_i such that $x \in U_i,\ y \notin U_i$ and $y \in V_i,\ x \notin V_i$. Then $\{x\} = \bigcap\{U_i :\ U_i\ \ is\ \ a\ \ R-\mathcal{I}-open\ \ nbd\ \ of\ \ x\}$. □

Theorem 3.2.8. *If for each $x \in X$, $\{x\}$ is $R-\mathcal{I}-$closed, then $\{x\} = \bigcap\{X - \{y\}\ \ for\ \ each\ \ y \neq x \in X\}$.*

Proof. Fix $x \in X$. Then for $y \neq x \in X$, $X - \{y\}$ is a $R-\mathcal{I}-$open nbd of x. Hence the statement. □

Remark 3.2.1. *None of the above defined separation axioms are hereditary and not even weakly hereditary as a subset A of an ideal topological space $(X, \tau, \mathcal{I})$ which is $R-\mathcal{I}-$open in X need not be $R-\mathcal{I}-$open with respect to the subspace topology $(Y, \tau_Y, \mathcal{I}_Y)$.*

Example 3.2.1. *Let $X = \{a, b, c, d\}, \tau = \{\phi, X, \{c\}, \{d\}, \{c, d\}, \{a, c\}, \{a, c, d\}\}$, $\mathcal{I} = \{\phi, \{b\}\}$. The $R-\mathcal{I}-$open sets are $\{d\}, \{a, c\}$ and X. Let $Y = \{a, b\}$. Then $\tau_Y = \{\phi, Y, \{a\}\}$ and $\mathcal{I}_Y = \{\phi, \{b\}\}$. Then $\{a\} = Y \cap \{a, c\}$ is not $R-\mathcal{I}-$open in $(Y, \tau_Y, \mathcal{I}_Y)$.*

Remark 3.2.2. *In an ideal topological space* $(X, \tau, \mathcal{I})$*,* $R - \mathcal{I}-$*normal and completely* $R - \mathcal{I}-$*normal are not comparable and the reason is as described as above.*

Remark 3.2.3. $R-\mathcal{I}-$*normal* $\Rightarrow$ $R-\mathcal{I}-$*regular* $\Rightarrow R-\mathcal{I}-T_2 \Rightarrow R-\mathcal{I}-T_1 \Rightarrow R-\mathcal{I}-T_0$

3.3 $R - \mathcal{I}$ compactness

Definition 3.3.1. *For a subset* A *of an ideal topological space* X*, a collection of* $R - \mathcal{I}-$*open sets* $\{A_\alpha : A_\alpha \subset X\}$ *is a* $R - \mathcal{I}-$*open cover of* A *if* $A \subset \cup A_\alpha$.

Definition 3.3.2. *An ideal topological space is said to be* $R - \mathcal{I}-$*compact (*$R - \mathcal{I}-$*Lindeloff) if every* $R - \mathcal{I}-$*open cover admits a finite (countable) subcover.*

Definition 3.3.3. *A subset* A *of an ideal topological* X *is called* $R - \mathcal{I}-$*compact relative to* X *if every collection* $\{A_\alpha\}$ *of* $R - \mathcal{I}-$*open sets of* X *such that* $A \subset \cup A_\alpha$*, has a finite subcover.*

Theorem 3.3.1. *Let* V *be a* $R-\mathcal{I}-$*closed subset of a* $R-\mathcal{I}-$*compact space* X*. Then* V *is* $R - \mathcal{I}-$*compact relative to* X*.*

Proof. Consider a covering $\mathcal{A}$ of V by $R-\mathcal{I}-$open sets in X. By constructing a $R-\mathcal{I}-$open covering $\mathcal{B}$ of X by $\mathcal{B} = \mathcal{A} \cup \{X - V\}$, clearly some finite subcover of $\mathcal{B}$ will cover X. To obtain a subcover of V, discard $X - V$ from the subcover of $\mathcal{B}$ of X. Thus, V is $R - \mathcal{I}-$compact relative to X. □

Theorem 3.3.2. *The image of a* $R - \mathcal{I}-$*compact space under an onto* $R^* - \mathcal{I}-$*irresolute function is* $R - \mathcal{I}-$*compact.*

Proof. Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be an onto $R^* - \mathcal{I}-$irresolute function and let X be a $R - \mathcal{I}-$compact space. Let $\mathcal{A}$ be a covering of $f(X)$ by $R-\mathcal{I}-$open sets in Y. Consider $\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\}$. This is a collection of $R-\mathcal{I}-$open sets in X which covers X, since f is $R^*-\mathcal{I}-$irresolute. Hence, finitely many members of $\mathcal{B}$, say, $f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)$ cover X. But then $A_1, A_2, \ldots, A_n$ will cover $f(X)$ since f is onto. □

Theorem 3.3.3. *The image of a $R - \mathcal{I}-$compact space under an onto $R^* - \mathcal{I}-$continuous function is compact.*

Proof. Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be an onto $R^* - \mathcal{I}-$continuous function and let X be a $R - \mathcal{I}-$compact space. Let $\mathcal{A}$ be a covering of $f(X)$ by open sets in Y. Consider $\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\}$. This is a collection of $R-\mathcal{I}-$open sets in X which covers X, since f is $R^*-\mathcal{I}-$continuous. Hence, finitely many members of $\mathcal{B}$, say, $f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)$ cover X. But then $A_1, A_2, \ldots, A_n$ will cover $f(X)$ since f is onto. □

Theorem 3.3.4. *The image of a compact space under an onto almost $R-\mathcal{I}-$continuous function is $R - \mathcal{I}-$compact.*

Proof. We are omitting the proof as it can be proved by the similar arguement as in the above theorem. □

4

Weak separation axioms

4.1 Introduction

We extended the notion of $R-\mathcal{I}-$open sets to develop weak separation axioms $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_1$. We studied certain characterisations of these axioms and analysed the relationship between them. Further, relations with the separation axioms discussed in chapter 3 are examined.

First let us memorise the definitions of R_0 and R_1 spaces.

Definition 4.1.1. *[3] A topological space (X, τ) is said to be R_0 if every open set contains the closure of each of its singletons.*

[3] A topological space (X, τ) is said to be R_1 if for $x, y \in X$ with $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U, V such that $Cl(\{x\}) \subset U$ and $Cl(\{y\}) \subset V$.

4.2 $R-\mathcal{I}-R_0$ spaces

Definition 4.2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. The $R-\mathcal{I}-$kernel of A is denoted by $\mathcal{I}_R Ker(A)$ and is defined to be the set $\mathcal{I}_R Ker(A) = \cap\{G \in R\mathcal{I}O(X) : A \subset G\}$.*

Lemma 4.2.1. *For subsets A, B of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

1. $A \subset \mathcal{I}_R Ker(A)$
2. *If* $A \subset B$*, then* $\mathcal{I}_R Ker(A) \subset \mathcal{I}_R Ker(B)$
3. *If* A *is* $R-\mathcal{I}-$*open, then* $\mathcal{I}_R Ker(A) = A$
4. $x \in \mathcal{I}_R Ker(A)$ *if and only if* $A \cap D \neq \phi$ *for any* $R-\mathcal{I}-$*closed set* D *of* X *such that* $x \in D$.

Proof. The proof follows directly from the definition of $R-\mathcal{I}-$kernel of A for $A \subset X$. □

Theorem 4.2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $x, y \in X$*. Then* $y \in \mathcal{I}_R Ker(\{x\})$ *if and only if* $x \in R-\mathcal{I}-Cl(\{y\})$.

Proof. Let $x, y \in X$. Suppose $y \notin \mathcal{I}_R Ker(\{x\})$. Then there exists $U \in R\mathcal{I}O(X, x)$ such that $y \notin U$. So $X - U$ is a $R-\mathcal{I}-$closed set containing y but not x. Therefore $x \notin R-\mathcal{I}-Cl(\{y\})$. Conversely suppose $x \notin R-\mathcal{I}-Cl(\{y\})$. Then there exists $V \in R\mathcal{I}C(X, y)$ such that $x \notin V$. So $X - V$ is a $R-\mathcal{I}-$open set containing x but not y. Hence $y \notin \mathcal{I}_R Ker(\{x\})$. □

Theorem 4.2.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* S *be a subset of* X*. Then* $\mathcal{I}_R Ker(S) = \{x \in X : R-\mathcal{I}-Cl(\{x\}) \cap S \neq \phi\}$.

Proof. Let $S \subset X$ and let $x \in \mathcal{I}_R Ker(S)$. Suppose $S \cap R-\mathcal{I}-Cl(\{x\}) = \phi$. We have $X - (R-\mathcal{I}-Cl(\{x\}))$ is a $R-\mathcal{I}-$open set not containing x. But $S \subset X - (R-\mathcal{I}-Cl(\{x\}))$. This implies $x \notin \mathcal{I}_R Ker(S)$, which is a contradiction. Hence $S \cap R-\mathcal{I}-Cl(\{x\}) \neq \phi$.

Now suppose $x \in X$ and $S \cap (R-\mathcal{I}-Cl(\{x\})) \neq \phi$ and suppose that $x \notin \mathcal{I}_R Ker(S)$. Then there exists a $R-\mathcal{I}-$open set U such that $S \subset U$ and $x \notin U$. Let $y \in S \cap (R-\mathcal{I}-Cl(\{x\}))$. Then $y \in S \subset U$ and also

$y \in R-\mathcal{I}-Cl(\{x\}) \subset X-U$. This will make a contradiction and hence the proof. □

Definition 4.2.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is called a $R-\mathcal{I}-R_0$ space if every $R-\mathcal{I}-$open set contains the $R-\mathcal{I}-$closure of each of its singletons.*

Theorem 4.2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then X is $R-\mathcal{I}-T_1$ if and only if it is $R-\mathcal{I}-T_0$ and $R-\mathcal{I}-R_0$.*

Proof. Let X be a $R-\mathcal{I}-T_1$ space. Then clearly X is a $R-\mathcal{I}-T_0$ space and also X is a $R-\mathcal{I}-R_0$ space.
Conversely, let X be both $R-\mathcal{I}-T_0$ and $R-\mathcal{I}-R_0$. Let x, y be any two distinct points of X . Since X is $R-\mathcal{I}-T_0$, there exists a $R-\mathcal{I}-$open set U such that $x \in U$ and $y \notin U$ or there exists a $R-\mathcal{I}-$open set V such that $y \in V$and $x \notin V$. Since X is $R-\mathcal{I}-R_0$, then $R-\mathcal{I}-Cl(\{x\}) \subset U$ for $x \in U$. Since $y \notin U, y \notin R-\mathcal{I}-Cl(\{x\})$. So $y \in X-(R-\mathcal{I}-Cl(\{x\})) = W, say.$ Thus U and W are $R-\mathcal{I}-$open sets containing x and y respectively. Also $x \notin W$ and $y \notin U$. Hence X is $R-\mathcal{I}-T_1$. □

Remark 4.2.1. *Every $R-\mathcal{I}-T_1$ space is $R-\mathcal{I}-R_0$, since in $R-\mathcal{I}-T_1$ every singletons are $R-\mathcal{I}-$closed. The converse is not true in general.*

Example 4.2.1. *Let $X = \{a, b, c\}, \tau = \{\phi, X, \{a, b\}, \{c\}\}, \mathcal{I} = \{\phi, \{a, b\}, \{a\}, \{b\}\}$. Then $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-R_0$, but not $R-\mathcal{I}-T_1$.*

Example 4.2.2. *Let $X = \{a, b, c\}, \tau = \{\phi, X, \{a, b\}\}, \mathcal{I} = \{\phi, \{a, b\}\}$. Then $(X, \tau, \mathcal{I})$ is not $R-\mathcal{I}-R_0$ and not $R-\mathcal{I}-T_0$.*

Remark 4.2.2. *From the above two examples it is clear that $R-\mathcal{I}-T_0$ and $R-\mathcal{I}-R_0$ are independent.*

Theorem 4.2.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for x, y in X, $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\}) \Longleftrightarrow R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$.*

Proof. Let $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\})$. Then there exists $z \in X$ such that $z \in \mathcal{I}_R Ker(\{x\})$ and $z \notin \mathcal{I}_R Ker(\{y\})$. Also by theorem 4.2.1, $y \notin R - \mathcal{I} - Cl(\{z\})$ and $x \in R - \mathcal{I} - Cl(\{z\})$. So $R - \mathcal{I} - Cl(\{x\}) \subset R - \mathcal{I} - Cl(\{z\})$ and so $y \notin R - \mathcal{I} - Cl(\{x\})$. Hence $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$. Conversely, let $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$. Then there exists $z \in X$ such that $z \in R - \mathcal{I} - Cl(\{x\})$ and $z \notin R - \mathcal{I} - Cl(\{y\})$. This implies that there exists a $R - \mathcal{I}-$open set containing z and x but not y. So $y \notin \mathcal{I}_R Ker(\{x\})$. Hence $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\})$. □

Theorem 4.2.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent.*

(i) $(X, \tau, \mathcal{I})$ *is a* $R - \mathcal{I} - R_0$ *space.*

(ii) For any $P \in R\mathcal{I}C(X), x \notin P$ *implies* $P \subset U$ *and* $x \notin U$ *for some* $U \in R\mathcal{I}O(X)$.

(iii) For any $P \in R\mathcal{I}C(X), x \notin P$ *implies* $P \cap (R - \mathcal{I} - Cl(\{x\})) = \phi$.

(iv) For any two distinct points x *and* y *of* X, $R - \mathcal{I} - Cl(\{x\}) = R - \mathcal{I} - Cl(\{y\})$ *or* $(R - \mathcal{I} - Cl(\{x\})) \cap (R - \mathcal{I} - Cl(\{y\})) = \phi$.

Proof. (i) $\Rightarrow$ (ii)

Let P be a $R - \mathcal{I}-$closed set of X and $x \notin P$. Since X is $R - \mathcal{I} - R_0$, $R - \mathcal{I} - Cl(\{x\}) \subset X - P$. Denote $U = X - (R - \mathcal{I} - Cl(\{x\}))$. Then U is a $R - \mathcal{I}-$open set and $P \subset U$ and $x \notin U$.

(ii) $\Rightarrow$ (iii)

Let $P \in R\mathcal{I}C(X)$ and $x \notin P$. Then $U = X - (R - \mathcal{I} - Cl(x)) \in R\mathcal{I}O(X)$ such that $P \subset U$ and $x \notin U$. Hence $P \cap (R - \mathcal{I} - Cl(\{x\})) = \phi$.

(iii) $\Rightarrow$ (iv)

Suppose $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$ for $x \neq y \in X$. Then there exists $z \in X$ with $z \in R - \mathcal{I} - Cl(\{x\})$ and $z \notin R - \mathcal{I} - Cl(\{y\})$. So we get,

$x \notin R-\mathcal{I}-Cl(\{y\})$. Hence $(R-\mathcal{I}-Cl(\{x\})) \cap (R-\mathcal{I}-Cl(\{y\})) = \phi$, by (iii) and taking $P = R-\mathcal{I}-Cl(\{y\})$.

(iv) $\Rightarrow$ (i)

Let V be a $R-\mathcal{I}-$open set in X and let $x \in V$. Then for each $y \notin V$, $x \neq y$ and $x \notin R-\mathcal{I}-Cl(\{y\})$. So by (iv), $(R-\mathcal{I}-Cl(\{x\})) \cap (R-\mathcal{I}-Cl(\{y\})) = \phi$ for each $y \notin V$. Hence $(R-\mathcal{I}-Cl(\{x\})) \cap (\cup_{y \in X-V} R-\mathcal{I}-Cl(\{y\})) = \phi$. Since V is $R-\mathcal{I}-$open and $y \in X-V$, $R-\mathcal{I}-Cl(\{y\}) \subset X-V$ and so $X-V = \cup_{y \in X-V} R-\mathcal{I}-Cl(\{y\})$. Therefore $(X-V) \cap R-\mathcal{I}-Cl(\{x\}) = \phi$ or $R-\mathcal{I}-Cl(\{x\}) \subset V$. Hence $(X,\tau,\mathcal{I})$ is a $R-\mathcal{I}-R_0$ space. □

Corollary 4.2.1. *An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$ if and only if for any two points $x, y \in X$, $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ implies $(R-\mathcal{I}-Cl(\{x\})) \cap (R-\mathcal{I}-Cl(\{y\})) = \phi$.*

Proof. Let $(X,\tau,\mathcal{I})$ be $R-\mathcal{I}-R_0$. Then by theorem 4.2.5, the statement holds.

Conversely, let U be a $R-\mathcal{I}-$open set of X containing x. We claim $R-\mathcal{I}-Cl(\{x\}) \subset U$. For that let $y \in X-U$. So $x \notin R-\mathcal{I}-Cl(\{y\})$. This implies $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$. By assumption, $(R-\mathcal{I}-Cl(\{x\})) \cap (R-\mathcal{I}-Cl(\{y\})) = \phi$. Thus $y \notin R-\mathcal{I}-Cl(\{x\})$ and hence the claim. □

Theorem 4.2.6. *An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$ if and only if for any two points $x, y \in X$, $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\})$ implies $\mathcal{I}_R Ker(\{x\}) \cap \mathcal{I}_R Ker(\{y\}) = \phi$.*

Proof. Let $(X,\tau,\mathcal{I})$ be $R-\mathcal{I}-R_0$. By theorem 4.2.4, for any two points $x, y \in X$, if $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\})$, then $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$. Contrary suppose $z \in \mathcal{I}_R Ker(\{x\}) \cap \mathcal{I}_R Ker(\{y\})$. Since $z \in \mathcal{I}_R Ker(\{x\})$, $x \in R-\mathcal{I}-Cl(\{z\})$. Also $x \in R-\mathcal{I}-Cl(\{x\})$. Then by theorem 4.2.5(iv), $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{z\})$. Thus, in a similar way, we get $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{z\}) = R-\mathcal{I}-Cl(\{y\})$, which is not true. From this contradiction, we have $\mathcal{I}_R Ker(\{x\}) \cap \mathcal{I}_R Ker(\{y\}) = \phi$. Now assume the converse. By theorem 4.2.4, if $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$, then $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\})$. So by assumption, $\mathcal{I}_R Ker(\{x\}) \cap \mathcal{I}_R Ker(\{y\}) = \phi$. Now let $z \in R-\mathcal{I}-Cl(\{x\})$ and this implies $x \in \mathcal{I}_R Ker(\{z\})$. Therefore $\mathcal{I}_R Ker(\{x\}) \cap \mathcal{I}_R Ker(\{z\}) \neq \phi$. Then by hypothesis, $\mathcal{I}_R Ker(\{x\}) = \mathcal{I}_R Ker(\{z\})$. So if $z \in R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Cl(\{y\})$, then $\mathcal{I}_R Ker(\{x\}) = \mathcal{I}_R Ker(\{z\}) = \mathcal{I}_R Ker(\{y\})$, which is a contradiction. Hence we get $R-\mathcal{I}-Cl(\{x\}) \cap R-\mathcal{I}-Cl(\{y\}) = \phi$. Thus by corollary, $(X,\tau,\mathcal{I})$ is a $R-\mathcal{I}-R_0$ space. □

Theorem 4.2.7. *Let $(X,\tau,\mathcal{I})$ be a $R-\mathcal{I}-R_0$ space. Then $x \in R-\mathcal{I}-Cl(\{y\})$ if and only if $y \in R-\mathcal{I}-Cl(\{x\})$ for any $x, y \in X$. The converse is also true.*

Proof. Let $(X,\tau,\mathcal{I})$ be $R-\mathcal{I}-R_0$. Let $x \in R-\mathcal{I}-Cl(\{y\})$ and let U be a $R-\mathcal{I}-$open set of X containing y. Then $R-\mathcal{I}-Cl(\{y\}) \subset U$ since X is $R-\mathcal{I}-R_0$. So, $x \in R-\mathcal{I}-Cl(\{y\})$ implies $x \in U$. That means, every $R-\mathcal{I}-$open set containing y contains x. Hence $y \in R-\mathcal{I}-Cl(\{x\})$. Assume the converse. Let U be a $R-\mathcal{I}-$open set in X containing x. If $y \notin U$, then $x \notin R-\mathcal{I}-Cl(\{y\})$ and hence $y \notin R-\mathcal{I}-Cl(\{x\})$. This means $R-\mathcal{I}-Cl(\{x\}) \subset U$. Then $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$. □

Theorem 4.2.8. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then the following are equivalent:*

(i) $(X,\tau,\mathcal{I})$ *is a* $R-\mathcal{I}-R_0$ *space.*

(ii) For any $\phi \neq P \in X$ *and* $U \in R\mathcal{I}O(X)$ *with* $P \cap U \neq \phi$, *there exists* $V \in R\mathcal{I}C(X)$ *such that* $P \cap V \neq \phi$ *and* $V \subset U$.

(iii) For any $U \in R\mathcal{I}O(X)$, $U = \cup\{V \in R\mathcal{I}C(X) : V \subset U\}$.

(iv) *For any* $V \in R\mathcal{I}C(X)$, $V = \cap\{U \in R\mathcal{I}O(X) : V \subset U\}$.

(v) *For any* $x \in X$, $R - \mathcal{I} - Cl(\{x\}) \subset \mathcal{I}_R Ker(\{x\})$.

Proof. (i) $\Rightarrow$ (ii)

Let $\phi \neq P \in X$ and U be an $R - \mathcal{I}-$open set with $P \cap U \neq \phi$. Let $x \in P \cap U$. Since $x \in U$ and X is $R - \mathcal{I} - R_0$, $R - \mathcal{I} - Cl(\{x\}) \subset U$. Let $V = R - \mathcal{I} - Cl(\{x\})$. Then $V \in R\mathcal{I}C(X)$ and $V \subset U$ and $P \cap V \neq \phi$.

(ii) $\Rightarrow$ (iii)

Let $U \in R\mathcal{I}O(X)$. Then clearly $\cup\{V \in R\mathcal{I}C(X) : V \subset U\} \subset U$. Now let $x \in U$. Then there exists $V \in R\mathcal{I}C(X)$ such that $x \in V \subset U$ as in (ii). Thus $x \in V \subset \cup\{V \in R\mathcal{I}C(X) : V \subset U\}$. Hence $U = \cup\{V \in R\mathcal{I}C(X) : V \subset U\}$.

(iii) $\Rightarrow$ (iv)

It directly follows from (iii).

(iv) $\Rightarrow$ (v)

Let $x \in X$ and let $y \notin \mathcal{I}_R Ker(\{x\})$. Then there exists $G \in R\mathcal{I}O(X)$ such that $x \in G$ and $y \notin G$. So $R - \mathcal{I} - Cl(\{y\}) \cap G = \phi$. Then by (iv), $(\cap\{U \in R\mathcal{I}O(X) : R - \mathcal{I} - Cl(\{y\}) \subset U\}) \cap G = \phi$. So there exists an $R - \mathcal{I}-$open set U such that $x \notin U$ and $R - \mathcal{I} - Cl(\{y\}) \subset U$. Hence $R - \mathcal{I} - Cl(\{x\}) \cap U = \phi$ and $y \notin R - \mathcal{I} - Cl(\{x\})$. Thus $R - \mathcal{I} - Cl(\{x\}) \subset \mathcal{I}_R Ker(\{x\})$.

(v) $\Rightarrow$ (i)

Let U be an $R - \mathcal{I}-$open set in X and $x \in U$. Let $y \in \mathcal{I}_R Ker(\{x\})$. Then $y \in U$. Then $\mathcal{I}_R Ker(\{x\}) \subset U$. Thus $x \in R - \mathcal{I} - Cl(\{x\}) \subset \mathcal{I}_R Ker(\{x\}) \subset U$. Hence X is a $R - \mathcal{I} - R_0$ space. $\square$

Theorem 4.2.9. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then the following are equivalent:*

(i) $(X, \tau, \mathcal{I})$ *is a* $R - \mathcal{I} - R_0$ *space.*

(ii) If V is a $R-\mathcal{I}-$closed subset of X, then $V = \mathcal{I}_R Ker(V)$.

(iii) If V is a $R-\mathcal{I}-$closed subset of X and $x \in V$, then $\mathcal{I}_R Ker(\{x\}) \subset V$.

(iv) If $x \in X$, then $\mathcal{I}_R Ker(\{x\}) \subset R-\mathcal{I}-Cl(\{x\})$.

Proof. (i) $\Rightarrow$ (ii)

Let V be a $R-\mathcal{I}-$closed subset of X and let $x \in X - V$. Since X is a $R-\mathcal{I}-R_0$ space and $X - V \in R\mathcal{I}O(X, x)$, $R-\mathcal{I}-Cl(\{x\}) \subset X - V$. Then $\mathcal{I}_R Ker(V) \subset X - (R-\mathcal{I}-Cl(\{x\}))$. Also $x \notin \mathcal{I}_R Ker(V)$. Thus $\mathcal{I}_R Ker(V) = V$.

(ii) $\Rightarrow$ (iii)

Since $U \subset V$ implies $\mathcal{I}_R Ker(U) \subset \mathcal{I}_R Ker(V)$, it follows that $\mathcal{I}_R Ker(\{x\}) \subset \mathcal{I}_R Ker(V)$ for $x \in V$. Therefore $\mathcal{I}_R Ker(\{x\}) \subset V$ from (ii).

(iii) $\Rightarrow$ (iv)

Clearly $x \in R-\mathcal{I}-Cl(\{x\})$. From (iii) $\mathcal{I}_R Ker(\{x\}) \subset R-\mathcal{I}-Cl(\{x\})$.

(iv) $\Rightarrow$ (i)

Let $x \in R-\mathcal{I}-Cl(\{y\})$. Then by theorem 4.2.1, $y \in \mathcal{I}_R Ker(\{x\})$. Thus we get $y \in \mathcal{I}_R Ker(\{x\}) \subset R-\mathcal{I}-Cl(\{x\})$ by (iv). Hence $x \in R-\mathcal{I}-Cl(\{y\})$ implies $y \in R-\mathcal{I}-Cl(\{x\})$. Clearly the reverse implication holds. Thus by theorem 4.2.8, X is a $R-\mathcal{I}-R_0$ space. $\square$

Corollary 4.2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $(X, \tau, \mathcal{I})$ is $R-\mathcal{I}-R_0$, then $\mathcal{I}_R Ker(\{x\}) = R-\mathcal{I}-Cl(\{x\})$ for all $x \in X$. The converse is also true.*

Proof. Suppose $(X, \tau, \mathcal{I})$ is a $R-\mathcal{I}-R_0$ space. By theorem 4.2.9(v) and theorem 4.2.10(iv) the statement is obvious. The converse is trivial by theorem 4.2.10. $\square$

4.3 $R-\mathcal{I}-R_1$ spaces

Definition 4.3.1. *An ideal topological space* $(X,\tau,\mathcal{I})$ *is called a* $R-\mathcal{I}-R_1$ *space if for* $x,y \in X$ *with* $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ *there exist disjoint* $R-\mathcal{I}-$*open sets* U,V *such that* $R-\mathcal{I}-Cl(\{x\}) \subset U$ *and* $R-\mathcal{I}-Cl(\{y\}) \subset V$.

Theorem 4.3.1. *Every* $R-\mathcal{I}-R_1$ *space is* $R-\mathcal{I}-R_0$.

Proof. Let $(X,\tau,\mathcal{I})$ be a $R-\mathcal{I}-R_1$ space and $x,y \in X$. Let U be a $R-\mathcal{I}-$open set containing x but not y. So $x \notin R-\mathcal{I}-Cl(\{y\})$. Then we have $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$. By hypothesis, then there exists $R-\mathcal{I}-$open set V such that $R-\mathcal{I}-Cl(\{y\}) \subset V$ and also $x \notin V$ and this implies $y \notin R-\mathcal{I}-Cl(\{x\})$. Thus $R-\mathcal{I}-Cl(\{x\}) \subset U$. Hence $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$. □

Remark 4.3.1. *The converse of the above theorem is not true in general.*

Example 4.3.1. *Let* $X=\{a,b,c\}, \tau=\{\phi,X,\{a\},\{b\},\{a,b\},\{b,c\}\}, \mathcal{I}=\{\phi,\{a\}\}$. *The* $R-\mathcal{I}-$*open sets are* $\{a\},\{b,c\},X$. *Then* $(X,\tau,\mathcal{I})$ *is* $R-\mathcal{I}-R_0$ *but not* $R-\mathcal{I}-R_1$.

Remark 4.3.2. R_0 *implies* $R-\mathcal{I}-R_0$ *but the converse is not true.*

Example 4.3.2. *Consider the same example written above (Example 4.3.1).* $(X,\tau,\mathcal{I})$ *is* $R-\mathcal{I}-R_0$ *but not* R_0.

Remark 4.3.3. R_1 *implies* $R-\mathcal{I}-R_1$ *but the converse is not true.*

Example 4.3.3. *Let* $X=\{a,b,c\}, \tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}, \mathcal{I}=\{\phi,\{a\}\}$. *The* $R-\mathcal{I}-$*open sets are* $\{a\},\{b,c\},X$. *Then* $(X,\tau,\mathcal{I})$ *is* $R-\mathcal{I}-R_1$ *but not* R_1.

Theorem 4.3.2. *An ideal topological space* $(X,\tau,\mathcal{I})$ *is* $R-\mathcal{I}-R_1$ *if and only if for any two points* $x,y \in X$, $\mathcal{I}_R Ker(\{x\}) \neq \mathcal{I}_R Ker(\{y\})$ *implies* *there exists disjoint* $R-\mathcal{I}-$*open sets* U,V *such that* $R-\mathcal{I}-Cl(\{x\})\subset U$ *and* $R-\mathcal{I}-Cl(\{y\})\subset V$*.*

Proof. By theorem 4.2.4, theorem directly follows. $\square$

Theorem 4.3.3. *Let* $(X,\tau,\mathcal{I})$ *be a* $R-\mathcal{I}-R_1$ *space. Then for* $x,y\in X$ *with* $R-\mathcal{I}-Cl(\{x\})\neq R-\mathcal{I}-Cl(\{y\})$*, there exists* $R-\mathcal{I}-$*closed sets* K_1 *and* K_2 *such that* $x\in K_1, y\in K_2, y\notin K_1, x\notin K_2$ *and* $K_1\cup K_2=X$*.*

Proof. Let $(X,\tau,\mathcal{I})$ be $R-\mathcal{I}-R_1$. Suppose $x,y\in X$ with $R-\mathcal{I}-Cl(\{x\})\neq R-\mathcal{I}-Cl(\{y\})$. Then there exists disjoint $R-\mathcal{I}-$open sets U,V such that $R-\mathcal{I}-Cl(\{x\})\subset U$ and $R-\mathcal{I}-Cl(\{y\})\subset V$. Let $K_1=X-V$ and $K_2=X-U$. Then K_1 and K_2 are $R-\mathcal{I}-$closed sets such that $x\in K_1, y\in K_2, y\notin K_1, x\notin K_2$ and $K_1\cup K_2=X$.

Assume the converse. To Show X is $R-\mathcal{I}-R_1$, we first prove X is $R-\mathcal{I}-R_0$. For that suppose U be a $R-\mathcal{I}-$open set containing x and suppose $R-\mathcal{I}-Cl(\{x\})$ is not a subset of U. So $R-\mathcal{I}-Cl(\{x\})\cap U^c\neq\phi$. Let $y\in R-\mathcal{I}-Cl(\{x\})\cap U^c$. Then $R-\mathcal{I}-Cl(\{x\})\neq R-\mathcal{I}-Cl(\{y\})$. Then by hypothesis there exists $R-\mathcal{I}-$closed sets K_1 and K_2 such that $x\in K_1, y\in K_2, y\notin K_1, x\notin K_2$ and $K_1\cup K_2=X$. Thus there exists a $R-\mathcal{I}-$closed set containing x but not y, which is a contradiction. Hence $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$. Now assume that $u,v\in X$ with $R-\mathcal{I}-Cl(\{u\})\neq R-\mathcal{I}-Cl(\{v\})$. **Then as earlier there exists** $R-\mathcal{I}-$**closed sets** L_1 **and** L_2 **such that** $x\in L_1, y\in L_2, y\notin L_1, x\notin L_2$ **and** $L_1\cup L_2=X$**. Thus** $u\in L_1-L_2$ **and** $v\in L_2-L_1$**. But** $L_1-L_2=X-L_2$ **and** $L_2-L_1=X-L_1$ **and both are** $R-\mathcal{I}-$**open. Since** X **is** $R-\mathcal{I}-R_0$**,** $R-\mathcal{I}-Cl(\{u\})\subset L_1-L_2$ **and** $R-\mathcal{I}-Cl(\{v\})\subset L_2-L_1$**.** Therefore $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_1$. $\square$

Theorem 4.3.4. *The following statements are equivalent:*

(i) $(X, \tau, \mathcal{I})$ *is a* $R - \mathcal{I} - R_1$ *space.*

(ii) *For each* $x, y \in X$ *either one of the following holds.*

(a) *if* U *is* $R - \mathcal{I}-$*open, then* $x \in U$ *if and only if* $y \in U$*.*

(b) *there exist disjoint* $R-\mathcal{I}-$*open sets* U *and* V *such that* $x \in U$ *and* $y \in V$*.*

(iii) *If* $x, y \in X$ *with* $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$*, there exists* $R-\mathcal{I}-$*closed sets* K_1 *and* K_2 *such that* $x \in K_1, x \notin K_2, y \notin K_1, y \in K_2$ *and* $K_1 \cup K_2 = X$*.*

Proof. (i) $\Rightarrow$ (ii)

Let $x, y \in X$. Case 1: $R - \mathcal{I} - Cl(\{x\}) = R - \mathcal{I} - Cl(\{y\})$.

Let U be an $R-\mathcal{I}-$open set. Then if $x \in U$ we have $y \in R-\mathcal{I}-Cl(\{x\}) \subset U$ and if $y \in U$ then we have $x \in R - \mathcal{I} - Cl(\{y\}) \subset U$. Thus $x \in U$ if and only if $y \in U$.

Case 2: $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$.

Then there exists disjoint $R-\mathcal{I}-$open sets U and V such that $x \in R-\mathcal{I}-Cl(\{x\}) \subset U$ and $y \in R - \mathcal{I} - Cl(\{y\}) \subset V$.

(ii) $\Rightarrow$ (iii)

Let $x, y \in X$ such that $R - \mathcal{I} - Cl(\{x\}) \neq R - \mathcal{I} - Cl(\{y\})$. Then either $x \notin R-\mathcal{I}-Cl(\{y\})$ or $y \notin R-\mathcal{I}-Cl(\{x\})$. Suppose $x \notin R-\mathcal{I}-Cl(\{y\})$. Then there exists a $R - \mathcal{I}-$open set S such that $x \in S$ and $y \notin S$. So by (ii) there exists disjoint $R - \mathcal{I}-$open sets U and V such that $x \in U$ and $y \in V$. Let $K_1 = V^c$ and $K_2 = U^c$. Then K_1 and K_2 are $R-\mathcal{I}-$closed sets such that $x \in K_1, y \in K_2, y \notin K_1, x \notin K_2$ and $K_1 \cup K_2 = X$.

(iii) $\Rightarrow$ (i)

This is the statement of theorem 4.3.3. □

Theorem 4.3.5. *An ideal topological space is* $R - \mathcal{I} - R_1$ *if and only if*

$x \notin R-\mathcal{I}-Cl(\{y\})$ implies that x and y have disjoint $R-\mathcal{I}-$open neighbourhoods.

Proof. Let $(X,\tau,\mathcal{I})$ be $R-\mathcal{I}-R_1$ and let $x \notin R-\mathcal{I}-Cl(\{y\})$. Then $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$. Then x and y have disjoint $R-\mathcal{I}-$open neighbourhoods.

Assume the converse. First, we prove that $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$. Let U be a $R-\mathcal{I}-$open set containing x. Suppose $y \notin U$. Then $R-\mathcal{I}-Cl(\{y\})\cap U = \phi$. Also $x \notin R-\mathcal{I}-Cl(\{y\})$. Then there exists disjoint $R-\mathcal{I}-$open sets V_1 and V_2 such that $x \in V_1$ and $y \in V_2$. Then $R-\mathcal{I}-Cl(\{x\}) \subset R-\mathcal{I}-Cl(V_1)$ and $(R-\mathcal{I}-Cl(\{x\})) \cap V_2 \subset (R-\mathcal{I}-Cl(V_1)) \cap V_2 = \phi$. Thus $y \notin R-\mathcal{I}-Cl(\{x\})$. Hence $R-\mathcal{I}-Cl(\{x\}) \subset U$ and $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$. Now suppose $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$. Then there exists an element $w \in R-\mathcal{I}-Cl(\{x\})$ and $w \notin R-\mathcal{I}-Cl(\{y\})$. By assumption there exists disjoint $R-\mathcal{I}-$open sets W_1 and W_2 such that $w \in W_1$ and $y \in W_2$. Since $w \in R-\mathcal{I}-Cl(\{x\})$, $x \in W_1$ for otherwise if $x \notin W_1$, then $w \in R-\mathcal{I}-Cl(\{x\}) \subset {W_1}^c$, a contradiction. Since $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_0$, $R-\mathcal{I}-Cl(\{x\}) \subset W_1$ and $R-\mathcal{I}-Cl(\{y\}) \subset W_2$. Thus $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_1$. □

5

Separation axioms weaker than $R-\mathcal{I}-R_0$

5.1 Introduction

We have discussed the weak separation axioms $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_1$. In this chapter, we concentrate on separation axioms which are weaker than $R-\mathcal{I}-R_0$. At first we will define $R-\mathcal{I}-$cluster point of a set and studied some of its properties. Also, the relation connecting the $R-\mathcal{I}-$cluster point and $R-\mathcal{I}-$regular space as well as the $R-\mathcal{I}-R_1$ space is examined. Afterwards, we will switch to the main purpose of the chapter: separation axioms namely $R-\mathcal{I}-R_S$, $R-\mathcal{I}-R_D$, $R-\mathcal{I}-R_T$, weakly $R-\mathcal{I}-R_0$ and weakly $R-\mathcal{I}-C_0$.

5.2 $R-\mathcal{I}-$cluster point

A point x of X is called a $R-\mathcal{I}-$cluster point of A if $R-\mathcal{I}-Cl(U)\cap A\neq\phi$ for every $U\in R\mathcal{I}O(X,x)$. The set of all $R-\mathcal{I}-$cluster points of A is called the $R-\mathcal{I}-\theta$-closure of A and is denoted by $R-\mathcal{I}-\theta Cl(A)$. A subset A is said to be $R-\mathcal{I}-\theta$-closed if $A=R-\mathcal{I}-\theta Cl(A)$. The complement of a $R-\mathcal{I}-\theta$-closed set is said to be $R-\mathcal{I}-\theta$-open.

Note 5.2.1. *For any subset A of an ideal topological space $(X,\tau,\mathcal{I})$, $R-\mathcal{I}-Cl(A)\subset R-\mathcal{I}-\theta Cl(A)$.*

Theorem 5.2.1. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent.*

(i) $(X, \tau, \mathcal{I})$ *is* $R-\mathcal{I}-$*regular.*

(ii) $R-\mathcal{I}-\theta Cl(A) = R-\mathcal{I}-Cl(A)$ *for every subset* A *of* X.

(iii) $R-\mathcal{I}-\theta Cl(A) = A$ *for every* $R-\mathcal{I}-$*closed subset* A *of* X.

Proof. (i) $\Rightarrow$ (ii)

It is enough to show that $R-\mathcal{I}-\theta Cl(A) \subset R-\mathcal{I}-Cl(A)$.

Assume $x \notin R-\mathcal{I}-Cl(A)$. Then by the $R-\mathcal{I}-$regularity of $(X, \tau, \mathcal{I})$, there exists disjoint $R-\mathcal{I}-$open sets U and V such that $x \in U$ and $R-\mathcal{I}-Cl(A) \subset V$. Hence $A \cap R-\mathcal{I}-Cl(U) \subset (R-\mathcal{I}-Cl(A)) \cap (R-\mathcal{I}-Cl(U)) \subset V \cap (R-\mathcal{I}-Cl(U)) = \phi$. Hence $x \notin R-\mathcal{I}-\theta Cl(A)$ and so (ii) holds.

(ii) $\Rightarrow$ (iii)

Let A be a $R-\mathcal{I}-$closed subset of X. Then $R-\mathcal{I}-Cl(A) = A$ and by (ii) $R-\mathcal{I}-\theta Cl(A) = A$.

(iii) $\Rightarrow$ (i)

Let $x \in X$ and A be a $R-\mathcal{I}-$closed set not containing x. By (iii) $R-\mathcal{I}-\theta Cl(A) = A$. Then there exists a $R-\mathcal{I}-$open set U containing x such that $(R-\mathcal{I}-Cl(U)) \cap A = \phi$. Thus $A \subset X-(R-\mathcal{I}-Cl(U))$. Hence (i) holds. □

Theorem 5.2.2. *An ideal topological space* $(X, \tau, \mathcal{I})$ *is* $R-\mathcal{I}-R_1$ *if and only if* $R-\mathcal{I}-\theta Cl(\{x\}) = R-\mathcal{I}-Cl(\{x\})$ *for each* $x \in X$.

Proof. Let $(X, \tau, \mathcal{I})$ be $R-\mathcal{I}-R_1$ and let $x \in X$. Suppose $y \notin R-\mathcal{I}-Cl(\{x\})$. Then by theorem 4.3.5, x and y belongs to disjoint $R-\mathcal{I}-$open sets U and V respectively. So $\{x\} \cap (R-\mathcal{I}-Cl(V)) \subset U \cap (R-\mathcal{I}-Cl(V)) = \phi$ and $y \notin R-\mathcal{I}-\theta Cl(\{x\})$. Hence $R-\mathcal{I}-\theta Cl(\{x\}) \subset R-\mathcal{I}-Cl(\{x\})$ and so $R-\mathcal{I}-\theta Cl(\{x\}) = R-\mathcal{I}-Cl(\{x\})$.

Conversely, let $x \neq y$ and $y \notin R-\mathcal{I}-Cl(\{x\})$. Then $y \notin R-\mathcal{I}-\theta Cl(\{x\})$. So there exists a $R-\mathcal{I}-$open set V containing y and $(R-\mathcal{I}-Cl(V)) \cap \{x\} = \phi$. Thus $x \in X-(R-\mathcal{I}-Cl(V)) = U(say)$. Thus U and V are disjoint $R-\mathcal{I}-$open sets containing x and y respectively. Hence $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_1$ by theorem 4.3.5. □

Remark 5.2.1. *Every $R-\mathcal{I}-$regular space is $R-\mathcal{I}-R_1$.*

5.3 Weaker than $R-\mathcal{I}-R_0$ spaces

Definition 5.3.1. *An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_S$ if and only if for any points $x, y \in X$, $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ implies $(R-\mathcal{I}-Cl(\{x\})) \cap (R-\mathcal{I}-Cl(\{y\})) = \phi$ or $\{x\}$ or $\{y\}$.*

Definition 5.3.2. *An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_D$ if and only if for $x \in X$, $(R-\mathcal{I}-Cl(\{x\})) \cap (I_R Ker(\{x\})) = \{x\}$ implies that $(R-\mathcal{I}-Cl(\{x\}) \setminus \{x\})$ is $R-\mathcal{I}-$closed.*

Definition 5.3.3. *An ideal topological space $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$ if and only if for each $x \in X$, both $I_R Ker(\{x\}) \setminus (R-\mathcal{I}-Cl(\{x\}))$ and $(R-\mathcal{I}-Cl(\{x\})) \setminus I_R Ker(\{x\})$ are degenerate.*

Definition 5.3.4. *An ideal topological space $(X,\tau,\mathcal{I})$ is weakly-$R-\mathcal{I}-R_0$ if $\bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\}) = \phi$.*

Definition 5.3.5. *An ideal topological space $(X,\tau,\mathcal{I})$ is weakly-$R-\mathcal{I}-C_0$ if $\bigcap_{x\in X} I_R Ker(\{x\}) = \phi$.*

Remark 5.3.1. *Every $R-\mathcal{I}-R_0$ space is weakly-$R-\mathcal{I}-R_0$.*

Remark 5.3.2. *Every $R-\mathcal{I}-R_0$ space is $R-\mathcal{I}-R_T$.*

Example 5.3.1. *The converses of the above remarks are not true.*

Let $X = \{a,b,c\}, \tau = \{\phi, X, \{b\}, \{c\}, \{b,c\}, \{a,b\}\}, \mathcal{I} = \{\phi, \{c\}\}$. The $R-$

$\mathcal{I}-$open sets are $\{b\}, \{c\}, \{a,b\}, X$. Then $(X,\tau,\mathcal{I})$ is not $R-\mathcal{I}-R_0$, but weakly $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_T$.

Theorem 5.3.1. *If $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$, then it is $R-\mathcal{I}-R_D$.*

Proof. Let X be $R-\mathcal{I}-R_T$ and denote $(x) = (R-\mathcal{I}-Cl(\{x\}))\cap(R-\mathcal{I}-Ker(\{x\}))$. Then $R-\mathcal{I}-Cl(\{x\}) = (x)\cup D$ and $R-\mathcal{I}-Ker(\{x\}) = (x)\cup E$, where D and E be degenerate sets such that D is not a subset of $R-\mathcal{I}-Ker(\{x\}$ and E is not a subset of $R-\mathcal{I}-Cl(\{x\})$. If $(x) = \{x\}$, then $R-\mathcal{I}-Cl(\{x\}) = \{x\}\cup D$ and $R-\mathcal{I}-Ker(\{x\}) = \{x\}\cup E$.
Let U be a $R-\mathcal{I}-$open set containing $R-\mathcal{I}-Ker(\{x\})$. Then $X-U$ is a $R-\mathcal{I}-$closed set and $(X-U)\cap(R-\mathcal{I}-Cl(\{x\})) = \phi$ or D. If $(X-U)\cap(R-\mathcal{I}-Cl(\{x\})) = D$, then D is a $R-\mathcal{I}-$closed set, being the intersection of two $R-\mathcal{I}-$closed sets. If $(X-U)\cap(R-\mathcal{I}-Cl(\{x\})) = \phi$, then $R-\mathcal{I}-Cl(\{x\})\subset U, D\subset U$. Since D is not a subset of $R-\mathcal{I}-Ker(\{x\}$, there exists a $R-\mathcal{I}-$open set V such that $x\in V$ and D is not a subset of V. Then $(R-\mathcal{I}-Cl(\{x\}))\cap(X-V) = D$ is a $R-\mathcal{I}-$closed set. Hence $(R-\mathcal{I}-Cl(\{x\})-\{x\})$ is $R-\mathcal{I}-$closed whenever $(x)=\{x\}$. Thus $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_D$. □

Theorem 5.3.2. *If $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-R_T$, then it is $R-\mathcal{I}-R_S$.*

Proof. Let X be $R-\mathcal{I}-R_T$ and let $x,y\in X$. If $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$ and there is an element $s\in X$ such that $s\neq x,\ s\neq y$ but $s\in(R-\mathcal{I}-Cl(\{x\}))\cap(R-\mathcal{I}-Cl(\{y\}))$, then $s\in R-\mathcal{I}-Cl(\{x\})$ and $s\in R-\mathcal{I}-Cl(\{y\})$. Hence $x\in I_RKer(\{s\})$ and $y\in I_RKer(\{s\})$. But $I_RKer(\{s\}) = (s)\cup E$ where E is a degenerate set and E is not a subset of $R-\mathcal{I}-Cl(\{s\})$.
Now four cases are possible as follows:
(i) $x\in(s)$ and $y\in(s)$.
Then $x\in R-\mathcal{I}-Cl(\{s\})$ and $y\in R-\mathcal{I}-Cl(\{s\})$. But $s\in R-\mathcal{I}-Cl(\{x\})$

and $s \in R-\mathcal{I}-Cl(\{y\})$. Hence $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{y\}) = R-\mathcal{I}-Cl(\{s\})$, which is impossible.

(ii) $\{x\} = E$, $y \in (s)$.

Then $x \notin R-\mathcal{I}-Cl(\{s\})$ and $y \in R-\mathcal{I}-Cl(\{s\})$. Since $s \in R-\mathcal{I}-Cl(\{y\})$, $R-\mathcal{I}-Cl(\{y\}) = R-\mathcal{I}-Cl(\{s\})$. Now either $y \in R-\mathcal{I}-Cl(\{x\})$ or $y \notin R-\mathcal{I}-Cl(\{x\})$. Let $y \in R-\mathcal{I}-Cl(\{x\})$. Since $x \notin R-\mathcal{I}-Cl(\{s\}), x \in X-(R-\mathcal{I}-Cl(\{s\}))$, which is $R-\mathcal{I}-$open, $I_R Ker(\{x\}) \subset X-(R-\mathcal{I}-Cl(\{s\}))$. Hence $(R-\mathcal{I}-Cl(\{s\})) \subset (R-\mathcal{I}-Cl(\{x\}))-I_R Ker(\{x\})$. But $\{s, y\} \subset R-\mathcal{I}-Cl(\{s\})$. Hence $(R-\mathcal{I}-Cl(\{x\}))-I_R Ker(\{x\})$ is not a degenerate set, a contradiction to the fact that X is $R-\mathcal{I}-R_T$. Now let $y \notin R-\mathcal{I}-Cl(\{x\})$. Since $y \in R-\mathcal{I}-Cl(\{s\})$ and $s \in R-\mathcal{I}-Cl(\{x\})$, we have $y \in R-\mathcal{I}-Cl(\{x\})$, a contradiction.

(iii) $x \in (s)$ and $\{y\} = E$.

The proof is similar to that of the above case.

(iv) $\{x\} = \{y\} = E$.

Then $R-\mathcal{I}-Cl(\{x\}) = R-\mathcal{I}-Cl(\{y\})$, which is not possible.

Thus if $R-\mathcal{I}-Cl(\{x\}) \neq R-\mathcal{I}-Cl(\{y\})$, then $(R-\mathcal{I}-Cl(\{x\})) \cap (R-\mathcal{I}-Cl(\{y\}))$ is either ϕ or $\{x\}$ or $\{y\}$. Hence X is $R-\mathcal{I}-R_S$.

□

The above discussed spaces hold the following relation:

$$R-\mathcal{I}-\mathrm{R}_1 \Rightarrow R-\mathcal{I}-R_0 \Rightarrow \text{weakly-}R-\mathcal{I}-R_0$$

$$\Downarrow$$

$$R-\mathcal{I}-R_D \Rightarrow R-\mathcal{I}-R_T \Rightarrow R-\mathcal{I}-R_S$$

Theorem 5.3.3. *An ideal topological space $(X, \tau, \mathcal{I})$ is weakly-$R-\mathcal{I}-R_0$ if and only if for each $x \in X$, $R-\mathcal{I}-Ker(\{x\}) \neq X$.*

Proof. Assume that X is weakly-$R-\mathcal{I}-R_0$. If there exists an element u which does not belong to any proper $R-\mathcal{I}-$open set, then u belongs to $R-\mathcal{I}-Cl(\{x\})$ for all $x \in X$. Hence $u \in \bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\})$, a contradiction.
Conversely, assume $R-\mathcal{I}-Ker(\{x\}) \neq X$ for each $x \in X$, which means for each $x \in X$, there exists a proper $R-\mathcal{I}-$open set U such that $x \in U$. If X is not weakly-$R-\mathcal{I}-R_0$, then there exists at least an element $u \in \bigcap_{x\in X} R-\mathcal{I}-Cl(\{x\})$. Then X is the only $R-\mathcal{I}-$open set containing u. This contradiction proves that X is weakly-$R-\mathcal{I}-R_0$.

□

In a similar manner, we will have the following theorem.

Theorem 5.3.4. *An ideal topological space* $(X,\tau,\mathcal{I})$ *is weakly-*$R-\mathcal{I}-C_0$ *if and only if for each* $x \in X$, $R-\mathcal{I}-Cl(\{x\}) \neq X$.

6

Supra ideal topological space via $R-\mathcal{I}-$open sets

6.1 Introduction

In this chapter, we present a new class of sets and functions in a supra ideal topological space. We have brought in via ideal, supra $R-\mathcal{I}-$open sets. We have introduced and studied certain properties of supra $R-\mathcal{I}-$continuous functions, supra* $R-\mathcal{I}-$continuous functions, supra $R-\mathcal{I}-$irresolute functions, supra $R-\mathcal{I}-$open maps, supra $R-\mathcal{I}-$closed maps and supra $R-\mathcal{I}-$homeomorphism. Separation axioms in terms of supra $R-\mathcal{I}-$open sets are also looked into.

[4] First let us recall what a supra topology is.

A subfamily μ of the power set $P(X)$ of a non-empty set X is called a supra topology on X if μ satisfies the following conditions:

1. μ contains ϕ and X.

2. μ is closed under the arbitrary union.

The pair (X, μ) is called a supra topological space. If (X, τ) is a topological space and $\tau \subset \mu$, then μ is known as supra topology associated with τ. The members of μ are called supra open sets in (X, μ). The complement of a supra open set is called a supra closed set.

[4] Let (X, μ) be a supra topological space and $A \subset X$. Then supra

interior and supra closure of A in (X, μ) are defined as $\cup\{U : U \subset A, U \in \mu\}$ and $\cap\{F : A \subset F, X - F \in \mu\}$ respectively. The supra interior and supra closure of A in (X, μ) are denoted as $Int^{\mu}(A)$ and $Cl^{\mu}(A)$ respectively. From definition, $Int^{\mu}(A)$ is a supra open set and $Cl^{\mu}(A)$ is a supra closed set.

Definition 6.1.1. *[43] A supra topological space (X, μ) with an ideal $\mathcal{I}$ on X is called an ideal supra topological space and is denoted as $(X, \mu, \mathcal{I})$.*

Definition 6.1.2. *[43] Let $(X, \mu, \mathcal{I})$ be an ideal supra topological space. A set operator $()^{*\mu} : P(X) \to P(X)$, called the μ-local function of $\mathcal{I}$ on X with respect to μ, is defined as $(A)^{*\mu}(\mathcal{I}, \mu) = \{x \in X : U \cap A \notin \mathcal{I}, for\ every\ U \in N(x)\}$, where $N(x) = \{U \in \mu : x \in U\}$.*
This is simply called μ-local function and denoted as $A^{*\mu}$.

6.2 Supra $R - \mathcal{I}-$open sets

Definition 6.2.1. *Let $(X, \mu, \mathcal{I})$ be an ideal supra topological space. A set A is called supra $R - \mathcal{I}-$open if $A = Int^{\mu}(Cl^{*}(A))$ and the complement of a supra $R - \mathcal{I}-$open set is called a supra $R - \mathcal{I}-$closed set.*

Example 6.2.1. *Consider $(X, \mu, \mathcal{I})$ where $X = \{a, b, c\}$,*
$\mu = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, \mathcal{I} = \{\phi, \{a\}\}$.
Then $\{a\}$ and $\{b\}$ are supra $R - \mathcal{I}-$open sets.

Remark 6.2.1. *Every $R - \mathcal{I}-$open set is supra $R - \mathcal{I}-$open.*

Remark 6.2.2. *Every regular open set is supra $R - \mathcal{I}-$open.*

Theorem 6.2.1. *Every supra $R - \mathcal{I}-$open set is supra open.*

Proof. Since every $R - \mathcal{I}-$open set is open, every supra $R - \mathcal{I}-$open set is supra open. □

Remark 6.2.3. *Converse of the above theorem need not be true.*

For example, in the example 6.2.1, $\{a, b\}$ *is supra open but not supra* $R - \mathcal{I}-$*open.*

Theorem 6.2.2. *If supra topology equals the discrete topology, then every supra open set is supra* $R - \mathcal{I}-$*open.*

Proof. In the discrete topology, every open set is $R - \mathcal{I}-$open and hence the result. □

Definition 6.2.2. *The supra* $R - \mathcal{I}-$*closure of a set* A*, denoted by supra* $R - \mathcal{I} - Cl(A)$ *is the intersection of all supra* $R - \mathcal{I}-$*closed sets containing* A*.*

Definition 6.2.3. *The supra* $R - \mathcal{I}-$*interior of a set* A*, denoted by supra* $R - \mathcal{I} - Int(A)$ *is the union of all supra* $R - \mathcal{I}-$*open sets contained in* A*.*

Example 6.2.2. *Consider* $(X, \mu, \mathcal{I})$ *where* $X = \{a, b, c\}$,
$\mu = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{b\}\}$.
Then supra $R - \mathcal{I} - Cl(\{b\}) = \{b, c\}$ *and supra* $R - \mathcal{I} - Int(\{b\}) = \phi$.

6.3 Supra $R - \mathcal{I}-$continuous functions

Definition 6.3.1. *Let* $(X, \tau, \mathcal{I})$ *be a* $R - \mathcal{I}-$ *space and* (Y, σ) *be a topological space and let* μ *be an associated supra topology with* τ*.*
A function $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ *is called a supra* $R - \mathcal{I}-$*continuous function if the inverse image of each open set in* Y *is supra* $R - \mathcal{I}-$*open in* X*.*

Example 6.3.1. *Let* $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$,
$\mu = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{b\}\}$, $Y = \{a, b, c\}$,
$\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.

Define $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ *by* $f(a) = a, f(b) = b, f(c) = b.$

Then f *is supra* $R - \mathcal{I}-$*continuous.*

Definition 6.3.2. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is said to be completely* $\mathcal{I}-$*continuous if* $f^{-1}(U)$ *is a* $R-\mathcal{I}-$*open set in* X *for every open set* $U \subset Y$.

Definition 6.3.3. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is said to be almost completely* $\mathcal{I}-$*continuous if* $f^{-1}(U)$ *is a* $R - \mathcal{I}-$*open set in* X *for every* $R - \mathcal{J}-$*open set* $U \subset Y$.

Definition 6.3.4. *A function* $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ *is said to be almost perfectly* $\mathcal{I}-$*continuous if* $f^{-1}(V)$ *is clopen for every* $R - \mathcal{J}-$*open set* V *in* Y.

Theorem 6.3.1. *Every completely* $\mathcal{I}-$*continuous function is supra* $R - \mathcal{I}-$*continuous.*

Remark 6.3.1. *Converse of the above theorem need not be true.*

Example 6.3.2. *Let* $X = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}, \mu = \{\phi, X, \{a\}, \{b\}, \{a, b\}\},$ $\mathcal{I} = \{\phi, \{b\}\}, \sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. *Define* $f : (X, \mu, \mathcal{I}) \to (X, \sigma)$ *by* $f(a) = a, f(b) = b, f(c) = b$. *Then* f *is supra* $R - \mathcal{I}-$*continuous but not completely* $\mathcal{I}-$*continuous.*

Theorem 6.3.2. *Let* Y *be a discrete space. If* $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ *is completely* $\mathcal{I}-$*continuous, then* f *is supra* $R - \mathcal{I}-$*continuous.*

Theorem 6.3.3. *Every almost perfectly* $\mathcal{I}-$*continuous function into a discrete space is supra* $R - \mathcal{I}-$*continuous.*

Theorem 6.3.4. *Every almost completely* $\mathcal{I}-$*continuous function into a discrete space is supra* $R - \mathcal{I}-$*continuous.*

Let $(X, \tau, \mathcal{I}), (Y, \sigma, \mathcal{J})$ be $R - \mathcal{I}-$ spaces and (Z, ρ) be a topological space. Let μ and ν be the associated supra topologies with τ and σ respectively.

Theorem 6.3.5. *If $f : (X, \mu, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is supra $R - \mathcal{I}-$continuous and $g : (Y, \sigma, \mathcal{J}) \to (Z, \rho)$ is continuous, then $g \circ f : (X, \mu, \mathcal{I}) \to (Z, \rho)$ is supra $R - \mathcal{I}-$continuous.*

Proof. Let V be open in ρ. Since g is continuous $g^{-1}(V)$ is open in σ. Since f is supra $R - \mathcal{I}-$continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is supra $R - \mathcal{I}-$open in X. Thus for any open set $V \in \rho$, $(g \circ f)^{-1}(V)$ is supra $R - \mathcal{I}-$open in μ. □

Theorem 6.3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and (Y, σ) be a topological space and let μ be the associated supra topology with τ. Let $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ be a bijective map. Then the following are equivalent:*
(i) f is supra $R - \mathcal{I}-$continuous.
(ii) inverse image of a closed set in Y is supra $R - \mathcal{I}-$closed in X.
(iii) supra $R - \mathcal{I}-Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \subset Y$.
(iv) f(supra $R - \mathcal{I}-Cl(U)) \subset Cl(f(U))$ for every $U \subset X$.
(v) $f^{-1}(Int(B)) \subset$ supra $R - \mathcal{I}-Int(f^{-1}(B))$ for every $B \subset Y$.

Proof. (i) ⇒ (ii)
Let V be a closed set in Y. Then $Y - V$ is open. Since f is supra $R - \mathcal{I}-$continuous, $f^{-1}(Y - V)$ is supra $R - \mathcal{I}-$open in X. That is, $X - f^{-1}(V)$ is supra $R - \mathcal{I}-$open in X. That is, $f^{-1}(V)$ is supra $R - \mathcal{I}-$closed in X.

(ii) ⇒ (iii)
Let $V \subset Y$. Then $Cl(V)$ is closed in Y. So by (ii) $f^{-1}(Cl(V))$ is supra $R - \mathcal{I}-$closed in X. So supra $R - \mathcal{I}-Cl(f^{-1}Cl(V)) = f^{-1}(Cl(V))$. Therefore $f^{-1}(Cl(V)) =$ supra $R-\mathcal{I}-Cl(f^{-1}Cl(V)) \supset$ supra $R-\mathcal{I}-Cl(f^{-1}(V))$.

(iii) $\Rightarrow$ (iv)

Let $U \subset X$ and $f(U) = V \subset Y$. By (iii) supra $R - \mathcal{I} - Cl(f^{-1}(f(U)) \subset f^{-1}(Cl(f(U)))$. That is, supra $R - \mathcal{I} - Cl(U) \subset f^{-1}(Cl(f(U)))$. That is, $f(\text{supra } R - \mathcal{I} - Cl(U)) \subset Cl(f(U))$.

(iv) $\Rightarrow$ (ii)

Let $W \subset Y$ be a closed set and $U = f^{-1}(W) \subset X$. By (iv) $f(\text{supra } R - \mathcal{I} - Cl(U)) \subset Cl(f(U)) = Cl(f(f^{-1}(W))) \subset Cl(W) = W$. So supra $R - \mathcal{I} - Cl(U) \subset f^{-1}(W) = U$. Thus U is supra $R - \mathcal{I} -$closed in X.

(ii) $\Rightarrow$ (i)

If $U \subset Y$ is an open set, then $Y - U$ is closed. By (ii) $f^{-1}(Y - U) = X - f^{-1}(U)$ is supra $R - \mathcal{I} -$closed in X. Hence $f^{-1}(U)$ is supra $R - \mathcal{I} -$open in X.

(i) $\Rightarrow$ (v)

Let $B \subset Y$. Then $Int(B)$ is open in Y and so $f^{-1}(Int(B))$ is supra $R - \mathcal{I} -$open in X. Hence $f^{-1}(Int(B)) = \text{supra } R - \mathcal{I} - Int(f^{-1}(Int(B))) \subset \text{supra } R - \mathcal{I} - Int(f^{-1}(B))$.

(v) $\Rightarrow$ (i)

Let $U \subset X$ be open. Then by (v) $f^{-1}(Int(U)) \subset \text{supra } R - \mathcal{I} - Int(f^{-1}(U))$ and so $f^{-1}(U) \subset \text{supra } R - \mathcal{I} - Int(f^{-1}(U))$. But supra $R - \mathcal{I} - Int(f^{-1}(U)) \subset f^{-1}(U)$. Hence supra $R - \mathcal{I} - Int(f^{-1}(U)) = f^{-1}(U)$. Thus $f^{-1}(U)$ is supra $R - \mathcal{I} -$open in X. $\square$

Theorem 6.3.7. *Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be ideal topological spaces and μ and ν be the associated supra topologies with τ and σ respectively. Then $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ is supra $R - \mathcal{I} -$continuous if $f^{-1}(Int(B)) \subset R - \mathcal{I} - Int(f^{-1}(B))$ for every $B \subset Y$.*

Definition 6.3.5. *Let* $(X, \tau, \mathcal{I})$ *and* $(Y, \sigma, \mathcal{J})$ *be ideal topological spaces and* μ *and* ν *be the associated supra topologies with* τ *and* σ *respectively. A function* $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ *is called supra** $R-\mathcal{I}-$*continuous if the inverse image of each supra open subset of* Y *is supra* $R-\mathcal{I}-$*open in* X.

Definition 6.3.6. *Let* $(X, \tau, \mathcal{I})$ *and* $(Y, \sigma, \mathcal{J})$ *be ideal topological spaces and* μ *and* ν *be the associated supra topologies with* τ *and* σ *respectively. A function* $f : (X, \mu, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ *is called supra* $R-\mathcal{I}-$*irresolute if the inverse image of each supra* $R-\mathcal{I}-$*open subset of* Y *is supra* $R-\mathcal{I}-$*open in* X.

Theorem 6.3.8. *Every supra** $R-\mathcal{I}-$*continuous function is supra* $R-\mathcal{I}-$*continuous.*

Remark 6.3.2. *Converse of the above theorem need not be true.*

Example 6.3.3. *Let* $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$,
$\mu = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{b\}\}$, $Y = \{a, b, c\}$,
$\sigma = \{\phi, X, \{a\}, \{b\}\}$, $\nu = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.
Define $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ *by* $f(a) = b, f(b) = a, f(c) = b$. *Then* f *is supra* $R-\mathcal{I}-$*continuous but not supra* * $R-\mathcal{I}-$*continuous since* $f^{-1}(b) = \{a, c\}$ *which is not supra* $R-\mathcal{I}-$*open in* X.

Theorem 6.3.9. *Every supra** $R-\mathcal{I}-$*continuous function is supra* $R-\mathcal{I}-$*irresolute.*

Remark 6.3.3. *Converse of the above theorem need not be true.*

Example 6.3.4. *Let* $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$,
$\mu = \{\phi, X, \{a\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{b\}\}$, $Y = \{a, b, c\}$,
$\sigma = \{\phi, X, \{a\}, \{b\}\}$, $\nu = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.
Define $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ *by* $f(a) = b, f(b) = a, f(c) = b$. *Then* f *is supra* $R-\mathcal{I}-$*irresolute but not supra* * $R-\mathcal{I}-$*continuous since* $f^{-1}(b) = \{a, c\}$ *which is not supra* $R-\mathcal{I}-$*open in* X.

Remark 6.3.4. *supra $R-\mathcal{I}-$continuous $\Leftarrow$ supra* $R-\mathcal{I}-$continuous $\Rightarrow$ supra $R-\mathcal{I}-$irresolute.*

Theorem 6.3.10. *If $f:(X,\mu,\mathcal{I})\to(Y,\nu,\mathcal{J})$ is supra* $R-\mathcal{I}-$continuous and $g:(Y,\nu,\mathcal{J})\to(Z,\rho)$ is supra $R-\mathcal{I}-$continuous, then*
$g\circ f:(X,\mu,\mathcal{I})\to(Z,\rho)$ is supra $R-\mathcal{I}-$continuous.

Theorem 6.3.11. *If $f:(X,\mu,\mathcal{I})\to(Y,\nu,\mathcal{J})$ is supra $R-\mathcal{I}-$irresolute and $g:(Y,\nu,\mathcal{J})\to(Z,\rho)$ is supra $R-\mathcal{I}-$continuous, then*
$g\circ f:(X,\mu,\mathcal{I})\to(Z,\rho)$ is supra $R-\mathcal{I}-$continuous.

Theorem 6.3.12. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be ideal topological spaces and let μ and γ be the corresponding associated supra topologies with τ and σ respectively. Let $f:(X,\mu,\mathcal{I})\to(Y,\sigma)$ be a bijective map. Then the following are equivalent:*

(i) f is supra* $R-\mathcal{I}-$continuous.

(ii) inverse image of a closed set in Y is supra $R-\mathcal{I}-$closed in X.

(iii) supra $R-\mathcal{I}-Cl(f^{-1}(V))\subset f^{-1}(Cl(V))$ for every $V\subset Y$.

(iv) f(supra $R-\mathcal{I}-Cl(U))\subset Cl(f(U))$ for every $U\subset X$.

(v) $f^{-1}(Int(B))\subset$ supra $R-\mathcal{I}-Int(f^{-1}(B))$ for every $B\subset Y$.

Theorem 6.3.13. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be ideal topological spaces and μ and γ be the corresponding associated supra topologies with τ and σ. Let $f:(X,\mu,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ be a bijective map. Then the following are equivalent:*

(i) f is supra $R-\mathcal{I}-$irresolute.

(ii) inverse image of a closed set in Y is supra $R-\mathcal{I}-$closed in X.

(iii) supra $R-\mathcal{I}-Cl(f^{-1}(V))\subset f^{-1}(Cl(V))$ for every $V\subset Y$.

(iv) f(supra $R-\mathcal{I}-Cl(U))\subset Cl(f(U))$ for every $U\subset X$.

(v) $f^{-1}(Int(B))\subset$ supra $R-\mathcal{I}-Int(f^{-1}(B))$ for every $B\subset Y$.

Theorem 6.3.14. *Let* $(X, \tau, \mathcal{I})$ *and* $(Y, \sigma, \mathcal{J})$ *be ideal topological spaces and* μ *and* γ *be the corresponding associated supra topologies with* τ *and* σ *respectively. Then* $f : (X, \mu, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *be supra* $R - \mathcal{I}-$*irresolute if* $f^{-1}(Int(B)) \subset R - \mathcal{I}-Int(f^{-1}(B))$ *for every* $B \subset Y$.

Theorem 6.3.15. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* (Y, σ) *be a topological space and* μ *be the corresponding associated supra topology with* τ. *Then* $f : (X, \mu, \mathcal{I}) \to (Y, \sigma)$ *be supra** $R - \mathcal{I}-$*continuous if* $f^{-1}(Int(B)) \subset R - \mathcal{I}-Int(f^{-1}(B))$ *for every* $B \subset Y$.

6.4 Supra $R - \mathcal{I}-$open maps and supra $R - \mathcal{I}-$closed maps

Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be $R - \mathcal{I}-$ spaces and $(X, \mu, \mathcal{I})$ and $(Y, \nu, \mathcal{J})$ be the associated supra ideal topological spaces.

Definition 6.4.1. *A map* $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ *is supra* $R - \mathcal{I}-$*open (resp. supra* $R - \mathcal{I}-$*closed) if the image of each open (resp. closed) set in* X *is supra* $R - \mathcal{I}-$*open (resp. supra* $R - \mathcal{I}-$*closed) in* Y.

Example 6.4.1. *Let* $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$, $\mu = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. *Define* $f : (X, \mu, \mathcal{I}) \to (X, \mu, \mathcal{I})$ *by* $f(a) = a, f(b) = b, f(c) = c$. *Then* f *is supra* $R - \mathcal{I}-$*open.*

Theorem 6.4.1. *A map* $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ *is supra* $R - \mathcal{I}-$*open if and only if* $f(Int(A)) \subset$ *supra* $R - \mathcal{I}-Int(f(A))$ *for each* $A \subset X$.

Proof. Suppose f is supra $R-\mathcal{I}-$open. Since $f(IntA)$ is a supra $R-\mathcal{I}-$open set contained in $f(A)$, $f(IntA) \subset$ supra $R - \mathcal{I}-Int(f(A))$ for each $A \subset X$. Conversely, suppose that A is open in X. Then $f(A) = f(IntA) \subset$ supra $R - \mathcal{I}-Int(f(A))$. Therefore $f(A) =$ supra $R - \mathcal{I}-Int(f(A))$. Thus f is a supra $R - \mathcal{I}-$open map. □

Theorem 6.4.2. *A map $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is supra $R - \mathcal{I}-$closed if and only if supra $R - \mathcal{I}-Cl(f(A)) \subset f(Cl(A))$ for each $A \subset X$.*

Proof. Suppose f is supra $R - \mathcal{I}-$closed. Since $f(Cl(A))$ is a supra $R - \mathcal{I}-$closed set containing $f(A)$, supra $R - \mathcal{I}-Cl(f(A)) \subset f(Cl(A))$ for each $A \subset X$.

Converse is obvious. □

Theorem 6.4.3. *Let $(X, \tau, \mathcal{I}), (Y, \sigma, \mathcal{J})$ and (Z, ρ, K) be ideal topological spaces. Let μ, ν and ξ be the associated supra topologies of τ, σ and ρ respectively. Then*

(i) if $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \xi, K)$ is supra $R - \mathcal{I}-$open and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a continuous surjection, then $g : (Y, \sigma, \mathcal{J}) \to (Z, \xi, K)$ is a supra $R - \mathcal{I}-$open map.

(ii) if $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \rho, K)$ is open and $g : (Y, \nu, \mathcal{J}) \to (Z, \xi, K)$ is a supra $R - \mathcal{I}-$continuous injection, then $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is a supra $R - \mathcal{I}-$open map.

(iii) if $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \xi, K)$ is supra $R - \mathcal{I}-$open and $g : (Y, \nu, \mathcal{J}) \to (Z, \xi, K)$ is supra $R - \mathcal{I}-$irresolute injection, then $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is a supra $R - \mathcal{I}-$open map.

Proof. Let V be open in Y. Then $f^{-1}(V)$ is open in X. So $(g \circ f)(f^{-1}(V))$ is supra $R - \mathcal{I}-$open in (Z, ρ, K). Since f is onto, $(g \circ f)(f^{-1}(V)) = g(f(f^{-1}(V))) = g(V)$. Hence (i).

Let U be open in X. Then $(g \circ f)(U)$ is open in Z. So $g^{-1}((g \circ f)(U))$ is supra $R - \mathcal{I}-$open in Y. Since g is one-one, $g^{-1}((g \circ f)(U)) = (g^{-1} \circ g)(f(U)) = f(U)$. Hence (ii).

Let U be open in X. Then $(g \circ f)(U)$ is supra $R - \mathcal{I}-$open in Z. Since g is supra $R - \mathcal{I}-$irresolute, $g^{-1}((g \circ f)(U))$ is supra $R - \mathcal{I}-$open in Y. Since g is one-one, $g^{-1}((g \circ f)(U)) = (g^{-1} \circ g)(f(U)) = f(U)$. Hence (iii) □

Theorem 6.4.4. *Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be ideal topological spaces and ν be the associated supra topology with σ. Let $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ be a bijective map. Then the following are equivalent:*
(i) f is supra $R-\mathcal{I}-$open.
(ii) f is supra $R-\mathcal{I}-$closed.
(iii) f^{-1} is supra $R-\mathcal{I}-$continuous.

Proof. (i) $\Rightarrow$ (ii)
Suppose that V is a closed subset of X. Then V^c is open in X and $f(V^c)$ is supra $R-\mathcal{I}-$open in Y. Since f is bijective, then $f(V^c) = Y - f(V)$. So $f(V)$ is supra $R-\mathcal{I}-$closed in Y. Thus f is supra $R-\mathcal{I}-$closed.

(ii) $\Rightarrow$ (iii)
Suppose f is a supra $R-\mathcal{I}-$closed map and V is a closed subset of X. Then $f(V)$ is supra $R-\mathcal{I}-$closed in Y. Since f is bijective, $(f^{-1})^{-1}(V) = f(V)$. Therefore f^{-1} is supra $R-\mathcal{I}-$continuous.

(iii) $\Rightarrow$ (i)
Suppose that U is an open subset of X. Since f^{-1} is a supra $R-\mathcal{I}-$continuous map, $(f^{-1})^{-1}(U)$ is supra $R-\mathcal{I}-$open in Y. Since f is bijective, $f(U) = (f^{-1})^{-1}(U)$. Thus f is supra $R-\mathcal{I}-$open. □

6.5 Supra $R-\mathcal{I}-$homeomorphism

Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be ideal topological spaces and $(X, \mu, \mathcal{I})$ and $(Y, \nu, \mathcal{J})$ be the associated supra ideal topological spaces.

Definition 6.5.1. *A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be supra $R-\mathcal{I}-$homeomorphism if f is supra $R-\mathcal{I}-$continuous and supra $R-\mathcal{I}-$open.*

Theorem 6.5.1. *Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be ideal topological spaces and $(X, \mu, \mathcal{I})$ and $(Y, \nu, \mathcal{J})$ be the associated supra ideal topological spaces. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a bijective and supra $R - \mathcal{I}-$continuous map. Then the following are equivalent:*

(i) f is supra $R - \mathcal{I}-$homeomorphism.

(ii) f^{-1} is supra $R - \mathcal{I}-$continuous.

(iii) f is supra $R - \mathcal{I}-$closed.

Proof. (i) $\Rightarrow$ (ii)

Assume (i). Then f is a supra $R - \mathcal{I}-$open map. So for any open set $U \in \tau$, $f(U)$ is supra $R - \mathcal{I}-$open in ν. Since f is bijective, $(f^{-1})^{-1}(U) = f(U)$. Now consider, $f^{-1} : (Y, \nu, \mathcal{J}) \to (X, \tau, \mathcal{I})$. So for any open set $U \in \tau$, $(f^{-1})^{-1}(U)$ is supra $R - \mathcal{I}-$open in $(Y, \nu, \mathcal{J})$. Hence f^{-1} is supra $R - \mathcal{I}-$continuous.

(ii) $\Rightarrow$ (iii)

Let $f^{-1} : (Y, \nu, \mathcal{J}) \to (X, \tau, \mathcal{I})$ be supra $R - \mathcal{I}-$continuous. Then for any open set $U \in \tau$, $(f^{-1})^{-1}(U)$ is supra $R - \mathcal{I}-$open in Y. Let V be closed in X. Then $X - V$ is open in X. So $(f^{-1})^{-1}(X - V) = f(X - V) = Y - f(V)$ is supra $R - \mathcal{I}-$open in Y. So $f(V)$ is supra $R - \mathcal{I}-$closed in Y. Thus for any closed set V in X, $f(V)$ is supra $R - \mathcal{I}-$closed in Y.

Hence $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ is a supra $R - \mathcal{I}-$closed map.

(iii) $\Rightarrow$ (i)

Let $f : (X, \tau, \mathcal{I}) \to (Y, \nu, \mathcal{J})$ be supra $R - \mathcal{I}-$closed and U be open in τ. Then $f(X - U)$ is supra $R - \mathcal{I}-$closed in Y. Since f is a bijection, $Y - f(U)$ is supra $R - \mathcal{I}-$closed in Y and so $f(U)$ is supra $R - \mathcal{I}-$open in Y. Since f is supra $R - \mathcal{I}-$continuous, f is a supra $R - \mathcal{I}-$homeomorphism. □

6.6 Separation axioms

Definition 6.6.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $(X, \mu, \mathcal{I})$ be the associated supra ideal topological space. Then $(X, \mu, \mathcal{I})$ is called*

(i) supra $R - \mathcal{I} - T_0$ if for every two distinct points of X, there exists a supra $R - \mathcal{I}-$open set which contains one, but not the other.

(ii) supra $R - \mathcal{I} - T_1$ if for every two distinct points x and y of X, there exists supra $R - \mathcal{I}-$open sets U and V such that $x \in U, y \notin U$ and $x \notin V, y \in V$.

(iii) supra $R - \mathcal{I} - T_2$ if for every two distinct points x and y of X, there exists supra $R - \mathcal{I}-$open sets U and V such that $x \in U, y \in V$ and $U \cap V = \phi$.

Definition 6.6.2. *A subset U of $(X, \mu, \mathcal{I})$ is called a supra $R - \mathcal{I}-$nbd of a point $x \in X$ if there exists a supra $R - \mathcal{I}-$open set V such that $x \in V \subset U$.*

Theorem 6.6.1. *Let $(X, \mu, \mathcal{I})$ be a supra ideal topological space. Then X is supra $R - \mathcal{I} - T_0$ if and only if supra $R - \mathcal{I} - Cl(\{x\}) \neq$ supra $R - \mathcal{I} - Cl(\{y\})$ for every distinct x and y in X.*

Proof. Suppose X is supra $R - \mathcal{I} - T_0$. Let $x \neq y \in X$. Then there exists a supra $R - \mathcal{I}-$open set U containing x but not y. So $X - U$ is a supra $R - \mathcal{I}-$closed set such that $x \notin X - U$ and $y \in X - U$. Thus supra $R - \mathcal{I} - Cl(\{x\}) \neq$ supra $R - \mathcal{I} - Cl(\{y\})$.
Conversely, suppose that supra $R - \mathcal{I} - Cl(\{x\}) \neq$ supra $R - \mathcal{I} - Cl(\{y\})$ for any $x \neq y \in X$. Assume X is not supra $R - \mathcal{I} - T_0$. Then every supra $R - \mathcal{I}-$closed set containing x always contains y. But then supra $R - \mathcal{I} - Cl(\{x\}) =$ supra $R - \mathcal{I} - Cl(\{y\})$ which is a contradiction. Hence there should exist at least one supra $R - \mathcal{I}-$open set containing x but not y. So X is supra $R - \mathcal{I} - T_0$. □

Theorem 6.6.2. *Let* $(X, \mu, \mathcal{I})$ *be a supra ideal topological space. Then* X *is supra* $R - \mathcal{I} - T_1$ *if and only if every singleton is supra* $R - \mathcal{I}-$*closed.*

Proof. Let X be supra $R - \mathcal{I} - T_1$ and $x \in X$. For any $y \in X, x \neq y$, there exists a supra $R - \mathcal{I}-$open set V such that $y \in V$ and $x \notin V$. Then $\{x\} \subset V^c$. So for any $y \neq x \in X$, $y \notin$ supra $R - \mathcal{I} - Cl(\{x\})$. Therefore supra $R - \mathcal{I} - Cl(\{x\}) = \{x\}$. Thus $\{x\}$ is supra $R - \mathcal{I}-$closed.
Conversely, let $x, y \in X$. Then $\{x\}, \{y\}$ are supra $R - \mathcal{I}-$closed sets and so $X - \{x\}$ and $X - \{y\}$ are supra $R - \mathcal{I}-$open sets. Then X is supra $R - \mathcal{I} - T_1$. □

Theorem 6.6.3. *Let* $(X, \mu, \mathcal{I})$ *be a supra ideal topological space. Then* X *is supra* $R - \mathcal{I} - T_2$ *if and only if for each* $x \in X$,
$\{x\} = \cap\{N : N \textit{ is a supra } R - \mathcal{I} - \textit{closed nbd of } x\}$.

Proof. Let X be supra $R - \mathcal{I} - T_2$. Then for $x \neq y \in X$, there exists two disjoint supra $R - \mathcal{I}-$open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$. Then $U \subset V^c$. So supra $R - \mathcal{I} - Cl(U) \subset V^c$. Let $N =$ supra $R - \mathcal{I} - Cl(U)$. Hence $\{x\} = \cap\{N : N \textit{ is a supra } R - \mathcal{I} - \textit{closed nbd of } x\}$.
Conversely, for $x \in X, \{x\} = \cap\{N : N \textit{ is a supra } R - \mathcal{I} - \textit{closed nbd of } x\}$. Then for $x \neq y$, there exists supra $R - \mathcal{I}-$closed nbd F such that $x \in F$ and $y \notin F$. So there exist supra $R - \mathcal{I}-$open sets U and V such that $x \in U \subset F$ and $y \in V = F^c$. Also $U \cap V = \phi$. Thus X is supra $R - \mathcal{I} - T_2$. □

Theorem 6.6.4. *Let* $(X, \mu, \mathcal{I})$ *be a supra ideal topological space. If* X *is supra* $R - \mathcal{I} - T_1$, *then for each* $x \in X, \{x\} = \cap\{U : U \textit{ is a supra } R - \mathcal{I} - \textit{open nbd of } x\}$.

Proof. Suppose X is supra $R - \mathcal{I} - T_1$. Fix $x \in X$. Then for each $y \in X$, there exists supra $R - \mathcal{I}-$open sets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Clearly each U is a supra $R - \mathcal{I}-$open nbd of x. Hence $\{x\} = \cap\{U : U \textit{ is a supra } R - \mathcal{I} - \textit{open nbd of } x\}$. □

Theorem 6.6.5. *Let $(X,\mu,\mathcal{I})$ be a supra ideal topological space. If singletons are supra $R-\mathcal{I}-$closed, then for each $x \in X$,*
$\{x\} = \cap\{X - \{y\} : for\ each\ y \neq x \in X\}$.

Proof. Suppose singletons are supra $R-\mathcal{I}-$closed. Fix $x \in X$. Then for any $y \neq x \in X$, $X - \{y\}$ is a supra $R-\mathcal{I}-$open nbd of x. Hence clearly $\{x\} = \cap\{X - \{y\} : for\ each\ y \neq x \in X\}$. □

Theorem 6.6.6. *Every supra $R-\mathcal{I}-T_i$, $(i = 0, 1, 2)$ is a supra T_i, $(i = 0, 1, 2)$ space.*

Proof. Since every supra $R-\mathcal{I}-$open set is supra open, the theorem follows. □

Remark 6.6.1. *Converse of the above theorem need not be true.*

For example, let $X = \{a, b, c\}$, $\mu = \{\phi, X, \{a, b\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$. Then $(X, \mu, \mathcal{I})$ is supra T_1 but not supra $R-\mathcal{I}-T_1$.

Note 6.6.1. *The axioms supra $R-\mathcal{I}-T_0$, supra $R-\mathcal{I}-T_1$, supra $R-\mathcal{I}-T_2$ form a hierarchy of progressively stronger conditions.*

Remark 6.6.2. *None of the implications in the above theorem is reversible.*
For example, let $X = \{a, b, c\}, \mu = \{\phi, X, \{a, b\}, \{a, c\}, \{b, c\}\}, \mathcal{I} = \{\phi, \{a\}, \{b\}, \{c\}\}$. Then (X, μ, I) is a supra $R-\mathcal{I}-T_1$ space but is not a supra $R-\mathcal{I}-T_2$ space.

Remark 6.6.3. *Every finite supra $R-\mathcal{I}-T_1$ space is not discrete.*

For example, let $X = \{a, b, c\}$, $\mu = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$. Then (X, μ, I) is a supra $R-\mathcal{I}-T_1$ space but is not discrete.

Theorem 6.6.7. *Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be ideal topological spaces and $(X, \mu, \mathcal{I})$ and $(Y, \nu, \mathcal{J})$ be the associated supra ideal topological spaces. Let*

$f : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ be a bijective and supra $R-\mathcal{I}-$open map. If $(X,\mu,\mathcal{I})$ is a supra $R-\mathcal{I}-T_0$ space, then $(Y,\nu,\mathcal{J})$ is a supra $R-\mathcal{I}-T_0$ space.

Proof. Suppose $(X,\mu,\mathcal{I})$ is a supra $R-\mathcal{I}-T_0$ space. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is bijective there exists $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $(X,\mu,\mathcal{I})$ is supra $R-\mathcal{I}-T_0$, there exists a supra $R-\mathcal{I}-$open set $U \subset X$ such that $x_1 \in U$ and $x_2 \notin U$. Since f is supra $R-\mathcal{I}-$open, $f(U) \subset Y$ is a supra $R-\mathcal{I}-$open set. Also $y_1 \in f(U)$ and $y_2 \notin f(U)$. Thus $(Y,\nu,\mathcal{J})$ is a supra $R-\mathcal{I}-T_0$ space. □

Theorem 6.6.8. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be ideal topological spaces and $(X,\mu,\mathcal{I})$ and $(Y,\nu,\mathcal{J})$ be the associated supra ideal topological spaces. Let $f : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ be a bijective and supra $R-\mathcal{I}-$continuous map. If $(Y,\nu,\mathcal{J})$ is a supra $R-\mathcal{I}-T_0$ space, then $(X,\mu,\mathcal{I})$ is a supra $R-\mathcal{I}-T_0$ space.*

Proof. Suppose $(Y,\nu,\mathcal{J})$ is a supra $R-\mathcal{I}-T_0$ space. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is bijective, there exists $y_1 \neq y_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $(Y,\nu,\mathcal{J})$ is supra $R-\mathcal{I}-T_0$, there exists a supra $R-\mathcal{I}-$open set $V \subset Y$ such that $y_1 \in V$ and $y_2 \notin V$. Since f is supra $R-\mathcal{I}-$continuous, f^{-1} is supra $R-\mathcal{I}-$open and so $f^{-1}(V) \subset X$ is a supra $R-\mathcal{I}-$open set. Also $x_1 \in f^{-1}(V)$ and $x_2 \notin f^{-1}(V)$. Thus $(X,\mu,\mathcal{I})$ is a supra $R-\mathcal{I}-T_0$ space. □

Remark 6.6.4. *The above two theorems are also true for supra $R-\mathcal{I}-T_i$ spaces for $i = 1, 2$.*

Remark 6.6.5. *The theorem 6.6.8 is also true for the cases:*

(i) the bijective function f is supra * $R-\mathcal{I}-$continuous.

(ii) the bijective function f is supra $R-\mathcal{I}-$irresolute.

7

Minimal $R-\mathcal{I}-$open sets

7.1 Introduction

Till now, we have explored mainly on separation axioms in various modes and also on supra space. From this chapter onwards we will pertain to $R-\mathcal{I}-$open sets and $R-\mathcal{I}-$continuous functions and its generalizations.

We intend to propose in this chapter, a new class of sets and a new class of continuous functions in ideal topological space incorporating the idea of minimal open sets and $R-\mathcal{I}-$open sets. We bring in minimal $R-\mathcal{I}-$open set in section 2. Further, we surveyed $R-\mathcal{I}-T_{min}$ and $R-\mathcal{I}-T_{max}$ spaces, in section 3. In section 4. we move on to the continuous functions namely minimal $R-\mathcal{I}-$continuous function and maximal $R-\mathcal{I}-$continuous function. The relation between the above defined continuous functions with certain other continuous functions is also investigated.

7.2 Minimal $R-\mathcal{I}-$open sets

Definition 7.2.1. *A non-empty proper $R-\mathcal{I}-$open subset M of X is called minimal $R-\mathcal{I}-$open if and only if any $R-\mathcal{I}-$open set contained in M is either ϕ or M.*

Definition 7.2.2. *A non-empty proper $R-\mathcal{I}-$open subset M of X is called maximal $R-\mathcal{I}-$open if and only if any $R-\mathcal{I}-$open set that contains M is either X or M.*

Definition 7.2.3. *A non-empty proper $R-\mathcal{I}-$closed subset P of X is called minimal $R-\mathcal{I}-$closed if and only if any $R-\mathcal{I}-$closed set contained in P is either ϕ or P.*

Definition 7.2.4. *A non-empty proper $R-\mathcal{I}-$closed subset P of X is called maximal $R-\mathcal{I}-$closed if and only if any $R-\mathcal{I}-$closed set that contains P is either X or P.*

Example 7.2.1. *Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{a, b\}, X\}, \mathcal{I} = \{\phi, \{a\}\}$. The proper $R-\mathcal{I}-$open sets are $\{a\}$ and $\{a, b\}$. So the proper $R-\mathcal{I}-$closed sets $\{b, c\}$ and $\{c\}$.*
Now $\{a\}$ is minimal $R-\mathcal{I}-$open and $\{a, b\}$ is maximal $R-\mathcal{I}-$open. Also $\{c\}$ is minimal $R-\mathcal{I}-$closed and $\{b, c\}$ is maximal $R-\mathcal{I}-$closed.

Theorem 7.2.1. *Let $R \subset X$.*

- *R is minimal $R-\mathcal{I}-$closed if and only if $X\setminus R$ is maximal $R-\mathcal{I}-$open.*
- *R is maximal $R-\mathcal{I}-$closed if and only if $X\setminus R$ is minimal $R-\mathcal{I}-$open.*

Proof. We have $X \setminus R$ is a $R-\mathcal{I}-$open set. Let $S \neq X$ be a $R-\mathcal{I}-$open set containing $X \setminus R$. Then $X \setminus S \subset R$ which contradicts the minimality of the $R-\mathcal{I}-$closed set R. Hence $X \setminus R$ is maximal $R-\mathcal{I}-$open.
In a similar manner the converse can be proved.
For the second statement, let $\phi \neq S$ be a $R-\mathcal{I}-$open set contained in $X\setminus R$. Then $R \subset X \setminus S$ which contradicts the maximality of the $R-\mathcal{I}-$closed set R. Hence $X \setminus R$ is minimal $R-\mathcal{I}-$open.
In a similar manner the converse can be proved. □

Lemma 7.2.1. *(i) Let M be a minimal $R-\mathcal{I}-$open set and U be any $R-\mathcal{I}-$open set of X. Then $M \cap U = \phi$ or $M \subset U$.*

(ii) If M and N are minimal $R-\mathcal{I}-$open sets of X, then $M \cap N = \phi$ or $M = N$.

Proof. (i) Let U be a $R-\mathcal{I}-$open set of X with $M \cap U \neq \phi$. M being minimal $R-\mathcal{I}-$open and $M \cap U \subset M$, we get $M \cap U = M$. Thus $M \subset U$.
(ii) If $M \cap N \neq \phi$, then as in (i), $M \subset N$ and $N \subset M$ so that $M = N$. □

Lemma 7.2.2. *Let M be a maximal $R-\mathcal{I}-$open set and U be any $R-\mathcal{I}-$open set of X such that $M \cap U \neq \phi$. Then $U \subset M$.*

Proof. Let U be a $R-\mathcal{I}-$open set of X with $M \cap U \neq \phi$. M being maximal $R-\mathcal{I}-$open, $M \cup U = M$. Thus $U \subset M$. □

Remark 7.2.1. *The union of every pair of different maximal $R-\mathcal{I}-$open sets need not be the whole space X.*

Example 7.2.2. *Consider the ideal topological space $(X, \tau, \mathcal{I})$ where*
$X = \{a, b, c, d\}$, $\tau = \{\phi, \{a, c\}, \{d\}, \{a, c, d\}, X\}$, $\mathcal{I} = \{\phi, \{d\}\}$.
The proper $R-\mathcal{I}-$open sets are $\{a, c\}$ and $\{d\}$ and both of them are maximal $R-\mathcal{I}-$open. But their union is not equal to X.

Lemma 7.2.3. *(i) Let P be a minimal $R-\mathcal{I}-$closed set and U be any $R-\mathcal{I}-$closed set of X. Then $P \cap U = \phi$ or $P \subset U$.*

(ii) If P and Q are minimal $R-\mathcal{I}-$closed sets of X, then $P \cap Q = \phi$ or $P = Q$.

Proof. The proof follows in a similar fashion as in lemma 7.2.1. □

Lemma 7.2.4. *Let P be a maximal $R-\mathcal{I}-$closed set and U be any $R-\mathcal{I}-$closed set of X such that $P \cap U \neq \phi$. Then $U \subset P$.*

Proof. The proof follows in a similar fashion as in lemma 7.2.2. □

Theorem 7.2.2. *Let M be a minimal $R-\mathcal{I}-$open set and $x \in M$. Then for any $R-\mathcal{I}-$open nbd U of x, $M \subset U$.*

Proof. Let $x \in M \subset X$. Suppose U be a $R-\mathcal{I}-$open nbd of x such that M is not a subset of U. Clearly $M \cap U$, a $R-\mathcal{I}-$open set, is such that $M \cap U \subsetneq M$ and $M \cap U \neq \phi$. This contradicts the fact that M is a minimal $R-\mathcal{I}-$open set. Hence the proof. □

Theorem 7.2.3. *Let M be a maximal $R-\mathcal{I}-$open set and $x \in M$. Then for any $R-\mathcal{I}-$open nbd U of x, $U \subset M$.*

Proof. The proof is a consequence of lemma 7.2.2. □

Theorem 7.2.4. *Let M be a minimal $R-\mathcal{I}-$open set. Then for $x \in M$, $M = \cap\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$.*

Proof. Clearly $\cap\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$ is a $R-\mathcal{I}-$open nbd of x. Since M is minimal $R-\mathcal{I}-$open, $M \subset \cap\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$. But since M itself is a $R-\mathcal{I}-$open nbd of x, $\cap\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\} \subset M$. Thus $M = \cap\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$. □

Theorem 7.2.5. *Let M be a maximal $R-\mathcal{I}-$open set. Then for $x \in M$, $M = \cup\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$.*

Proof. Let $\mathcal{U} = \cup\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$. M being a $R-\mathcal{I}-$open nbd of x, $M \subset \mathcal{U}$. Since M is maximal $R-\mathcal{I}-$open, $U \subset M$ for each $R-\mathcal{I}-$open nbd U of x. Hence $M = \cup\{U : U\ is\ a\ R-\mathcal{I}-open\ nbd\ of\ x\}$. □

Theorem 7.2.6. *Let M be a proper non-empty $R-\mathcal{I}-$open subset of X. Then the following are equivalent:*

(i) M is minimal $R-\mathcal{I}-$open.

(ii) $M \subset R - \mathcal{I} - Cl(V)$ *for any non-empty subset* V *of* M.

(iii) $R - \mathcal{I} - Cl(M) = R - \mathcal{I} - Cl(V)$ *for any non-empty subset* V *of* M.

Proof. (i) $\Rightarrow$ (ii)

Let V be a non-empty subset of M. Now for any $x \in M$ and any R–$\mathcal{I}$–open nbd U of x, by theorem 7.2.2., $M \subset U$. So $V = V \cap M \subset V \cap U$. That is $V \cap U \neq \phi$ and thus $x \in R - \mathcal{I} - Cl(V)$. Hence $M \subset R - \mathcal{I} - Cl(V)$ for any non-empty subset V of M.

(ii) $\Rightarrow$ (iii)

Let V be a non-empty subset of M. Then $(R–\mathcal{I}–Cl(V)) \subset (R–\mathcal{I}–Cl(M))$. But we have $M \subset (R–\mathcal{I}–Cl(V))$. Then $(R–\mathcal{I}–Cl(M)) \subset (R–\mathcal{I}–Cl(V))$. Thus $R - \mathcal{I} - Cl(M) = R - \mathcal{I} - Cl(V)$ for any non-empty subset V of M.

(iii) $\Rightarrow$ (i)

Assume that M is not minimal $R - \mathcal{I}$–open. Then there exists a non-empty $R - \mathcal{I}$–open set $S \subset M$ such that $S \neq M$. So there is at least one element $m \in M$ such that $m \notin S$. Then $m \in X \setminus S$ and hence $R–\mathcal{I}–Cl(\{m\}) \subset X \setminus S$. This implies $R–\mathcal{I}–Cl(\{m\}) \neq R–\mathcal{I}–Cl(M)$. □

Theorem 7.2.7. *Let* M *be a minimal* $R - \mathcal{I}-$*open subset of* X *and let* $x \notin M$. *Then for any* $R - \mathcal{I}-$*open nbd* U *of* x, *either* $M \cap U = \phi$ *or* $M \subset U$.

Proof. We have $x \in X \setminus M$ and U is a $R - \mathcal{I}$–open nbd of x. Then U is a $R - \mathcal{I}$–open set in X. Then by lemma 7.2.1., either $M \cap U = \phi$ or $M \subset U$. □

Corollary 7.2.1. *Let* M *be a minimal* $R - \mathcal{I}-$*open subset of* X *and* $x \in X \setminus M$. *Let* $M_x = \cap\{U : U \text{ is a } R - \mathcal{I} - \text{open nbd of } x\}$. *Then* $M_x \cap M = \phi$ *or* $M \subset M_x$.

Proof. If $M \subset U$ for any $R - \mathcal{I}$–open nbd U of x, then $M \subset M_x$. If not, there exists a $R-\mathcal{I}-$open set U of x such that $M \cap U = \phi$. This implies $M_x \cap M = \phi$. □

Theorem 7.2.8. *Let M be maximal $R-\mathcal{I}-$open subset of X and let $x \notin M$. Then for any $R-\mathcal{I}-$open nbd U of x, $U \subset X \setminus M$.*

Proof. We have $x \in X \setminus M$. Then for any $R-\mathcal{I}-$open nbd U of x, $U \cap M = \phi$, otherwise by lemma 7.2.2., $U \subset M$, which is not true. Thus $U \subset X \setminus M$. □

7.3 $R-\mathcal{I}-T_{min}$ space and $R-\mathcal{I}-T_{max}$ space

Definition 7.3.1. *A topological space $(X, \tau, \mathcal{I})$ is called a $R-\mathcal{I}-T_{min}$ space if every proper non-empty $R-\mathcal{I}-$open subset of X is a minimal $R-\mathcal{I}-$open set.*

Definition 7.3.2. *A topological space $(X, \tau, \mathcal{I})$ is called a $R-\mathcal{I}-T_{max}$ space if every proper non-empty $R-\mathcal{I}-$open subset of X is a maximal $R-\mathcal{I}-$open set.*

Example 7.3.1. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a, c\}, \{d\}, \{a, c, d\}, X\}$, $I = \{\phi, \{d\}\}$.*
Then proper $R-\mathcal{I}-$open sets are $\{a, c\}$ and $\{d\}$. Both of these sets are minimal $R-\mathcal{I}-$open and maximal $R-\mathcal{I}-$open. Hence this ideal topological space is a $R-\mathcal{I}-T_{min}$ space as well a $R-\mathcal{I}-T_{max}$ space.

Theorem 7.3.1. *A topological space $(X, \tau, \mathcal{I})$ is a $R-\mathcal{I}-T_{min}$ space (resp. $R-\mathcal{I}-T_{max}$ space) if and only if every non-empty proper $R-\mathcal{I}-$closed subset of X is a maximal $R-\mathcal{I}-$closed (resp. minimal $R-\mathcal{I}-$closed) set in X.*

Proof. The proof directly follows from the definition and theorem 7.2.1. □

Remark 7.3.1. *The $R-\mathcal{I}-T_{min}$ (resp. $R-\mathcal{I}-T_{max}$) and $R-\mathcal{I}-T_0$ (resp. $R-\mathcal{I}-T_1$, $R-\mathcal{I}-T_2$) spaces are independent of each other.*

Example 7.3.2. *Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}$, $\mathcal{I}=\{\phi,\{a\}\}$. Then proper $R-\mathcal{I}-$open sets are $\{a\},\{b,c\}$ and both are minimal $R-\mathcal{I}-$open and maximal $R-\mathcal{I}-$open. Hence X is a $R-\mathcal{I}-T_{min}$ space as well a $R-\mathcal{I}-T_{max}$ space. But X is not a $R-\mathcal{I}-T_0$ (resp. $R-\mathcal{I}-T_1$, $R-\mathcal{I}-T_2$) space.*

Example 7.3.3. *Let $X=\{a,b,c\}$, $\tau=\mathcal{P}(X)$, $\mathcal{I}=\{\{a\}\}$. Here not all $R-\mathcal{I}-$open sets are minimal $R-\mathcal{I}-$open and as well not maximal $R-\mathcal{I}-$open. But $(X,\tau,\mathcal{I})$ is $R-\mathcal{I}-T_0$, $R-\mathcal{I}-T_1$ and $R-\mathcal{I}-T_2$.*

Definition 7.3.3. *An ideal topological space $(X,\tau,\mathcal{I})$ is called a $R-\mathcal{I}-$door space if every subset of X is either $R-\mathcal{I}-$open or $R-\mathcal{I}-$closed.*

Definition 7.3.4. *A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be $R-\mathcal{I}-$dense in X if $R-\mathcal{I}-Cl(A)=X$.*

Definition 7.3.5. *An ideal topological space $(X,\tau,\mathcal{I})$ is called a sub maximal $R-\mathcal{I}-$space if every $R-\mathcal{I}-$dense subset of X is $R-\mathcal{I}-$open.*

Remark 7.3.2. *The $R-\mathcal{I}-T_{min}$ (resp. $R-\mathcal{I}-T_{max}$) and $R-\mathcal{I}-$door spaces are independent of each other.*

Example 7.3.4. *Let $X=\{a,b,c\}$, $\tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}$, $\mathcal{I}=\{\phi,\{a\}\}$. Then proper $R-\mathcal{I}-$open sets are $\{a\},\{b,c\}$ and both are minimal $R-\mathcal{I}-$open and maximal $R-\mathcal{I}-$open. Hence X is a $R-\mathcal{I}-T_{min}$ space as well a $R-\mathcal{I}-T_{max}$ space. But X is not a $R-\mathcal{I}-$door space.*

Example 7.3.5. *Let $X=\{a,b,c\}$, $\tau=\mathcal{P}(X)$, $\mathcal{I}=\{\{a\}\}$. Here not all $R-\mathcal{I}-$open sets are minimal $R-\mathcal{I}-$open but maximal $R-\mathcal{I}-$open. But $(X,\tau,\mathcal{I})$ is a $R-\mathcal{I}-$door space.*

Remark 7.3.3. *The $R-\mathcal{I}-T_{min}$ space (resp. $R-\mathcal{I}-T_{max}$) and sub maximal $R-\mathcal{I}-$space are independent of each other.*

Example 7.3.6. *Let $X=\{a,b,c,d\}$, $\tau=\{\phi,\{a,c\},\{d\},\{a,c,d\},X\}$,
$I=\{\phi,\{d\}\}$.
Then proper $R-\mathcal{I}-$open sets are $\{a,c\}$ and $\{d\}$. Also X is $R-\mathcal{I}-T_{min}$ and $R-\mathcal{I}-T_{max}$ space.
But the $R-\mathcal{I}-$dense subset $\{b\}$ is not $R-\mathcal{I}-$open and hence X is not sub maximal $R-\mathcal{I}-$space.*

Example 7.3.7. *Let $X=\{a,b,c\}$, $\tau=\mathcal{P}(X)$, $\mathcal{I}=\{\{a\}\}$. Then $(X,\tau,\mathcal{I})$ is not a $R-\mathcal{I}-T_{min}$ space. But it is a sub maximal $R-\mathcal{I}-$space.*

7.4 Minimal $R-\mathcal{I}-$continuous and maximal $R-\mathcal{I}-$continuous functions

Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be ideal topological spaces and $f:(X,\tau,I)\rightarrow(Y,\sigma,J)$ be a continuous function.

Then:

Definition 7.4.1. *f is minimal R–$\mathcal{I}-$continuous, if $f^{-1}(M)$ is R–$\mathcal{I}-$open in X for every minimal $R-\mathcal{J}-$open set M in Y.*

Definition 7.4.2. *f is maximal $R-\mathcal{I}-$continuous, if $f^{-1}(M)$ is $R-\mathcal{I}-$open in X for every maximal $R-\mathcal{J}-$open set M in Y.*

Definition 7.4.3. *f is minimal $R-\mathcal{I}-$irresolute, if $f^{-1}(M)$ is minimal $R-\mathcal{I}-$open in X for every minimal $R-\mathcal{J}-$open set M in Y.*

Definition 7.4.4. *f is maximal $R-\mathcal{I}-$irresolute, if $f^{-1}(M)$ is maximal $R-\mathcal{I}-$open in X for every maximal $R-\mathcal{J}-$open set M in Y.*

Definition 7.4.5. *f is minimal–maximal $R-\mathcal{I}-$continuous if $f^{-1}(M)$ is maximal $R-\mathcal{I}-$open in X for every minimal $R-\mathcal{J}-$open set M in Y.*

Definition 7.4.6. *f is maximal–minimal $R-\mathcal{I}-$continuous if $f^{-1}(M)$ is minimal $R-\mathcal{I}-$open in X for every maximal $R-\mathcal{J}-$open set M in Y.*

Now relation between the above defined continuous functions and some continuous functions which are already defined are studied as follows. All the proofs directly follow from the definitions and the basic concepts of $R-\mathcal{I}-$open sets.

Theorem 7.4.1. *Every $R^*-\mathcal{I}-$irresolute function is minimal $R-\mathcal{I}-$continuous.*

Remark 7.4.1. *The converse of the above theorem need not be true.*

Example 7.4.1. *Consider $X=Y=\{a,b,c\}, \tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}$, $\sigma=\{\phi,X,\{a\},\{a,b\}\}$, $\mathcal{I}=\{\phi,\{a\}\}$, $\mathcal{J}=\{\phi,\{a\}\}$. Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ be such that $f(a)=a, f(b)=c, f(c)=b$. Then f is minimal $R-\mathcal{I}-$continuous, but not $R^*-\mathcal{I}-$irresolute.*

Theorem 7.4.2. *If Y is a $R-\mathcal{J}-T_{min}$ space and $f:X\to Y$ is minimal $R-\mathcal{I}-$continuous surjective function, then f is $R^*-\mathcal{I}-$irresolute.*

Theorem 7.4.3. *Every $R^*-\mathcal{I}-$irresolute function is maximal $R-\mathcal{I}-$continuous.*

Remark 7.4.2. *The converse of the above theorem need not be true.*

Example 7.4.2. *Consider $X=Y=\{a,b,c\}, \tau=\{\phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}$, $\sigma=\{\phi,X,\{a\},\{a,b\}\}$, $\mathcal{I}=\{\phi,\{a\}\}$, $\mathcal{J}=\{\phi,\{a\}\}$. Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ be such that $f(a)=c, f(b)=b, f(c)=a$. Then f is maximal $R-\mathcal{I}-$continuous, but not $R^*-\mathcal{I}-$irresolute.*

Theorem 7.4.4. *If* Y *is a* $R-\mathcal{J}-T_{max}$ *space and* $f : X \to Y$ *is maximal* $R-\mathcal{I}-$*continuous surjective function, then* f *is* $R^*-\mathcal{I}-$*irresolute.*

Theorem 7.4.5. *Every totally* $\mathcal{I}-$*continuous function is minimal* $R-\mathcal{I}-$*continuous.*

Remark 7.4.3. *The converse of the above theorem need not be true.*

For example, consider the example 7.4.1. which is minimal $R-\mathcal{I}-$*continuous but not totally* $\mathcal{I}-$*continuous.*

Definition 7.4.7. *A topological space is said to be locally indiscrete if every open set is closed.*

Theorem 7.4.6. *Let* X *and* Y *be locally indiscrete spaces. If* $f : X \to Y$ *is minimal* $R-\mathcal{I}-$*continuous where* Y *is a* $R-\mathcal{J}-T_{min}$ *space, then* f *is totally* $\mathcal{I}-$*continuous.*

Theorem 7.4.7. *Every totally* $\mathcal{I}-$*continuous function is maximal* $R-\mathcal{I}-$*continuous.*

Remark 7.4.4. *The converse of the above theorem need not be true.*

For example, consider the example 7.4.2. which is maximal $R-\mathcal{I}-$*continuous but not totally* $\mathcal{I}-$*continuous.*

Theorem 7.4.8. *Let* X *and* Y *be locally indiscrete spaces. If* $f : X \to Y$ *is maximal* $R-\mathcal{I}-$*continuous where* Y *is is a* $R-\mathcal{J}-T_{max}$ *space, then* f *is totally* $\mathcal{I}-$*continuous.*

Theorem 7.4.9. *If* $f : X \to Y$ *is minimal* $R-\mathcal{I}-$*continuous where* Y *is a* $R-\mathcal{J}-T_{min}$ *space, then* f *is almost* $R-\mathcal{I}-$*continuous.*

Theorem 7.4.10. *If* $f : X \to Y$ *is maximal* $R-\mathcal{I}-$*continuous where* Y *is is a* $R-\mathcal{J}-T_{max}$ *space, then* f *is almost* $R-\mathcal{I}-$*continuous.*

Theorem 7.4.11. *Let Y be a locally indiscrete space. If $f : X \to Y$ is almost $R-\mathcal{I}-$continuous, then f is minimal $R-\mathcal{I}-$continuous as well as maximal $R-\mathcal{I}-$continuous.*

Theorem 7.4.12. *Every $R^*-\mathcal{I}-$continuous function is minimal $R-\mathcal{I}-$continuous.*

Remark 7.4.5. *The converse of the above theorem need not be true.*

For example, consider the example 7.4.1 which is minimal $R-\mathcal{I}-$continuous but not $R^*-\mathcal{I}-$continuous.

Theorem 7.4.13. *Let Y be a locally indiscrete $R-\mathcal{J}-T_{min}$ space. Then every minimal $R-\mathcal{I}-$continuous surjective function is $R^*-\mathcal{I}-$continuous.*

Theorem 7.4.14. *Every $R^*-\mathcal{I}-$continuous function is maximal $R-\mathcal{I}-$continuous.*

Remark 7.4.6. *The converse of the above theorem need not be true.*

For example, consider the example 7.4.2. which is maximal $R-\mathcal{I}-$continuous but not $R^*-\mathcal{I}-$continuous.

Theorem 7.4.15. *Let Y be a locally indiscrete $R-\mathcal{J}-T_{max}$ space. Then every maximal $R-\mathcal{I}-$continuous surjective function is $R^*-\mathcal{I}-$continuous.*

Theorem 7.4.16. *Every almost perfectly $\mathcal{I}-$continuous function is minimal $R-\mathcal{I}-$continuous.*

Remark 7.4.7. *The converse of the above theorem need not be true.*

For example, consider the example 7.4.1. which is minimal $R-\mathcal{I}-$continuous but not almost perfectly $\mathcal{I}-$continuous.

Theorem 7.4.17. *Every almost perfectly $\mathcal{I}-$continuous function is maximal $R-\mathcal{I}-$continuous.*

Remark 7.4.8. *The converse of the above theorem need not be true.*

For example, consider the example 7.4.2. which is maximal $R-\mathcal{I}-$continuous but not almost perfectly $\mathcal{I}-$continuous.

Theorem 7.4.18. *Let X be a locally indiscrete space and Y be a $R-\mathcal{J}-T_{min}$ space. If $f: X \to Y$ is minimal $R-\mathcal{I}-$continuous, then f is almost perfectly $\mathcal{I}-$continuous.*

Theorem 7.4.19. *Let X be a locally indiscrete space and Y be a $R-\mathcal{J}-T_{max}$ space. If $f: X \to Y$ is maximal $R-\mathcal{I}-$continuous, then f is almost perfectly $\mathcal{I}-$continuous.*

Remark 7.4.9. *Minimal $R-\mathcal{I}-$continuous and maximal $R-\mathcal{I}-$continuous functions are independent of each other.*

Example 7.4.3. *Let $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$, $\mathcal{J} = \{\phi, \{a\}\}$.*
Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be such that $f(a) = a, f(b) = c, f(c) = b$. Then f is minimal $R-\mathcal{I}-$continuous but not maximal $R-\mathcal{I}-$continuous.
If f is defined as $f(a) = c, f(b) = b, f(c) = a$, then $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is maximal $R-\mathcal{I}-$continuous but not minimal $R-\mathcal{I}-$continuous.

Theorem 7.4.20. *Every minimal $R-\mathcal{I}-$irresolute map is minimal $R-\mathcal{I}-$continuous.*

Remark 7.4.10. *The converse of the above theorem need not be true.*

Example 7.4.4. *Consider $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\mathcal{I} = \{\phi, \{c\}\}$, $\mathcal{J} = \{\phi, \{a\}\}$.*
Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be such that $f(a) = c, f(b) = a, f(c) = a$. Then f is minimal $R-\mathcal{I}-$continuous but not minimal $R-\mathcal{I}-$irresolute.

Theorem 7.4.21. *Every maximal $R-\mathcal{I}-$irresolute map is maximal $R-\mathcal{I}-$continuous.*

Remark 7.4.11. *The converse of the above theorem need not be true.*

Example 7.4.5. *Consider* $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{c\}, \{b, c\}\}$,
$\sigma = \{\phi, X, \{a\}, \{a, b\}\}, \quad \mathcal{I} = \{\phi, \{c\}\}, \quad \mathcal{J} = \{\phi, \{a\}\}$.
Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *be such that* $f(a) = c, f(b) = c, f(c) = a$. *Then* f *is maximal* $R-\mathcal{I}-$*continuous but not maximal* $R-\mathcal{I}-$*irresolute.*

Theorem 7.4.22. *If* $f : X \to Y$ *is minimal* $R-\mathcal{I}-$*continuous where* X *is a* $R-\mathcal{I}-T_{min}$ *space, then* f *is minimal* $R-\mathcal{I}-$*irresolute.*

Theorem 7.4.23. *If* $f : X \to Y$ *is maximal* $R-\mathcal{I}-$*continuous where* X *is a* $R-\mathcal{I}-T_{max}$ *space, then* f *is maximal* $R-\mathcal{I}-$*irresolute.*

Theorem 7.4.24. *If* $f : X \to Y$ *is a minimal* $R-\mathcal{I}-$*irresolute surjective function where* Y *is* $R-\mathcal{J}-T_{min}$, *then* f *is* $R^*-\mathcal{I}-$*irresolute.*

Theorem 7.4.25. *If* $f : X \to Y$ *is a maximal* $R-\mathcal{I}-$*irresolute surjective function where* Y *is* $R-\mathcal{J}-T_{max}$, *then* f *is* $R^*-\mathcal{I}-$*irresolute.*

Theorem 7.4.26. *If* X *is* $R-\mathcal{I}-T_{min}$ *space, then every* $R^*-\mathcal{I}-$*irresolute function is minimal* $R-\mathcal{I}-$*irresolute.*

Theorem 7.4.27. *If* X *is* $R-\mathcal{I}-T_{max}$ *space, then every* $R^*-\mathcal{I}-$*irresolute function is maximal* $R-\mathcal{I}-$*irresolute.*

Theorem 7.4.28. *Every minimal–maximal* $R-\mathcal{I}-$*continuous function is minimal* $R-\mathcal{I}-$*continuous.*

Theorem 7.4.29. *Every minimal* $R-\mathcal{I}-$*continuous function from a* $R-\mathcal{I}-T_{max}$ *space is minimal-maximal* $R-\mathcal{I}-$*continuous.*

Theorem 7.4.30. *Every maximal–minimal* $R-\mathcal{I}-$*continuous function is maximal* $R-\mathcal{I}-$*continuous.*

Theorem 7.4.31. *Every maximal* $R-\mathcal{I}-$*continuous function from a* $R-\mathcal{I}-T_{min}$ *space is maximal-minimal* $R-\mathcal{I}-$*continuous.*

Theorem 7.4.32. *A continuous surjective function $f : X \to Y$ is minimal $R-\mathcal{I}-$continuous if and only if the inverse image of each maximal $R-\mathcal{J}-$closed set in Y is a $R-\mathcal{I}-$closed set in X.*

Theorem 7.4.33. *A continuous surjective function $f : X \to Y$ is maximal $R-\mathcal{I}-$continuous if and only if the inverse image of each minimal $R-\mathcal{J}-$closed set in Y is a $R-\mathcal{I}-$closed set in X.*

Theorem 7.4.34. *A continuous surjective function $f : X \to Y$ is minimal $R-\mathcal{I}-$irresolute if and only if the inverse image of each maximal $R-\mathcal{J}-$closed set in Y is a maximal $R-\mathcal{I}-$closed set in X.*

Theorem 7.4.35. *A continuous surjective function $f : X \to Y$ is maximal $R-\mathcal{I}-$irresolute if and only if the inverse image of each minimal $R-\mathcal{J}-$closed set in Y is a minimal $R-\mathcal{I}-$closed set in X.*

Theorem 7.4.36. *A continuous surjective function $f : X \to Y$ is maximal–minimal $R-\mathcal{I}-$continuous if and only if the inverse image of each minimal $R-\mathcal{J}-$closed set in Y is a maximal $R-\mathcal{I}-$closed set in X.*

Theorem 7.4.37. *A continuous surjective function $f : X \to Y$ is minimal–maximal $R-\mathcal{I}-$continuous if and only if the inverse image of each maximal $R-\mathcal{J}-$closed set in Y is a minimal $R-\mathcal{I}-$closed set in X.*

Remark 7.4.12. *Composition of minimal $R-\mathcal{I}-$continuous functions need not be minimal $R-\mathcal{I}-$continuous.*

Example 7.4.6. *Let $X = Y = Z = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\eta = \{\phi, X, \{b, c\}\}$ $\mathcal{I} = \{\phi, \{a\}\}$, $\mathcal{J} = \{\phi, \{a\}\}$, $\mathcal{K} = \{\phi, \{a\}\}$.*
Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be such that $f(a) = a, f(b) = c, f(c) = b$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta, \mathcal{K})$ be such that $g(a) = b, g(b) = c, g(c) = a$. Then $g \circ f : X \to Z$ is not minimal $R-\mathcal{I}-$continuous even though f is minimal

$R-\mathcal{I}-$continuous and g is minimal $R-\mathcal{J}-$continuous.

Here $(g\circ f)^{-1}(\{b,c\}) = f^{-1}(g^{-1}(\{b,c\})) = f^{-1}(\{a,b\}) = \{a,c\}$ which is not $R-\mathcal{I}-$open in X.

Remark 7.4.13. *Composition of maximal $R-\mathcal{I}-$continuous functions need not be maximal $R-\mathcal{I}-$continuous.*

Example 7.4.7. *Let $X = Y = Z = \{a,b,c\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a,b\}\}$, $\eta = \{\phi, X, \{a\}\}$ $\mathcal{I} = \{\phi, \{a\}\}$, $\mathcal{J} = \{\phi, \{a\}\}$, $\mathcal{K} = \{\phi, \{a\}\}$.*

Let $f : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ be such that $f(a) = c, f(b) = b, f(c) = a$ and $g : (Y,\sigma,\mathcal{J}) \to (Z,\eta,\mathcal{K})$ be such that $g(a) = a, g(b) = b, g(c) = c$. Then $g\circ f : X \to Z$ is not maximal $R-\mathcal{I}-$continuous even though f is maximal $R-\mathcal{I}-$continuous and g is maximal $R-\mathcal{J}-$continuous.

Here $(g\circ f)^{-1}(\{a\}) = f^{-1}(g^{-1}(\{a\})) = f^{-1}(\{a\}) = \{c\}$ which is not $R-\mathcal{I}-$open in X.

Let $(X,\tau,\mathcal{I})$, $(Y,\sigma,\mathcal{J})$ and $(Z,\eta,\mathcal{K})$ be ideal topological spaces.

Theorem 7.4.38. *If $f : X \to Y$ is $R^* - \mathcal{I}-$irresolute and $g : Y \to Z$ is minimal $R-\mathcal{J}-$continuous, then $g\circ f : X \to Z$ is minimal $R-\mathcal{I}-$continuous.*

Theorem 7.4.39. *If $f : X \to Y$ is $R^* - \mathcal{I}-$irresolute and $g : Y \to Z$ is maximal $R-\mathcal{J}-$continuous, then $g\circ f : X \to Z$ is maximal $R-\mathcal{I}-$continuous.*

Theorem 7.4.40. *If $f : X \to Y$ is minimal $R-\mathcal{I}-$irresolute continuous function and $g : Y \to Z$ is minimal $R-\mathcal{J}-$irresolute continuous function, then $g\circ f : X \to Z$ is minimal $R-\mathcal{I}-$irresolute.*

Theorem 7.4.41. *If $f : X \to Y$ is maximal $R - \mathcal{I}-$irresolute continuous function and $g : Y \to Z$ is maximal $R - \mathcal{J}-$irresolute continuous function, then $g \circ f : X \to Z$ is maximal $R - \mathcal{I}-$irresolute.*

Remark 7.4.14. *Composition of minimal maximal $R-\mathcal{I}-$continuous functions need not be minimal maximal $R - \mathcal{I}-$continuous.*

Example 7.4.8. *Consider $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{b\}, Y\}$, $\mathcal{I} = \{\phi, \{a\}\}$, $\mathcal{J} = \{\phi, \{b\}\}$.*
Then $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ defined as $f(a) = b, f(b) = a, f(c) = c$ is minimal maximal $R - \mathcal{I}-$continuous. But $f \circ f$ is not minimal maximal $R - \mathcal{I}-$continuous.

Remark 7.4.15. *Composition of maximal minimal $R-\mathcal{I}-$continuous functions need not be maximal minimal $R - \mathcal{I}-$continuous.*

Example 7.4.9. *Consider $X = Y = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}X\}$, $\sigma = \{\phi, \{b\}, \{b, c\}Y\}$, $\mathcal{I} = \{\phi, \{a\}\}$, $\mathcal{J} = \{\phi, \{b\}\}$.*
Then $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ defined as $f(a) = b, f(b) = a, f(c) = b$ is maximal minimal $R - \mathcal{I}-$continuous. But $f \circ f$ is not maximal minimal $R - \mathcal{I}-$continuous.

Theorem 7.4.42. *If $f : X \to Y$ is minimal maximal $R-\mathcal{I}-$continuous and $g : Y \to Z$ is minimal $R-\mathcal{J}-$continuous function where Y is a $R-\mathcal{J}-T_{min}$ space, then $g \circ f : X \to Z$ is minimal maximal $R - \mathcal{I}-$continuous.*

Theorem 7.4.43. *If $f : X \to Y$ is maximal minimal $R - \mathcal{I}-$continuous function and $g : Y \to Z$ is maximal $R - \mathcal{J}-$continuous function where Y is a $R - \mathcal{J} - T_{max}$ space, then $g \circ f : X \to Z$ is maximal minimal $R - \mathcal{I}-$continuous.*

Theorem 7.4.44. *If $f : X \to Y$ is maximal $R - \mathcal{I}-$irresolute and $g : Y \to Z$ is minimal maximal $R - \mathcal{J}-$continuous, then $g \circ f : X \to Z$ is minimal maximal $R - \mathcal{I}-$continuous.*

Theorem 7.4.45. *If* $f : X \to Y$ *is maximal* $R - \mathcal{I}-$*continuous and* $g : Y \to Z$ *is minimal maximal* $R - \mathcal{J}-$*continuous, then* $g \circ f : X \to Z$ *is minimal* $R - \mathcal{I}-$*continuous.*

Theorem 7.4.46. *If* $f : X \to Y$ *is minimal* $R - \mathcal{I}-$*irresolute and* $g : Y \to Z$ *is maximal minimal* $R - \mathcal{I}-$*continuous, then* $g \circ f : X \to Z$ *is maximal minimal* $R - \mathcal{I}-$*continuous.*

Theorem 7.4.47. *If* $f : X \to Y$ *is minimal* $R - \mathcal{I}-$*continuous and* $g : Y \to Z$ *is maximal minimal* $R - \mathcal{I}-$*continuous, then* $g \circ f : X \to Z$ *is maximal* $R - \mathcal{I}-$*continuous.*

8

Somewhat $R-\mathcal{I}-$continuous and Somewhat $R-\mathcal{I}-$open Functions

8.1 Introduction

Now, we put forward the notions of somewhat $R-\mathcal{I}-$continuous functions and somewhat $R-\mathcal{I}-$open functions. In section 2, we will study about somewhat $R-\mathcal{I}-$continuous functions and its relationship with other classes of functions. Also some of its characterizations and properties are obtained besides giving examples and counter examples. In section 4, we will study about somewhat $R-\mathcal{I}-$open functions and get results which go parallel with the results of somewhat $R-\mathcal{I}-$continuous functions.
Some definitions which are related in this topic:

Definition 8.1.1. *[25] A function $f:(X,\tau)\to(Y,\sigma)$ is said to be somewhat continuous if for $U\in\sigma$ and $f^{-1}(U)\neq\phi$, there exists an open set V in X such that $V\neq\phi$ and $V\subset f^{-1}(U)$.*

Definition 8.1.2. *[32] A function $f:(X,\tau)\to(Y,\sigma)$ is said to be somewhat r-continuous, if $U\in\sigma$ and $f^{-1}(U)\neq\phi$, then there exists a regular open set V in X such that $V\neq\phi$ and $V\subset f^{-1}(U)$.*

Definition 8.1.3. *[2] A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is said to be $R-\mathcal{I}-$continuous if for each $x\in X$ and for open set $V\in\sigma$ containing $f(x)$,* *there exists a* $R-\mathcal{I}-$*open set* $U \subset X$ *containing* x *such that* $f(U) \subset V$.

Definition 8.1.4. *[18] A function* $f : X \to Y$ *is said to be completely continuous if* $f^{-1}(V)$ *is a regular open set in* X*, for every open set* V *in* Y.

Definition 8.1.5. *[29] A function* $f : X \to Y$ *is said to be almost completely continuous if* $f^{-1}(V)$ *is a regular open set in* X *for every regular open set* V *in* Y.

Definition 8.1.6. *[17] A function* $f : X \to Y$ *is said to be perfectly continuous if* $f^{-1}(V)$ *is clopen in* X *for every open set* V *in* Y.

Definition 8.1.7. *[17] A function* $f : X \to Y$ *is said to be almost perfectly continuous if* $f^{-1}(V)$ *is clopen for every regular open set* V *in* Y.

Definition 8.1.8. *[1] A function* $f : X \to Y$ *is said to be almost cl-supercontinuous if for each* $x \in X$ *and each regular open set* V *containing* $f(x)$ *there is a clopen set* U *containing* x *such that* $f(U) \subset V$.

Definition 8.1.9. *[1] A function* $f : X \to Y$ *is said to be cl-supercontinuous, if for each* $x \in X$ *and for each open set* V *containing* $f(x)$*, there exists a clopen set* U *containing* x *such that* $f(U) \subset V$.

8.2 Somewhat $R-\mathcal{I}-$continuous function

Definition 8.2.1. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is said to be completely* $\mathcal{I}-$*continuous if* $f^{-1}(U)$ *is a* $R-\mathcal{I}-$*open set in* X *for every open set* $U \subset Y$.

Definition 8.2.2. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is said to be almost completely* $\mathcal{I}-$*continuous if* $f^{-1}(U)$ *is a* $R-\mathcal{I}-$*open set in* X *for every* $R-\mathcal{J}-$*open set* $U \subset Y$.

Definition 8.2.3. *A function* $f : X \to Y$ *is said to be almost perfectly* $\mathcal{I}-$*continuous if* $f^{-1}(V)$ *is* $R-\mathcal{I}-$ *clopen for every* $R-\mathcal{I}-$*open set* V *in* Y.

Definition 8.2.4. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is said to be almost* $\mathcal{I} - cl-$*supercontinuous if for each* $x \in X$ *and each* $R - \mathcal{J}-$*open set* V *containing* $f(x)$ *there is a clopen set* U *containing* x *such that* $f(U) \subset V$*.*

Definition 8.2.5. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* (Y, σ) *be any topological space. A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is said to be somewhat* $R - \mathcal{I}-$*continuous if for any* $U \in \sigma$ *such that* $f^{-1}(U) \neq \phi$ *there exists a* $R - \mathcal{I}-$*open set* V *in* X *such that* $V \neq \phi$ *and* $V \subset f^{-1}(U)$*.*

Example 8.2.1. *Let* $X = Y = \{a, b, c\}, \quad \tau = \{\phi, X, \{a\}, \{b, c\}\},$
$\mathcal{I} = \{\phi, \{a\}\}, \quad \sigma = \{\phi, X, \{b, c\}\}.$
Define $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ *by* $f(a) = a, f(b) = c, f(c) = b.$
Then f *is somewhat* $R - \mathcal{I}-$*continuous and the* $R - \mathcal{I}-$*open sets are* $\{a\}, \{b, c\}$ *and* X*.*

Theorem 8.2.1. *Every somewhat* $R - \mathcal{I}-$*continuous function is somewhat continuous.*
The converse does not hold.

Example 8.2.2. *Let* $X = \{a, b, c, d\}, \quad Y = \{e, f, g\},$
$\tau = \{\phi, X, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}, \quad \sigma = \{\phi, Y, \{f\}, \{g\}, \{f, g\}\},$
$\mathcal{I} = \{\phi, \{b\}\}$. *Define* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *by* $f(a) = e, f(d) = e, f(c) = g, f(b) = g$. *Then* f *is somewhat continuous, but not somewhat* $R - \mathcal{I}-$*continuous.*

Theorem 8.2.2. *Every* $R - \mathcal{I}-$*continuous function is somewhat* $R - \mathcal{I}-$*continuous.*
The converse does not hold.

Example 8.2.3. *Let* $X = \{a, b, c\}, \quad \tau = \{\phi, X, \{a\}, \{b, c\}\},$
$\mathcal{I} = \{\phi, \{b, c\}\}, \quad \sigma = \{\phi, X, \{a, b\}\}.$

Let $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ *be the identity function.*

Then f *is somewhat* $R - \mathcal{I}-$*continuous, but not* $R - \mathcal{I}-$*continuous.*

Theorem 8.2.3. *If* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is somewhat* $r-$*continuous then* f *is somewhat* $R - \mathcal{I}-$*continuous.*

Theorem 8.2.4. *Every* $cl-$*supercontinuous function is somewhat* $R - \mathcal{I}-$*continuous.*

The converse does not hold.

Example 8.2.4 (2, Example 2.5)**.** *Let* $X = \{a, b, c, d\}$,

$\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, $\sigma = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$,

$\mathcal{I} = \{\phi, \{d\}\}$. *Let* $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ *be the identity function.*

The clopen sets of X *are* ϕ *and* X *only. The* $R - \mathcal{I}-$*open sets are* $\{a, c\}, \{d\}$ *and* X. *Clearly* f *is somewhat* $R - \mathcal{I}-$*continuous, but is not* $cl-$*supercontinuous.*

Theorem 8.2.5. *Every almost* $\mathcal{I} - cl-$*supercontinuous function is somewhat* $R - \mathcal{I}-$*continuous.*

The converse does not hold.

Example 8.2.5. *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$,

$\sigma = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, $\mathcal{I} = \{\phi, \{d\}\}$.

Let $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ *be the identity function.*

The clopen sets of X *are* ϕ *and* X *only. The* $R-\mathcal{I}-$*open sets are* $\{a, c\}, \{d\}$ *and* X. *Clearly* f *is somewhat* $R - \mathcal{I}-$*continuous, but is not almost* $\mathcal{I} - cl-$*supercontinuous.*

Theorem 8.2.6. *Every completely* $\mathcal{I}-$*continuous function is somewhat* $R - \mathcal{I}-$*continuous.*

The converse does not hold.

Example 8.2.6. *Let* $X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b, c\}\}, \mathcal{I} = \{\phi, \{b, c\}\}$, $\sigma = \{\phi, X, \{a, b\}\}$.
Let $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ *be the identity function. Clearly* f *is somewhat* $R - \mathcal{I}-$*continuous, but is not completely* $\mathcal{I}-$*continuous.*

Theorem 8.2.7. *Every almost perfectly* $\mathcal{I}-$*continuous function is almost* $\mathcal{I} - cl-$*supercontinuous.*

Theorem 8.2.8. *If* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is somewhat continuous and* X *is locally indiscrete, then* f *is somewhat* $R - \mathcal{I}-$*continuous.*

Proof. Since clopen sets are $R - \mathcal{I}-$open, the result follows. □

Theorem 8.2.9. *If* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is somewhat* $R - \mathcal{I}-$*continuous and* X *is locally indiscrete, then* f *is* $cl-$*supercontinuous and hence almost* $\mathcal{I} - cl-$*supercontinuous.*

Proof. Given f is somewhat $R-\mathcal{I}-$continuous. So for any $U \in \sigma$ such that $f^{-1}(U) \neq \phi$, there exists a $R - \mathcal{I}-$open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$. Clearly $V \in \tau$ and since X is locally indiscrete, V is clopen. So f is $cl-$supercontinuous. Hence f is almost $\mathcal{I} - cl-$supercontinuous. □

Theorem 8.2.10. *If* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *is almost completely* $\mathcal{I}-$*continuous and* X *is locally indiscrete, then* f *is almost perfectly* $\mathcal{I}-$*continuous.*

Proof. Since $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is almost completely $\mathcal{I}-$continuous, $f^{-1}(U)$ is a $R-\mathcal{I}-$open set in X for every $R-\mathcal{J}-$open set $U \subset Y$. Clearly $f^{-1}(U)$ is open and since X is locally indiscrete, $f^{-1}(U)$ is clopen. Hence f is almost perfectly $\mathcal{I}-$continuous. □

Theorem 8.2.11. *If* X *is discrete and* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is somewhat* $R - \mathcal{I}-$*continuous, then* f *is completely* $\mathcal{I}-$*continuous and hence almost completely* $\mathcal{I}-$*continuous.*

Proof. Since X is discrete, every subset is clopen. Thus every subset of X is $R-\mathcal{I}-$open. Since f is somewhat $R-\mathcal{I}-$continuous, for any $U \in \sigma$ such that $f^{-1}(U) \neq \phi$ there exists a $R-\mathcal{I}-$open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(U)$. Then clearly $f^{-1}(U)$ is $R-\mathcal{I}-$open for every $U \in \sigma$. Thus f is completely $\mathcal{I}-$continuous and hence almost completely $\mathcal{I}-$continuous. □

Corollary 8.2.1. *If X is finite and T_1 and $f:(X,\tau,\mathcal{I}) \to (Y,\sigma)$ is somewhat $R-\mathcal{I}-$continuous, then f is completely $\mathcal{I}-$continuous and hence almost completely $\mathcal{I}-$continuous.*

Theorem 8.2.12. *If $f:(X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ is almost completely $\mathcal{I}-$continuous and Y is locally indiscrete, then f is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let $V \in \sigma$. Since Y is locally indiscrete, V is clopen and hence V is $R-\mathcal{J}-$open. Since f is almost completely $\mathcal{I}-$continuous, $f^{-1}(V)$ is $R-\mathcal{I}-$open. Let $U = f^{-1}(V)$. Then for each $V \in \sigma$ such that $f^{-1}(V) \neq \phi$ there exists a $R-\mathcal{I}-$open set U in X such that $U \neq \phi$ and $U \subset f^{-1}(V)$. Thus f is somewhat $R-\mathcal{I}-$continuous. □

Theorem 8.2.13. *If $f:(X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ is somewhat $R-\mathcal{I}-$continuous and $g:(Y,\sigma,\mathcal{J}) \to (Z,\eta)$ is continuous, then $g \circ f$ is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let $U \in \eta$ and $g^{-1}(U) \neq \phi$. Since g is continuous, $g^{-1}(U) \in \sigma$. Suppose that $f^{-1}(g^{-1}(U)) \neq \phi$. Since f is somewhat $R-\mathcal{I}-$continuous, for $g^{-1}(U) \in \sigma$ there exists a $R-\mathcal{I}-$open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$. Hence $g\circ f$ is somewhat $R-\mathcal{I}-$continuous. □

Theorem 8.2.14. *If $f:(X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ is somewhat $R-\mathcal{I}-$continuous surjection and $g:(Y,\sigma,\mathcal{J}) \to (Z,\eta)$ is somewhat continuous, then $g \circ f$ is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let $U \in \eta$ and $(g \circ f)^{-1}(U) \neq \phi$. Then $g^{-1}(U) \neq \phi$. Since g is somewhat continuous, there exists an open set V in Y such that $V \neq \phi$ and $V \subset g^{-1}(U)$. Then $\phi \neq f^{-1}(V) \subset f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Since f is somewhat $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{I}-$open set $W \in \tau$ such that $\phi \neq W \subset f^{-1}(V) \subset (g \circ f)^{-1}(U)$. Hence $g \circ f$ is somewhat $R-\mathcal{I}-$continuous. □

Theorem 8.2.15. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is somewhat $r-$continuous and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ is continuous, then $g \circ f$ is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let $U \in \eta$ and $g^{-1}(U) \neq \phi$. Since g is continuous, $g^{-1}(U) \in \sigma$. Suppose that $f^{-1}(g^{-1}(U)) \neq \phi$. Since f is somewhat $r-$continuous, for $g^{-1}(U) \in \sigma$ there exists a regular open set V in X such that $V \neq \phi$ and $V \subset f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Since V is regular open, V is $R-\mathcal{I}-$open. Hence $g \circ f$ is somewhat $R-\mathcal{I}-$continuous. □

Theorem 8.2.16. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ are somewhat $R-\mathcal{I}-$continuous functions, then $g \circ f$ is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let $U \in \eta$ and $(g \circ f)^{-1}(U) \neq \phi$. Then $g^{-1}(U) \neq \phi$. Since g is somewhat $R-\mathcal{J}-$continuous, there exists a $R-\mathcal{J}-$open set V in Y such that $V \neq \phi$ and $V \subset g^{-1}(U)$. Then $\phi \neq f^{-1}(V) \subset f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Since f is somewhat $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{I}-$open set $W \in \tau$ such that $\phi \neq W \subset f^{-1}(V) \subset (g \circ f)^{-1}(U)$. Hence $g \circ f$ is somewhat $R-\mathcal{I}-$continuous. □

Remark 8.2.1. *If f is continuous and g is somewhat $R-\mathcal{I}-$continuous, then it is not necessarily true that $g \circ f$ is somewhat $R-\mathcal{I}-$continuous.*

Example 8.2.7. *Let $X = Y = Z = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, c\}\}, \sigma = \{\phi, X, \{a, b\}\}, \eta = \{\phi, X, \{a, b\}\}, \mathcal{I} = \{\phi, \{c\}\}, \mathcal{J} = \{\phi, \{a, b\}\}$.*

Define $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ *by* $f(a) = a, f(b) = c, f(c) = a$ *and* $g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta)$ *by* $g(a) = a, g(b) = b, g(c) = c$.

Here f *is continuous and* g *is somewhat* $R-\mathcal{I}-$*continuous, but* $g \circ f$ *is not somewhat* $R-\mathcal{I}-$*continuous.*

Definition 8.2.6. *Let* M *be a subset of an ideal topological space* $(X, \tau, \mathcal{I})$. *Then* M *is said to be* $R-\mathcal{I}-$*dense in* X *if there is no proper* $R-\mathcal{I}-$*closed set* C *in* X *such that* $M \subset C \subset X$.

Example 8.2.8. *Let* $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, $\mathcal{I} = \{\phi, \{d\}\}$. *The* $R-\mathcal{I}-$*open sets are* $\{a, c\}, \{d\}$ *and* X. *Then* $\{b\}$ *is* $R-\mathcal{I}-$*dense in* X *since there exists no* $R-\mathcal{I}-$*closed set* C *in* X *such that* $\{b\} \subset C \subset X$.

Theorem 8.2.17. *Let* $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ *be a surjection. Then the following are equivalent:*

(i) f *is somewhat* $R-\mathcal{I}-$*continuous.*

(ii) If C *is a closed subset of* Y *such that* $f^{-1}(C) \neq X$, *then there is a proper* $R-\mathcal{I}-$*closed subset* D *of* X *such that* $D \supset f^{-1}(C)$.

(iii) If M *is a* $R-\mathcal{I}-$*dense subset of* X, *then* $f(M)$ *is a dense subset of* Y.

Proof. (i) $\Rightarrow$ (ii)

Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y - C$ is open in Y such that $f^{-1}(Y - C) = X - f^{-1}(C) \neq \phi$. Since f is somewhat $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{I}-$open set $V \neq \phi$ in X such that $V \subset f^{-1}(Y - C) = X - f^{-1}(C)$. Thus $f^{-1}(C) \subset X - V$. Since V is $R-\mathcal{I}-$open, $X - V = D$ is $R-\mathcal{I}-$closed. That is, there exists a $R-\mathcal{I}-$closed set D such that $D \supset f^{-1}(C)$.

(ii) $\Rightarrow$ (iii)

Let M be $R-\mathcal{I}-$dense in X. Suppose $f(M)$ is not dense in Y. Then there exists a proper closed set C in Y such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. Hence by (ii), there exists a proper $R-\mathcal{I}-$closed set D of X such that $D \supset f^{-1}(C)$. That is, $M \subset f^{-1}(C) \subset D \subset X$. This is a contradiction to the fact that M is $R-\mathcal{I}-$dense in X. So $f(M)$ is dense in Y.

(iii) $\Rightarrow$ (ii)

Suppose (ii) is not true. This means that there exists a closed set C in Y such that $f^{-1}(C) \neq X$, but there is no proper $R-\mathcal{I}-$closed set D in X such that $f^{-1}(C) \subset D$. This implies $f^{-1}(C)$ is $R-\mathcal{I}-$dense in X. But by (iii), $f(f^{-1}(C)) = C$ must be dense in Y, a contradiction to the choice of C. So (ii) is true.

(ii) $\Rightarrow$ (i)

Let $U \in \sigma$ and $f^{-1}(U) \neq \phi$. Then $Y-U$ is closed in Y and $f^{-1}(Y-U) = X - f^{-1}(U) \neq X$. So by (ii), there exists a proper $R-\mathcal{I}-$closed subset D of X such that $D \supset f^{-1}(Y-U) = X - f^{-1}(U)$. That is, $\phi \neq X - D \subset f^{-1}(U)$ and $X-D$ is a $R-\mathcal{I}-$open subset. So f is somewhat $R-\mathcal{I}-$continuous. □

Theorem 8.2.18. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be any two ideal topological spaces. Let A be a $R-\mathcal{I}-$open subset of X and $f : (A, \tau/A) \to (Y,\sigma)$ be somewhat $R-\mathcal{I}-$continuous such that $f(A)$ is dense in Y. Then any extension F of f is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let U be any open set in Y such that $F^{-1}(U) \neq \phi$. Since $f(A) \subset Y$ is dense in Y, $U \cap f(A) \neq \phi$. So $F^{-1}(U) \cap A \neq \phi$. So $f^{-1}(U) \cap A \neq \phi$. Since f is somewhat $R-\mathcal{I}-$continuous, by hypothesis there exists a $R-\mathcal{I}-$open set V such that $V \subset f^{-1}(U)$. Since V is a $R-\mathcal{I}-$open subset of A and A is a $R-\mathcal{I}-$open subset of X, V is a $R-\mathcal{I}-$open subset of X. Hence $V \subset F^{-1}(U)$. Thus corresponding to the open set U in Y such that $F^{-1}(U) \neq \phi$, there exists a $R-\mathcal{I}-$open set V such that $V \subset F^{-1}(U)$. So F is somewhat

$R-\mathcal{I}-$continuous. □

Theorem 8.2.19. *Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be any two ideal topological spaces. Let $X = A\cup B$ where A and B are $R-\mathcal{I}-$open subsets of X. Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ be a function such that f/A and f/B are somewhat $R-\mathcal{I}-$continuous. Then f is somewhat $R-\mathcal{I}-$continuous.*

Proof. Let U be any open set in Y such that $f^{-1}(U)\neq\phi$. Then either $(f/A)^{-1}(U)\neq\phi$ or $(f/B)^{-1}(U)\neq\phi$ or both $(f/A)^{-1}(U)$ and $(f/B)^{-1}(U)\neq\phi$.
Case(1): $(f/A)^{-1}(U)\neq\phi$.
Since f/A is somewhat $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{I}-$open set V in A such that $V\neq\phi$ and $V\subset(f/A)^{-1}(U)\subset f^{-1}(U)$. Since V is $R-\mathcal{I}-$open in A and A is $R-\mathcal{I}-$open in X, V is $R-\mathcal{I}-$open in X. So f is somewhat $R-\mathcal{I}-$continuous.
Case(2): $(f/B)^{-1}(U)\neq\phi$.
This can be proved by using the same argument as in case(1).
Case(3): $(f/A)^{-1}(U)$ and $(f/B)^{-1}(U)\neq\phi$.
The proof follows from the proof of the cases(1) and (2). □

Definition 8.2.7. *Let (X,τ) and (X,σ) be topological spaces with same ideal I. Then τ and σ are said to be $R-\mathcal{I}-$weakly equivalent if both the conditions below hold:*
(i) if for every non empty open set $U\in\tau$ there is a non empty $R-\mathcal{I}-$open set V in $(X,\sigma,\mathcal{I})$ such that $V\subset U$.
(ii) if for every non empty open set $U\in\sigma$ there is a non empty $R-\mathcal{I}-$open set V in $(X,\tau,\mathcal{I})$ such that $V\subset U$.

Example 8.2.9. *Let $X=\{a,b,c\}$, $\mathcal{I}=\{\phi,\{a\}\}$, $\tau=\{\phi,X,\{a\},\{a,b\}\}$, $\sigma=\{\phi,X,\{a\},\{a,c\}\}$. Then τ is $R-\mathcal{I}-$weakly equivalent to σ.*

Theorem 8.2.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat continuous function and let τ^* be a topology for X which is $R - \mathcal{I}-$weakly equivalent to τ. Then the function $f : (X, \tau^*, \mathcal{I}) \to (Y, \sigma)$ is somewhat $R - \mathcal{I}-$continuous.*

Proof. Since τ^* is $R - \mathcal{I}-$weakly equivalent to τ, $i : (X, \tau^*, \mathcal{I}) \to (X, \tau, \mathcal{I})$ is somewhat $R - \mathcal{I}-$continuous. Then by theorem 8.2.14, $f = f \circ i : (X, \tau^*, \mathcal{I}) \to (Y, \sigma)$ is somewhat $R - \mathcal{I}-$continuous. □

Theorem 8.2.21. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ be a somewhat $R - \mathcal{I}-$continuous surjective function and let σ^* be a topology for Y which is $R - \mathcal{I}-$weakly equivalent to σ. Then $f : (X, \tau, \mathcal{I}) \to (Y, \sigma^*)$ is somewhat $R - \mathcal{I}-$continuous.*

Proof. Since σ^* is $R - \mathcal{I}-$weakly equivalent to σ, $i : (Y, \sigma, \mathcal{I}) \to (Y, \sigma^*, I)$ is somewhat $R - \mathcal{I}-$continuous. Then by theorem 8.2.16, $f = f \circ i : (X, \tau, \mathcal{I}) \to (Y, \sigma^*)$ is somewhat $R - \mathcal{I}-$continuous. □

8.3 Somewhat $R - \mathcal{I}-$open functions

Definition 8.3.1. *[25] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat open if for each non empty open set $U \in \tau$ there exists a non empty open set V in σ such that $V \subset f(U)$.*

Definition 8.3.2. *A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is said to be somewhat $R - \mathcal{I}-$open if for each non empty open set $U \in \tau$ there exists a non empty $R - \mathcal{I}-$open set V in σ such that $V \subset f(U)$.*

Example 8.3.1. *Let $X = \{a, b, c\}$, $\mathcal{I} = \{\phi, \{a\}\}$, $\tau = \{\phi, X, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{b, c\}\}$. The $R - \mathcal{I}-$open sets in (X, σ) are $X, \{a\}, \{b, c\}$. Let $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is defined by $f(a) = a, f(b) = c, f(c) = b$. Then f is somewhat $R - \mathcal{I}-$open.*

Definition 8.3.3. *[25] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat clopen provided for $U \in \tau$ there exists clopen set V in Y such that $V \subset f(U)$.*

Theorem 8.3.1. *Every somewhat clopen function is somewhat $R{-}\mathcal{I}{-}$open.*

Remark 8.3.1. *The converse does not hold.*

Example 8.3.2. *Let $X = \{a, b, c\}$, $\mathcal{I} = \{\phi, \{b\}\}$, $\tau = \{X, \phi, \{a\}\}$, $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma, \mathcal{I})$ by $f(a) = b, f(b) = c, f(c) = a$.*
Then f is not somewhat clopen.

Theorem 8.3.2. *Every somewhat r-open function is somewhat $R{-}\mathcal{I}{-}$open.*

Theorem 8.3.3. *If $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is somewhat $R - \mathcal{I}-$open, then f is somewhat open.*

Remark 8.3.2. *The converse does not hold.*

Example 8.3.3. *Let $X = \{a, b, c\}$, $\mathcal{I} = \{\phi, \{c\}\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, X, \{c\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (X, \sigma, \mathcal{I})$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is somewhat open but not somewhat $R - \mathcal{I}-$open.*

Theorem 8.3.4. *If $f : (X, \tau) \to (Y, \sigma)$ is somewhat open and $g : (Y, \sigma) \to (Z, \eta, \mathcal{I})$ is somewhat $R{-}\mathcal{I}{-}$open, then $g \circ f : (X, \tau) \to (Z, \eta, \mathcal{I})$ is somewhat $R - \mathcal{I}-$open.*

Proof. Let $U \neq \phi \in \tau$. Since f is somewhat open, there exists a non empty open set $W \in \sigma$ such that $W \subset f(U)$. Since g is somewhat $R{-}\mathcal{I}{-}$open, for $W \in \sigma$ there exists a non empty $R{-}\mathcal{I}{-}$open set $V \in \eta$ such that $V \subset g(W)$. Then, $V \subset g(f(U)) = (g \circ f)(U)$. So $g \circ f$ is somewhat $R - \mathcal{I}-$open. $\square$

Theorem 8.3.5. *If $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is somewhat $R - \mathcal{I}-$open and $g : (Y, \sigma, \mathcal{I}) \to (Z, \eta, \mathcal{J})$ is somewhat $R - \mathcal{J}-$open, then $g \circ f : (X, \tau) \to (Z, \eta, \mathcal{J})$ is somewhat $R - \mathcal{J}-$open.*

Proof. Let $U \neq \phi \in \tau$. Since f is somewhat $R - \mathcal{I}-$open, there exists a $R - \mathcal{I}-$open set $V \neq \phi \in \sigma$ such that $V \subset f(U)$. Since g is somewhat $R - \mathcal{J}-$open, for $V \neq \phi \in \sigma$ there exists a non empty $R - \mathcal{J}-$open set $W \in \eta$ such that $W \subset g(V)$. Then, $\phi \neq W \subset g(V) \subset g(f(U)) = (g \circ f)(U)$. So $g \circ f$ is somewhat $R - \mathcal{J}-$open. □

Theorem 8.3.6. *If $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is a bijection, then the following are equivalent.*

(i) f is somewhat $R - \mathcal{I}-$open.

(ii) If C is a closed subset of X such that $f(C) \neq Y$, then there is a $R - \mathcal{I}-$closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$.

Proof. (i) $\Rightarrow$ (ii)
Let C be a closed subset of X such that $f(C) \neq Y$. Then $X - C$ is open in X and $X - C \neq \phi$. Since f is somewhat $R - \mathcal{I}-$open, there exists a non empty $R - \mathcal{I}-$open set V in Y such that $V \subset f(X - C)$. Put $D = Y - V$. Then clearly D is $R - \mathcal{I}-$closed in Y. Also $D \neq Y$, for if $D = Y$, then $V = \phi$, a contradiction. Since $V \subset f(X - C)$ and f is bijective, $D = Y - V \supset Y - f(X - C) = f(C)$.
(ii) $\Rightarrow$ (i)
Let U be a non empty open set in X. Put $C = X - U$. Then C is a proper closed subset of X and $f(C) = f(X - U) = Y - f(U)$. Then $f(C) \neq Y$. So by (ii), there exists a $R - \mathcal{I}-$closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$. Let $V = Y - D$. Then V is $R - \mathcal{I}-$open in Y and non empty. Further $V = Y - D \subset Y - f(C) = f(U)$. So f is somewhat $R - \mathcal{I}-$open. □

Theorem 8.3.7. *Let $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ be somewhat $R-\mathcal{I}-$open and A be any open subset of X. Then $f/A : (A, \tau/A) \to (Y, \sigma, \mathcal{I})$ is also somewhat $R - \mathcal{I}-$open.*

Proof. Let $U \in \tau/A$ and $U \neq \phi$. Since U is open in A and A is open in (X, τ), U is open in (X, τ). By hypothesis, $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is somewhat $R - \mathcal{I}-$open and so there exists a non empty $R - \mathcal{I}-$open set V in Y such that $V \subset f(U)$. Thus for any non empty open set U in $(A, \tau/A)$ with U, there exists a non empty $R - \mathcal{I}-$open set V in Y such that $V \subset (f/A)(U)$. So f/A is somewhat $R - \mathcal{I}-$open. □

Theorem 8.3.8. *Let (X, τ) and (Y, σ) be any two topological spaces and $X = A \cup B$ where A and B are open subsets of X. Let $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ be a function such that f/A and f/B are somewhat $R - \mathcal{I}-$open. Then f is somewhat $R - \mathcal{I}-$open.*

Proof. Let U be any non empty open subset of (X, τ). Since $X = A \cup B$, there are three cases.

(1) $A \cap U \neq \phi$ (2) $B \cap U \neq \phi$ (3) both $A \cap U \neq \phi$ and $B \cap U \neq \phi$.

Case(1): Since $A \cap U \in \tau/A$ and f/A is somewhat $R-\mathcal{I}-$open, there exists a non empty $R-\mathcal{I}-$open set $V \in \sigma$ such that $V \subset ((f/A)(U)) \subset f(U)$. So f is somewhat $R - \mathcal{I}-$open.

By a similar argument, the other cases can be proved. □

9

Contra $R-\mathcal{I}-$continuous functions

9.1 Introduction

The aim of this chapter is to set forth a class of functions using the concepts in ideal topological space called the contra $R-\mathcal{I}-$continuous functions. We also introduce almost contra $R-\mathcal{I}-$continuous functions and investigate certain properties and several characterisations of such concepts. Further, we will deal with contra $R-\mathcal{I}-$closed graphs and strongly contra $R-\mathcal{I}-$closed graphs. In section 2., using the notion of $R-\mathcal{I}-$open sets, the contra $R-\mathcal{I}-$continuous functions are presented and studied. In section 3., almost contra $R-\mathcal{I}-$continuous functions are investigated. Section 4. handles with $R-\mathcal{I}-$closed, contra $R-\mathcal{I}-$closed and strongly contra $R-\mathcal{I}-$closed graphs.

Definition 9.1.1. *[30] Let A be a subset of a topological space (X, τ). Then the set $\cap\{U \in \tau : A \subset U\}$ is called the kernel of A and is denoted by $Ker(A)$.*

Definition 9.1.2. *[15] A function $f : (X, \tau) \to (Y, \sigma)$ is called a contra continuous function if $f^{-1}(V)$ is closed in X for every open set V in Y.*

Lemma 9.1.1. *[41] Let A be a subset of a topological space (X, τ). Then,*

1. *$x \in Ker(A)$ if and only if $A \cap F \neq \phi$ for any closed subset F of X containing x.*

2. *$A \subset Ker(A)$ and $A = Ker(A)$ if A is open in X.*

3. *if $A \subset B$, then $Ker(A) \subset Ker(B)$.*

Lemma 9.1.2. *The following properties hold for a subset A of an ideal topological space $(X, \tau, \mathcal{I})$.*

1. *$R-\mathcal{I}-Int(A) = X \setminus R-\mathcal{I}-Cl(X-A)$.*

2. *$x \in R-\mathcal{I}-Cl(A)$ if and only if $A \cap U \neq \phi$ for each $U \in R\mathcal{I}O(X, x)$.*

Recollect the definitions of $R-\mathcal{I}-$open function (3.1.2), $R^*-\mathcal{I}-$irresolute function (3.1.4), $R-\mathcal{I}-T_1$ space (3.2.1), $R-\mathcal{I}-T_2$ space(3.2.1), $R-\mathcal{I}-$normal space (3.2.3) and $R-\mathcal{I}-$compact space (3.3.2) from the chapter 3.

9.2 Contra $R-\mathcal{I}-$continuous functions

Definition 9.2.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called contra $R-\mathcal{I}-$continuous if $f^{-1}(V)$ is $R-\mathcal{I}-$closed in X for every open set V in Y.*

Example 9.2.1. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$. Let $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ be such that $f(a) = c, f(b) = a, f(c) = a$. Then f is contra $R-\mathcal{I}-$continuous.*

Example 9.2.2. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$. Let $g : (X, \tau, \mathcal{I}) \to (X, \sigma)$ be such that $g(a) = a, g(b) = b, g(c) = c$. Then g is contra continuous, but not contra $R-\mathcal{I}-$continuous.*

Theorem 9.2.1. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following conditions are equivalent:*

1. *f is contra $R-\mathcal{I}-$continuous.*

2. *for each $x \in X$ and each closed set F in Y containing $f(x)$ there exist a $R-\mathcal{I}-$open set U containing x such that $f(U) \subset F$.*

3. *for each $x \in X$ and each closed set F in Y containing $f(x)$, $f^{-1}(F)$ is $R-\mathcal{I}-$open in X.*

4. *$f(R-\mathcal{I}-Cl(A)) \subset Ker(f(A))$ for every $A \subset X$.*

5. *$R-\mathcal{I}-Cl(f^{-1}(B)) \subset f^{-1}(Ker(B))$ for every $B \subset Y$.*

Proof. 1. $\Rightarrow$ 2.

Let $x \in X$ and F be any closed subset in Y containing $f(x)$. Then by the contra $R-\mathcal{I}-$continuity of f, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is $R-\mathcal{I}-$closed in X. Hence by taking $U = f^{-1}(F)$, we get there exists a $R-\mathcal{I}-$open set U containing x such that $f(U) \subset F$.

2. $\Rightarrow$ 3.

Let F be any closed subset in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$ and for each $x \in f^{-1}(F)$ there exists a $R-\mathcal{I}-$open set U_x containing x such that $f(U_x) \subset F$. Hence $f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}$. So $f^{-1}(F)$ is $R-\mathcal{I}-$open in X.

We use Lemma 9.1.1. in all the following implications.

3. $\Rightarrow$ 4.

Let $A \subset X$ and assume that $y \notin Ker(f(A))$. Then there exists a closed set F in Y with $y \in F$ such that $f(A) \cap F = \phi$. This implies $A \cap f^{-1}(F) = \phi$ and so $R-\mathcal{I}-Cl(A) \subset X \setminus f^{-1}(F)$. Thus $(R-\mathcal{I}-Cl(A)) \cap f^{-1}(F) = \phi$ and we obtain $f(R-\mathcal{I}-Cl(A)) \cap F = \phi$. So $y \notin f(R-\mathcal{I}-Cl(A))$. Hence $f(R-\mathcal{I}-Cl(A)) \subset Ker(f(A))$ for every $A \subset X$.

4. $\Rightarrow$ 5.

Let $B \subset Y$. Then $f(R-\mathcal{I}-Cl(f^{-1}(B))) \subset Ker(f(f^{-1}(B))) \subset Ker(B)$. Thus $R-\mathcal{I}-Cl(f^{-1}(B)) \subset f^{-1}(Ker(B))$ for every $B \subset Y$.

5. $\Rightarrow$ 1.

Let V be a open set in Y. Then $R-\mathcal{I}-Cl(f^{-1}(V)) \subset f^{-1}(Ker(V)) = f^{-1}(V)$ and so $R-\mathcal{I}-Cl(f^{-1}(V)) = f^{-1}(V)$. Thus $f^{-1}(V)$ is $R-\mathcal{I}-$closed in X and hence f is contra $R-\mathcal{I}-$continuous. □

Theorem 9.2.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function which is contra $R-\mathcal{I}-$continuous where Y is a regular space, then f is $R-\mathcal{I}-$continuous.*

Proof. Let $x \in X$ and V be an open set in Y with $f(x) \in V$. By the regularity of Y, there exists an open set H in Y containing $f(x)$ such that $Cl(H) \subset V$. Also, by theorem 9.2.1(2), there exists a $R-\mathcal{I}-$open set U in X containing x such that $f(U) \subset Cl(H)$, since f is contra $R-\mathcal{I}-$continuous. Thus, $f(U) \subset V$ and hence f is $R-\mathcal{I}-$continuous. □

Definition 9.2.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $R-\mathcal{I}-$connected if X cannot be written as the union of two non-empty disjoint $R-\mathcal{I}-$open subsets of X.*

Definition 9.2.3. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called almost-$R-\mathcal{I}-$continuous if for each $x \in X$ and for each open set V in Y containing $f(x)$, there exist a $R-\mathcal{I}-$open set U in X containing x such that $f(U) \subset R-\mathcal{J}-Int(Cl(V))$.*

Definition 9.2.4. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called almost weakly–$R-\mathcal{I}-$continuous if for each $x \in X$ and for each open set $V \subset Y$ containing $f(x)$, there exist a $R-\mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset Cl(V)$.*

Theorem 9.2.3. *Let X be a $R-\mathcal{I}-$connected space and the function*

$f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be contra $R - \mathcal{I}-$continuous, then Y is not a discrete space.

Proof. On the contrary, assume that Y is discrete. Let S be a proper non-empty clopen set in Y. Since f is contra $R - \mathcal{I}-$continuous, from definition and theorem 9.2.1(3), we get $f^{-1}(S)$ is a proper non-empty $R - \mathcal{I}-$clopen set in X. This contradicts the fact that X is $R - \mathcal{I}-$connected, since X can be written as the union of $f^{-1}(S)$ and $X - f^{-1}(S)$. □

Theorem 9.2.4. *Let X be a $R - \mathcal{I}-$connected space. If the function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $R - \mathcal{I}-$continuous and surjective, then Y is connected.*

Proof. Assume that Y is not connected. Then, $Y = A \cup B$, where A and B are disjoint non-empty open subsets of Y. Since f is contra $R - \mathcal{I}-$continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty $R - \mathcal{I}-$closed sets in X. Also, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \phi$. This contradicts the fact that X is $R - \mathcal{I}-$connected and hence Y is connected. □

Theorem 9.2.5. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is $R - \mathcal{I}-$open and contra-$R - \mathcal{I}-$continuous, then f is almost-$R - \mathcal{I}-$continuous.*

Proof. Let $x \in X$ and $V \subset Y$ be an open set containing $f(x)$. Since f is contra $R - \mathcal{I}-$continuous, by theorem 9.2.1(3), there exists a $R - \mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset Cl(V)$. Since f is $R - \mathcal{I}-$open, $f(U)$ is $R - \mathcal{J}-$open in Y. So $f(U) = R - \mathcal{J} - Int(f(U))$. Hence $f(U) \subset R - \mathcal{J} - Int(Cl(V))$ and f is almost-$R - \mathcal{I}-$continuous. □

Theorem 9.2.6. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $R - \mathcal{I}-$continuous, then f is almost weakly$-R - \mathcal{I}-$continuous.*

Proof. Let $x \in X$ and $V \subset Y$ be an open set containing $f(x)$. Since f is contra $R-\mathcal{I}-$continuous, $f^{-1}(Cl(V))$ is $R-\mathcal{I}-$open in X by theorem 9.2.1(3). Take $U = f^{-1}(Cl(V))$. Then $f(U) \subset Cl(V)$. Thus f is almost weakly-$R-\mathcal{I}-$continuous. □

Theorem 9.2.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a contra $R-\mathcal{I}-$continuous function and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ be a continuous function.*
Then $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \eta)$ is contra $R-\mathcal{I}-$continuous.

Proof. Let $x \in X$ and W be a closed set in Z containing $(g \circ f)(x)$. Then $V = g^{-1}(W)$ is closed in Y by the continuity of g. Also, since f is contra $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset V = g^{-1}(W)$. Thus $(g \circ f)(U) \subset W$. Hence
$g \circ f : (X, \tau, \mathcal{I}) \to (Z, \eta)$ is contra $R-\mathcal{I}-$continuous. □

Theorem 9.2.8. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $R^* - \mathcal{I}-$irresolute function and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ be a contra $R-\mathcal{I}-$continuous function. Then $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \eta)$ is contra $R-\mathcal{I}-$continuous.*

Proof. Let $x \in X$ and W be a closed set in Z containing $(g \circ f)(x)$. Since g is contra $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{J}-$open set V in Y containing $f(x)$ such that $g(V) \subset W$. Also, since f is $R^* - \mathcal{I}-$irresolute, $f^{-1}(V)$ is $R-\mathcal{I}-$open in X containing x. Take $U = f^{-1}(V)$. Thus $(g \circ f)(U) \subset W$. Hence $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \eta)$ is contra $R-\mathcal{I}-$continuous. □

Theorem 9.2.9. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a surjective $R^* - \mathcal{I}-$irresolute and $R-\mathcal{I}-$open function and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ be any function. Then $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \eta)$ is contra $R-\mathcal{I}-$continuous if and only if g is contra $R-\mathcal{I}-$continuous.*

Proof. If g is contra $R-\mathcal{I}-$continuous, then the proof follows from theorem 9.2.8. Now assume $g \circ f$ is contra $R-\mathcal{I}-$continuous. Let W be closed in

Z. Then $(g \circ f)^{-1}(W)$ is $R-\mathcal{I}-$open in X. But f is $R-\mathcal{I}-$open and so $f(f^{-1}(g^{-1}(W)))$ is $R-\mathcal{J}-$open in Y. Hence $g^{-1}(W)$ is $R-\mathcal{J}-$open in Y and so g is contra $R-\mathcal{I}-$continuous. □

Definition 9.2.5. *[38] A space (X, τ) is said to be ultra normal if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.*

Definition 9.2.6. *[38] A topological space (X, τ) is ultra Hausdorff if for each pair of distinct points x and y of X there exist closed sets U and V such that $x \in U, y \in V$ and $U \cap V = \phi$.*

Definition 9.2.7. *[46] A topological space (X, τ) is said to be weakly Hausdorff if each element of X is the intersection of regular closed sets of X.*

Definition 9.2.8. *[28] A topological space (X, τ) is called Urysohn if, for each x, $y \in X$ with $x \neq y$, there exist open sets U and V of X containing x and y respectively such that $Cl(U) \cap Cl(V) = \phi$.*

Theorem 9.2.10. *Let (X, τ, I) be a $R-\mathcal{I}-$connected space and (Y, σ) a T_1 space. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $R-\mathcal{I}-$continuous, then f is constant.*

Proof. Let X be a $R-\mathcal{I}-$connected space and Y be a T_1 space. Then $\mathcal{P} = \{f^{-1}(y) : y \in Y\}$ is a disjoint collection of $R-\mathcal{I}-$open sets which partitions X. If $|\mathcal{P}| \geq 2$, then X is the union of two disjoint non-empty $R-\mathcal{I}-$open sets. This implies $|\mathcal{P}| = 1$ and hence f is constant. □

Theorem 9.2.11. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a contra $R-\mathcal{I}-$continuous and one-one function. If Y is a Urysohn space, then X is $R-\mathcal{I}-T_2$.*

Proof. Let $x \neq y \in X$. Then $f(x) \neq f(y)$. Since Y is a Urysohn space, there exist open sets P, Q in Y such that $f(x) \in P$, $f(y) \in Q$ and $Cl(P) \cap Cl(Q) = \phi$. Since f is contra $R-\mathcal{I}-$continuous, by theorem

9.2.1(2), there exist $R-\mathcal{I}-$open sets $U,\ V$ in X such that $x \in U,\ y \in V$ and $f(U) \subset Cl(P),\ f(V) \subset Cl(Q)$. Thus $U \cap V = \phi$, since $f(U) \cap f(V) = \phi$. Hence X is $R - \mathcal{I} - T_2$. □

Theorem 9.2.12. *If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a contra $R - \mathcal{I}-$continuous, closed and one-one function where Y is ultra normal, then X is $R - \mathcal{I}-$normal.*

Proof. Let $A,\ B$ be disjoint closed subsets in X. Since f is closed and one-one, $f(A)$ and $f(B)$ are disjoint closed subsets in Y. Since Y is ultra normal, $f(A)$ and $f(B)$ are separated by disjoint clopen sets, U and V. Since f is contra $R - \mathcal{I}-$continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $R - \mathcal{I}-$open. Also, $A \subset f^{-1}(U),\ B \subset f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. So X is $R - \mathcal{I}-$normal. □

Theorem 9.2.13. *If the function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is contra $R - \mathcal{I}-$continuous and one-one where Y is ultra Hausdorff, then X is $R-\mathcal{I}-T_2$.*

Proof. Let $x \neq y \in X$. Then $f(x) \neq f(y)$. Since Y is ultra Hausdorff, there exists closed sets U and V in Y such that $f(x) \in U,\ f(y) \in V$ and $U \cap V = \phi$. Since f is contra $R - \mathcal{I}-$continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $R-\mathcal{I}-$open in X containing x and y respectively. Also, $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence X is $R - \mathcal{I} - T_2$. □

9.3 Almost Contra $R - \mathcal{I}-$continuous functions

Definition 9.3.1. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called almost contra $R - \mathcal{I}-$continuous if $f^{-1}(V)$ is $R - \mathcal{I}-$closed in $(X, \tau, \mathcal{I})$ for every regular open set V in (Y, σ).*

Definition 9.3.2. *An ideal topological space is said to be countably $R-\mathcal{I}-$compact if every countable $R-\mathcal{I}-$open cover admits a finite subcover.*

Definition 9.3.3. *An ideal topological space is said to be $R-\mathcal{I}-$closed compact ($R-\mathcal{I}-$closed Lindeloff) if every $R-\mathcal{I}-$closed cover admits a finite (countable) subcover.*

Definition 9.3.4. *An ideal topological space is said to be countably $R-\mathcal{I}-$closed compact if every countable $R-\mathcal{I}-$closed cover admits a finite subcover.*

Definition 9.3.5. *A topological space X is called S-closed [16] (resp. countably S-closed [21], S-Lindeloff [9] if every regular closed (resp. countably regular closed, regular closed) cover of X has a finite (resp. finite, countable) subcover.*

Definition 9.3.6. *[28] A topological space X is said to be nearly compact (resp. nearly countably compact, nearly Lindeloff) if every regular open (resp. countable regular open, regular open) cover of X has a finite (resp. finite, countable) subcover.*

Theorem 9.3.1. *For a function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ the following statements are equivalent:*

1. *f is almost contra $R-\mathcal{I}-$continuous.*
2. *$f^{-1}(F)$ is $R-\mathcal{I}-$open in X for each regular closed set F in Y.*
3. *for each $x\in X$ and each regular closed set F in Y containing $f(x)$, there exist a $R-\mathcal{I}-$open set U in X containing x such that $f(U)\subset F$.*
4. *for each $x\in X$ and each regular open set V in Y not containing $f(x)$, there exist a $R-\mathcal{I}-$closed set C in X not containing x such that $f^{-1}(V)\subset C$.*

Proof. 1. $\Leftrightarrow$ 2.

Let F be a regular closed set in Y. Then $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is $R - \mathcal{I}-$closed in X. Hence $f^{-1}(F)$ is $R - \mathcal{I}-$open in X.

In a similar manner, the converse part can be proved.

2. $\Leftrightarrow$ 3.

Let $x \in X$ and F be a regular closed set in Y containing $f(x)$. Then $f^{-1}(F)$ is $R - \mathcal{I}-$open in X and contains x. Take $U = f^{-1}(F)$. Then U is a $R - \mathcal{I}-$open set in X containing x such that $f(U) \subset F$.

Conversely, let F be a regular closed set in Y and let $x \in f^{-1}(F)$. Then there exists a $R - \mathcal{I}-$open set U_x containing x such that $f(U_x) \subset F$. Hence $f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}$. So $f^{-1}(F)$ is $R - \mathcal{I}-$open in X.

3. $\Leftrightarrow$ 4.

Let $x \in X$ and V be a regular open set not containing $f(x)$. Then $Y \setminus V$ is a regular closed set containing $f(x)$. So there exist a $R - \mathcal{I}-$open set U in X containing x such that $f(U) \subset (Y \setminus V)$. Hence $U \subset f^{-1}(Y \setminus V) \subset X \setminus f^{-1}(V)$. So $f^{-1}(V) \subset X \setminus U$. Taking $C = X \setminus U$ gives a $R - \mathcal{I}-$closed set in X not containing x such that $f^{-1}(V) \subset C$.

The converse can be proved in a similar manner. $\square$

Theorem 9.3.2. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is almost contra $R - \mathcal{I}-$continuous, then f is almost weakly$-R - \mathcal{I}-$continuous.*

Proof. Let $x \in X$ and V be an open set in Y containing $f(x)$. Obviously $Cl(V)$ is regular closed in Y containing $f(x)$. Then, since f is almost contra $R - \mathcal{I}-$continuous, there exist a $R - \mathcal{I}-$open set U in X containing x such that $f(U) \subset Cl(V)$ by theorem 9.3.1(3). Hence f is almost weakly$-R - \mathcal{I}-$continuous. $\square$

Lemma 9.3.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is almost $R - \mathcal{I}-$continuous if and only if for each $x \in X$ and for each regu-*

lar open set V of Y containing $f(x)$, there exists a $R-\mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset V$.

Proof. The 'if part' is trivial. The 'only if part' can be proved by taking V as an open set in Y and hence $R-\mathcal{J}-Int(Cl(V))$ will be regular open. □

Remark 9.3.1. *A topological space (X, τ) is said to be extremally disconnected if the closure of every open set in X is open in X.*

Theorem 9.3.3. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a function where Y is extremally disconnected. Then f is almost contra $R-\mathcal{I}-$continuous if and only if f is almost $R-\mathcal{I}-$continuous.*

Proof. Let $x \in X$ and V be a regular open set containing $f(x) \in Y$. Since Y is extremally disconnected, $Cl(V)$ is open which implies $Int(Cl(V)) = V$ and so $Cl(V) = V$. Thus, V is regular closed. Since f is almost contra $R-\mathcal{I}-$continuous, there exist a $R-\mathcal{I}-$open set U in X containing x such that $f(U) \subset V$ by theorem 9.3.1.(3). Then by the lemma 9.3.1, f is almost $R-\mathcal{I}-$continuous.

Conversely, assume f is almost $R-\mathcal{I}-$continuous. Let $x \in X$ and V be a regular closed set containing $f(x) \in Y$. This implies $Cl(Int(V)) = V$. Since Y is extremally disconnected, we get V is open in Y and thus is regular open in Y. Since f is almost $R-\mathcal{I}-$continuous, there exist a $R-\mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset V$ by lemma 9.3.1. Thus f is almost contra $R-\mathcal{I}-$continuous by theorem 9.3.1.(3). □

Theorem 9.3.4. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an almost contra $R-\mathcal{I}-$continuous and one-one function where Y is weakly Hausdorff, then X is $R-\mathcal{I}-T_1$.*

Proof. Let $x \neq y \in X$. Since f is one-one, $f(x) \neq f(y)$ in Y. Since Y is weakly Hausdorff, there exist disjoint regular closed sets C and D such that $f(x) \in C$, $f(y) \notin C$ and $f(y) \in D$, $f(x) \notin D$. But f is almost contra $R-\mathcal{I}-$continuous. So $f^{-1}(U)$ and $f^{-1}(V)$ are $R-\mathcal{I}-$open sets in X containing x and y respectively. Also $x \notin f^{-1}(V)$ and $y \notin f^{-1}(U)$. Hence X is $R-\mathcal{I}-T_1$. □

Corollary 9.3.1. *If $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is a contra $R-\mathcal{I}-$continuous and one-one function where Y is weakly Hausdorff, then X is $R-\mathcal{I}-T_1$.*

Proof. Since every contra $R-\mathcal{I}-$continuous function is almost contra $R-\mathcal{I}-$continuous, the proof follows. □

Theorem 9.3.5. *If $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is an almost contra $R-\mathcal{I}-$continuous and onto function where X is $R-\mathcal{I}-$connected, then Y is connected.*

Proof. On the contrary assume that Y is disconnected. Then $Y = A \cup B$, where A and B are disjoint non-empty clopen subsets of Y. Then A and B are regular open in Y since they are clopen. Since f is almost contra $R-\mathcal{I}-$continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $R-\mathcal{I}-$open in X. Since f is onto, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \phi$ which contradicts to the fact that X is $R-\mathcal{I}-$connected. Hence Y is connected. □

Theorem 9.3.6. *Let $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a surjective almost contra-$R-\mathcal{I}-$continuous function. Then the following statements are true:*

1. *X is $R-\mathcal{I}-$compact implies Y is S-closed*
2. *X is $R-\mathcal{I}-$Lindeloff implies Y is S-Lindeloff*
3. *X is countably $R-\mathcal{I}-$compact implies Y is countably S-closed*

Proof. 1. Let $\{V_j : j \in J\}$ be a regular closed cover of Y. Then $\{f^{-1}(V_j) : j \in J\}$ is a R–$\mathcal{I}-$open cover of X since f is almost contra-R–$\mathcal{I}-$continuous and surjective. But X is $R-\mathcal{I}-$compact. Hence there exists a finite subset J_κ of J such that $X = \cup_{j\in J_\kappa}\{(f^{-1}(V_j))\}$. So $Y = \cup\{V_j : j \in J_\kappa\}$ and thus Y is S-closed.

2. and 3. can be proved in a similar fashion. □

Theorem 9.3.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a surjective almost contra-$R-\mathcal{I}-$continuous function. Then the following statements are true:*

1. *X is $R-\mathcal{I}-$ closed compact implies Y is nearly compact*
2. *X is $R-\mathcal{I}-$closed Lindeloff implies Y is nearly Lindeloff*
3. *X is countably $R-\mathcal{I}-$closed compact implies Y is nearly countably compact*

Proof. 1. Let $\{V_j : j \in J\}$ be an regular open cover of Y. Then $\{f^{-1}(V_j) : j \in J\}$ is a $R-\mathcal{I}-$closed cover of X since f is almost contra-$R-\mathcal{I}-$continuous and surjective. But X is $R-\mathcal{I}-$closed compact. Hence there exists a finite subset J_κ of J such that $X = \cup_{j\in J_\kappa}\{(f^{-1}(V_j))\}$. So $Y = \cup\{V_j : j \in J_\kappa\}$ and thus Y is nearly compact.

2. and 3. can be proved in a similar fashion. □

9.4 Graphs

Definition 9.4.1. *[36] For a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.*

Definition 9.4.2. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $R-\mathcal{I}-$closed (resp. contra $R-\mathcal{I}-$closed) if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R-\mathcal{I}-$open set U in X containing x and an open (resp. a closed) set V in Y containing y such that $(U \times V) \cap G(f) = \phi$.*

Definition 9.4.3. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be strongly contra $R - \mathcal{I}-$closed if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}-$open set U in X containing x and a regular closed set V in Y containing y such that $(U \times V) \cap G(f) = \phi$.*

Lemma 9.4.1. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $R - \mathcal{I}-$closed (resp. contra $R - \mathcal{I}-$closed) if and only if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}-$open set U in X containing x and an open (resp. a closed) set V in Y containing y such that $f(U) \cap V = \phi$.*

Proof. Suppose that $(U \times V) \cap G(f) \neq \phi$. Then there exists at least one $(x, y) \in X \times Y$ such that $(x, y) \in U \times V$ and $(x, y) \in G(f)$. That means, $x \in U$, $y \in V$ and $f(x) \in f(U)$. Hence $f(U) \cap V \neq \phi$. The other part of the proof is clear from the definition. □

Lemma 9.4.2. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly contra $R - \mathcal{I}-$closed if and only if for each $(x, y) \in X \times Y \setminus G(f)$, there exist a $R - \mathcal{I}-$open set U in X containing x and a regular closed set V in Y containing y such that $f(U) \cap V = \phi$.*

Proof. The proof follows as in the similar manner as above. □

Theorem 9.4.1. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $R - \mathcal{I}-$continuous and Y is Urysohn, then $G(f)$ is contra $R - \mathcal{I}-$closed in $X \times Y$.*

Proof. Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Y is Urysohn implies that there exist open sets P and Q of Y containing $f(x)$ and y respectively such that $Cl(P) \cap Cl(Q) = \phi$. Since $Cl(P)$ is a closed set containing $f(x)$ and since f is contra $R - \mathcal{I}-$continuous, there exist a $R - \mathcal{I}-$open set U in X containing x such that $f(U) \subset Cl(P)$ by theorem 9.2.1(2). Hence $f(U) \cap Cl(Q) = \phi$ and so $G(f)$ is contra $R - \mathcal{I}-$closed in $X \times Y$ by lemma 9.4.1. □

Theorem 9.4.2. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $R - \mathcal{I}-$continuous and Y is T_1, then $G(f)$ is contra $R - \mathcal{I}-$closed in $X \times Y$.*

Proof. Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Since Y is T_1, there exists an open set V in Y such that $f(x) \in V$, $y \notin V$. Since f is $R - \mathcal{I}-$continuous, there exists an $R - \mathcal{I}-$open set U in X containing x such that $f(U) \subset V$. Then $f(U) \cap (Y \setminus V) = \phi$. But $Y \setminus V$ is a closed set in Y containing y. Thus $G(f)$ is contra $R - \mathcal{I}-$closed in $X \times Y$ by lemma 9.4.1. □

Theorem 9.4.3. *Let Y be a Urysohn space. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ and $g : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ are contra $R - \mathcal{I}-$continuous functions, then $S = \{x \in X : f(x) = g(x)\}$ is $R - \mathcal{I}-$closed in X.*

Proof. Let $x \notin S$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exists open sets P and Q of Y containing $f(x)$ and $g(x)$ respectively such that $Cl(P) \cap Cl(Q) = \phi$. Since f and g are contra $R - \mathcal{I}-$continuous, $f^{-1}(Cl(P))$ and $g^{-1}(Cl(Q))$ are $R - \mathcal{I}-$open in X containing x. Name $f^{-1}(Cl(P)) = A$ and $f^{-1}(Cl(Q)) = B$ and let $R = A \cap B$. Then R is a $R-\mathcal{I}-$open set in X containing x. Also, $f(R) \cap g(R) = f(A \cap B) \cap g(A \cap B) \subset f(A) \cap g(B) = Cl(P) \cap Cl(Q) = \phi$. Hence $S \cap R = \phi \Rightarrow S \subset (X \setminus R) \Rightarrow R - \mathcal{I} - Cl(S) \subset (X \setminus R) \Rightarrow x \notin R - \mathcal{I} - Cl(S)$. Thus $R - \mathcal{I} - Cl(S) \subset S$ and so $S = R - \mathcal{I} - Cl(S)$. Hence $S = \{x \in X : f(x) = g(x)\}$ is $R - \mathcal{I}-$closed in X. □

Theorem 9.4.4. *Let Y be a Urysohn space. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ almost weakly $R - \mathcal{I}-$continuous, then $G(f)$ is strongly contra $R - \mathcal{I}-$closed in $X \times Y$.*

Proof. Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Y is Urysohn implies that there exist open sets P and Q of Y containing $f(x)$ and y respectively such that $Cl(P) \cap Cl(Q) = \phi$. Since f is almost weakly $R - \mathcal{I}-$continuous, there exist a $R - \mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset Cl(P)$.

Hence $f(U) \cap Cl(Q) = \phi \Rightarrow f(U) \cap Cl(Int(Q)) = \phi$. Since $Cl(Int(Q))$ is regular closed in Y, $G(f)$ is strongly contra $R-\mathcal{I}-$closed in $X \times Y$ by lemma 9.4.2. □

Theorem 9.4.5. *If a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is almost $R-\mathcal{I}-$continuous and Y is rT_2, then $G(f)$ is strongly contra $R-\mathcal{I}-$closed in $X \times Y$.*

Proof. Let $(x, y) \in X \times Y \setminus G(f)$. So $f(x) \neq y$. Since Y is rT_2, there exists disjoint regular open sets A and B containing $f(x)$ and y respectively. Since f is almost $R-\mathcal{I}-$continuous, by lemma 9.3.1., there exists a $R-\mathcal{I}-$open set $U \subset X$ containing x such that $f(U) \subset A$. Hence $f(U) \cap Cl(B) = \phi$ and thus $G(f)$ is strongly contra $R-\mathcal{I}-$closed in $X \times Y$ by lemma 9.4.2. □

Theorem 9.4.6. *Let Y be a Urysohn space and $D \subset X$ be a $R-\mathcal{I}-$dense set. Let f, $g : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be any two contra $R-\mathcal{I}-$continuous functions. If $f = g$ on D, then $f = g$ on X.*

Proof. By theorem 9.4.3., $S = \{x \in X : f(x) = g(x)\}$ is $R-\mathcal{I}-$closed in X. Clearly $D \subset S$. Since D is $R-\mathcal{I}-$dense, $R-\mathcal{I}-Cl(D) = X$. Thus $X = R-\mathcal{I}-Cl(D) \subset R-\mathcal{I}-Cl(S) = S$. Hence $f = g$ on X. □

10

$R-\mathcal{I}-$continuous multifunctions

10.1 Introduction

We now widen the concept of continuity to set multifunctions in ideal topological space by extending the class of $R-\mathcal{I}-$open sets. We introduce a class of continuous multifunctions namely upper and lower $R-\mathcal{I}-$continuous multifunctions and explored several characterisations of the same. This is done in section 2. We also present and examined two weaker forms of the upper and lower $R-\mathcal{I}-$continuous multifunctions. In section 3., the weaker form of $R-\mathcal{I}-$continuous multifunction, namely almost $R-\mathcal{I}-$continuous multifunction is developed. In section 4., another weaker form of $R-\mathcal{I}-$continuous multifunction, namely weakly $R-\mathcal{I}-$continuous multifunction is studied.

[26] By a multifunction $F : X \to Y$, we mean a correspondence from each point $x \in X$ to a non-empty set $F(x)$ of Y. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset B of Y are denoted by $F^{+}(B)$ and $F^{-}(B)$ respectively, where $F^{+}(B) = \{x \in X : F(x) \subset B\}$ and $F^{-}(B) = \{x \in X : F(x) \cap B \neq \phi\}$.

In particular, $F^{-}(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A multifunction $F : X \to Y$ is called a surjection if $F(X) = Y$.

A multifunction $F : X \to Y$ is called upper semi-continuous (rename upper continuous) (resp. lower semi-continuous (rename lower continuous) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y. [26]

10.2 $R-\mathcal{I}-$continuous multifunctions

Definition 10.2.1. *A multifunction $F:(X,\tau,\mathcal{I})\rightarrow(Y,\sigma,\mathcal{J})$ is said to be:*

1. *upper $R-\mathcal{I}-$continuous at a point $x\in X$ if for each $R-\mathcal{J}-$open set V of Y such that $F(x)\subset V$, there exists $U\in R\mathcal{I}O(X,x)$ such that $F(U)\subset V$.*

2. *lower $R-\mathcal{I}-$continuous at a point $x\in X$ if for each $R-\mathcal{J}-$open set V of Y such that $F(x)\cap V\neq\phi$, there exists $U\in R\mathcal{I}O(X,x)$ such that $F(u)\cap V\neq\phi$ for each $u\in U$.*

3. *upper/lower $R-\mathcal{I}-$continuous if F is both upper $R-\mathcal{I}-$continuous and lower $R-\mathcal{I}-$continuous at each point x of X.*

Theorem 10.2.1. *For a multifunction $F:(X,\tau,\mathcal{I})\rightarrow(Y,\sigma,\mathcal{J})$, the following statements are equivalent:*

1. *F is upper $R-\mathcal{I}-$continuous.*

2. *$F^+(V)\in R\mathcal{I}O(X,\tau)$ for every $R-\mathcal{J}-$open set V of Y.*

3. *$F^-(H)\in R\mathcal{I}C(X,\tau)$ for every $R-\mathcal{J}-$closed set H of Y.*

4. *$R-\mathcal{I}-Cl(F^-(B))\subset F^-(R-\mathcal{J}-Cl(B))$ for every $B\subset Y$.*

5. *For each point $x\in X$ and each $R-\mathcal{J}-$nbd V of $F(x)$, $F^+(V)$ is a $R-\mathcal{I}-$nbd of x.*

6. *For each point $x\in X$ and each $R-\mathcal{J}-$nbd V of $F(x)$, there is a $R-\mathcal{I}-$nbd U of x such that $F(U)\subset V$.*

Proof. 1. $\Rightarrow$ 2.

Let V be $R-\mathcal{J}-$open in Y and let $x \in F^+(V)$. Then $F(x) \subset V$. Also there exists a $R-\mathcal{I}-$open set U containing x such that $F(U) \subset V$. Thus $x \in U \subset F^+(V) \Rightarrow x \in R-\mathcal{I}-Int(F^+(V)) \Rightarrow F^+(V) \subset R-\mathcal{I}-Int(F^+(V))$. Hence $F^+(V) \in R\mathcal{I}O(X,\tau)$.

2. $\Rightarrow$ 3.

Let $x \in F^+(Y-A) \Longleftrightarrow F(x) \subset Y-A \Longleftrightarrow F(x) \cap A = \phi \Longleftrightarrow x \notin F^-(A) \Longleftrightarrow x \in X - F^-(A)$. Thus, $F^+(Y-A) = X - F^-(A)$.

Similarly, $F^-(Y-A) = X - F^+(A)$.

3. $\Rightarrow$ 4.

Let $B \subset Y$. Then $F^-(R-\mathcal{J}-Cl(B))$ is $R-\mathcal{I}-$closed. Hence $R-\mathcal{I}-Cl(F^-(B)) \subset F^-(R-\mathcal{J}-Cl(B))$.

4. $\Rightarrow$ 3.

Let H be $R-\mathcal{J}-$closed in Y. Then $R-\mathcal{I}-Cl(F^-(H)) \subset F^-(R-\mathcal{J}-Cl(H)) = F^-(H)$. This shows that $F^-(H)$ is $R-\mathcal{I}-$closed in X.

2. $\Rightarrow$ 5.

Let $x \in X$ and V be a $R-\mathcal{J}-$nbd of $F(x)$. Then there exists a $R-\mathcal{J}-$open set W of Y such that $F(x) \subset W \subset V$. Thus $x \in F^+(W) \subset F^+(V)$. Since W is $R-\mathcal{J}-$open, $F^+(W)$ is $R-\mathcal{I}-$open in X. Hence $F^+(V)$ is a $R-\mathcal{I}-$nbd of x.

5. $\Rightarrow$ 6.

Let $x \in X$ and V be a $R-\mathcal{J}-$nbd of $F(x)$. Then $F^+(V)$ is a $R-\mathcal{I}-$nbd of x. Set $U = F^+(V)$. Hence U is a $R-\mathcal{I}-$nbd of x such that $F(U) \subset V$.

6. $\Rightarrow$ 1.

Let $x \in X$ and V be any $R-\mathcal{J}-$open set of Y such that $F(x) \subset V$. Then V is a $R-\mathcal{J}-$nbd of $F(x)$. Then there is a $R-\mathcal{I}-$nbd N of x such that $F(N) \subset V$. Therefore, there exists a $R-\mathcal{I}-$open set U of $(X,\tau,\mathcal{I})$ such that $x \in U \subset N$. Hence $F(U) \subset V$. $\square$

Theorem 10.2.2. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following statements are equivalent:*

1. *F is lower $R - \mathcal{I}-$continuous.*
2. *$F^-(V) \in R\mathcal{I}O(X, \tau)$ for every $R - \mathcal{J}-$open set V of Y.*
3. *$F^+(H) \in R\mathcal{I}C(X, \tau)$ for every $R - \mathcal{J}-$closed set H of Y.*
4. *$R - \mathcal{I} - Cl(F^+(B)) \subset F^+(R - \mathcal{J} - Cl(B))$ for every $B \subset Y$.*
5. *$F(R - \mathcal{I} - Cl(A)) \subset R - \mathcal{J} - Cl(F(A))$ for every $A \subset X$.*

Proof. 4. $\Rightarrow$ 5.
Let $A \subset X$. Then, we get $R - \mathcal{I} - Cl(A) \subset R - \mathcal{I} - Cl(F^+(F(A)))$
$\subset F^+(R - \mathcal{J} - Cl(F(A)))$. Thus $F(R - \mathcal{I} - Cl(A)) \subset R - \mathcal{J} - Cl(F(A))$.

The proofs of other implications are similar to those of Theorem 10.2.1. □

Theorem 10.2.3. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $G : (Y, \sigma, \mathcal{J}) \to (Z, \eta, \mathcal{K})$ be multifunctions. If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is upper(lower) $R - \mathcal{I}-$continuous and $G : (Y, \sigma, \mathcal{J}) \to (Z, \eta, \mathcal{K})$ is upper(lower) $R - \mathcal{J}-$continuous, then $G \circ F : X \to Z$ is upper(lower) $R - \mathcal{I}-$continuous.*

Proof. Let V be a $R - \mathcal{K}-$open subset of Z. From the definition of $G \circ F$, we get $(G \circ F)^+(V) = F^+(G^+(V))$ and $(G \circ F)^-(V) = F^-(G^-(V))$. $(G^+(V))$ $(G^-(V))$ is a $R - \mathcal{J}-$open set since G is upper(lower) $R - \mathcal{J}-$continuous. $F^+(G^+(V))/F^-(G^-(V))$ is $R - \mathcal{I}-$open since F is upper(lower) $R - \mathcal{I}-$continuous. Hence $G \circ F : X \to Z$ is an upper(lower) $R - \mathcal{I}-$continuous multifunction.

□

[36] For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the graph multifunction denoted as G_F, $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = (x, F(x))$ for every $x \in X$.

Lemma 10.2.1. *[44] For a multifunction $F : (X,\tau,\mathcal{I}) \to (Y,\sigma)$, the following hold: $G_F^+(A \times B) = A \cap F^+(B)$ and $G_F^-(A \times B) = A \cap F^-(B)$ for any subsets $A \subset X$ and $B \subset Y$.*

Theorem 10.2.4. *Let $F : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ be a multifunction such that $F(x)$ is $R-\mathcal{J}-$compact for each $x \in X$. Then if F is upper $R-\mathcal{I}-$continuous, then $G_F : X \to X \times Y$ is upper $R-\mathcal{I}-$continuous.*

Proof. Suppose that $F : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ is upper $R-\mathcal{I}-$continuous. Let $x \in X$ and B be any $R-\mathcal{J}-$open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist $R-\mathcal{I}-$open set $U_y \subset X$ and $R-\mathcal{J}-$open set $V_y \subset Y$ such that $(x,y) \in U_y \times V_y \subset B$. The family $\{V_y : y \in F(x)\}$ is an $R-\mathcal{J}-$open cover of $F(x)$. Since $F(x)$ is $R-\mathcal{J}-$compact, there exist, $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \cup\{V_{y_i} : 1 \leq i \leq n\}$. Set $U = \cap\{U_{y_i} : 1 \leq i \leq n\}$ and $V = R-\mathcal{J}-Int(R-\mathcal{J}-Cl(\cup\{V_{y_i} : 1 \leq i \leq n\}))$. Then U is $R-\mathcal{I}-$open in X and V is $R-\mathcal{J}-$open in Y and $(x, F(x)) \subset U \times V \subset B$. Since F is upper $R-\mathcal{I}-$continuous, there exists a $R-\mathcal{I}-$open set W containing x such that $F(W) \subset V$. Clearly $U \cap W$ is $R-\mathcal{I}-$open in X containing x. By Lemma 10.2.1., we have $U \cap W \subset U \cap F^+(V) = G_F^+(U \times V) \subset G_F^+(B)$. Hence $G_F(U \cap W) \subset B$. Thus G_F is upper $R-\mathcal{I}-$continuous. □

Theorem 10.2.5. *If a multifunction $F : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ is lower $R-\mathcal{I}-$continuous, then $G_F : X \to X \times Y$ is lower $R-\mathcal{I}-$continuous.*

Proof. Suppose that $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is lower $R - \mathcal{I}-$continuous. Let $x \in X$ and B be any $R - \mathcal{J}-$open set of $X \times Y$ such that $(x, F(x)) \in G_F^-(B)$. Since $B \cap (x, F(x)) \neq \phi$, there exists $y \in F(x)$ such that $(x, y) \in B$ and so $(x, y) \in U \times V \subset B$ where U is $R-\mathcal{I}-$open in X and V is $R-\mathcal{J}-$open in Y. Since $F(x) \cap V \neq \phi$, there exists a $R - \mathcal{I}-$open set W containing x such that $F(W) \cap V \neq \phi$ or $W \subset F^-(V)$ by the definition of lower $R - \mathcal{I}-$continuous multifunction. $U \cap W \subset U \cap F^-(V) = G_F^-(U \times V) \subset G_F^-(B)$ by Lemma 10.2.1.. Also $U \cap W$ is a $R - \mathcal{I}-$open set containing x. Hence G_F is lower $R - \mathcal{I}-$continuous. □

Theorem 10.2.6. *Let $(X, \tau, \mathcal{I})$ be $R - \mathcal{I}-$compact. If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be an upper $R - \mathcal{I}-$continuous multifunction which is onto and such that $F(x)$ is $R - \mathcal{J}-$compact for each $x \in X$, then $(Y, \sigma, \mathcal{J})$ is $R - \mathcal{J}-$compact.*

Proof. Let $\{V_j : j \in J\}$ be an $R - \mathcal{J}-$open cover of Y. Since $F(x)$ is $R - \mathcal{J}-$compact for each $x \in X$, there exist a finite subset J_x of J such that $F(x) \subset \cup\{V_j : j \in J_x\}$. Let $V_x = \cup\{V_j : j \in J_x\}$. Then there exists a $R - \mathcal{I}-$open set U_x containing x such that $F(U_x) \subset V_x$, since F is upper $R - \mathcal{I}-$continuous. Since $(X, \tau, \mathcal{I})$ is $R - \mathcal{I}-$compact and since $\{U_x : x \in X\}$ is a $R - \mathcal{I}-$open cover of X, there exist $x_1, x_2, \ldots, x_n$ in X such that $X \subset \cup\{U_{x_i} : 1 \leq i \leq n\}$. Hence $Y = F(X) \subset F(\cup_{i=1}^n U_{x_i}) = \cup_{i=1}^n F(U_{x_i}) \subset \cup_{i=1}^n V_{x_i} = \cup_{i=1}^n \cup_{j \in J_{x_i}} V_j$. Thus Y is $R - \mathcal{J}-$compact. □

Theorem 10.2.7. *Let $(X, \tau, \mathcal{I})$ be a $R - \mathcal{I}-$connected space. If $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an upper $R - \mathcal{I}-$continuous multifunction which is onto and such that $F(x)$ is $R - \mathcal{J}-$connected for each $x \in X$, then Y is $R - \mathcal{J}-$connected.*

Proof. Suppose that Y is not $R - \mathcal{J}-$connected. Then there exist disjoint non-empty $R - \mathcal{J}-$open sets U, V of Y such that $U \cup V = Y$. Since $F(x)$

is $R-\mathcal{J}-$connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^{+}(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^{+}(U) \cup F^{+}(V)$. Since F is onto, there exist x and y in X with $F(x) \subset U$ and $F(y) \subset V$. Hence $x \in F^{+}(U)$ and $y \in F^{+}(V)$. Thus we obtain $F^{+}(U) \cup F^{+}(V) = F^{+}(U \cup V) = X$, $F^{+}(U) \cap F^{+}(V) = F^{+}(U \cap V) = \phi$ and $F^{+}(U) \neq \phi$, $F^{+}(V) \neq \phi$.
If $F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is an upper $R-\mathcal{I}-$continuous multifunction, then $F^{+}(U)$ and $F^{+}(V)$ are $R-\mathcal{I}-$open in X by theorem 10.2.1. This imply $(X, \tau, \mathcal{I})$ is not $R-\mathcal{I}-$connected. This contradiction proves Y is $R-\mathcal{J}-$connected. □

Theorem 10.2.8. *Let* $F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ *be a multifunction. If* $F^{+}(I_{R}Ker(B)) \subset R-\mathcal{I}-Int(F^{+}(B))$ *for* $B \subset Y$, *then* F *is upper* $R-\mathcal{I}-$*continuous.*

Proof. Let V be a $R-\mathcal{J}-$open subset of Y. Then we have, $F^{+}(V) = F^{+}(I_{R}Ker(V)) \subset R-\mathcal{I}-Int(F^{+}(V))$. This implies $R-\mathcal{I}-Int(F^{+}(V)) = F^{+}(V)$. Hence F is upper $R-\mathcal{I}-$continuous. □

Theorem 10.2.9. *Let* $F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ *be a multifunction. If* $F^{-}(I_{R}Ker(B)) \subset R-\mathcal{I}-Int(F^{-}(B))$ *for* $B \subset Y$, *then* F *is lower* $R-\mathcal{I}-$*continuous.*

Proof. The proof is as similar to the proof of theorem 10.2.8 □

10.3 Almost $R-\mathcal{I}-$continuous Multifunctions

Definition 10.3.1. *A multifunction* $F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ *is said to be:*

1. *upper almost* $R-\mathcal{I}-$*continuous at a point* $x \in X$, *if for each* $R-\mathcal{J}-$*open set* V *of* Y *such that* $F(x) \subset V$, *there exists* $U \in R\mathcal{I}O(X, x)$ *such that* $F(U) \subset Int(R-\mathcal{J}-Cl(V))$.

2. *lower almost* $R-\mathcal{I}-$*continuous at a point* $x \in X$, *if for each* $R-\mathcal{J}-$*open set* V *of* Y *such that* $F(x) \cap V \neq \phi$, *there exists* $U \in R\mathcal{I}O(X,x)$ *such that* $F(w) \cap Int(R-\mathcal{J}-Cl(V)) \neq \phi$ *for each* $w \in U$.

3. *almost* $R-\mathcal{I}-$*continuous at* $x \in X$, *if it is both upper almost* $R-\mathcal{I}-$*continuous and lower almost* $R-\mathcal{I}-$*continuous at* x.

4. *almost* $R-\mathcal{I}-$*continuous, if* F *is both upper almost* $R-\mathcal{I}-$*continuous and lower almost* $R-\mathcal{I}-$*continuous at each point* x *of* X.

Theorem 10.3.1. *For a multifunction* $F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$, *the following statements are equivalent:*

1. F *is upper almost* $R-\mathcal{I}-$*continuous at* $x \in X$.

2. *For every* $R-\mathcal{J}-$*open set* V *of* Y *with* $F(x) \subset V$, $x \in R-\mathcal{I}-Int(F^+(Int(R-\mathcal{J}-Cl(V))))$ *holds.*

3. *For every* $R-\mathcal{J}-$*open set* V *of* Y *with* $F(x) \subset V$, $x \in R-\mathcal{I}-Int(F^+(V))$ *holds.*

4. *For every* $R-\mathcal{J}-$*open set* V *of* Y *with* $F(x) \subset V$, *there exists a* $R-\mathcal{I}-$*open set* U *containing* x *such that* $F(U) \subset V$.

Proof. 1. $\Rightarrow$ 2.

Let $V \subset Y$ be a $R-\mathcal{J}-$open set with $F(x) \subset V$. Then there exists $U \in R\mathcal{I}O(X,x)$ such that $F(U) \subset Int(R-\mathcal{J}-Cl(V))$. This implies $x \in U \subset F^+(Int(R-\mathcal{J}-Cl(V)))$. Thus $x \in R-\mathcal{I}-Int(F^+(Int(R-\mathcal{J}-Cl(V))))$.

2. $\Rightarrow$ 3.

The proof is trivial.

3. $\Rightarrow$ 4.

Let $V \subset Y$ be a $R-\mathcal{J}-$open set with $F(x) \subset V$. Then $x \in R-\mathcal{I}-$

$Int(F^+(V))$. Let $U = R-\mathcal{I}-Int(F^+(V))$, a $R-\mathcal{I}-$open set containing x. So $x \in U \subset F^+(V)$ and thus $F(U) \subset V$.

4. $\Rightarrow$ 1.

Let $V \subset Y$ be a $R-\mathcal{J}-$open set with $F(x) \subset V$. Since $V \subset Int(R-\mathcal{J}-Cl(V))$, there exists a $R-\mathcal{I}-$open set U containing x such that $F(U) \subset Int(R-\mathcal{J}-Cl(V))$. Hence F is upper almost $R-\mathcal{I}-$continuous at $x \in X$. □

Theorem 10.3.2. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following statements are equivalent:*

1. *F is upper almost $R-\mathcal{I}-$continuous.*
2. *For every $R-\mathcal{J}-$open set V of Y, $F^+(V) \subset R-\mathcal{I}-Int(F^+(Int(R-\mathcal{J}-Cl(V))))$.*
3. *For every $R-\mathcal{J}-$open set V of Y, $F^+(V)$ is $R-\mathcal{I}-$open in X.*
4. *For every $R-\mathcal{J}-$closed set C of Y, $F^-(C)$ is $R-\mathcal{I}-$closed in X.*

Proof. 1. $\Rightarrow$ 2.

Let V be $R-\mathcal{J}-$open in Y and let $x \in F^+(V)$. So $F(x) \subset V$ and by theorem 10.3.1.(2), $x \in R-\mathcal{I}-Int(F^+(Int(R-\mathcal{J}-Cl(V))))$. Thus $F^+(V) \subset R-\mathcal{I}-Int(F^+(Int(R-\mathcal{J}-Cl(V))))$.

2. $\Rightarrow$ 3.

Let V be $R-\mathcal{J}-$open in Y. Then we know that $F^+(Int(R-\mathcal{J}-Cl(V))) \subset F^+(V)$. But using 2., we have $F^+(V) \subset R-\mathcal{I}-Int(F^+(Int(R-\mathcal{J}-Cl(V))))$. Hence $F^+(V)$ is $R-\mathcal{I}-$open in X.

3. $\Rightarrow$ 4.

We know that for any $A \subset Y$, $F^+(Y \setminus A) = X \setminus F^-(A)$. From this fact, the proof follows.

4. $\Rightarrow$ 1.

Let $x \in X$ and V be any $R-\mathcal{J}-$open set of Y with $F(x) \subset V$. Then $Y \setminus V$ is $R-\mathcal{J}-$closed in Y. Then $X \setminus F^{+}(V) = F^{-}(Y \setminus V)$ is $R-\mathcal{I}-$closed in X. This means $F^{+}(V)$ is $R-\mathcal{I}-$open in X. Set $U = F^{+}(V)$. Then $U \in R\mathcal{I}O(X,x)$ such that $F(U) \subset V$. Hence F is upper almost $R-\mathcal{I}-$continuous by theorem 10.3.1. □

Theorem 10.3.3. *For a multifunction $F : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma,\mathcal{J})$, the following statements are equivalent:*

1. *F is lower almost $R-\mathcal{I}-$continuous at $x \in X$.*
2. *For every $R-\mathcal{J}-$open set V of Y with $F(x) \cap V \neq \phi$, $x \in R-\mathcal{I}-Int(F^{-}(Int(R-\mathcal{J}-Cl(V))))$ holds.*
3. *For every $R-\mathcal{J}-$open set V of Y with $F(x) \cap V \neq \phi$, $x \in R-\mathcal{I}-Int(F^{-}(V))$ holds.*
4. *For every $R-\mathcal{J}-$open set V of Y with $F(x) \cap V \neq \phi$, there exists a $R-\mathcal{I}-$open set U containing x such that $F(U) \subset V$.*

Proof. As in theorem 10.3.1., it can be proved in a similar manner. □

Theorem 10.3.4. *For a multifunction $F : (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma,\mathcal{J})$, the following statements are equivalent:*

1. *F is lower almost $R-\mathcal{I}-$continuous.*
2. *For every $R-\mathcal{J}-$open set V of Y, $F^{-}(V) \subset R-\mathcal{I}-Int(F^{-}(Int(R-\mathcal{J}-Cl(V))))$.*
3. *For every $R-\mathcal{J}-$open set V of Y, $F^{-}(V)$ is $R-\mathcal{I}-$open in X.*
4. *For every $R-\mathcal{J}-$closed set C of Y, $F^{+}(C)$ is $R-\mathcal{I}-$closed in X.*

Proof. As in theorem 10.3.2., it can be proved in a similar manner.

□

Remark 10.3.1. *Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a multifunction. Then always upper $R - \mathcal{I}-$continuous implies upper almost $R - \mathcal{I}-$continuous.*

10.4 Weakly $R - \mathcal{I}-$continuous Multifunctions

Definition 10.4.1. *A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be:*

1. *upper weakly $R - \mathcal{I}-$continuous at a point $x \in X$ if for each $R - \mathcal{J}-$open set V of Y with $F(x) \subset V$, there exists $U \in R\mathcal{I}O(X, x)$ such that $F(U) \subset R - \mathcal{J} - Cl(V)$.*

2. *lower weakly $R - \mathcal{I}-$continuous at a point $x \in X$ if for each $R - \mathcal{J}-$open set V of Y with $F(x) \cap V \neq \phi$, there exists $U \in R\mathcal{I}O(X, x)$ such that $F(w) \cap R - \mathcal{J} - Cl(V) \neq \phi$ for each $w \in U$.*

3. *weakly $R - \mathcal{I}-$continuous at $x \in X$ if it is both upper weakly $R - \mathcal{I}-$continuous and lower weakly $R - \mathcal{I}-$continuous at x.*

4. *weakly $R - \mathcal{I}-$continuous if F has this property at each point x of X.*

Theorem 10.4.1. *For a multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following statements are equivalent:*

1. *F is upper weakly $R - \mathcal{I}-$continuous.*

2. *For every $R - \mathcal{J}-$open set V of Y, $F^+(V) \subset R - \mathcal{I} - Int(F^+(R - \mathcal{J} - Cl(V)))$.*

3. *For every $R - \mathcal{J}-$closed set C of Y, $R - \mathcal{I} - Cl(F^-(R - \mathcal{J} - Int(C))) \subset F^-(C)$.*

4. *For* $B \subset Y$, $R-\mathcal{I}-Cl(F^-(R-\mathcal{J}-Int(R-\mathcal{J}-Cl(B)))) \subset F^-(R-\mathcal{J}-Cl(B))$.

5. *For* $B \subset Y$, $F^+(R-\mathcal{J}-Int(B)) \subset R-\mathcal{I}-Int(F^+(R-\mathcal{J}-Cl(R-\mathcal{J}-Int(B))))$.

Proof. 1. $\Rightarrow$ 2.

Let V be a $R-\mathcal{J}-$open subset of Y and let $x \in F^+(V)$. So $F(x) \subset V$ and there exists $U \in R\mathcal{I}O(X, x)$ such that $F(U) \subset R-\mathcal{J}-Cl(V)$. Then $x \in U \subset F^+(R-\mathcal{J}-Cl(V))$. Thus $x \in R-\mathcal{I}-Int(F^+(R-\mathcal{J}-Cl(V)))$. Hence $F^+(V) \subset R-\mathcal{I}-Int(F^+(R-\mathcal{J}-Cl(V)))$.

2. $\Rightarrow$ 3.

Let C be a $R-\mathcal{J}-$closed subset of Y. Then $Y \setminus C$ is $R-\mathcal{J}-$open in Y. Also, $X \setminus F^-(C) = F^+(Y \setminus C) \subset R-\mathcal{I}-Int(F^+(R-\mathcal{J}-Cl(Y \setminus C))) = R-\mathcal{I}-Int(F^+(Y \setminus R-\mathcal{J}-Int(C))) = R-\mathcal{I}-Int(X \setminus F^-(R-\mathcal{J}-Int(C))) = X \setminus R-\mathcal{I}-Cl(F^-(R-\mathcal{J}-Int(C)))$. Hence 3.

3. $\Rightarrow$ 4.

Let $B \subset Y$. Since $R-\mathcal{J}-Cl(B)$ is a $R-\mathcal{J}-$closed subset of Y, from 3., we get $R-\mathcal{I}-Cl(F^-(R-\mathcal{J}-Int(R-\mathcal{J}-Cl(B)))) \subset F^-(R-\mathcal{J}-Cl(B))$.

4. $\Rightarrow$ 5.

Let $B \subset Y$. Then, $X \setminus R-\mathcal{I}-Int(F^+(R-\mathcal{J}-Cl(R-\mathcal{J}-Int(B)))) = R-\mathcal{I}-Cl(X \setminus F^+(R-\mathcal{J}-Cl(R-\mathcal{J}-Int(B)))) = R-\mathcal{J}-Cl(F^-(Y \setminus R-\mathcal{J}-Cl(R-\mathcal{J}-Int(B)))) = R-\mathcal{J}-Cl(F^-(R-\mathcal{J}-Int(R-\mathcal{J}-Cl(Y \setminus B)))) \subset F^-(R-\mathcal{J}-Cl(Y \setminus B)) = X \setminus F^+(R-\mathcal{J}-Int(B))$. Hence 5.

5. $\Rightarrow$ 1.

Let $x \in X$ and $V \subset Y$ be $R-\mathcal{J}-$open with $F(x) \subset V$. So $x \in F^+(V)$. Also $F^+(V) \subset R-\mathcal{I}-Int(F^+(R-\mathcal{J}-Cl(V)))$. Then there exists a $R-\mathcal{I}-$open set $U \in X$ containing x with $U \subset F^+(R-\mathcal{J}-Cl(V))$. Thus $F(U) \subset R-\mathcal{J}-Cl(V)$ and hence F is upper weakly $R-\mathcal{I}-$continuous. $\square$

Theorem 10.4.2. *For a multifunction* $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$*, the following statements are equivalent:*

1. F *is lower weakly* $R - \mathcal{I}-$*continuous.*

2. *For every* $R - \mathcal{J}-$*open set* V *of* Y, $F^-(V) \subset R - \mathcal{I} - Int(F^-(R - \mathcal{J} - Cl(V)))$.

3. *For every* $R - \mathcal{J}-$*closed set* C *of* Y, $R - \mathcal{I} - Cl(F^+(R - \mathcal{J} - Int(C))) \subset F^+(C)$.

4. *For* $B \subset Y$, $R - \mathcal{I} - Cl(F^+(R - \mathcal{J} - Int(R - \mathcal{J} - Cl(B)))) \subset F^+(R - \mathcal{J} - Cl(B))$.

5. *For* $B \subset Y$, $F^-(R - \mathcal{J} - Int(B)) \subset R - \mathcal{I} - Int(F^-(R - \mathcal{J} - Cl(R - \mathcal{J} - Int(B))))$.

Proof. The proof is as similar as in the theorem 10.4.1.

□

Remark 10.4.1. *Let* $F : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *be a multifunction. Then always upper almost* $R - \mathcal{I}-$*continuous implies upper weakly* $R - \mathcal{I}-$*continuous.*

$$\text{upper } R - \mathcal{I}-\text{continuous} \Rightarrow \text{upper almost } R - \mathcal{I}-\text{continuous}$$
$$\Rightarrow \text{upper weakly } R - \mathcal{I}-\text{continuous}$$

$$\text{lower } R - \mathcal{I}-\text{continuous} \Rightarrow \text{lower almost } R - \mathcal{I}-\text{continuous}$$
$$\Rightarrow \text{lower weakly } R - \mathcal{I}-\text{continuous}$$

11

Conclusions

In this chapter the conclusions figured out in this study is outlined. Finally, we sketch certain proposals for further research related to this area.

In this thesis, we study ideal topological space in terms of a class of sets called $R-\mathcal{I}-$open sets and hence defined the $R-\mathcal{I}-$space. We render certain characterisations and properties of this space.

The main outcomes are given below:

1. We expanded a generalization of separation axioms in $R-\mathcal{I}-$space and defined $R-\mathcal{I}-T_i$ $i = 0, 1, 2$, $R-\mathcal{I}-$regular, $R-\mathcal{I}-$normal, completely $R-\mathcal{I}-$normal spaces. We initiated the concept of compactness in this space. Certain characterisations in this context are obtained. Moreover, none of the above defined separation axioms are hereditary and not even weakly hereditary because a subset A of a $R-\mathcal{I}-$space $(X, \tau, \mathcal{I})$ which is $R-\mathcal{I}-$open in X need not be $R-\mathcal{I}-$open with respect to the subspace topology $(Y, \tau_Y, \mathcal{I}_Y)$. The hierarchy $R-\mathcal{I}-$normal $\Rightarrow$ $R-\mathcal{I}-$regular $\Rightarrow$ $R-\mathcal{I}-T_2 \Rightarrow R-\mathcal{I}-T_1 \Rightarrow R-\mathcal{I}-T_0$ preserves, but completely $R-\mathcal{I}-$normal and $R-\mathcal{I}-$normal are not comparable.

2. In $R-\mathcal{I}-$space we defined weak separation axioms $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-R_1$. We studied some characterisations and analysed the relationship between them. We concluded that every $R-\mathcal{I}-T_1$ space is $R-\mathcal{I}-R_0$ and $R-\mathcal{I}-T_0$ and $R-\mathcal{I}-R_0$ are independent. Also, every $R-\mathcal{I}-R_1$ space is $R-\mathcal{I}-R_0$, but not the converse.

3. Later, we concentrate on separation axioms $R-\mathcal{I}-R_S$, $R-\mathcal{I}-R_D$, $R-\mathcal{I}-R_T$, weakly $R-\mathcal{I}-R_0$, weakly $R-\mathcal{I}-C_0$ which are weaker than $R-\mathcal{I}-R_0$. We also obtained that every $R-\mathcal{I}-$regular space is $R-\mathcal{I}-R_1$. We attain the following relation:

$$R-\mathcal{I}-\mathrm{R}_1 \Rightarrow R-\mathcal{I}-R_0 \Rightarrow \text{weakly-}R-\mathcal{I}-R_0$$
$$\Downarrow$$
$$R-\mathcal{I}-R_D \Rightarrow R-\mathcal{I}-R_T \Rightarrow R-\mathcal{I}-R_S$$

4. We moved to supra ideal topological space and have brought in supra $R-\mathcal{I}-$open sets. Henceforth, introduced supra $R-\mathcal{I}-$continuous functions, supra* $R-\mathcal{I}-$continuous functions, supra $R-\mathcal{I}-$irresolute functions, supra $R-\mathcal{I}-$open maps, supra $R-\mathcal{I}-$closed maps, supra $R-\mathcal{I}-$homeomorphism and their characterizations. The relation between these continuous functions are discussed.

5. We proposed a class of sets and continuous functions in $R-\mathcal{I}-$space called minimal $R-\mathcal{I}-$open sets and minimal $R-\mathcal{I}-$continuous functions. A small discussion on maximal $R-\mathcal{I}-$open sets and maximal $R-\mathcal{I}-$continuous functions is given. The relation between these continuous functions and some continuous functions which are defined in previous chapters is studied. Also, we surveyed $R-\mathcal{I}-T_{min}$ and $R-\mathcal{I}-T_{max}$ spaces and obtained that $R-\mathcal{I}-T_{min}$ (resp. $R-\mathcal{I}-T_{max}$) and $R-\mathcal{I}-T_i$ $i=0,1,2$ spaces are independent of each other. Sim-

ilarly, $R-\mathcal{I}-T_{min}$ (resp. $R-\mathcal{I}-T_{max}$) and $R-\mathcal{I}-$door spaces are independent of each other.

6. We introduced and studied weak notions of functions namely somewhat $R-\mathcal{I}-$continuous functions and somewhat $R-\mathcal{I}-$open functions. Some of their characterizations and properties are analysed.

7. We introduced and studied strong notions of functions namely contra $R-\mathcal{I}-$continuous functions and almost contra $R-\mathcal{I}-$continuous functions and investigated some of their characterizations and properties. Also, we dealt with contra $R-\mathcal{I}-$closed graphs.

8. We expand the concept of continuity to set multifunctions in $R-\mathcal{I}-$space. We developed upper and lower $R-\mathcal{I}-$continuous multifunctions and examined two of their weaker forms namely almost and weakly $R-\mathcal{I}-$continuous multifunctions and obtained the relation:

upper $R-\mathcal{I}-$continuous $\Rightarrow$ upper almost $R-\mathcal{I}-$continuous $\Rightarrow$ upper weakly $R-\mathcal{I}-$continuous

lower $R-\mathcal{I}-$continuous $\Rightarrow$ lower almost $R-\mathcal{I}-$continuous $\Rightarrow$ lower weakly $R-\mathcal{I}-$continuous

Further work

- Using the idea of simple extension topology and $R-\mathcal{I}-$space in ideal topology, one can try to define a class of open sets in this simple ideal extension topological space and can generalise this concept and investigate the properties.

- One can incorporate the class of continuous functions called faint continuous functions in $R-\mathcal{I}-$space. Also, to find out the basic properties of this continuous function and its relationship with other functions.

- The concept of paraopen sets and some of their properties are already studied. One can try to extend this concept in $R-\mathcal{I}-$space in ideal topology.

- One can extend the notion of open sets and continuous functions discussed in this work to bitopological spaces. The main question is how to define the local function in bitopological spaces.

- One can study operators and its applications in ideal topological group.

References

[1] A. Kanbir and I.L.Reily,*On almost cl-supercontinuous functions*, Appl. Gen. Topol.,**11(1)**(2010), 57-65.

[2] A. R. Prasannan, Jayanta Biswas,*On strong form of $\delta - I-$continuous functions and $R-I-$continuous functions*, Journal of Advanced Studies in Topology, **7:1**(2016), 21-24.

[3] A. S. Davis. *Indexed systems of neighborhoods for general topological spaces*, American Mathematical Society, **68**(1961), 886–893.

[4] A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, *On supra topological spaces*, Indian J. Pure and Appl. Math., **14(4)**(1983), 502 – 510.

[5] D. Jankovic and T.R.Hamlett,*Compatible extensions of ideals*, Boll. Un. Mat.Ital., **7(6)**(1992), 453-465.

[6] D. Jankovic and T.R.Hamlett,*New topologies from old via ideals*, Amer.Math.Monthly, **97**(1990), 295-310.

[7] D. N. Mishra and K. K. Dube, *Some axioms weaker than the R_0 axiom*, Glasnik Mat. Ser. III, **8**(1973), 145-147.

[8] D. W. Hall, S. K. Murphy, and J. Rozycki. *On spaces which are essentially T_1*, Journal of the Australian Mathematical Society, **12**(1971), 451–455.

[9] E. Ekici, *Almost contra-precontinuous functions*, Bull. Malaysian Math. Soc., 27(1) (2004), 53-65

[10] F. G. Arenas, J. Dontchev, and M. L. Puertas. *Idealization of some weak separation axioms*, Acta Mathematica Hungarica, **89(1)**(2000), 47–53.

[11] F. Nakaoka and N. Oda, *Some applications of minimal open sets*, International Journal of Mathematics and Mathematical Sciences **27, no. 8**(2001), 471–476.

[12] F. Nakaoka and N. Oda, *Some properties of maximal open sets*, International Journal of Mathematics and Mathematical Sciences **2003, no. 21**(2003), 1331–1340.

[13] F. Nakaoka and N. Oda, *Minimal closed sets and maximal closed sets* International Journal of Mathematics and Mathematical Sciences **2006**(2006), 1-8.

[14] Giuseppe Di Maio, *A separation axiom weaker than R_0*, Indian J. pure appl. Math., **16(4)**(1985), 373-375.

[15] J. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Internat. J. Math. Math. Sci. 19(2) (1996), 303-310

[16] J. E. Joseph and M.H. Kwack, *On S-closed spaces*, Proceedings Amer. Math. Soc., 82(2) (1980), 341-348

[17] J. K. Kohli and D.Singh,*Almost cl-supercontinuous functions*, Appl.Gen.Topol., **10(1)**(2009), 1-12.

[18] J. K. Kohli and D.Singh, *Between strong continuity and almost continuity*, Appl.Gen.Topol., **11(1)**(2010), 29-42.

[19] James R. Munkres, *Topology*, Prentice Hall India, 2002.

[20] Jingcheng Tong, *On the separation axiom* R_0, Glasnik Matematicki, **18(38)**(1983), 149-152.

[21] K. Dlaska, N. Ergun and M. Ganster, *Countably S-closed spaces*, Mathematica Slovaca, 44(3) (1994), 337-348

[22] K. K. Dube. *A note on* R_0 *topological spaces*, Matematicki vesnik, **11**(1974), 203–208.

[23] K. K. Dube. *A note on* R_1 *topological spaces*, Period. Math. Hungar., **13(4)**(1982), 267-271.

[24] K. Kuratowski, *Topology*, NewYork: Academic Press, (1966).

[25] K.R. Gentry, H.B. Hoyle,*Somewhat continuous functions*, Zech.Math.J., **21(1)**(1971), 5-12.

[26] M. E. Abd El-Monsef and A. A. Nasef, *On Multifunctions*, Chaos, Solitons and Fractals, 12(2001), 2387-2394.

[27] M. E. Abd El-Monsef, E.F.Lashien and A.A.Nasef,*On* $I-$*open sets and* $I-$*continuous functions*, Kyunpook Math.J., **32**(1992), 21-30.

[28] M. K. Singal and A. Mathur, *On nearly compact spaces*, Boll. Un. Mat. Ital., 2(1969), 702-710

[29] M. K. Singal and A. R. Singal, *Almost continuous mappings*, Yokohama Math. J. 16(1968), 67–73.

[30] M. Mrsevic, *On pairwise* R_0 *and pairwise* R_1 *bitopological spaces*, Bulletin Mathematique de la Societe des Sciences Mathematiques de la

Republique Socialiste de Roumanie. Nouvelle Serie, vol. 30(78), no. 2, (1986), 141-148

[31] M. N. Mukherjee, R. Bishwambhar, and R. Sen. *On extension of topological spaces in terms of ideals*, Topology and its Applications, **154(18)**(2007), 3167–3172.

[32] N. Anuradha, B.Chacko, *Somewhat r-continuous and Somewhat r-open functions*9 International Journal of Pure and Applied Mathematics, **100(4)**(2015), 507-524.

[33] N. A. Shanin. *On separation in topological spaces*, Doklady Akademii Nauk SSSR, **38**(1943), 110– 113.

[34] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Mon. 68(1961), 44-46.

[35] O. R. Sayed and T. Noiri, *On supra b-open sets and supra b-continuity on topological spaces*, European J. Pure and Appl. Math., **vol. 3 no. 2.**(2010), 295-302.

[36] R. E. Smithson, *Almost and weak continuity for multifunctions*, Bull. Calcutta Math. Soc. 70(1978), 383–390.

[37] R. Manoharan and P.Thangavelu,*Some New Sets and Topologies in Ideal Topological Spaces*,Chinese Journal of Mathematics, **Article ID 973608**(2013).

[38] R. Staum, *The algebra of bounded continuous functions into a non-archimedean field*, Pacific J. Math. 50(1)(1974), 169-185

[39] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, 1960

[40] S. A. Naimpally. *On R_0-topological spaces.* Annales Universitatis Scientiarum Budapestinensis, Sectio Mathematica, **10**(1967), 53–54.

[41] S. Jafari, and T. Noiri, *Contra-super-continuous functions*, Annales Univ. Sci. Budapest,42 (1999), 27-34

[42] S. Yuksel, A.Acikgoz and T.Noiri,*On $\delta - I$ continuous functions*, Turk J Math, **29**(2005), 39-51.

[43] Shyamapada Modak, Sukalyan Mistry,*Ideals on Supra Topological Space*, Int.Journal of Math.Analysis, **vol.6(1)**(2012), 1-10.

[44] T. Noiri and V. Popa, *Almost weakly continuous multifunctions*, Demonstratio Math., 26(1993), 363-380.

[45] T. R. Hamlett and D. Jankovic, *Ideals in topological spaces and the set operator* Ψ, Bull. U.M.I., **7** 4-B(1990), 863 – 874.

[46] T.soundararajan, *Weakly hausdorff spaces and the cardinality of topological spaces*, General topology and its relations to modern analysis and algebra, Praha: Academia Publishing House of the Czechoslovak Academy of Sciences **vol. 1**(1971), 301–306.

[47] V. Popa, *A note on weakly and almost continuous multifunctions*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 21 (1991), 31-38.

[48] V. Popa, *Almost continuous multifunctions*, Matematicki Vesnik, 6(19)(34), 1982, 75-84

[49] V. Popa, *Some properties of almost continuous multifunctions*, Matematicki Vesnik, 35, 1983, 425-432

[50] V. Popa, *Weakly continuous multifunctions*, Boll. Un. Mat. Ital. V, Ser. A 15(1978), 379–388 .

PUBLICATIONS

- Sangeetha M. V. and Baby Chacko, *Somewhat* $R-\mathcal{I}-$*continuous and Somewhat* $R-\mathcal{I}-$*open Functions in an Ideal Topological Space*, Journal of Computer and Mathematical Sciences, 9(4)(2018), 282-292.

- Sangeetha M. V. and Baby Chacko, *On supra ideal topological space via* $R-\mathcal{I}-$*open sets*, Malaya Journal of Matematik, Vol. S, No. 1, 2019, 433-440.

- Sangeetha M. V. and Baby Chacko, *Weak separation axioms in terms of* $R-\mathcal{I}-$*open sets*, IOSR Journal of Mathematics Volume 16, Issue 4 Ser. II (Jul.-Aug. 2020), 63-67.

- Sangeetha M. V. and Baby Chacko, *Some properties of separation axioms in ideal topological spaces*, International Journal of Research and Analytical Reviews, July 2020, Volume 7, Issue 3 323-326.

PRESENTATIONS

- MESMAC International Conference 2019 - "PEOPLE FIRST? MAN MACHINE MILIEU"organized by MES Mampad College (Autonomous), Malappuram, Kerala, India on January 15-17, 2019. Presented a paper on the topic "On Supra Ideal Topological space via $R-\mathcal{I}-$open sets".

- International Virtual Conference on "RECENT APPLICATIONS OF MATHEMATICAL SCIENCES (RAMS-2021)"organized by the Department of Mathematics and Statistics, Mohanlal Sukhadia University, Udaipur (Raj.) India on July 15-16, 2021. Presented a paper on the topic "Separation Axioms Weaker than $R-\mathcal{I}-R_0$".

- Advancement in Mathematics and its Emerging Areas (AIMEA-2021) 23-24 July 2021 Organized by Department of Mathematics, Faculty of Engineering and Computing Sciences Teerthanker Mahaveer University, Moradabad. Presented a paper on the topic "Minimal open sets in ideal topological spaces".