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NEIGHBOURHOOD AND STAR V_4-MAGIC LABELING OF GRAPHS

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CERTIFICATE

I hereby certify that the thesis entitled "NEIGHBOURHOOD AND STAR V_4-MAGIC LABELING OF GRAPHS" is a bonafide work carried out by **Sri. Vineesh K. P.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**NEIGHBOURHOOD AND STAR V_4-MAGIC LABELING OF GRAPHS**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

List of Symbols

G	Simple, connected and undirected graph
$V(G)$	Vertex set of G
$E(G)$	Edge set of G
I_G	The incidence relation.
$deg(v)$	Degree of the vertex v
$N(u)$	Neighbour set of the vertex u
P_n	Path on n vertices
C_n	Cycle on n vertices
W_n	Wheel graph
H_n	Helm graph
K_n	Complete graph
$S'(G)$	Subdivision graph of G
$K_{1,n}$	Star graph or n-star
$K_{m,n}$	Complete bipartite graph
$F_m(or D_3^m)$	Friendship graph or Dutch windmill graph
SF_n	Sunflower graph

$B_{m,n}$	Bistar
$Bt(n,k)$	The $(n,k)-$banana tree
$J(m,n)$	Jelly fish graph
L_n	Ladder graph
CB_n	Comb graph
QS_n	Quadrilateral snake
C_n^*	Crown graph
B_n	Book graph
$BP(n)$	Bipyramid graph
G_n	Gear graph
Fl_n	Flower graph
$\mathbb{F}_n$	Fan graph
$U_{n,m}$	Umbrella graph
$U_{n,m,k}$	Extended umbrella graph
$J_{n,m}$	Jahangir graph
$W(2,n)$	Web graph
J_n	Jewel graph
$G_1 \vee G_2$	Join of G_1 and G_2
$G_1 \odot G_2$	Corona of G_1 and G_2
$G_1 \Box G_2$	Cartesian product of G_1 and G_2
$G_1 \times G_2$	Cartesian product of G_1 and G_2
$G_1[G_2]$	Composition of G_1 and G_2
$S(G)$	Splitting graph of G
$Sh(G)$	Shadow graph of G
$M(G)$	Middle graph of G

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Introduction

Graph theory at the outset, an intellectual endeavor confined to the academists soon attained momentum and began to sprinkle its drops in vide vistas of both theoretical and practical aspects. Branches of Mathematics like Number theory, Algebra, Algebraic topology, Algebraic geometry, Numerical analysis, Matrix theory, Probability and Representation theory flourished in consonance with the advanced studies based on graph theory. The wide range of application of graph theory brought benefit to Chemistry, Electrical engineering, Geography, Sociology and Architecture. Genetics, the branch of life science also includes. Linguistics cannot be set apart. Revolution in the field of communication which is enjoyed by all stratas of society is much blessed by studies in graph theory.

Graph labeling has wide range of application in communication networks, addressing database management, circuit design, Astronomy, radar, X-ray crystallographic analysis, etc. Most graph labeling methods trace their origin to one introduced by Rosa in 1967 or one given by Graham and Sloane in 1980. A magic graph is a graph whose edges are labeled by positive integers, so that the sum over the edges incident with any vertex is the same, independent of the choice of vertex; or it is a graph that has such a labeling. A graph is vertex-magic if its vertices can be labeled so that the sum on any edge is the same. Graph labeling such as graceful, harmonious, prime and magic has many applications.

An overview of the thesis

The thesis introduces new types of labelings namely Neighbourhood V_4-magic labeling, Neighbourhood barycentric V_4-magic labeling and Star V_4-magic labeling in graphs. In this work we consider graphs that are connected, finite, simple and undirected. The Klein 4-group, denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$, where $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. For a graph $G = (V(G), E(G))$, a labeling $f : V(G) \to V_4 \backslash \{0\}$ is said to be Neighbourhood V_4-magic if the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(v) = \sum_{u \in N(v)} f(u)$ is constant. If such labeling f exists, we say G is a neighbourhood V_4-magic graph. A labeling $f : V(G) \to V_4 \backslash \{0\}$ is said to be Neighbourhood barycentric V_4-magic if the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(u) = \sum_{v \in N(u)} f(v)$ satisfies the following conditions:

(i) N_f^+ is a constant map, and

(ii) For each $u \in V(G)$, $N_f^+(u) = deg(u)f(v_u)$ for some vertex $v_u \in N(u)$.

If such a labeling exists, we say G is a Neighbourhood barycentric V_4-magic graph. A graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$,is said to be star V_4-magic if there exists a labeling $f : V(G) \to V_4 \backslash \{0\}$ such that the induced mapping $V_f^+ : V(G) \to V_4$ defined by $V_f^+(v) = \sum_{u \in N(v)} f^*(uv)$, where $f^*(uv) = f(u) + f(v)$ is a constant map.

Through out this thesis we will use the following notations:

(a) $\Omega_a :=$ the class of all a-neighbourhood V_4-magic graphs

(b) $\Omega_0 :=$ the class of all 0-neighbourhood V_4-magic graphs

(c) $\Omega_{a,0} := \Omega_a \cap \Omega_0$.

(d) $\Lambda_a :=$ the class of all a-neighbourhood barycentric V_4-magic graphs

(e) $\Lambda_0 :=$ the class of all 0-neighbourhood barycentric V_4-magic graphs

(f) $\Psi_a :=$ the class of all a-star V_4-magic graphs

(g) $\Psi_0 :=$ the class of all 0-star V_4-magic graphs

(h) $\Psi_{a,0} := \Psi_a \cap \Psi_0$.

The thesis comprises an introductory chapter and nine other chapters. In the introductory chapter, we deal with the motivation for the study of Neighbourhood V_4-magic labeling, Neighbourhood barycentric V_4-magic labeling and Star V_4-magic labeling of graphs and a literature survey on it.

In **Chapter One**, we include preliminary definitions and theorems from the areas of graph theory and group theory which are the pre-requisites of the forthcoming chapters in the thesis. **Chapter Two** introduces the concept of Neighbourhood V_4-magic labeling in graphs. The first section of the chapter gives the definition of Neighbourhood V_4-magic labeling in graphs and some definitions of cycle related graphs. The second section proves an important Lemma:

If $f : V(C_n) \to V_4 \backslash \{0\}$ is any labeling of C_n, then $\sum_{v \in V} N_f^+(v) = 0$. Then we proved necessary and sufficient conditions for the cycle C_n, the helm graph H_n and the sunflower SF_n belongs to the above said classes (a),(b) and (c). The chapter includes admissibility of neighbourhood V_4-magic labeling of some other graphs namely the friendship graph F_m, the corona $C_n \odot K_2$ and $C_n \odot \overline{K}_m$, the wheel graph W_n and the flower graph Fl_n.

Chapter Three investigates Neighbourhood V_4-magic labeling of star and path related graphs. The first section contains definitions of some star and path related graphs. Second section discusses the star graph $K_{1,n}$, bistar $B_{m,n}$, subdivision graph $S'(K_{1,n})$, banana tree $Bt(n, k)$, jelly fish $J(m, n)$, the graph $K_{1,n}^*$ and the graph $< K_{1,n} : m >$ admits neighbourhood V_4-magic labeling or not. Third section of the chapter investigates the neighbourhood V_4-magic labeling of path related graphs like ladders L_n, L_{n+2}, the comb CB_n and the quadrilateral snake QS_n.

Chapter Four discusses the neighbourhood V_4-magic labeling of some special graphs like $K_{m,n}$, $P_2 \Box C_n$, the crown graph C_n^*, $P_2 \Box C_n^*$, the book graph B_n, the corona $C_m \odot C_n$, the n-gon book $B(n, k)$, the one point union of k cycles $C_n(k)$, the bipyramid $BP(n)$, the gear graph G_n and the carona on cycles $C_m(C_n)$. A necessary and sufficient condition for a-neighbourhood V_4-magic labeling of the complete graph K_n is also discussed in the same chapter.

In **Chapter Five**, we provide the definitions of splitting graph of a graph, shadow and middle graph of a graph. Continuing section discusses neighbourhood V_4-magic labeling of splitting graphs like $S(C_n)$, $S(P_n)$, $S(Bm, n)$, $S(K_{1,n})$, $S(K_{m,n})$, $S(F_m)$, $S(QS_n)$ an $S(B_n)$ respectively. Neighbourhood V_4-magic labeling of shadow graphs like $Sh(C_n)$, $Sh(P_n)$, $Sh(K_{1,n})$, $Sh(B_{m,n})$, $Sh(W_n)$, $Sh(H_n)$,

$Sh(SF_n)$, $Sh(C_n \odot K_2)$, $Sh(C_n \odot \overline{K}_m)$, $Sh(J(m,n))$, $Sh(L_n)$, $Sh(L_{n+2})$, $Sh(CB_n)$, $Sh(K_{m,n})$, $Sh(B_n)$, $Sh(G_n)$ and middle graphs like $M(C_n)$, $M(P_n)$, $M(K_{1,n})$, $M(F_m)$, $M(B_{m,n})$ are discussed in the continuing sections.

Chapter Six introduces the concept of Neighbourhood barycentric V_4-magic labeling of graphs. The first section of the chapter gives definition of the concept Neighbourhood barycentric V_4-magic labeling in graphs. Second section investigates neighbourhood barycentric V_4-magic labeling of some general graphs and some special graphs.

The first section of the **Chapter Seven** introduces the new concept, star V_4-magic labeling in graphs. Next section of the chapter discusses star V_4-magic labeling of the cycle C_n, the path P_n, the complete graph K_n, the star $K_{1,n}$, the complete graph $K_{m,n}$, the bistar $B_{m,n}$, the wheel graph W_n, the helm H_n, jelly fish $J(m,n)$, the crown C_n^*, the flower graph Fl_n, the friendship graph F_m and the book graph B_n.

Chapter Eight discusses star V_4-magic labeling of fan $\mathbb{F}_n$ and fan related graphs like the umbrellas $U_{n,m}$, extended umbrellas $U_{n,m,k}$, the jahangir graph $J_{n,m}$. It also discusses star V_4-magic labeling of graphs like $Bt(n,k)$, $< K_{1,n} : m >$, the ladder L_n, the comb CB_n, the gear graph G_n, the web graph $W(2,n)$, the jewel graph J_n, the corona $C_n \odot K_2$,$P_n \odot \overline{K}_2$ and the planar grid $P_m \Box P_n$.

Chapter Nine gives a brief summary and further scope of the research.

Chapter 1

Preliminaries

This chapter gives a brief account of the preliminary definitions of graph theory and some group theory which are used in the forthcoming chapters. For the notations and terminologies not defined in this thesis we used to refer readers [2] and [6].

1.1 Basic Definitions

Definition 1.1.1. [2] *Graph is an ordered triple $G = (V(G), E(G), I_G)$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and I_G is an "incidence" relation that associates with each element of $E(G)$, an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the vertices (or nodes or points) of G, and elements of $E(G)$ are called the edges (or lines) of G. If, for the edge e of G, $I_G(e) = \{u, v\}$, we write $I_G(e) = uv$.*

Definition 1.1.2. *If in a graph, $I_G(e) = \{u, v\}$, then we say that the vertices u and v are adjacent or e is incident to the vertices u and v. Also u and v are called end vertices of e.*

Definition 1.1.3. *If two or more edges have same end points, then such edges are called parallel edges and an edge e with end vertices are same is called a loop.*

Definition 1.1.4. *A vertex v is called a neighbour of a vertex u, if v is adjacent to u. The set of all neighbours of a vertex u is called neighbour set of u, denoted by $N(u)$. That is, $N(u) = \{v \in V(G) : uv \in E(G)\}$.*

Definition 1.1.5. [2] *A graph G is called a simple graph if it has no parallel edges or loop in it. Thus for a simple graph G, the incidence function I_G is injection. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph therefore may be considered as an ordered pair $(V(G), E(G))$, where $V(G)$ is a non empty set and $E(G)$ is a set of unordered pairs of elements of $V(G)$.*

Definition 1.1.6. *In a graph G, the number of elements in $V(G)$ and $E(G)$ are finite, then G is called a finite graph. A graph which is not finite is called an infinite graph.*

Definition 1.1.7. *In a simple graph, if every pair of vertices are adjacent, then such a graph is called a complete graph. A complete graph on n vertices is usually denoted by K_n.*

Definition 1.1.8. *A graph with one vertex and no edges is called a trivial graph.*

Definition 1.1.9. *A graph G is called bipartite if its vertex set can be partitioned into two sets V_1 and V_2 such that each edge in G has one end in V_1 and other end in V_2.*

Definition 1.1.10. *A bipartite graph G is called a complete bipartite graph if each vertex of V_1 is adjacent to all the vertices of V_2, where V_1 and V_2 are bipartition of V. If $|V_1| = m$ and $|V_2| = n$, then G is usually denoted by $K_{m,n}$.*

1.2 Subgraphs and Supergraphs

Definition 1.2.1. [2] *A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and I_H is the restriction of I_G to $E(H)$. If H is a subgraph of G, then G is said to be a supergraph of H.*

Definition 1.2.2. *If H is a subgraph of G with $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then H is called a proper subgraph of G.*

Definition 1.2.3. *The degree of a vertex v in a graph G is the number of vertices adjacent to v in G, it is usually denoted by $deg(v)$ or $d(v)$ or $d_G(v)$.*

Definition 1.2.4. *A vertex v in a graph G is called an even vertex if $deg(v)$ is even and is said to be odd if $deg(v)$ is odd.*

Definition 1.2.5. *A vertex v in a graph G is called a pendant vertex if $deg(v) = 1$. The unique edge incident to such a vertex is called a pendant edge.*

Definition 1.2.6. *A graph G is called a k-regular graph if $deg(v) = k$ for all $v \in V(G)$. A graph is called regular if it is k-regular for some k.*

Theorem 1.2.7. [2] *The sum of the degrees of the vertices of a graph is equal to twice the number of its edges.*

1.3 Walks and Connectedness

Definition 1.3.1. [6] *A walk of a graph G is finite sequence of vertices and edges $W := v_0e_1v_1e_2v_2e_3 \ldots e_nv_n$, beginning and ending with vertices, in which each edge is incident with two vertices immediately preceding and succeeding it. The walk $W := v_0e_1v_1e_2v_2e_3 \ldots e_nv_n$ is sometimes called $v_0 - v_n$ walk.*

Definition 1.3.2. *A walk $v_0 - v_n$ is called a closed walk if $v_0 = v_n$. otherwise it is called an open walk.*

Definition 1.3.3. [6] *If every edges in a walk are distinct, then it is called a trail. Length of a walk is the number of edges involved in the walk.*

Definition 1.3.4. [6] *If every vertices in a walk are distinct, then it is called a path. A path on n vertices is usually denoted by P_n.*

Definition 1.3.5. [2] *A cycle is a closed trail in which all the vertices are distinct. A cycle on n vertices is usually denoted by C_n.*

Definition 1.3.6. *Two vertices u and v in a graph G are connected if there is a $u - v$ path in G. A graph G is called connected if every pair of vertices are connected.*

Definition 1.3.7. *A graph is called acyclic or forest if it has no cycle involved in it. A connected acyclic graph is called a tree.*

1.4 Operations on Graphs

Definition 1.4.1. [2] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. Then the graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the union of graphs G_1 and G_2 and is denoted by $G_1 \cup G_2$. If $V_1 \cap V_2 = \phi$, then $G_1 \cup G_2$ is usually denoted by $G_1 + G_2$, called the sum of the graphs G_1 and G_2.*

Definition 1.4.2. [2] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 \neq \phi$. Then the graph $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$ is called the intersection of graphs G_1 and G_2 and is denoted by $G_1 \cap G_2$.*

Definition 1.4.3. [2] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 = \phi$. Then the join, $G_1 \vee G_2$, of G_1 and G_2 is the super graph of $G_1 + G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2.*

Definition 1.4.4. [2] *The Cartesian product of two simple graphs G_1 and G_2, commonly denoted by $G_1 \square G_2$ or $G_1 \times G_2$, has the vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \square G_2$ are adjacent if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$.*

Definition 1.4.5. [2] *The Composition of lexicographic product of two simple graphs G_1 and G_2, commonly denoted by $G_1[G_2]$ has the vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1[G_2]$ are adjacent if either u_1 is adjacent to u_2 or $u_1 = u_2$, and v_1 is adjacent to v_2.*

Definition 1.4.6. *The Corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, which has p_1 vertices, and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2.*

1.5 Groups

Definition 1.5.1. *Let S be a nonempty set. Then, a binary operation $*$ on S is a mapping from $S \times S$ into S.*

Definition 1.5.2. *A binary operation $*$ on a set S is associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$.*

Definition 1.5.3. *A binary operation $*$ on a set S is commutative if $x * y = y * x$ for all $x, y \in S$.*

Definition 1.5.4. *A set S together with a binary operation $*$ is called a binary algebraic structure or simply binary structure, denoted by $< S, * >$.*

Definition 1.5.5. *A group $< G, * >$ is a binary structure satisfying the following conditions:*

(i) The operation $*$ is associative.

(ii) There exists an element $e \in G$ such that $e * g = g = g * e$ for all $g \in G$. (The element e is called the identity element)

(iii) For each $g \in G$, there is an element $g^{-1} \in G$ such that $g * g^{-1} = e = g^{-1} * g$ for all $g \in G$. (Existence of the inverse element)

Definition 1.5.6. *A group $< G, * >$ is called an abelian group or a commutative group if $g_1 * g_2 = g_2 * g_1$ for all $g_1, g_2 \in G$.*

Definition 1.5.7. *The Klein-4-group $V_4 = \{0, a, b, c\}$ is an abelian group with identity 0, where $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$.*

Remark 1.5.8. *The Klein-4-group V_4 is not cyclic, since every element is of order two (except possibly the identity).*

Chapter 2

Neighbourhood V_4-magic Labeling of Cycle Related Graphs

The first section of this chapter introduces the concept of neighbourhood V_4-magic labeling in graphs, and then defines some cycle related graphs. The second section of this chapter discusses neighbourhood V_4-magic labeling of such cycle related graphs.

2.1 Introduction

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0. For any graph $G = (V(G), E(G))$, a mapping $f : V(G) \to V_4 \backslash \{0\}$ is said to be Neighbourhood V_4-magic labeling if the induced mapping $N_f^+ : V(G) \to V_4$ defined by

$$N_f^+(v) = \sum_{u \in N(v)} f(u)$$

is a constant map. If this constant is p, where p is any non zero element in V_4,

[1]This chapter has been published in *Far East Journal of Mathematical Sciences(FJMS)*, Volume 111, Number 2, 2019, Pages 263-272.

we say that f is a p-neighbourhood V_4-magic labeling of G and G is said to be a p-neighbourhood V_4-magic graph. If this constant is 0, then we say that f is a 0-neighbourhood V_4-magic labeling of G and G is said to be a 0-neighbourhood V_4-magic graph.

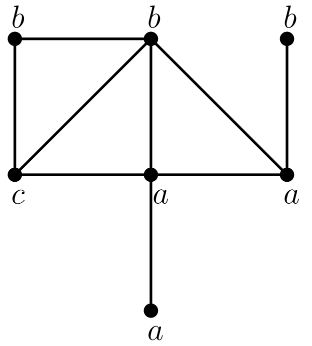

Figure 2.1: An a-neighbourhood V_4-magic labeling of a graph G_1

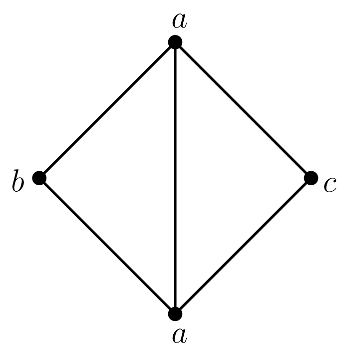

Figure 2.2: A 0-neighbourhood V_4-magic labeling of a graph G_2

Note that the labeling function defined above is not unique. That is there may have more than one neighbourhood V_4-magic labeling for a graph G. Figure 2.3 shows another 0-neighbourhood V_4-magic labeling of the above graph G_2.

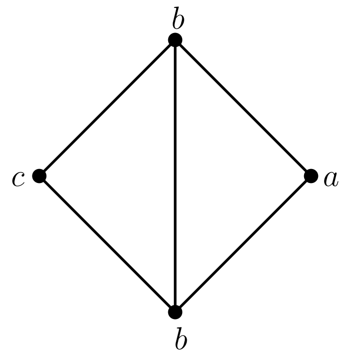

Figure 2.3: Another 0-neighbourhood V_4-magic labeling of the graph G_2

This chapter investigates the Neighbourhood V_4-magic labeling of some cycle related graphs that belongs to the following categories:

(i) $\Omega_a :=$ the class of all a-neighbourhood V_4-magic graphs,

(ii) $\Omega_0 :=$ the class of all 0-neighbourhood V_4-magic graphs, and

(iii) $\Omega_{a,0} := \Omega_a \cap \Omega_0$.

Definition 2.1.1. [27] *The friendship graph or the Dutch windmill graph, denoted by* F_m *(or* $D_3^{(m)}$*) is the graph obtained by taking* m *copies of* C_3 *with one vertex in common.*

Definition 2.1.2. [2] *The wheel graph* W_n *is defined as* $W_n = C_n \vee K_1$*, where* C_n *for* $n \geq 3$ *is a cycle of length* n*.*

Definition 2.1.3. [22] *The helm* H_n *is the graph obtained from the wheel graph* W_n *by attaching a pendant edge at each vertex of the cycle* C_n*.*

Definition 2.1.4. [26] *The flower graph* Fl_n *is the graph obtained from a helm* H_n *by joining each pendant vertex to the central vertex of the helm.*

Definition 2.1.5. [16] *The Sunflower* SF_n *is obtained from a wheel* W_n *with the central vertex* w_0 *and cycle* $C_n = w_1w_2w_3\cdots w_nw_1$ *and additional vertices* $v_1, v_2, v_3, \ldots, v_n$ *where* v_i *is joined by edges to* w_i *and* w_{i+1} *where* $i + 1$ *is taken over modulo* n.

2.2 Cycle related graphs

Lemma 2.2.1. *If* $f : V(C_n) \to V_4\backslash\{0\}$ *is any labeling of* C_n*, then* $\sum_{v\in V} N_f^+(v) = 0$.

Proof. Let C_n be the cycle with vertices $v_1, v_2, v_3, \ldots, v_n$ and let f be any neighbourhood V_4-magic labeling on it. Then we have

$$N_f^+(v_i) = f(v_{i-1}) + f(v_{i+1}) \text{ for } 1 < i < n,$$
$$N_f^+(v_1) = f(v_2) + f(v_n),\ N_f^+(v_n) = f(v_1) + f(v_{n-1}).$$

Therefore,

$$\sum_{v\in V} N_f^+(v) = 2\sum f(v_i) = 0.$$

This completes the proof. □

Theorem 2.2.2. $C_n \in \Omega_0$ *for all* $n \geq 3$.

Proof. If we label all the vertices of C_n by a, we obtain $N_f^+(v_i) = 0$ for $1 \leq i \leq n$. This completes the proof. □

Theorem 2.2.3. $C_n \in \Omega_a$ *if and only if* $n \equiv 0 (\text{mod } 4)$.

Proof. Suppose that $C_n \in \Omega_a$ with a labeling f. Then by Lemma 2.2.1, we have

$na = 0$. Therefore $n \equiv 0 (\text{mod } 2)$. Then either $n \equiv 0 (\text{mod } 4)$ or $n \equiv 2 (\text{mod } 4)$. We prove that the case $n \equiv 2 (\text{mod } 4)$ is impossible. For if $n \equiv 2 (\text{mod } 4)$, then $n = 4k+2$ for some integer k and let $v_1, v_2, v_3, \ldots, v_{4k}, v_{4k+1}, v_{4k+2}$ be the vertices of C_n in order. Since $C_n \in \Omega_a$, we should have $f(v_1)$ is either b or c. Without loss of generality, we assume that $f(v_1) = b$ (The case $f(v_1) = c$ can be treated similar way). If $f(v_1) = b$, then $f(v_3) = c$, $f(v_5) = b$, $f(v_7) = c$, $f(v_9) = b$. Proceeding like this we will get $f(v_{4k+1}) = b$. Then, $N_f^+(v_{4k+2}) = b + b = 0$, a contradiction. Therefore the case where $n \equiv 2 (\text{mod } 4)$ is impossible. Hence $n \equiv 0 (\text{mod } 4)$. Conversely, assume that C_n be the cycle with vertices $v_1, v_2, v_3, \ldots, v_n$ with $n \equiv 0 (\text{mod } 4)$. Define $f : V(C_n) \to V_4 \backslash \{0\}$ as :

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

Then obviously $N_f^+(v_i) = a \quad \text{for} \;\; 1 \leq i \leq n$. □

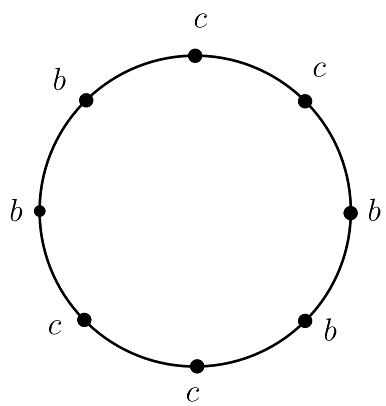

Figure 2.4: An a-neighbourhood V_4-magic labeling of C_8

Corollary 2.2.4. *$C_n \in \Omega_{a,0}$ if and only if $n \equiv 0 (\text{mod } 4)$.*

Proof. Proof follows from Theorem 2.2.2 and Theorem 2.2.3. □

Theorem 2.2.5. *The friendship graph $F_m \notin \Omega_a$ for any m.*

Proof. Note that F_m is the one-point union of m copies of a rooted triangle. Let the vertices of i^{th} copy of C_3 in F_m be w, u_i and v_i where w is the common vertex of the triangles. Suppose that $F_m \in \Omega_a$ with a labeling f. Then $N_f^+(u_i) = N_f^+(v_i) = a$, implies that $f(u_i) = f(v_i) = b \quad$ for $1 \le i \le n$ and $f(w) = c$ or $f(u_i) = f(v_i) = c \quad$ for $1 \le i \le n$ and $f(w) = b$. In either case $N_f^+(w) = 0$. This is a contradiction. □

Theorem 2.2.6. *The friendship graph $F_m \in \Omega_0$ for all m.*

Proof. By Labeling all the vertices by a, we get $F_m \in \Omega_0$. □

Corollary 2.2.7. *The friendship graph $F_m \notin \Omega_{a,0}$ for any m.*

Proof. Proof directly follows from Theorem 2.2.5. □

Theorem 2.2.8. *$C_n \odot K_2 \in \Omega_a$ for $n \equiv 0 (\text{mod } 4)$.*

Proof. Let C_n be the cycle with vertices $u_1, u_2, u_3, \ldots, u_n$ and let v_k and w_k be the vertices of k^{th} copy of K_2. Then $\mid V(C_n \odot K_2) \mid = 3n$. Define $f : V(C_n \odot K_2) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = \begin{cases} c & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \end{cases}$$

Then f gives an a-neighbourhood V_4-magic labeling of $C_n \odot K_2$. □

Theorem 2.2.9. *$C_n \odot K_2 \in \Omega_0$ for all $n \geq 3$.*

Proof. Proof follows if we label all the vertices by a. □

Corollary 2.2.10. *$C_n \odot K_2 \in \Omega_{a,0}$ for $n \equiv 0(\text{mod } 4)$.*

Proof. Proof follows from Theorem 2.2.8 and Theorem 2.2.9. □

Theorem 2.2.11. *$C_n \odot \overline{K}_m \in \Omega_a$ for all $n \geq 3$ and $m \in \mathbb{N}$.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the rim vertices and let $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{im}\}$ be the set of pendant vertices adjacent to u_i for $1 \leq i \leq n$. Here we consider the following two cases:

Case 1: m is even.

Define $f : V(C_n \odot \overline{K}_m) \to V_4 \backslash \{0\}$ as:

$f(u_i) = a \quad \text{for} \quad 1 \leq i \leq n$

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

Case 2: m is odd.

Define $f : V(C_n \odot \overline{K}_m) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(u_{ij}) = a \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m$$

In either case, f is an a-neighbourhood V_4-magic labeling of $C_n \odot \overline{K}_m$. □

Theorem 2.2.12. *$C_n \odot \overline{K}_m \notin \Omega_0$ for any $n \geq 3$ and $m \in \mathbb{N}$.*

Proof. It is obvious due to the presence of pendant vertices in $C_n \odot \overline{K}_m$. □

Corollary 2.2.13. *$C_n \odot \overline{K}_m \notin \Omega_{a,0}$ for any $n \geq 3$ and $m \in \mathbb{N}$.*

Proof. Proof directly follows from Theorem 2.2.12. □

Theorem 2.2.14. *$W_n \in \Omega_0$ for $n \equiv 0 (\text{mod } 4)$.*

Proof. Consider the wheel W_n with vertex set $V = \{u, u_i : 1 \leq i \leq n\}$ where u be the central vertex and let $n \equiv 0(\text{mod } 4)$. We define $f : V(W_n) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3(\text{mod } 4) \end{cases}$$

$$f(u) = a.$$

Then $N_f^+(u) = N_f^+(u_i) = 0$ for $1 \leq i \leq n$. Hence the theorem is proved. □

Theorem 2.2.15. *$W_n \in \Omega_a$ for $n \equiv 1 (\text{mod } 2)$.*

Proof. Consider the wheel W_n with vertex set $V = \{u, u_i : 1 \leq i \leq n\}$ where u be the central vertex and let $n \equiv 1(\text{mod } 2)$. Then $d(u) = n$ and $d(u_i) = 3$ for

$1 \leq i \leq n$. If we label all the vertices by a, we will get $N_f^+(u) = N_f^+(u_i) = a$ for $1 \leq i \leq n$. This completes the proof of the theorem. □

Theorem 2.2.16. *$W_n \in \Omega_a$ for $n \equiv 2(\text{mod } 4)$.*

Proof. Consider the wheel W_n with vertex set $V = \{u, u_i : 1 \leq i \leq n\}$ where u be the central vertex and let $n \equiv 2(\text{mod } 4)$. We define $f : V(W_n) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 2(\text{mod } 4) \end{cases}$$

$$f(u) = a.$$

Then $N_f^+(u) = N_f^+(u_i) = a$ for $1 \leq i \leq n$. Hence $W_n \in \Omega_a$. □

Theorem 2.2.17. *$H_n \in \Omega_a$ if and only if n is odd.*

Proof. Suppose that $H_n \in \Omega_a$. Let v be central vertex, $v_1, v_2, v_3, \ldots, v_n$ be the rim vertices and $u_1, u_2, u_3, \ldots, u_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$ respectively in H_n. Then $N(u_i) = \{v_i\}$ for $1 \leq i \leq n$ and $N(v) = \{v_1, v_2, v_3, \ldots, v_n\}$. Since $N_f^+(u_i) = a$ for $1 \leq i \leq n$, we should have $f(v_i) = a$ for $1 \leq i \leq n$. Therefore, $N_f^+(v) = a$ implying that $na = a$ and hence n is odd. Conversely, suppose that n is odd. We define $f : V(H_n) \to V_4 \backslash \{0\}$ as :

$$f(w) = \begin{cases} b & \text{if} \quad w = v \\ a & \text{if} \quad w = v_1, v_2, v_3, \ldots, v_n \\ c & \text{if} \quad w = u_1, u_2, u_3, \ldots, u_n \end{cases}$$

Then f is an a-neighbourhood V_4-magic labeling of H_n. Hence the theorem. □

Theorem 2.2.18. *$H_n \notin \Omega_0$ for any n.*

Proof. Proof is obvious, since H_n has pendant vertices. □

Corollary 2.2.19. *$H_n \notin \Omega_{a,0}$ for any n.*

Proof. It directly follows from Theorem 2.2.18. □

Theorem 2.2.20. *The flower graph $Fl_n \in \Omega_0$ for all n.*

Proof. Proof is obvious by labeling all the vertices by a. □

Theorem 2.2.21. *$Fl_n \notin \Omega_a$ for any n.*

Proof. Suppose that $Fl_n \in \Omega_a$ for some n with a labeling f. Then $\sum_{i=1}^{n} N_f^+(u_i) + \sum_{i=1}^{n} N_f^+(v_i) + N_f^+(v) = 0$. This implies that $(2n+1)a = 0$. Thus $a = 0$, a contradiction. Hence $Fl_n \notin \Omega_a$ for any n. □

Corollary 2.2.22. *$Fl_n \notin \Omega_{a,0}$ for any n.*

Proof. Proof obviously follows from Theorem 2.2.21. □

Theorem 2.2.23. *The sunflower $SF_n \in \Omega_a$ if and only if $n \equiv 2 (\text{mod } 4)$.*

Proof. Consider the sunflower SF_n with vertex set $V = \{w_0, w_i, v_i : 1 \le i \le n\}$ where w_0 is the central vertex, $w_1, w_2, w_3, \dots, w_n$ are vertices of the cycle and v_i is the vertex joined by edges to w_i and w_{i+1} where $i+1$ is taken over modulo n. Assume that $SF_n \in \Omega_a$ for some n with a labeling f. Since $N_f^+(v_1) = a$, we have $f(w_1) = b$ or c. Suppose $f(w_1) = b$ (The case where $f(w_1) = c$ can be treated similarly), then $f(w_3) = f(w_5) = f(w_7) = \cdots = b$ and $f(w_2) = f(w_4) =$

$f(w_6) = \cdots = c$. Since $N_f^+(v_n) = a$, $f(w_n) = c$, therefore $n \equiv 0(\text{mod } 2)$, implies that either $n \equiv 0(\text{mod } 4)$ or $n \equiv 2(\text{mod } 4)$. If $n \equiv 0(\text{mod } 4)$, then $n = 4k$ some $k \in \mathbb{N}$. Let $w_1, w_2, w_3, \ldots, w_{4k}$ be the vertices of on C_n. Therefore $N_f^+(w_0) = a$ implying that $2k(b + c) = a$ and hence $a = 0$, a contradiction. Thus the case $n \equiv 0(\text{mod } 4)$ is not possible. Hence $n \equiv 2(\text{mod } 4)$. Conversely, assume that $n \equiv 2(\text{mod } 4)$. We define $f : V(SF_n) \to V_4 \backslash \{0\}$ as :

$$f(w_0) = f(v_i) = a \quad \text{for } \ 1 \leq i \leq n,$$

$$f(w_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 2(\text{mod } 4) \end{cases}$$

Clearly f is an a-neighbourhood V_4-magic labeling of SF_n. This completes the proof of the theorem. □

Theorem 2.2.24. *$SF_n \in \Omega_0$ if and only if $n \equiv 0(\text{mod } 2)$.*

Proof. Assume that $SF_n \in \Omega_0$ for some n. Then $f(w_i) = a$ (or b or c) for $1 \leq i \leq n$. Since $N_f^+(w_0) = 0$, we have $na = 0$, implying that $n \equiv 0(mod\ 2)$. Conversely, assume that $n \equiv 0(\text{mod } 2)$. Now define $f : V(SF_n) \to V_4 \backslash \{0\}$ as :

$$f(w_0) = f(w_i) = a \quad \text{for } 1 \leq i \leq n$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 2(\text{mod } 4) \end{cases}$$

Then $N_f^+(u) = 0$ for all $u \in V(SF_n)$. This completes the proof of the theorem.

□

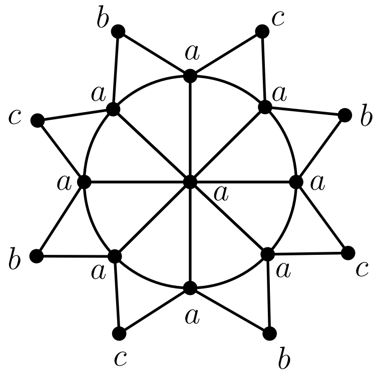

Figure 2.5: A 0-neighbourhood V_4-magic labeling of SF_8

Corollary 2.2.25. $SF_n \in \Omega_{a,0}$ *if and only if* $n \equiv 2 (\text{mod } 4)$.

Proof. Proof directly follows from Theorem 2.2.23 and Theorem 2.2.24. □

Chapter 3

Neighbourhood V_4-magic Labeling of Star and Path Related Graphs

The first section of this chapter provides definitions of some star and path related graphs. Second section of this chapter discusses neighbourhood V_4-magic labeling of star related graphs and the final section investigates the neighbourhood V_4-magic labeling of some path related graphs.

3.1 Introduction

Here we consider the following definitions.

Definition 3.1.1. [2] *A complete bipartite graph of the form $K_{1,n}$ is called a star graph. A star graph $K_{1,n}$ is sometimes called an n-star.*

Definition 3.1.2. [10] *The Bistar $B_{m,n}$ is the graph obtained by joining the*

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central vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.

Definition 3.1.3. *The graph $S'(G)$ obtained by subdividing each edge of G by a vertex is called the subdivision graph of G.*

Definition 3.1.4. [10] *The (n, k)-Banana tree $Bt(n, k)$ is the graph obtained by starting with n number of $k-$stars and connecting one end vertex from each to a new vertex.*

Definition 3.1.5. [10] *Jelly fish graph $J(m, n)$ is obtained from a 4-cycle $w_1w_2w_3w_4w_1$ by joining w_1 and w_3 with an edge and appending the central vertex of $K_{1,m}$ to w_2 and appending the central vertex of $K_{1,n}$ to w_4.*

Definition 3.1.6. [10] *Let $< K_{1,n} : m >$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the m copies of $K_{1,n}$.*

Definition 3.1.7. *The graph $P_2 \Box P_n$ is called a Ladder. It is denoted by L_n.*

Definition 3.1.8. *The graph with vertex set $\{u_i, v_i : 0 \leq i \leq n + 1\}$ and edge set $\{u_iu_{i+1}, v_iv_{i+1} : 0 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\}$ is called the ladder L_{n+2}.*

Definition 3.1.9. *The Corona $P_n \odot K_1$ is called the comb graph CB_n.*

Definition 3.1.10. [19] *A quadrilateral snake QS_n is the graph obtained from a path $v_1v_2v_3 \ldots v_n$ by joining each pair v_i, v_{i+1} to the new vertices u_i, w_i respectively and then joining u_i and w_i by an edge.*

3.2 Star related graphs

Theorem 3.2.1. *The star $K_{1,n} \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof. Consider the star graph with vertex set $V = \{v, v_i : 1 \leq i \leq n\}$ where v be the central vertex of $K_{1,n}$. Here we consider the following cases.

Case 1: n is even.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(v) = a \qquad \text{and} \quad f(v_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases}$$

Case 2: n is odd.

Define $f : V \to V_4 \backslash \{0\}$ as: $f(v) = f(v_i) = a \quad \text{for} \ \ 1 \leq i \leq n.$

In either case, f gives an a-neighbourhood V_4-magic labeling of $K_{1,n}$. □

Theorem 3.2.2. $K_{1,n} \notin \Omega_0$ *for any* $n \in \mathbb{N}$.

Proof. Proof is obvious, since $K_{1,n}$ has pendant vertices. □

Corollary 3.2.3. $K_{1,n} \notin \Omega_{a,0}$ *for all* $n \in \mathbb{N}$.

Proof. It directly follows from Theorem 3.2.2. □

Theorem 3.2.4. $B_{m,n} \in \Omega_a$ *for all* $m > 1$ *and* $n > 1$.

Proof. Consider the bistar $B_{m,n}$ with vertex set $V = \{u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ where $u_i(1 \leq i \leq m)$ and $v_j(1 \leq j \leq n)$ are pendant vertices adjacent to u and v respectively. Here we consider the following cases.

Case 1: Both m and n are even.

Define $f : V \to V_4\backslash\{0\}$ as:

$f(u) = f(v) = f(u_i) = f(v_j) = a$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Case 2: m is even and n is odd.

Define $f : V \to V_4\backslash\{0\}$ as:

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$$f(u) = f(v) = f(u_i) = a \quad \text{for } 1 \leq i \leq m.$$

Case 3: m is odd and n is even.

Define $f : V \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases}$$

$$f(u) = f(v) = f(v_j) = a \quad \text{for } 1 \leq j \leq n.$$

Case 4: Both m and n are odd.

Define $f : V \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases}$$

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$$f(u) = f(v) = a.$$

In all the above cases, f is an a-neighbourhood V_4-magic labeling of $B_{m,n}$. $\square$

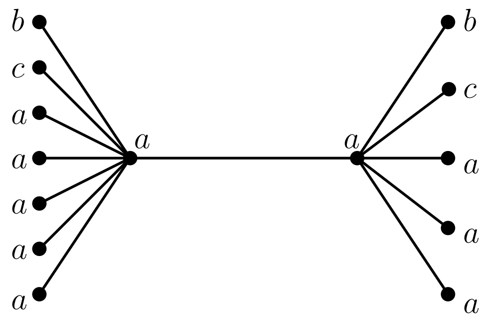

Figure 3.1: An a-neighbourhood V_4-magic labeling of the bistar $B_{7,5}$

Remark 3.2.5. *$B_{m,n} \notin \Omega_a$ for $m = 1$ or $n = 1$. Because if $m = 1$(other case is similar), since $f(u) = f(v) = a$, $N_f^+(u) = a$, implying that $f(u_1) = 0$.*

Theorem 3.2.6. *$B_{m,n} \notin \Omega_0$ for any $m, n \in \mathbb{N}$.*

Proof. The proof is obvious due to presence of pendant vertices in $B_{m,n}$. □

Corollary 3.2.7. *$B_{m,n} \notin \Omega_{a,0}$ for any $n \in \mathbb{N}$.*

Proof. It directly follows from Theorem 3.2.6. □

Theorem 3.2.8. *$S'(K_{1,n}) \in \Omega_a$ if and only if n is odd.*

Proof. Let $V = \{v, u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of $S'(K_{1,n})$ with central vertex u and $v_i(1 \leq i \leq n)$ be the pendant vertices which are adjacent to $u_i(1 \leq i \leq n)$ respectively. Suppose that $S'(K_{1,n}) \in \Omega_a$ with the labeling f. Since $N(v_i) = \{u_i\}$ for $1 \leq i \leq n$, we should have $f(u_i) = a$ for $1 \leq i \leq n$. Now $N_f^+(v) = a$ implies that $na = a$, hence n is odd. Conversely, suppose that n is odd. We define $f : V \to V_4 \backslash \{0\}$ as: $f(v) = b$, $f(u_i) = a$, $f(v_i) = c$ for $1 \leq i \leq n$. Then f is an a-neighbourhood V_4-magic labeling of $S'(K_{1,n})$. □

Theorem 3.2.9. *$S'(K_{1,n}) \notin \Omega_0$ for any $n \in \mathbb{N}$.*

Proof. Proof is obvious due to the presence of pendant vertices in $S'(K_{1,n})$. □

Corollary 3.2.10. *$S'(K_{1,n}) \notin \Omega_{a,0}$ for any $n \in \mathbb{N}$.*

Proof. It follows from Theorem 3.2.9. □

Theorem 3.2.11. *The (n, k)-banana tree $Bt(n, k) \notin \Omega_a$ for any n and k.*

Proof. Assume that $Bt(n, k) \in \Omega_a$ for some n and k. Let $V(Bt(n, k))$=$\{u, u_i, u_{ij} : 1 \leq i \leq n\ , 1 \leq j \leq k\}$ and $E(Bt(n, k))$=$\{uu_i, u_iu_{i1}, u_{i1}u_{ij} : 1 \leq i \leq n, 2 \leq j \leq k\}$. Then $|\ V(Bt(n, k))\ |$=$n(k + 1) + 1$ and $|\ E(Bt(n, k))\ |$=$n(k + 1)$. Since $N_f^+(u_{12}) = a$, we should have $f(u_{11}) = a$. Now $N_f^+(u_1) = a$, implies that $f(u) = 0$, which is a contradiction. Therefore $Bt(n,k) \notin \Omega_a$ for any n and k. This completes the proof. □

Theorem 3.2.12. *$Bt(n,k) \notin \Omega_0$ for any n and k.*

Proof. Proof is obvious, since $Bt(n,k)$ has pendant vertices. □

Corollary 3.2.13. *$Bt(n,k) \notin \Omega_{a,0}$ for all n and k.*

Proof. Proof directly follows from Theorem 3.2.12. □

Theorem 3.2.14. *The Jelly fish $J(m,n) \in \Omega_a$ for all m and n.*

Proof. Let G be the jelly fish $J(m,n)$. Then G has $(m+n+4)$ vertices and $(m+n+5)$ edges. Let $V(G) = V_1 \cup V_2$ where $V_1 = \{w_1, w_2, w_3, w_4\}$, $V_2 = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = E_1 \cup E_2$, where $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\}$, $E_2 = \{w_2u_i, w_4v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Here we consider the following cases:

Case 1: Both m and n are even.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = a \quad \text{for} \quad 1 \leq i \leq 4$$

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

Case 2: m is even and n is odd.

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases}$$

$$f(w_i) = f(v_j) = a \quad \text{for} \quad 1 \leq i \leq 4, \ 1 \leq j \leq n.$$

Case 3: m is odd and n is even.

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(w_i) = f(u_j) = a \quad \text{for} \quad 1 \leq i \leq 4, 1 \leq j \leq m.$$

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

Case 4: Both m and n are odd.

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(w_i) = f(u_j) = f(v_k) = a \quad \text{for} \quad 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n.$$

In all the above cases, f is an a-neighbourhood V_4-magic labeling of $J(m, n)$. □

Theorem 3.2.15. *$J(m, n) \notin \Omega_0$ for any m and n.*

Proof. Proof is obvious due to the presence of pendant vertices in $J(m, n)$. □

Corollary 3.2.16. *$J(m, n) \notin \Omega_{a,0}$ for any m and n.*

Proof. Proof directly follows from Theorem 3.2.15. □

Theorem 3.2.17. *The graph* $< K_{1,n} : m >\in \Omega_a$ *if and only if* m *is odd.*

Proof. Let G be the graph $< K_{1,n} : m >$ and let $V_i = \{u_i, u_{ij} : 1 \leq j \leq n\}$ be the vertex set of i^{th} copy of $K_{1,n}$ with apex u_i and let u be the unique vertex adjacent to the central vertices $u_i (1 \leq i \leq m)$ in G. Then $V(G) = V_1 \cup V_2 \cup \ldots \cup V_m \cup \{u\}$. Suppose that $G \in \Omega_a$ with a labeling f. Then $N_f^+(u_{ij}) = a$ implies that $f(u_i) = a$ for $1 \leq i \leq m$. Also $N_f^+(u) = a$ implies that $ma = a$. Hence m is odd. Conversely, suppose that m is odd. Here we consider the following cases:

Case 1: $n = 1$.

In this case G is $S'(K_{1,m})$, which is in Ω_a when m is odd.

Case 2: $n \geq 2$ and n is odd.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$$f(u) = f(u_i) = a \ \text{ for } 1 \leq i \leq m$$

Case 3: $n \geq 2$ and n is even.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$f(u) = f(u_i) = f(u_{ij}) = a$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In all the above cases, f gives an a-neighbourhood V_4-magic labeling of G. $\square$

Theorem 3.2.18. *$< K_{1,n} : m > \notin \Omega_0$ for any m and n.*

Proof. It is obvious, since $< K_{1,n} : m >$ has pendant vertices in it. $\square$

Corollary 3.2.19. *$< K_{1,n} : m > \notin \Omega_{a,0}$ for any m and n.*

Proof. Proof directly follows from Theorem 3.2.18. $\square$

Definition 3.2.20. *The graph obtained by attaching central vertices (or apex)of n-copies of $K_{1,n}$ by a unique vertex u by n distinct edges is denoted by $K^*_{1,n}$.*

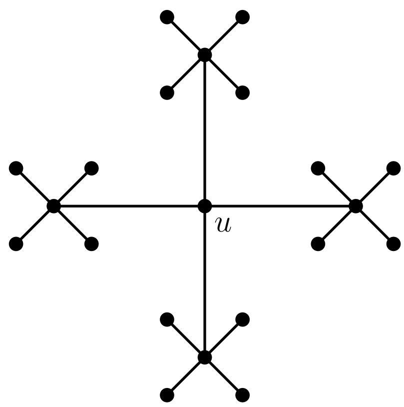

Figure 3.2: Graph $K^*_{1,4}$

Theorem 3.2.21. *The graph $K^*_{1,n} \in \Omega_a$ if and only if n is odd.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the central vertices of each of the n-copies of $K_{1,n}$, let $\{v_{ij} : 1 \leq j \leq n\}$ be the set of pendant vertices adjacent to the vertex v_i for $1 \leq i \leq n$. Also let v be the unique vertex connecting each v_i for $1 \leq i \leq n$ by an edge. Assume that $K^*_{1,n} \in \Omega_a$ with the labeling f. Since $N(v_{i,j}) = \{v_i\}$

for all $1 \leq i, j \leq n$, we should have $f(v_i) = a$ for $1 \leq i \leq n$. Now $N_f^+(v) = a$ implies that $na = a$, hence n is odd. Conversely, assume that n is odd. For $i = 1, 2, 3, \ldots, n$, we define $f : V(K_{1,n}^*) \to V_4 \backslash \{0\}$ as:

$$f(v_{ij}) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

and $f(v) = f(v_i) = a$ for $1 \leq i \leq n$. Obviously f is an a-neighbourhood V_4-magic labeling of $K_{1,n}^*$. This completes the proof of the theorem. □

Theorem 3.2.22. *$K_{1,n}^* \notin \Omega_0$ for any $n \in \mathbb{N}$.*

Proof. Proof is obvious due to the presence of pendant vertices in $K_{1,n}^*$. □

Corollary 3.2.23. *$K_{1,n}^* \notin \Omega_{a,0}$ for any $n \in \mathbb{N}$.*

Proof. It follows from Theorem 3.2.22. □

3.3 Path related graphs

Theorem 3.3.1. *The Ladder $L_n \notin \Omega_a$ for all $n \geq 3$.*

Proof. Consider L_n with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. Assume that $L_n \in \Omega_a$ for some $n \geq 3$ with a labeling f. Then $N_f^+(u_1) = f(u_2) + f(v_1) = a$. Therefore, $N_f^+(v_2) = a$ implies that $f(v_3) = 0$, a contradiction. □

Theorem 3.3.2. *$L_n \notin \Omega_0$ for any $n \geq 3$.*

Proof. Consider L_n with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Suppose that $L_n \in \Omega_0$ for some $n \geq 3$ with a labeling f. Then $N_f^+(u_1) = f(u_2) + f(v_1) = 0$. Also $N_f^+(v_2) = 0$ implies that $f(v_3) = 0$, a contradiction. □

Corollary 3.3.3. *$L_n \notin \Omega_{a,0}$ for any $n \geq 3$.*

Proof. Proof directly follows from Theorem 3.3.1. □

Theorem 3.3.4. *The Ladder $L_2 = P_2 \Box P_2 \in \Omega_a$.*

Proof. Consider L_2 with vertex set $V = \{u_1, u_2, v_1, v_2\}$ and edge set $E = \{u_1u_2, u_2v_2, v_1v_2, u_1v_1\}$. Define $f : V(L_2) \to V_4 \backslash \{0\}$ as: $f(u_1) = f(u_2) = b$, $f(v_1) = f(v_2) = c$. Then f is an a-neighbourhood V_4-magic labeling of L_2. □

Theorem 3.3.5. *$L_2 = P_2 \Box P_2 \in \Omega_0$.*

Proof. If we label all the vertices by a, we get $L_2 \in \Omega_0$. □

Corollary 3.3.6. *$L_2 = P_2 \Box P_2 \in \Omega_{a,0}$.*

Proof. Proof follows from Theorem 3.3.4 and Theorem 3.3.5. □

Theorem 3.3.7. *The Ladder $L_{n+2} \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof. Consider the Ladder L_{n+2} with vertex set $\{u_i, v_i : 0 \leq i \leq n+1\}$ and edge set $\{u_iu_{i+1}, v_iv_{i+1} : 0 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Then degree of each vertex is either 1 or 3. By labeling all the vertices by a, we get $L_{n+2} \in \Omega_a$. □

Theorem 3.3.8. *$L_{n+2} \notin \Omega_0$ for any $n \in \mathbb{N}$.*

Proof. Proof is obvious due to the presence of pendant vertices in L_{n+2}. □

Corollary 3.3.9. *$L_{n+2} \notin \Omega_{a,0}$ for any $n \in \mathbb{N}$.*

Proof. Proof directly follows from Theorem 3.3.8. □

Theorem 3.3.10. *The Comb graph $CB_n \notin \Omega_a$ for any $n \geq 2$.*

Proof. Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of CB_n where $v_i(1 \leq i \leq n)$ are the pendant vertices adjacent to $u_i(1 \leq i \leq n)$. Then $N(v_i) = \{u_i\}$ for $1 \leq i \leq n$. Suppose that $CB_n \in \Omega_a$ for some $n \geq 2$ with a labeling f. Then $f(u_i) = a$ for $1 \leq i \leq n$. Also $N_f^+(u_1) = a$ implies that $f(v_1) + f(u_2) = a$, hence $f(v_1) = 0$, a contradiction. Hence $CB_n \notin \Omega_a$ for all $n \geq 2$. □

Theorem 3.3.11. *$CB_n \notin \Omega_0$ for any $n \in \mathbb{N}$.*

Proof. It is obvious, since CB_n has pendant vertices. □

Corollary 3.3.12. *$CB_n \notin \Omega_{a,0}$ for any $n \in \mathbb{N}$.*

Proof. Proof follows from Theorem 3.3.11. □

Theorem 3.3.13. *The quadrilateral snake $QS_n \notin \Omega_a$ for any $n > 2$.*

Proof. Let QS_n be the quadrilateral snake obtained from the path $v_1v_2v_3 \ldots v_n$ by joining each pair v_i, v_{i+1} to the new vertices u_i, w_i respectively and then joining u_i and w_i by an edge. Suppose that $QS_n \in \Omega_a$ for some $n > 2$ with labeling f. Then, $N_f^+(u_1) = f(v_1) + f(w_1) = a$ and $N_f^+(w_2) = f(u_2) + f(v_3) = a$. Therefore

$N_f^+(v_2) = f(v_1) + f(w_1) + f(u_2) + f(v_3) = a + a = 0$, a contradiction. This completes the proof of the theorem. □

Theorem 3.3.14. *$QS_n \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof. The degree of each vertex in QS_n is either 2 or 4. If we label all the vertices by a, we get $QS_n \in \Omega_0$. □

Corollary 3.3.15. *$QS_n \notin \Omega_{a,0}$ for any $n > 2$.*

Proof. Proof follows from Theorem 3.3.13. □

Chapter 4

Neighbourhood V_4-magic Labeling of Some More Graphs

The first section of this chapter provides definitions of some special graphs like Crown graph, Book graph, n-gon book of k pages and Gear graph. The next section discusses neighbourhood V_4-magic labeling of such graphs and some other special graphs.

4.1 Introduction

Definition 4.1.1. [30] *A Crown graph C_n^* is obtained from C_n by attaching a pendant edge at each vertex of the cycle C_n.*

Definition 4.1.2. [25] *The Book graph B_n is the graph $S_n \Box P_2$, where S_n is the star with $n+1$ vertices and P_2 is the path on 2 vertices.*

Definition 4.1.3. [14] *When k copies of C_n share a common edge it will form*

[1]This chapter has been published in *Far East Journal of Mathematical Sciences(FJMS)*, Volume 113, Number 1, 2019, Pages 47-64.

the n-gon book of k pages and is denoted by $B(n, k)$.

Definition 4.1.4. [29] *A Gear graph G_n is obtained from the wheel graph W_n by adding a vertex between every pair of adjacent vertices of the cycle. G_n has $2n + 1$ vertices and $3n$ edges.*

4.2 Some more graphs

Lemma 4.2.1. *Let G be a graph such that all of its vertices are of even degree, then G is 0-neighbourhood V_4-magic graph.*

Proof. If we label all the vertices by a, we get $N_f^+(u) = 0$. Thus G is a 0-neighbourhood V_4-magic graph. □

Lemma 4.2.2. *Let G be a graph such that all of its vertices are of odd degree, then G is a-neighbourhood V_4-magic graph.*

Proof. If we label all the vertices by a, we get $N_f^+(u) = a$. Thus G is an a-neighbourhood V_4-magic graph. □

Theorem 4.2.3. *The complete bipartite graph $K_{m,n} \in \Omega_a$ for all m, n.*

Proof. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. We consider the following cases:

Case 1: Both m and n are even.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

Case 2: m is even and n is odd.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = a \text{ for } 1 \leq j \leq n.$$

Case 3: m is odd and n is even.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases} \qquad f(u_i) = a \text{ for } 1 \leq i \leq m.$$

Case 4: Both m and n are odd.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$f(u_i) = f(v_j) = a$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In each case, f gives an a-neighbourhood V_4-magic labeling of $K_{m,n}$. □

Theorem 4.2.4. *$K_{m,n} \in \Omega_0$ for $m > 1$ and $n > 1$.*

Proof. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. We consider the following cases:

Case 1: Both m and n are even.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$f(u_i) = f(v_j) = a$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Case 2: m is even and n is odd.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases} \qquad f(u_i) = a \ \text{ for } \ 1 \leq i \leq m.$$

Case 3: m is odd and n is even.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = a \ \text{ for } \ 1 \leq j \leq n.$$

Case 4: Both m and n are odd.

Define $f : V(K_{m,n}) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

In all the above cases, f gives 0-neighbourhood V_4-magic labeling of $K_{m,n}$. □

Corollary 4.2.5. *$K_{m,n} \in \Omega_{a,0}$ for $m > 1$ and $n > 1$.*

Proof. Proof follows from Theorem 4.2.3 and Theorem 4.2.4. □

Theorem 4.2.6. *The graph $P_2 \Box C_n \in \Omega_a$ for all $n \geq 3$.*

Proof. Consider $P_2 \Box C_n$ with vertex set $V = \{(u_i, v_j) : 1 \leq i \leq 2, 1 \leq j \leq n\}$. Then degree of each vertex is 3. Label all the vertices by a, then $N_f^+(u_i, v_j) = a$ for all $1 \leq i \leq 2$ and $1 \leq j \leq n$. This completes the proof of the theorem. □

Theorem 4.2.7. *$P_2 \Box C_n \in \Omega_0$ if and only if $n \equiv 0 (\text{mod } 3)$.*

Proof. Consider $P_2 \Box C_n$ with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_1 u_n, v_1 v_n\}$. If $n \equiv 0 (\text{mod } 3)$, we define $f : V \to V_4\backslash\{0\}$ as:

$$f(u_i) = f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 0 (\text{mod } 3) \\ b & \text{if} \quad i \equiv 1 (\text{mod } 3) \\ a & \text{if} \quad i \equiv 2 (\text{mod } 3) \end{cases}$$

Then $N_f^+(u_i) = N_f^+(v_i) = 0$ for all $1 \leq i \leq n$. Conversely, if $n \not\equiv 0 (\text{mod } 3)$, then $n \equiv 1 (\text{mod } 6)$ or $n \equiv 2 (\text{mod } 6)$ or $n \equiv 4 (\text{mod } 6)$ or $n \equiv 5 (\text{mod } 6)$.

Case 1: $n \equiv 1 (\text{mod } 6)$

In this case $n = 6k+1$ for some $k \in \mathbb{N}$. If $P_2 \Box C_n \in \Omega_0$, then $N_f^+(u_1) = 0$, consequently $f(v_1), f(u_2)$ and $f(u_{6k+1})$ are distinct non-zero elements in V_4. Without loss of generality we assume that $f(v_1) = a, f(u_2) = b$ and $f(u_{6k+1}) = c$. Then $f(v_3) = c, f(u_4) = a, f(v_5) = b, f(u_6) = c, f(v_7) = a, \ldots, f(v_{6k+1}) = a$. Now $N_f^+(u_{6k+1}) = 0$ implies that $f(u_1) = b, f(v_2) = c, f(u_3) = a, f(v_4) = b, \ldots, f(v_{6k}) = a$. Therefore, $N_f^+(u_{6k}) = f(u_{6k+1}) + f(u_{6k-1}) + f(v_{6k}) = c+c+a = a$, which is a contradiction. Hence $P_2 \Box C_n \notin \Omega_0$.

Case 2: $n \equiv 2 (\text{mod } 6)$

In this case $n = 6k+2$ for some $k \in \mathbb{N}$. If $P_2 \Box C_n \in \Omega_0$, then $N_f^+(u_1) = 0$, consequently $f(v_1), f(u_2)$ and $f(u_{6k+2})$ are distinct non-zero elements in V_4. Without loss of generality we assume that $f(v_1) = a, f(u_2) = b$ and $f(u_{6k+2}) = c$. Then $f(v_3) = c, f(u_4) = a, f(v_5) = b, f(u_6) = c, f(v_7) = a, \ldots, f(u_{6k}) = c$. Now $N_f^+(u_{6k+1}) = 0$ implies that $f(v_{6k+1}) = 0$, which is a contradiction. Hence $P_2 \Box C_n \notin \Omega_0$.

Case 3: $n \equiv 4 (\text{mod } 6)$

In this case $n = 6k+4$ for some $k \in \mathbb{N}$. If $P_2 \Box C_n \in \Omega_0$, then $N_f^+(u_1) = 0$, consequently $f(v_1), f(u_2)$ and $f(u_{6k+4})$ are distinct non-zero elements in V_4. Without loss of generality we assume that $f(v_1) = a, f(u_2) = b$ and $f(u_{6k+4}) = c$.Then $f(v_3) = c, f(u_4) = a, f(v_5) = b, f(u_6) = c, f(v_7) =$

$a, f(u_8) = b, f(v_9) = c \ldots, f(v_{6k+3}) = c$.Now $N_f^+(v_{6k+4}) = 0$ implies that $f(v_1) = 0$, which is a contradiction. Hence $P_2 \Box C_n \notin \Omega_0$.

Case 4: $n \equiv 5 (\text{mod } 6)$

In this case $n = 6k + 5$ for some $k \in \mathbb{N}$. If $P_2 \Box C_n \in \Omega_0$, then $N_f^+(u_1) = 0$, consequently $f(v_1), f(u_2)$ and$f(u_{6k+5})$ are distinct non-zero elements in V_4. Without loss of generality we assume that $f(v_1) = a, f(u_2) = b$ and $f(u_{6k+5}) = c$. Then $f(v_3) = c, f(u_4) = a, f(v_5) = b, f(u_6) = c, f(v_7) = a, f(u_8) = b, f(v_9) = c, f(u_{10}) = a, \ldots, f(v_{6k+5}) = b$. Now $N_f^+(v_{6k+5}) = 0$ implies that $f(v_{6k+4}) = b$, which again implies that $f(u_{4k+3}) = a, f(v_{6k+2}) = c, f(u_{6k+1}) = b, f(v_{6k}) = a, \ldots, f(v_2) = c$. Therefore, $N_f^+(v_1) = c$, a contradiction. Hence $P_2 \Box C_n \notin \Omega_0$.

Thus in each of the above cases, we have $P_2 \Box C_n \notin \Omega_0$. Completes the proof. □

Corollary 4.2.8. *$P_2 \Box C_n \in \Omega_{a,0}$ if and only if $n \equiv 0 (\text{mod } 3)$.*

Proof. Proof follows from Theorem 4.2.6 and Theorem 4.2.7. □

Theorem 4.2.9. *The crown graph $C_n^* \in \Omega_a$ for all $n \geqslant 3$.*

Proof. The degree of vertices of a crown graph is either 1 or 3.

Define $f : V(C_n^*) \to V_4 \backslash \{0\}$ as :

$f(v) = a$ for each $v \in V(C_n^*)$

Clearly f is an a-neighbourhood V_4-magic labeling of C_n^*. □

Theorem 4.2.10. *$C_n^* \notin \Omega_0$ for any n.*

Proof. Proof is obvious due to the presence of pendant vertices in C_n^*. □

Theorem 4.2.11. *$C_n^* \notin \Omega_{a,0}$ for any n.*

Proof. Proof directly follows from Theorem 4.2.10. □

Theorem 4.2.12. *The graph obtained by duplicating all pendant vertices in a Crown graph C_n^* is a-neighbourhood V_4-magic graph.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the rim vertices and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices in C_n^*. Let G be the graph obtained by duplicating all the pendant vertices in C_n^*. Suppose $w_1, w_2, w_3, \ldots, w_n$ be the new vertices in G by duplicating the vertices $v_1, v_2, v_3, \ldots, v_n$. Then $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$. We define $f : V(G) \to V_4 \backslash \{0\}$ as :

$$f(u) = \begin{cases} a & \text{if} \quad u = u_1, u_2, u_3, \ldots, u_n \\ b & \text{if} \quad u = v_1, v_2, v_3, \ldots, v_n \\ c & \text{if} \quad u = w_1, w_2, w_3, \ldots, w_n \end{cases}$$

Clearly f is an a-neighbourhood V_4-magic labeling of G. □

Theorem 4.2.13. *The graph G obtained by duplicating all the pendant vertices in a Crown C_n^* is not 0-neighbourhood V_4-magic.*

Proof. Proof is obvious, since G has pendant vertices. □

Theorem 4.2.14. *The graph G obtained by duplicating all the pendant vertices in a Crown C_n^* is not in $\Omega_{a,0}$.*

Proof. Proof follows from Theorem 4.2.12 and Theorem 4.2.13. □

Theorem 4.2.15. *The graph $P_2 \square C_n^* \in \Omega_a$ for all $n \geqslant 3$.*

Proof. Let $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_i, w_i : 1 \le i \le n\}$ be the vertex sets of P_2 and C_n^* respectively, where $v_1, v_2, v_3, \ldots, v_n$ are the rim variables and $w_i' s$ are pendant vertices adjacent to v_i for $1 \le i \le n$ in C_n^*. Define $f : V(P_2 \Box C_n^*) \to V_4 \backslash \{0\}$ as:

$$f(u_i, v_j) = \begin{cases} c & \text{if} \quad i = 1 \text{ and } 1 \le j \le n \\ b & \text{if} \quad i = 2 \text{ and } 1 \le j \le n \end{cases}$$

$$f(u_i, w_j) = \begin{cases} c & \text{if} \quad i = 1 \text{ and } 1 \le j \le n \\ b & \text{if} \quad i = 2 \text{ and } 1 \le j \le n \end{cases}$$

Then $N_f^+(u_i, v_j) = N_f^+(u_i, w_j) = a$ for $1 \le i \le 2$ and $1 \le j \le n$. Thus f is an a-neighbourhood V_4-magic labeling for $P_2 \Box C_n^*$. Completing the proof. □

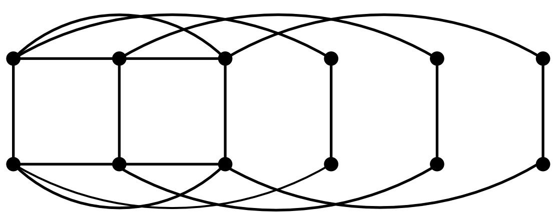

Figure 4.1: Graph $P_2 \Box C_3^*$

Theorem 4.2.16. *$P_2 \Box C_n^* \in \Omega_0$ for all $n \geqslant 3$.*

Proof. In $P_2 \Box C_n^*$, the degree of each vertex is either 2 or 4. Labeling all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(P_2 \Box C_n^*)$. □

Corollary 4.2.17. *$P_2 \Box C_n^* \in \Omega_{a,0}$ for all $n \geqslant 3$.*

Proof. It directly follows from Theorems 4.2.15 and 4.2.16. □

Theorem 4.2.18. *The book graph $B_n \in \Omega_a$ if and only if n is odd.*

Proof. Let $V_1 = \{u, u_1, u_2, u_3, \ldots, u_n\}$ and $V_2 = \{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, where u be the central vertex and $u_i's$ are pendant vertices in S_n. Then $V(B_n) = \{(u, v_j), (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$. Assume that $B_n \in \Omega_a$ for some n with a labeling f. Since $N_f^+(u_1, v_1) = a$, we should have either $f(u, v_1) = b$ or $f(u, v_1) = c$. If $f(u, v_1) = b$, then $f(u_1, v_2) = f(u_2, v_2) = f(u_3, v_2) = \cdots = f(u_n, v_2) = c$. Then $N_f^+(u, v_2) = a$ implies that $b + nc = a$, hence n is odd. If $f(u, v_1) = c$, then $f(u_1, v_2) = f(u_2, v_2) = f(u_3, v_2) = \cdots = f(u_n, v_2) = b$. Then $N_f^+(u, v_2) = a$ implies that $c + nb = a$, hence n is odd. Thus, in either case, n is an odd number. Conversely, assume that n is an odd number. We define $f : V(B_n) \to V_4 \backslash \{0\}$ as:

$$f(u, v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \end{cases} \qquad \text{and} \qquad f(u_i, v_j) = \begin{cases} b & \text{if} \quad j = 1 \text{ and } 1 \leq i \leq n \\ c & \text{if} \quad j = 2 \text{ and } 1 \leq i \leq n \end{cases}$$

Obviously, f is an a-neighbourhood V_4-magic labeling of B_n. This completes the proof of the theorem. □

Theorem 4.2.19. *$B_n \in \Omega_0$ if and only if n is odd.*

Proof. Let $V_1 = \{u, u_1, u_2, u_3, \ldots, u_n\}$ and $V_2 = \{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, where u be the central vertex and $u_i's$ are pendant vertices in S_n. Then $V(B_n) = \{(u, v_j), (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$. Assume that $B_n \in \Omega_0$ for some n with a labeling f. Since $N_f^+(u_1, v_1) = 0$, we should have either $f(u, v_1) = f(u_1, v_2) = a$ or $f(u, v_1) = f(u_1, v_2) = b$ or $f(u, v_1) = f(u_1, v_2) = c$.

Case 1: $f(u, v_1) = f(u_1, v_2) = a$.

Since $B_n \in \Omega_0$ and $f(u, v_1) = a$ implies that $f(u_i, v_2) = a$ for $1 \leq i \leq n$. Therefore, $N_f^+(u, v_2) = 0$ implies that $(n+1)a = 0$. Hence n is odd.

Case 2: $f(u, v_1) = f(u_1, v_2) = b$.

Since $B_n \in \Omega_0$ and $f(u, v_1) = b$ implies that $f(u_i, v_2) = b$ for $1 \leq i \leq n$. Therefore, $N_f^+(u, v_2) = 0$ implies that $(n+1)b = 0$. Hence n is odd.

Case 3: $f(u, v_1) = f(u_1, v_2) = c$.

Since $B_n \in \Omega_0$ and $f(u, v_1) = c$ implies that $f(u_i, v_2) = c$ for $1 \leq i \leq n$. Therefore, $N_f^+(u, v_2) = 0$ implies that $(n+1)c = 0$. Hence n is odd.

Conversely, assume that n is odd. Then $deg(u, v_1) = deg(u, v_2) = n + 1$ and $deg(u_i, v_1) = deg(u_i, v_2) = 2$ for $1 \leq i \leq n$. If we label all the vertices by a, we will get $B_n \in \Omega_0$. This completes the proof of the theorem. □

Corollary 4.2.20. *$B_n \in \Omega_{a,0}$ if and only if n is odd.*

Proof. Proof follows from Theorem 4.2.18 and Theorem 4.2.19. □

Theorem 4.2.21. *$C_m \odot C_n \in \Omega_a$ for $n \equiv 1 (\text{mod } 2)$.*

Proof. Consider $C_m \odot C_n$ with $n \equiv 1 (\text{mod } 2)$. Then degree of each vertex in $C_m \odot C_n$ is either $n + 2$ or 3. In either case vertices are odd. If we label all the vertices by a, we get $C_m \odot C_n \in \Omega_a$. □

Theorem 4.2.22. *$C_m \odot C_n \in \Omega_a$ for $m, n \equiv 0 (\text{mod } 4)$.*

Proof. Let the vertex set of C_m be $\{u_1, u_2, u_3, \ldots, u_m\}$ and vertex set of k^{th} copy of C_n is $\{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{kn}\}$ in order. Define $f : V(C_m \odot C_n) \rightarrow V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \end{cases}$$

$$f(v_{ij}) = \begin{cases} a & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \text{ and } \ j \equiv 1,2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \text{ and } \ j \equiv 0,3 (\text{mod } 4) \\ a & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \text{ and } \ j \equiv 1,2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \text{ and } \ j \equiv 0,3 (\text{mod } 4) \end{cases}$$

Then, f gives an a-neighbourhood V_4-magic labeling of $C_m \odot C_n$. Hence the theorem is proved. □

Theorem 4.2.23. $C_m \odot C_n \in \Omega_0$ *for* $n \equiv 0(\text{mod } 4)$.

Proof. Let the vertex set of C_m be $\{u_1, u_2, u_3, \ldots, u_m\}$ and vertex set of k^{th} copy of C_n is $\{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{kn}\}$ in order. Define $f : V(C_m \odot C_n) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = c \quad \text{for} \quad 1 \leq i \leq m$$

$$f(v_{ij}) = \begin{cases} a & \text{if} \quad 1 \leq i \leq m \ \text{ and } \ j \equiv 1,2 (\text{mod } 4) \\ b & \text{if} \quad 1 \leq i \leq m \ \text{ and } \ j \equiv 0,3 (\text{mod } 4) \end{cases}$$

Clearly f is a 0-neighbourhood V_4-magic labeling of $C_m \odot C_n$. □

Theorem 4.2.24. $C_m \odot C_n \in \Omega_{a,0}$ *for* $m, n \equiv 0(\text{mod } 4)$.

Proof. Proof directly follows from Theorems 4.2.22 and 4.2.23. □

Theorem 4.2.25. $B(n, k) \in \Omega_a$ *for* $n \equiv 0(\text{mod } 4)$ *and* $k \equiv 1(\text{mod } 2)$.

Proof. Consider the *n-gon* book $B(n,k)$ with $n \equiv 0(\text{mod } 4)$ and $k \equiv 1(\text{mod } 2)$. Let $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{in}\}$ be the vertex set of i^{th} copy of C_n and $u_{i1}u_{in}$ $(1 \leq i \leq k)$ are the common edges in $B(n,k)$. We define $f : V(B(n,k)) \to V_4 \backslash \{0\}$ as:

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad 1 \leq i \leq k \text{ and } \; j \equiv 1,2(\text{mod } 4) \\ c & \text{if} \quad 1 \leq i \leq k \text{ and } \; j \equiv 0,3(\text{mod } 4) \end{cases}$$

Obviously, $N_f^+(u_{ij}) = a$ for $1 \leq i \leq k$ and $1 \leq j \leq n$. Completes the proof. □

Theorem 4.2.26. *$B(n,k) \in \Omega_0$ for all $n \geq 3$ and $k \equiv 1(\text{mod } 2)$.*

Proof. Proof is obvious if we label all the vertices by a. □

Corollary 4.2.27. *$B(n,k) \in \Omega_{a,0}$ for $n \equiv 0(mod\ 4)$ and $k \equiv 1(\text{mod } 2)$.*

Proof. Proof follows from Theorem 4.2.25 and Theorem 4.2.26. □

Definition 4.2.28. [25] One point union of k cycles each of length n is denoted by $C_n(k)$.

Theorem 4.2.29. *$C_n(k) \in \Omega_a$ for all $n \equiv 0(\text{mod } 4)$ and $k \equiv 1(\text{mod } 2)$.*

Proof. Let $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{in}\}$ be the vertex set of i^{th} copy of the cycle in $C_n(k)$ for $n \equiv 0(\text{mod } 4)$ and $k \equiv 1(\text{mod } 2)$, where $u_{11} = u_{21} = u_{31} = \cdots = u_{k1}$. Define $f : V(C_n(k)) \to V_4 \backslash \{0\}$ as:

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad 1 \leq i \leq k \text{ and } \; j \equiv 1,2(\text{mod } 4) \\ c & \text{if} \quad 1 \leq i \leq k \text{ and } \; j \equiv 0,3(\text{mod } 4) \end{cases}$$

Then, f is an a-neighbourhood V_4-magic labeling of $C_n(k)$. □

Theorem 4.2.30. *$C_n(k) \in \Omega_0$ for all n and k.*

Proof. Labeling all the vertices by a, we get $C_n(k) \in \Omega_0$. □

Corollary 4.2.31. *$C_n(k) \in \Omega_{a,0}$ for all $n \equiv 0 (\text{mod } 4)$ and $k \equiv 1 (\text{mod } 2)$.*

Proof. Proof follows from Theorem 4.2.29 and Theorem 4.2.30. □

Definition 4.2.32. *Let $N_2 = \{v_1, v_2\}$ be the disconnected graph of order two. Then for any graph G, the graph $BP(G) = G \vee N_2$ is called the bipyramid based on G. The graph $C_n \vee N_2$ is called the bipyramid based on C_n and is denoted by $BP(n)$.*

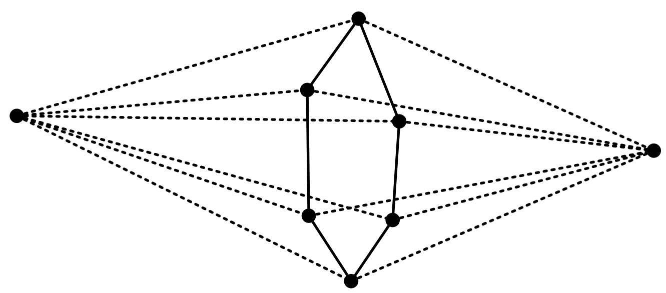

Figure 4.2: The Bipyramid $BP(6)$

Theorem 4.2.33. *$BP(n) \in \Omega_a$ for $n \equiv 1 (\text{mod } 2)$.*

Proof. Consider $BP(n)$ with vertex set $V = \{u_1, u_2, u_3, \ldots, u_n, v_1, v_2\}$ where $u_i's$ are vertices on C_n. We define $f : V \to V_4 \backslash \{0\}$ as: $f(v_1) = b, f(v_2) = c$ and $f(u_i) = a$ for $1 \leq i \leq n$. Then f gives an a-neighbourhood V_4-magic labeling of $BP(n)$. This completes the proof of the theorem. □

Theorem 4.2.34. *$BP(n) \in \Omega_a$ for $n \equiv 2 (\text{mod } 4)$.*

Proof. Consider $BP(n)$ with vertex set $V = \{u_1, u_2, u_3, \dots, u_n, v_1, v_2\}$ where $u_i's$ are vertices on C_n. We define $f : V \to V_4 \backslash \{0\}$ as:

$$f(v_1) = b \ , \ f(v_2) = c \ \text{ and}$$

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 3 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 2 (\text{mod } 4) \end{cases}$$

Then, f is an a-neighbourhood V_4-magic labeling of $BP(n)$. □

Theorem 4.2.35. *$BP(n) \in \Omega_0$ for $n \equiv 0(\text{mod } 2)$.*

Proof. Proof is obvious if we label all the vertices by a. □

Corollary 4.2.36. *$BP(n) \in \Omega_{a,0}$ for $n \equiv 2(\text{mod } 4)$.*

Proof. Proof follows from Theorem 4.2.34 and Theorem 4.2.35. □

Theorem 4.2.37. *The complete graph $K_n \in \Omega_a$ if and only if $n \equiv 0(\text{mod } 2)$.*

Proof. If $n \equiv 0(\text{mod } 2)$, then degree of each vertex in K_n is odd. By labeling all the vertices by a, we get $K_n \in \Omega_a$. If $n \not\equiv 0(\text{mod } 2)$, then $n \equiv 1(\text{mod } 2)$. If possible $K_n \in \Omega_a$, then $N_f^+(u) = a$ for all $u \in V(K_n)$. Then $\sum_{u \in V} N_f^+(u) = na$, implies that $0 = na$, implying that $a = 0$, a contradiction. Therefore, $K_n \notin \Omega_a$. Hence the theorem is proved. □

Theorem 4.2.38. *$K_n \in \Omega_0$ for $n \equiv 1(\text{mod } 2)$.*

Proof. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(K_n)$. □

Theorem 4.2.39. *Let G be a $k-$regular graph. Then G admits a-neighbourhood V_4-magic labeling for $k \equiv 1 (\text{mod } 2)$.*

Proof. Proof is obvious if we label all the vertices by a. □

Theorem 4.2.40. *Let G be a $k-$regular graph. Then G admits 0-neighbourhood V_4-magic labeling for $k \equiv 0 (\text{mod } 2)$.*

Proof. Proof is obvious if we label all the vertices by a. □

Theorem 4.2.41. $G_n \in \Omega_0$ *for* $n \equiv 0 (\text{mod } 2)$.

Proof. Consider the gear graph with vertex set $V(G_n) = \{u, u_i : 1 \leq i \leq 2n\}$ and edge set $E(G_n) = \{uu_{2i-1} : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_{2n} u_1\}$. Define $f : V(G_n) \to V_4 \backslash \{0\}$ as:

$$f(u) = a \quad \text{and}$$

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 0 (\text{mod } 4) \\ a & \text{if} \quad i \equiv 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2 (\text{mod } 4) \\ a & \text{if} \quad i \equiv 3 (\text{mod } 4) \end{cases}$$

Then, f gives a 0-neighbourhood V_4-magic labeling of G_n. This completes the proof of the theorem. □

Theorem 4.2.42. $G_n \in \Omega_a$ *for* $n \equiv 2 (\text{mod } 4)$.

Proof. Consider the gear graph with vertex set $V(G_n) = \{u, u_i : 1 \leq i \leq 2n\}$ and edge set $E(G_n) = \{uu_{2i-1} : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_{2n} u_1\}$.

Define $f : V(G_n) \to V_4\backslash\{0\}$ as:

$$f(u) = a \quad \text{and}$$

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0 (\bmod\ 4) \\ b & \text{if} \quad i \equiv 1 (\bmod\ 4) \\ a & \text{if} \quad i \equiv 2 (\bmod\ 4) \\ c & \text{if} \quad i \equiv 3 (\bmod\ 4) \end{cases}$$

Then f gives an a-neighbourhood V_4-magic labeling of G_n. This completes the proof of the theorem. □

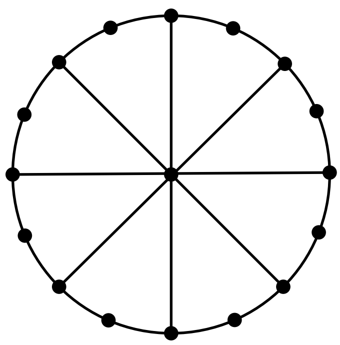

Figure 4.3: The gear graph G_8

Corollary 4.2.43. *$G_n \in \Omega_{a,0}$ for $n \equiv 2(\bmod\ 4)$.*

Proof. Proof follows from Theorem 4.2.41 and Theorem 4.2.42. □

Definition 4.2.44. [12] *Consider a cycle C_n on n vertices, call it the prime cycle and attach n cycles, each of length m, called the auxiliary cycles, at each* *vertex of the prime cycle. This new graph is called Carona on cycle, denoted by $C_m(C_n)$ (read as C_m on C_n).*

Theorem 4.2.45. *$C_m(C_n) \in \Omega_a$ for all $m \geq 3$ and $n \geq 3$.*

Proof. If we label all the vertices by a, we get $C_m(C_n) \in \Omega_a$. □

Theorem 4.2.46. *If $C_m(C_n)$admits a-neighbourhood V_4-magic labeling. Then either $n \equiv 0(\text{mod } 2)$ or $m \equiv 0(\text{mod } 2)$ or both.*

Proof. Assume that $C_m(C_n)$ admits a-neighbourhood V_4-magic labeling. Since $|V(C_m(C_n))| = nm$, we should have $nm.a = 0$ implies that $nm \equiv 0(\text{mod } 2)$. Hence $n \equiv 0(\text{mod } 2)$ or $m \equiv 0(\text{mod } 2)$ or both. □

Theorem 4.2.47. *$C_m(C_n)$admits a-neighbourhood V_4-magic labeling for $m \equiv 1(\text{mod } 4)$ and $n \equiv 0(\text{mod } 4)$.*

Proof. Consider $C_m(C_n)$ with vertex set $V = \{u_{ij} : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ where $\{u_{11}, u_{21}, u_{31}, \ldots, u_{n1}\}$ be the vertex set of the prime cycle C_n and $\{u_{k1}, u_{k2}, u_{k3}, \ldots, u_{km}\}$ be the vertex set of k^{th} copy of the auxiliary cycle C_m in order. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad i \equiv 1,2(\text{mod } 4) \text{ and } j \equiv 1,2(\text{mod } 4) \\ c & \text{if} \quad i \equiv 1,2(\text{mod } 4) \text{ and } j \equiv 0,3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0,3(\text{mod } 4) \text{ and } j \equiv 1,2(\text{mod } 4) \\ b & \text{if} \quad i \equiv 0,3(\text{mod } 4) \text{ and } j \equiv 0,3(\text{mod } 4) \end{cases}$$

Then, $N_f^+(u_{ij}) = a$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence the theorem. □

Theorem 4.2.48. *$C_m(C_n)$ admits a-neighbourhood V_4-magic labeling for $m \equiv 3 (\bmod\ 4)$ and $n \equiv 0 (\bmod\ 4)$.*

Proof. Consider $C_m(C_n)$ with vertex set $V = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$, where $\{u_{11}, u_{21}, u_{31}, \ldots, u_{n1}\}$ be the vertex set of the prime cycle C_n and $\{u_{k1}, u_{k2}, u_{k3}, \ldots, u_{km}\}$ be the vertex set of k^{th} copy of the auxiliary cycle C_m in order. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\bmod\ 4) \text{ and } j \equiv 0, 1 (\bmod\ 4) \\ c & \text{if} \quad i \equiv 1, 2 (\bmod\ 4) \text{ and } j \equiv 2, 3 (\bmod\ 4) \\ c & \text{if} \quad i \equiv 0, 3 (\bmod\ 4) \text{ and } j \equiv 0, 1 (\bmod\ 4) \\ b & \text{if} \quad i \equiv 0, 3 (\bmod\ 4) \text{ and } j \equiv 2, 3 (\bmod\ 4) \end{cases}$$

Clearly, $N_f^+(u_{ij}) = a$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence $C_m(C_n) \in \Omega_a$. □

Chapter 5

Neighbourhood V_4-magic Labeling of Splitting, Shadow and Middle of Graphs

This chapter studies neighbourhood V_4-magic labeling of splitting, shadow and middle of some graphs. The first section gives an introduction to the above said graphs. The Second section investigates neighbourhood V_4-magic labeling of splitting graph of some graphs. Third section studies neighbourhood V_4-magic labeling in shadow graph of some graphs and the final section discusses neighbourhood V_4-magic labeling of some middle graphs.

5.1 Introduction

The Splitting graph $S(G)$ of a connected graph G is obtained by adding to each vertex u in G, a new vertex u' such that u' is adjacent to the neighbours of u

[1]The second section of this chapter has been published in *International Journal of Research in Advent Technology,* Volume 7, Number 1, January 2019, Pages 530-535.

[2]The third section of this chapter has been published in *International Journal of Mathematical Combinatorics,* Volume 2, 2019, Pages 86-98.

in G [21]. The shadow graph $Sh(G)$ of a connected graph G is constructed by taking two copies of G say G_1 and G_2, join each vertex u in G_1; to the neighbours of the corresponding vertex v in G_2. The middle graph of a graph G, denoted by $M(G)$, is the graph obtained from G by inserting a new vertex into every edge of G and by joining those pairs of these new vertices with edges which lie on adjacent edges of G [1].

5.2 Splitting graphs

Theorem 5.2.1. *The graph* $S(C_n) \in \Omega_a$ *if and only if* $n \equiv 0(\text{mod } 4)$.

Proof. Consider the splitting graph $S(C_n)$, let $u_1, u_2, u_3, \dots, u_n$ be the vertices of C_n and let $u'_1, u'_2, u'_3, \dots, u'_n$ be the new vertices in $S(C_n)$. Assume that $n \not\equiv 0(\text{mod } 4)$. Then either $n \equiv 1(\text{mod } 4)$ or $n \equiv 2(\text{mod } 4)$ or $n \equiv 3(\text{mod } 4)$. We show that in each of these cases $S(C_n) \notin \Omega_a$.

Case 1: $n \equiv 1(\text{mod } 4)$

In this case $n = 4k + 1$ for some $k \in \mathbb{N}$. Then $V(S(C_n)) = \{u_i, u'_i : 1 \leq i \leq 4k + 1\}$. If possible let $S(C_n) \in \Omega_a$ with a labeling f. Then $N^+_f(u'_2) = a$ implies that $f(u_1) + f(u_3) = a$, which implies that either $f(u_1) = b$ or $f(u_1) = c$. Without loss of generality assume that $f(u_1) = b$. Then $f(u_3) = c, f(u_5) = b, f(u_7) = c, f(u_9) = b, f(u_{11}) = c, f(u_{13}) = b$. Proceeding like this, we get $f(u_{4k+1}) = b$. Also $N^+_f(u'_1) = a$ and $f(u_{4k+1}) = b$ implies that $f(u_2) = c, f(u_4) = b, f(u_6) = c, f(u_8) = b, f(u_{10}) = c, f(u_{12}) = b$. Proceeding like this we get $f(u_{4k}) = b$. Therefore, $N^+_f(u'_{4k+1}) = b + b = 0$, a contradiction. Thus if $n \equiv 1 (\text{mod } 4)$, we have $S(C_n) \notin \Omega_a$.

Case 2: $n \equiv 2 (\text{mod } 4)$

In this case $n = 4k + 2$ for some $k \in \mathbb{N}$. Then $V(S(C_n)) = \{u_i, u'_i : 1 \leq i \leq 4k + 2\}$. If possible let $S(C_n) \in \Omega_a$ with a labeling f. Then $N_f^+(u'_2) = a$ implies that $f(u_1) + f(u_3) = a$, which implies that either $f(u_1) = b$ or $f(u_1) = c$. Without loss of generality assume that $f(u_1) = b$. Then $f(u_3) = c, f(u_5) = b, f(u_7) = c, f(u_9) = b, f(u_{11}) = c, f(u_{13}) = b$. Proceeding like this, we get $f(u_{4k+1}) = b$. Now $N_f^+(u'_{4k+2}) = f(u_1) + f(u_{4k+1}) = b + b = 0$, a contradiction. Thus if $n \equiv 2 (\text{mod } 4)$, we have $S(C_n) \notin \Omega_a$.

Case 3: $n \equiv 3 (\text{mod } 4)$

In this case $n = 4k + 3$ for some $k \in \mathbb{N}$. Then $V(S(C_n)) = \{u_i, u'_i : 1 \leq i \leq 4k + 3\}$. If possible let $S(C_n) \in \Omega_a$ with a labeling f. Then $N_f^+(u'_2) = a$ implies that $f(u_1) + f(u_3) = a$, which implies that either $f(u_1) = b$ or $f(u_1) = c$. Without loss of generality assume that $f(u_1) = b$. Then $f(u_3) = c, f(u_5) = b, f(u_7) = c, f(u_9) = b, f(u_{11}) = c, f(u_{13}) = b$. Proceeding like this, we get $f(u_{4k+3}) = c$. Now $N_f^+(u'_1) = a$ implies that $f(u_2) = b, f(u_4) = c, f(u_6) = b, f(u_8) = c, f(u_{10}) = b, f(u_{12}) = c$. Proceeding like this, we get $f(u_{4k+2}) = b$. Therefore $N_f^+(u'_{4k+3}) = f(u_1) + f(u_{4k+2}) = b + b = 0$, a contradiction. Thus if $n \equiv 3 (\text{mod } 4)$, we also have $S(C_n) \notin \Omega_a$.

Hence if $n \not\equiv 0 (\text{mod } 4)$, $S(C_n) \notin \Omega_a$. Conversely assume that $n \equiv 0 (\text{mod } 4)$.

Define $f : V(S(C_n)) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases} \qquad f(u_i') = a \quad \text{for} \quad 1 \leq i \leq n.$$

Then f is an a-neighbourhood V_4- magic labeling for $S(C_n)$. This completes the proof of the theorem. □

Theorem 5.2.2. $S(C_n) \in \Omega_0$ *for all* $n \geq 3$.

Proof. The degree of each vertex in $S(C_n)$ is either 2 or 4. By labeling all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(S(C_n))$. □

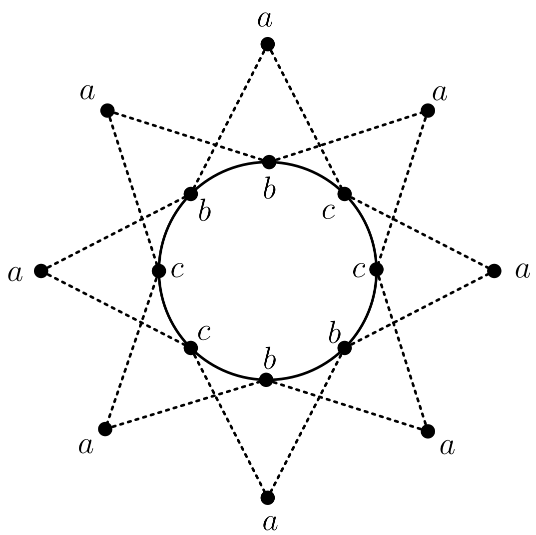

Figure 5.1: An a-neighbourhood V_4-magic labeling of $S(C_8)$.

Corollary 5.2.3. $S(C_n) \in \Omega_{a,0}$ *if and only if* $n \equiv 0(\text{mod } 4)$.

Proof. It obviously follows from Theorem 5.2.1 and Theorem 5.2.2. □

Theorem 5.2.4. *The graph* $S(P_n) \notin \Omega_0$ *for any* $n \geq 2$.

Proof. Proof is obvious due to the presence of pendant vertices in $S(P_n)$. □

Theorem 5.2.5. $S(P_n) \notin \Omega_a$ *for any* $n \geq 2$.

Proof. Consider the splitting graph $S(P_n)$, let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of P_n and let $u'_1, u'_2, u'_3, \ldots, u'_n$ be the new vertices in $S(P_n)$. Suppose that $S(P_n) \in \Omega_a$ for some $n \geq 2$ with a labeling f. Then $N_f^+(u'_1) = a$ implies that $f(u_2) = a$. Also $N_f^+(u'_1) = a$ gives $f(u_2) + f(u'_2) = a$, which implies that $f(u'_2) = 0$, a contradiction. This completes the proof of the theorem. □

Corollary 5.2.6. $S(P_n) \notin \Omega_{a,0}$ *for* $n \geq 2$.

Proof. Proof directly follows from Theorems 5.2.4 and 5.2.5. □

Theorem 5.2.7. $S(B_{m,n}) \notin \Omega_a$ *for any* $m > 1$ *and* $n > 1$.

Proof. Consider the bistar $B_{m,n}$ with vertex set $V = \{u, v, u_i, v_j : 1 \leq i \leq m\ ,\ 1 \leq j \leq n\}$ where $u_i (1 \leq i \leq m)$ and $v_j (1 \leq j \leq n)$ are pendant vertices adjacent to u and v respectively. Let $V' = \{u', v', u'_i, v'_j : 1 \leq i \leq m\ ,\ 1 \leq j \leq n\}$ be the corresponding set of new vertices in $S(B_{m,n})$. Then $V(S(B_{m,n})) = V \cup V'$. Suppose that $S(B_{m,n}) \in \Omega_a$ for some $m > 1$ and $n > 1$ with a labeling f. Then $N_f^+(v'_1) = a$ implies that $f(v) = a$. Now $N_f^+(v_1) = a$ gives $f(v) + f(v') = a$, which implies that $f(v') = 0$, a contradiction. This completes the proof of the theorem. □

Theorem 5.2.8. $S(B_{m,n}) \notin \Omega_0$ *for any* $m > 1$ *and* $n > 1$.

Proof. Proof is obvious, since $S(B_{m,n})$ has pendant vertices. □

Corollary 5.2.9. *$S(B_{m,n}) \notin \Omega_{a,0}$ for any $m > 1$ and $n > 1$.*

Proof. Proof directly follows from Theorem 5.2.7. □

Theorem 5.2.10. *$S(K_{1,n}) \notin \Omega_a$ for any $n \in \mathbb{N}$.*

Proof. Consider $K_{1,n}$ with vertex set $V = \{u, u_i : 1 \leq i \leq n\}$ and let $V' = \{u', u'_i : 1 \leq i \leq n\}$ be the corresponding set of new vertices in $S(K_{1,n})$. Assume that $S(K_{1,n}) \in \Omega_a$ for some $n \in \mathbb{N}$ with a labeling f. Now $N_f^+(u'_1) = a$ gives $f(u) = a$. Also $N_f^+(u_1) = a$ implies that $f(u) + f(u') = a$, which implies that $f(u') = 0$, a contradiction. Hence the theorem is proved. □

Theorem 5.2.11. *$S(K_{1,n}) \notin \Omega_0$ for any $n \in \mathbb{N}$.*

Proof. Proof is obvious due to the presence of pendant vertices in $S(K_{1,n})$. □

Corollary 5.2.12. *$S(K_{1,n}) \notin \Omega_{a,0}$ for any $n \in \mathbb{N}$.*

Proof. It follows from Theorem 5.2.10. □

Theorem 5.2.13. *$S(K_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof. Consider $K_{m,n}$ with $m > 1$ and $n > 1$. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. Also let $X' = \{u'_1, u'_2, u'_3, \ldots, u'_m\}$ and $Y' = \{v'_1, v'_2, v'_3, \ldots, v'_n\}$ be the corresponding sets of new vertices in $S(K_{m,n})$. Then $V(S(K_{m,n})) = X \cup Y \cup X' \cup Y'$. We consider the following cases:

Case 1: Both m and n are even.

Define $f : V(S(K_{m,n})) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$f(u'_i) = a$ for $1 \leq i \leq m$ $\qquad$ $f(v'_j) = a$ for $1 \leq j \leq n$.

Case 2: m is even and n is odd.

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v'_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$f(u'_i) = a$ for $1 \leq i \leq m$ $\qquad$ $f(v_j) = a$ for $1 \leq j \leq n$.

Case 3: m is odd and n is even.

$$f(u'_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$f(u_i) = a$ for $1 \leq i \leq m$ $\qquad$ $f(v'_j) = a$ for $1 \leq j \leq n$.

Case 4: Both m and n are odd.

$$f(u_i') = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(v_j') = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$f(u_i) = a$ for $1 \leq i \leq m$ $\qquad$ $f(v_j) = a$ for $1 \leq j \leq n$.

In each of the above cases, f gives a-neighbourhood V_4-magic labeling of $S(K_{m,n})$. This completes the proof of the theorem. □

Theorem 5.2.14. *$S(K_{m,n}) \in \Omega_0$ for all $m > 1$ and $n > 1$.*

Proof. Consider $K_{m,n}$ with $m > 1$ and $n > 1$. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. Also let $X' = \{u_1', u_2', u_3', \ldots, u_m'\}$ and $Y' = \{v_1', v_2', v_3', \ldots, v_n'\}$ be the corresponding sets of new vertices in $S(K_{m,n})$. Then $V(S(K_{m,n})) = X \cup Y \cup X' \cup Y'$. We consider the following cases:

Case 1: Both m and n are even.

Define $f : V(S(K_{m,n})) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(u_i') = a \quad \text{if} \quad i = 1, 2, 3, \ldots, m$$

$$f(v_j) = f(v_j') = a \quad \text{if} \quad j = 1, 2, 3, \ldots, n$$

Case 2: m is even and n is odd.

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases} \qquad f(v'_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$$f(u_i) = a \text{ for } 1 \leq i \leq m \qquad f(u'_i) = a \text{ for } 1 \leq i \leq m.$$

Case 3: m is odd and n is even.

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(u'_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases}$$

$$f(v_j) = a \text{ for } 1 \leq j \leq n \qquad f(v'_j) = a \text{ for } 1 \leq j \leq n.$$

Case 4: Both m and n are odd.

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad f(u'_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases}$$

$$f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases} \qquad f(v'_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

In each of the above cases, f gives 0-neighbourhood V_4-magic labeling of $S(K_{m,n})$.

This completes the proof of the theorem. □

Corollary 5.2.15. *$S(K_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof. Proof directly follows from Theorems 5.2.13 and 5.2.14. □

Theorem 5.2.16. *$S(F_m) \in \Omega_0$ for all $m \in \mathbb{N}$.*

Proof. If we label all the vertices of $S(F_m)$ by a, we get $S(F_m) \in \Omega_0$. □

Theorem 5.2.17. *$S(F_m) \notin \Omega_a$ for any $m \in \mathbb{N}$.*

Proof. Consider the friendship graph F_m. Let the vertices of i^{th} copy of C_3 in F_m be w, u_i and v_i where w is the common vertex of the triangles and let $\{w', u'_i, v'_i : 1 \leq i \leq m\}$ be the corresponding set of vertices in $S(F_m)$. Assume that $S(F_m) \in \Omega_a$ for some $m \in \mathbb{N}$ with a labeling f. Since $N_f^+(u'_1) = a$, either $f(w) = b$ or $f(w) = c$. Without loss of generality assume that $f(w) = b$. If $f(w) = b$, $f(u_i) = f(v_i) = c$ for all $1 \leq i \leq m$. Therefore, $N_f^+(w') = 2mc = 0$, a contradiction. Hence $S(F_m) \notin \Omega_a$ for all $m \in \mathbb{N}$. □

Corollary 5.2.18. *$S(F_m) \notin \Omega_{a,0}$ for any $m \in \mathbb{N}$.*

Proof. Proof directly follows from Theorem 5.2.16. □

Theorem 5.2.19. *$S(QS_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof. By labeling all the vertices of $S(QS_n)$ by a, we get $S(QS_n) \in \Omega_0$. □

Theorem 5.2.20. *$S(QS_n) \notin \Omega_a$ for any $n > 2$.*

Proof. Let QS_n be the quadrilateral snake obtained from the path $v_1v_2v_3 \ldots v_n$ by joining each pair v_i, v_{i+1} to the new vertices u_i, w_i respectively and then joining

u_i and w_i by an edge. Now consider $S(QS_n)$. Let v'_i, u'_i, w'_i be the new vertices corresponding to v_i, u_i, w_i. Suppose $S(QS_n) \in \Omega_a$ for some $n > 2$ with a labeling f. Then, $N_f^+(u'_1) = a$ gives $f(v_1) + f(w_1) = a$. Also $N_f^+(w'_2) = a$ implies that $f(u_2) + f(v_3) = a$, Therefore, $N_f^+(v'_2) = f(v_1) + f(w_1) + f(u_2) + f(v_3) = 0$, a contradiction. Hence, $S(QS_n) \notin \Omega_a$ for all $n > 2$. □

Corollary 5.2.21. *$S(QS_n) \notin \Omega_{a,0}$ for all $n > 2$.*

Proof. Proof directly follows from Theorem 5.2.20. □

Theorem 5.2.22. *$S(B_n) \in \Omega_a$ if and only if n is odd.*

Proof. Consider B_n with vertex set $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ and edge set $\{uv, uu_i, vv_i, u_iv_i : 1 \leq i \leq n\}$. Let $\{u', v', u'_i, v'_i : 1 \leq i \leq n\}$ be the set of new vertices in $S(B_n)$. Assume that $S(B_n) \in \Omega_a$ for some $n \in \mathbb{N}$ with a labeling f. Since $N_f^+(u'_1) = a$, we have $f(u) = b$ or $f(u) = c$. Without loss of generality we assume that $f(u) = b$. Then $f(v_i) = c$ for all $i = 1, 2, 3, \ldots, n$. Now $N_f^+(v') = a$ implies that $f(u) + \sum_{i=1}^{n} f(v_i) = b + nc = a$. Hence n is odd. Conversely, assume that n is odd. Define a label $f : V(S(B_n)) \to V_4 \backslash \{0\}$ as:

$$f(u) = f(u_i) = b \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$

$$f(v) = f(v_i) = c \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$

$$f(u') = fu'_i) = a \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$

$$f(v) = f(v'_i) = a \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$

Then, f is an a-neighbourhood V_4-magic labeling of $S(B_n)$. This completes the proof of the theorem. □

Theorem 5.2.23. *$S(B_n) \in \Omega_0$ if and only if n is odd.*

Proof. Consider B_n with vertex set $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ and edge set $\{uv, uu_i, vv_i, u_iv_i : 1 \leq i \leq n\}$. Let $\{u', v', u'_i, v'_i : 1 \leq i \leq n\}$ be the set of new vertices in $S(B_n)$. Assume that $S(B_n) \in \Omega_0$ for some $n \in \mathbb{N}$ with a labeling f. Since $N_f^+(u'_1) = 0$, we should have $f(u) = f(v_1) = a$ or $f(u) = f(v_1) = b$ or $f(u) = f(v_1) = c$. Without loss of generality we assume that $f(u) = f(v_1) = a$. Then $f(v_i) = a$ for all $i = 1, 2, 3, \ldots, n$. Now $N_f^+(v') = 0$ implies that $f(u) + \sum_{i=1}^{n} f(v_i) = a + na = 0$. Hence n is odd. Conversely, assume that n is odd. We define $f : V(S(B_n)) \to V_4\backslash\{0\}$ as: $f(w) = a$ for all $w \in V(S(B_n))$. Then, f is a 0-neighbourhood V_4-magic labeling of $S(B_n)$. □

Corollary 5.2.24. *$S(B_n) \in \Omega_{a,0}$ if and only if n is odd.*

Proof. It directly follows from Theorems 5.2.22 and 5.2.23. □

5.3 Shadow graphs

Theorem 5.3.1. *The graph $Sh(C_n) \in \Omega_a$ if and only if $n \equiv 0 (\text{mod } 4)$.*

Proof. Consider the shadow graph $Sh(C_n)$, let$\{u_1, u_2, u_3, \ldots, u_n\}$ be the vertex set of first copy of C_n and let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the corresponding vertex set of second copy of C_n in order. Assume that $n \not\equiv 0 (\text{mod } 4)$. Then either $n \equiv 1 (\text{mod } 4)$ or $n \equiv 2 (\text{mod } 4)$ or $n \equiv 3 (\text{mod } 4)$. We show that in each of these cases $Sh(C_n) \notin \Omega_a$.

Case 1: $n \equiv 1(\text{mod } 4)$

In this case $n = 4k+1$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k+1\}$. If possible let $Sh(C_n) \in \Omega_a$ with a labeling f. Then $N_f^+(u_2) = a$ implies that $f(u_1) + f(v_1) + f(u_3) + f(v_3) = a$, $N_f^+(u_4) = a$ implies that $f(u_3) + f(v_3) + f(u_5) + f(v_5) = a$, proceeding like this, $N_f^+(u_{4k}) = a$ implies that $f(u_{4k-1}) + f(v_{4k-1}) + f(u_{4k+1}) + f(v_{4k+1}) = a$. Now consider $f(u_1) + f(v_1)$, then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 1: $f(u_1) + f(v_1) = 0$

If $f(u_1)+f(v_1) = 0$, then $f(u_3)+f(v_3) = a, f(u_5)+f(v_5) = 0, f(u_7)+f(v_7) = a$, implies that $f(u_{4k+1}) + f(v_{4k+1}) = 0$. Now $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = a, f(u_4) + f(v_4) = 0, f(u_6) + f(v_6) = a$, proceeding like this we get $f(u_{4k})+f(v_{4k}) = 0$. Therefore $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = 0 + 0 = 0$, a contradiction.

Subcase 2: $f(u_1) + f(v_1) = a$

If $f(u_1)+f(v_1) = a$, then proceeding as in Subcase 1, we get $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = a + a = 0$, a contradiction.

Subcase 3: $f(u_1) + f(v_1) = b$

If $f(u_1)+f(v_1) = b$, then $f(u_3)+f(v_3) = c, f(u_5)+f(v_5) = b, f(u_7)+f(v_7) = c$, implies that$f(u_{4k+1})+f(v_{4k+1}) = b$. Now $N_f^+(u_1) = a$ gives $f(u_2) + f(v_2) = c, f(u_4) + f(v_4) = b, f(u_{4k}) + f(v_{4k}) = b$. Therefore $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = b + b = 0$, which is a contradiction.

Subcase 4: $f(u_1) + f(v_1) = c$

If $f(u_1)+f(v_1) = c$, then proceeding as in Subcase 3, we get $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = c + c = 0$, a contradiction.

Thus if $n \equiv 1 (\text{mod } 4)$, we have $Sh(C_n) \notin \Omega_a$.

Case 2: $n \equiv 2 (\text{mod } 4)$

In this case $n = 4k + 2$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k + 2\}$. If possible let $Sh(C_n) \in \Omega_a$ with a labeling f. Consider $f(u_1) + f(v_1)$, then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 1: $f(u_1) + f(v_1) = 0$

If $f(u_1) + f(v_1) = 0$, then $N_f^+(u_2) = a, f(u_3) + f(v_3) = a, f(u_5) + f(v_5) = 0$, which implies that $f(u_{4k+1}) + f(v_{4k+1}) = 0$ Therefore, $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = 0 + 0 = 0$, a contradiction.

Subcase 2: $f(u_1) + f(v_1) = a$

If $f(u_1)+f(v_1) = a$, then proceeding as in Subcase 1, we get $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = a + a = 0$, which is a contradiction.

Subcase 3: $f(u_1) + f(v_1) = b$

If $f(u_1) + f(v_1) = b$, then $N_f^+(u_2) = a$ implies that $f(u_3) + f(v_3) = c, f(u_5) + f(v_5) = b$, implies that $f(u_{4k+1}) + f(v_{4k+1}) = b$. Therefore, $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = b + b = 0$, which is a contradiction.

Subcase 4: $f(u_1) + f(v_1) = c$

If $f(u_1)+f(v_1) = c$, then proceeding as in Subcase 3, we get $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = c + c = 0$, a contradiction.

Thus if $n \equiv 2(\text{mod } 4)$, $Sh(C_n) \notin \Omega_a$.

Case 3: $n \equiv 3(\text{mod } 4)$

In this case $n = 4k + 3$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k + 3\}$. If possible let $Sh(C_n) \in \Omega_a$ with a labeling f. Consider $f(u_1) + f(v_1)$, then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 1: $f(u_1) + f(v_1) = 0$

If $f(u_1) + f(v_1) = 0$, then $N_f^+(u_2) = a$ gives $f(u_3) + f(v_3) = a$, $f(u_5) + f(v_5) = 0$, $f(u_{4k+1}) + f(v_{4k+1}) = 0$, $f(u_{4k+3}) + f(v_{4k+3}) = a$. Now, $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = 0$, $f(u_4) + f(v_4) = a$, $f(u_{4k+2}) + f(v_{4k+2}) = 0$. Therefore $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = 0 + 0 = 0$, which is a contradiction.

Subcase 2: $f(u_1) + f(v_1) = a$

If $f(u_1)+f(v_1) = a$, then proceeding as in Subcase 1, we get $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = a + a = 0$, a contradiction.

Subcase 3: $f(u_1) + f(v_1) = b$

If $f(u_1) + f(v_1) = b$, then $N_f^+(u_2) = a$ implies that $f(u_3) + f(v_3) = c, f(u_5) + f(v_5) = b, f(u_{4k+1}) + f(v_{4k+1}) = b, f(u_{4k+3}) + f(v_{4k+3}) = c$. Now, $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = b, f(u_4) + f(v_4) =$

$c, f(u_{4k+2}) + f(v_{4k+2}) = b$. Therefore, $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = b + b = 0$, which is a contradiction.

Subcase 4: $f(u_1) + f(v_1) = c$

If $f(u_1)+f(v_1) = c$, then proceeding as in Subcase 3, we get $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = c + c = 0$, a contradiction.

Thus if $n \equiv 3(\text{mod } 4)$, we also have $Sh(C_n) \notin \Omega_a$.

Therefore, $n \not\equiv 0(\text{mod } 4)$ implies that $Sh(C_n) \notin \Omega_a$. Conversely, if $n \equiv 0(\text{mod } 4)$,

We define $f : V(Sh(C_n)) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3(\text{mod } 4) \end{cases} \qquad f(v_i) = a \quad \text{for} \quad 1 \le i \le n.$$

Then, f is an a-neighbourhood V_4-magic labeling for $Sh(C_n)$. This completes the proof of the theorem. □

Theorem 5.3.2. *$Sh(C_n) \in \Omega_0$ for all $n \geq 3$.*

Proof. The degree of each vertex in $Sh(C_n)$is 4. By labeling all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(C_n))$. □

Corollary 5.3.3. *$Sh(C_n) \in \Omega_{a,0}$ if and only if $n \equiv 0(\text{mod } 4)$.*

Proof. Proof is obviously follows from Theorem 5.3.1 and Theorem 5.3.2. □

Theorem 5.3.4. *The graph $Sh(P_n) \in \Omega_0$ for all $n \geq 2$.*

Proof. If we label all the vertices by a, we get $G \in \Omega_0$. □

Theorem 5.3.5. *$Sh(P_n) \in \Omega_a$ for $n \equiv 0, 2, 3 (\text{mod } 4)$.*

Proof. Let G be the shadow graph $Sh(P_n)$, and let $\{u_i : 1 \leq i \leq n\}and\{v_i : 1 \leq i \leq n\}$ be the vertex sets of first and second copy of P_n respectively.

Case 1: $n \equiv 0(\text{mod } 4)$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1(\text{mod } 4) \\ b & \text{if} \quad i \equiv 2, 3(\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1(\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3(\text{mod } 4) \end{cases}$$

Case 2: $n \equiv 2(mod\ 4)$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 3(\text{mod } 4) \\ b & \text{if} \quad i \equiv 1, 2(\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 1, 2(\text{mod } 4) \end{cases}$$

Case 3: $n \equiv 3(\text{mod } 4)$

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ a & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ a & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

In all the above cases, we have $N_f^+(u_i) = N_f^+(v_i) = a$ for $1 \leq i \leq n$. Therefore, $Sh(P_n) \in \Omega_a$ for $n \equiv 0, 2, 3(\text{mod } 4)$. □

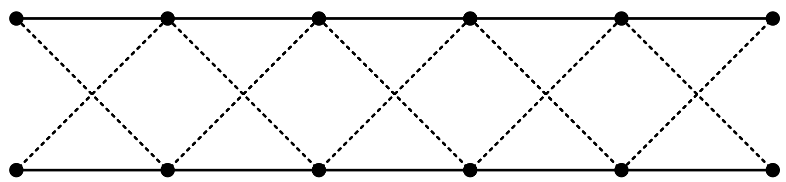

Figure 5.2: The shadow graph $Sh(P_6)$

Theorem 5.3.6. *$Sh(P_n) \notin \Omega_a$ for $n \equiv 1(\text{mod } 4)$.*

Proof. Consider the shadow graph $Sh(P_n)$ with $n \equiv 1(\text{ mod } 4)$. Let $\{u_i : 1 \leq i \leq 4k+1\}$ and$\{v_i : 1 \leq i \leq 4k+1\}$ be the vertex sets of first and second copy of P_n respectively. Assume that $Sh(P_n) \in \Omega_a$ with a labeling f. Since $N_f^+(u_1) = a$, we have either $f(u_2) = b$ and $f(v_2) = c$ or $f(u_2) = c$ and $f(v_2) = b$. Without loss of generality assume that $f(u_2) = b$ and $f(v_2) = c$. Then $f(u_{4k}) = f(v_{4k})$, implies that $N_f^+(u_{4k+1}) = 0$, a contradiction. Therefore, $Sh(P_n) \notin \Omega_a$. □

Corollary 5.3.7. *$Sh(P_n) \in \Omega_{a,0}$ for $n \equiv 0, 2, 3 (\text{mod } 4)$.*

Proof. Proof directly follows from Theorems 5.3.4 and 5.3.5. □

Theorem 5.3.8. *$Sh(K_{1,n}) \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof. Let $V = \{u_i, v_i : 0 \leq i \leq n\}$ be the vertex set of $Sh(K_{1,n})$ where $\{u_i : 0 \leq i \leq n\}$ and $\{v_i : 0 \leq i \leq n\}$ are the vertex sets of first and second copy of $K_{1,n}$ with apex u_0, v_0 respectively. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0, 1 \\ a & \text{if} \quad i = 2, 3, \dots, n \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i = 0, 1 \\ a & \text{if} \quad i = 2, 3, \dots, n \end{cases}$$

Then, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $0 \leq i \leq n$. This completes the proof. □

Theorem 5.3.9. *$Sh(K_{1,n}) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof. If we label all the vertices by a, we get $Sh(K_{1,n}) \in \Omega_0$. □

Corollary 5.3.10. *$Sh(K_{1,n}) \in \Omega_{a,0}$ for all $n \in \mathbb{N}$.*

Proof. Proof obviously follows from Theorems 5.3.8 and 5.3.9. □

Theorem 5.3.11. *$Sh(B_{m,n}) \in \Omega_0$ for all m and n.*

Proof. Labeling all the vertices by a, we get $Sh(B_{m,n}) \in \Omega_0$ for all m and n. □

Theorem 5.3.12. *$Sh(B_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof. Let $V_1 = \{u, v, u_1, u_2, \ldots, u_m, v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of first copy of $B_{m,n}$ and $V_2 = \{u', v', u'_1, u'_2, \ldots, u'_m, v'_1, v'_2, v'_3, \ldots, v'_n\}$ be the corresponding vertex set of second copy of $B_{m,n}$, where u_i, v_i are pendant vertices adjacent to u, v respectively. Then $V(Sh(B_{m,n})) = V_1 \cup V_2$.

Define $f : V(Sh(B_{m,n})) \to V_4 \backslash \{0\}$ as:

$$f(u) = f(v) = b$$

$$f(u') = f(v') = c$$

$$f(u_i) = f(u'_i) = a \text{ for } 1 \leq i \leq m$$

$$f(v_i) = f(v'_i) = a \text{ for } 1 \leq i \leq n$$

Then, f is an a-neighbourhood labeling of $Sh(B_{m,n})$. Completes the proof. □

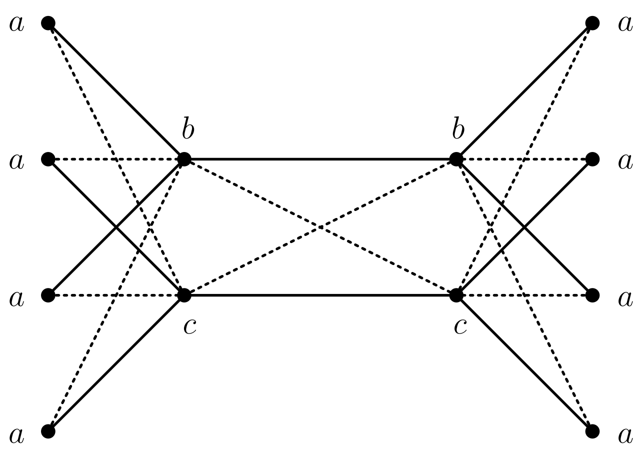

Figure 5.3: An a-neighbourhod V_4-magic labling of $Sh(B_{2,2})$

Corollary 5.3.13. *$Sh(B_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof. Proof follows from Theorems 5.3.11 and 5.3.12. □

Theorem 5.3.14. *$Sh(W_n) \in \Omega_0$ for all $n \geq 3$.*

Proof. Degree of vertex in $Sh(W_n)$ is either 6 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(W_n))$. □

Theorem 5.3.15. *$Sh(W_n) \in \Omega_a$ for all $n \equiv 1 (\mathrm{mod}\ 2)$.*

Proof. Consider $Sh(W_n)$ with $n \equiv 1(\mathrm{mod}\ 2)$. Let $V_1 = \{u_0, u_1, u_2, \ldots, u_n\}$ be the vertex set of first copy of W_n with central vertex u_0 and let $V_2 = \{v_0, v_1, v_2, \ldots, v_n\}$ be the corresponding vertex set of second copy of W_n with central vertex v_0. Then, $V = V(Sh(W_n)) = V_1 \cup V_2$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{if} \quad i = 0, 1, 2, 3, \ldots, n$$

$$f(v_i) = c \quad \text{if} \quad i = 0, 1, 2, 3, \ldots, n$$

Then, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $i = 0, 1, 2, \ldots, n$. □

Corollary 5.3.16. *$Sh(W_n) \in \Omega_{a,0}$ for all $n \equiv 1(\mathrm{mod}\ 2)$.*

Proof. Proof directly follows from Theorems 5.3.14 and 5.3.15. □

Theorem 5.3.17. *$Sh(W_n) \in \Omega_a$ for all $n \equiv 2(\mathrm{mod}\ 4)$.*

Proof. Let $V_1 = \{u_0, u_1, u_2, \ldots, u_n\}$ be the vertex set of first copy of W_n with central vertex u_0 and let $V_2 = \{v_0, v_1, v_2, \ldots, v_n\}$ be the vertex set of second copy with central vertex v_0. Then $V(Sh(W_n)) = V_1 \cup V_2$. Define $f : V(Sh(W_n)) \to$

$V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 1, 3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 2(\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 1, 3(\text{mod } 4) \\ b & \text{if} \quad i \equiv 0, 2(\text{mod } 4) \end{cases}$$

Clearly, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $i = 0, 1, 2, \ldots, n$. Hence $Sh(W_n) \in \Omega_a$. □

Corollary 5.3.18. *$Sh(W_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 4)$.*

Proof. Proof directly follows from Theorems 5.3.14 and 5.3.17. □

Theorem 5.3.19. *$Sh(H_n) \in \Omega_0$ for all $n \geq 3$.*

Proof. In $Sh(H_n)$, degree of vertices are either 2 or 8 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(H_n))$. □

Theorem 5.3.20. *$Sh(H_n)$ admits a-neighbourhood V_4-magic labeling for all $n \equiv 1(\text{mod } 2)$.*

Proof. Consider the shadow graph $Sh(H_n)$ with $n \equiv 1(\text{mod } 2)$. Let v be central vertex, $v_1, v_2, v_3, \ldots, v_n$ be the rim vertices and $u_1, u_2, u_3, \ldots, u_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$ in the first copy of H_n and let $v', v_1', v_2', v_3', \ldots, v_n', u_1', u_2', u_3', \ldots, u_n'$ be the corresponding vertices in the second copy of H_n. Then $V(Sh(H_n)) = \{v, v', v_i, v_i', u_i, u_i' : 1 \leq i \leq n\}$. We define $f : V(Sh(H_n)) \to V_4\backslash\{0\}$ as:

$$f(v) = a \quad \text{and} \quad f(v_i) = f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

$$f(v') = a \quad \text{and} \quad f(v_i') = f(u_i') = c \quad \text{for} \quad i = 1, 2, 3, \dots, n$$

Obviously, f is an a-neighbourhood V_4-magic labeling of $Sh(H_n)$. □

Corollary 5.3.21. *$Sh(H_n) \in \Omega_{a,0}$ for all $n \equiv 1(\text{mod } 2)$.*

Proof. Proof directly follows from Theorems 5.3.19 and 5.3.20. □

Theorem 5.3.22. *$Sh(SF_n)$ admits a-neighbourhood V_4-magic labeling for all $n \equiv 2(\text{mod } 4)$.*

Proof. Consider $Sh(SF_n)$, let the vertex set of first copy of SF_n be $V_1 = \{w, w_i, v_i : 1 \leq i \leq n\}$ where w is the central vertex, $w_1, w_2, w_3, \dots, w_n$ are vertices of the cycle and v_i is the vertex joined by edges to w_i and w_{i+1} where $i+1$ is taken over modulo n. Let $V_2 = \{w', w_i', v_i' : 1 \leq i \leq n\}$ be the corresponding vertex set of second copy of SF_n. Then $V(Sh(SF_n)) = V_1 \cup V_2$. Define $f : V(Sh(SF_n)) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = \begin{cases} b & \text{if} \quad i \equiv 1(\text{mod } 2) \\ c & \text{if} \quad i \equiv 0(\text{mod } 2) \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1(\text{mod } 2) \\ c & \text{if} \quad i \equiv 0(\text{mod } 2) \end{cases}$$

$f(w) = f(w') = f(w_i') = f(v_i') = a$ for $i = 1, 2, 3, \dots, n$. Then f is an a-neighbourhood V_4-magic labeling of $Sh(SF_n)$. □

Theorem 5.3.23. *$Sh(SF_n)$ admits 0-neighbourhood V_4-magic labeling for all n.*

Proof. Labeling all the vertices of $Sh(SF_n)$ by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(SF_n))$. □

Theorem 5.3.24. *$Sh(SF_n) \in \Omega_{a,0}$ for all $n \equiv 2 (\text{mod } 4)$.*

Proof. Proof directly follows from Theorems 5.3.22 and 5.3.23. □

Theorem 5.3.25. *$Sh(C_n \odot K_2) \in \Omega_a$ for all $n \equiv 0 (\text{mod } 4)$.*

Proof. Let G bet the shadow graph $Sh(C_n \odot K_2)$ with $n \equiv 0 (\text{mod } 4)$. Let $V_1 = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ be the vertex set of first copy of $C_n \odot K_2$, where $u_i's$ are vertices of C_n and v_j, w_j are the vertices of j^{th} copy of K_2 and let $V_2 = \{u_i', v_i', w_i' : 1 \leq i \leq n\}$ be the corresponding vertex set of second copy of $C_n \odot K_2$. Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$f(u_i') = f(v_i') = f(w_i') = a$ for $i = 1, 2, 3, \ldots, n$. Then f is an a-neighbourhood V_4-magic labeling of $Sh(C_n \odot K_2)$. □

Theorem 5.3.26. *$Sh(C_n \odot K_2) \in \Omega_0$ for all n.*

Proof. By labeling all the vertices of $Sh(C_n \odot K_2)$ by a, we get $N_f^+(u) = 0$. □

Corollary 5.3.27. *$Sh(C_n \odot K_2) \in \Omega_{a,0}$ for all $n \equiv 0 (\text{mod } 4)$.*

Proof. Proof directly follows from Theorem 5.3.25 and Theorem 5.3.26. □

Theorem 5.3.28. *$Sh(C_n \odot \overline{K}_m) \in \Omega_a$ for all m and $n \geq 3$.*

Proof. Let G be the shadow graph $Sh(C_n \odot \overline{K}_m)$. Let $u_1, u_2, u_3, \ldots, u_n$ be the rim vertices of first copy of $C_n \odot \overline{K}_m$ and $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{im}\}$ be the set of pendant vertices adjacent to u_i for $1 \leq i \leq n$ in $C_n \odot \overline{K}_m$ and let $u'_1, u'_2, u'_3, \ldots, u'_n$ be the rim vertices of second copy of $C_n \odot \overline{K}_m$ and $\{u'_{i1}, u'_{i2}, u'_{i3}, \ldots, u'_{im}\}$ be the set of pendant vertices adjacent to u'_i for $1 \leq i \leq n$ in second copy of $C_n \odot \overline{K}_m$. Here we consider two cases.

Case 1: $m = 1$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(u_{i,1}) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

$$f(u'_i) = f(u'_{i,1}) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

Case 2: $m \geq 2$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

$$f(u'_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

$$f(u'_{ij}) = a \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

$$f(u_{ij}) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

Obviously, f is an a-neighbourhood V_4-magic labeling of $Sh(C_n \odot \overline{K}_m)$. □

Theorem 5.3.29. *$Sh(C_n \odot \overline{K}_m) \in \Omega_0$ for all m and $n \geq 3$.*

Proof. Labeling all the vertices by a, we get $Sh(C_n \odot \overline{K}_m) \in \Omega_0$. □

Corollary 5.3.30. *$Sh(C_n \odot \overline{K}_m) \in \Omega_{a,0}$ for all m and $n \geq 3$.*

Proof. Proof directly follows from Theorems 5.3.28 and 5.3.29. □

Theorem 5.3.31. *$Sh(J(m, n)) \in \Omega_0$ for all m and n.*

Proof. Labeling all the vertices by a, we get $Sh(J(m, n)) \in \Omega_0$. □

Theorem 5.3.32. *$Sh(J(m, n)) \in \Omega_a$ for all m and n.*

Proof. Let G be the graph $Sh(J(m, n))$. Let $V_1 = \{w_i, u_j, v_k : 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n\}$ and $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\} \cup \{w_2u_j : 1 \leq j \leq m\} \cup \{w_4v_j : 1 \leq j \leq n\}$ be the vertex and edge set of first copy of $J(m, n)$. Let $V_2 = \{w'_i, u'_j, v'_k : 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n\}$ be the corresponding vertex set of second copy of $J(m, n)$. Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = b \quad \text{for} \quad i = 1, 2, 3, 4$$

$$f(w_i') = c \quad \text{for} \quad i = 1, 2, 3, 4$$

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ a & \text{if} \quad i \geq 2 \end{cases} \qquad f(u_i') = \begin{cases} c & \text{if} \quad i = 1 \\ a & \text{if} \quad i \geq 2 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i = 1 \\ a & \text{if} \quad i \geq 2 \end{cases} \qquad f(v_i') = \begin{cases} c & \text{if} \quad i = 1 \\ a & \text{if} \quad i \geq 2 \end{cases}$$

Then, f is an a-neighbourhood V_4-magic labeling of $Sh(J(m,n))$. □

Corollary 5.3.33. *$Sh(J(m,n)) \in \Omega_{a,0}$ for all m and n.*

Proof. Proof directly follows from Theorems 5.3.31 and 5.3.32. □

Theorem 5.3.34. *$Sh(L_n) \in \Omega_0$ for all n.*

Proof. By labeling all the vertices by a, we get $Sh(L_n) \in \Omega_0$ for all n. □

Theorem 5.3.35. *$Sh(L_n) \in \Omega_a$ for all $n \equiv 2 (\text{mod } 3)$.*

Proof. Consider $Sh(L_n)$ with $n \equiv 2(mod\ 3)$. Let $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of first copy of L_n with edge set $E_1 = \{u_iu_{i+1}, v_iv_{i+1}, u_jv_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Also let $V_2 = \{u_i', v_i' : 1 \leq i \leq n\}$ be the corresponding set of vertices in second copy of L_n. Then $V = V(Sh(L_n)) = V_1 \cup V_2$. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 6) \\ c & \text{if} \quad i \equiv 4, 5 (\text{mod } 6) \\ a & \text{if} \quad i \equiv 0, 3 (\text{mod } 6) \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2 (\text{mod } 6) \\ b & \text{if} \quad i \equiv 4, 5 (\text{mod } 6) \\ a & \text{if} \quad i \equiv 0, 3 (\text{mod } 6) \end{cases}$$

$$f(u_i') = a \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

$$f(v_i') = a \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Then f is an a-neighbourhood V_4-magic labeling of $Sh(L_n)$. □

Corollary 5.3.36. *$Sh(L_n) \in \Omega_{a,0}$ for all $n \equiv 2 (\text{mod } 3)$.*

Proof. Proof directly follows from Theorem 5.3.34 and Theorem 5.3.35. □

Theorem 5.3.37. *$Sh(L_{n+2}) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof. By labeling all the vertices by a, we get $Sh(L_{n+2}) \in \Omega_0$ for all n. □

Theorem 5.3.38. *$Sh(L_{n+2}) \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof. Let G be the shadow graph $Sh(L_{n+2})$. Let $V_1 = \{u_i, v_i : 0 \leq i \leq n+1\}$ and $E_1 = \{u_i u_{i+1}, v_i v_{i+1} : 0 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\}$ be the vertex and edge set of first copy of L_{n+2} and let $V_2 = \{u_i', v_i' : 0 \leq i \leq n+1\}$ be the corresponding set of vertices in second copy of L_{n+2}. Define $f : V(Sh(L_{n+2})) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(v_i) = b \quad \text{for} \quad i = 0, 1, 2, 3, \ldots, n+1$$

$$f(u_i') = f(v_i') = c \quad \text{for} \quad i = 0, 1, 2, 3, \ldots, n+1$$

Then, $N_f^+(u) = a$ for all vertices u in $Sh(L_{n+2})$. □

Corollary 5.3.39. *$Sh(L_{n+2}) \in \Omega_{a,0}$ for all $n \in \mathbb{N}$.*

Proof. Proof directly follows from Theorem 5.3.37 and Theorem 5.3.38. □

Theorem 5.3.40. *$Sh(CB_n) \in \Omega_a$ for all $n > 1$.*

Proof. Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of first copy of CB_n where $v_i\,(1 \leq i \leq n)$ are the pendant vertices adjacent to $u_i\,(1 \leq i \leq n)$. Let $\{u'_i, v'_i : 1 \leq i \leq n\}$ be the corresponding set of vertices in second copy of CB_n.

Define $f : V(Sh(CB_n)) \rightarrow V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{if} \quad 1 \leq i \leq n \qquad\qquad f(u'_i) = c \quad \text{if} \quad 1 \leq i \leq n$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i = 1 \;\&\; n \\ b & \text{if} \quad 1 < i < n \end{cases} \qquad\qquad f(v'_i) = \begin{cases} a & \text{if} \quad i = 1 \;\&\; n \\ c & \text{if} \quad 1 < i < n \end{cases}$$

Then f is an a-neighbourhood V_4-magic labeling of CB_n. □

Theorem 5.3.41. *$Sh(CB_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof. By labeling all the vertices by a, we get $Sh(CB_n) \in \Omega_0$. □

Corollary 5.3.42. *$Sh(CB_n) \in \Omega_{a,0}$ for all $n > 1$.*

Proof. Proof directly follows from Theorems 5.3.40 and 5.3.41. □

Theorem 5.3.43. *$Sh(K_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof. Let G be the shadow graph $Sh(K_{m,n})$. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of the first copy of $K_{m,n}$ and let $X' = \{u'_1, u'_2, u'_3, \ldots, u'_m\}$ and $Y' = \{v'_1, v'_2, v'_3, \ldots, v'_n\}$ be the corresponding bipartition second copy of $K_{m,n}$.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i = 2 \\ a & \text{if} \quad i > 2 \end{cases} \qquad\qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \\ a & \text{if} \quad j > 2 \end{cases}$$

$$f(u'_i) = a \ \text{ for } 1 \leq i \leq m \qquad\qquad f(v'_j) = a \ \text{ for } 1 \leq j \leq n$$

Then f is an a-neighbourhood V_4-magic labeling of $Sh(K_{m,n})$. This completes the proof of the theorem. □

Theorem 5.3.44. *$Sh(K_{m,n}) \in \Omega_0$ for all $m, n \in \mathbb{N}$.*

Proof. Labeling all the vertices by a, we get $Sh(K_{m,n}) \in \Omega_0$. □

Corollary 5.3.45. *$Sh(K_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof. Proof directly follows from Theorems 5.3.43 and 5.3.44. □

Theorem 5.3.46. *$Sh(B_n) \in \Omega_a$ for all $n \equiv 1 (\text{mod } 2)$.*

Proof. Let G be the shadow graph $Sh(B_n)$ with $n \equiv 1(\text{mod } 2)$. Let vertex set of first copy of B_n be $V_1 = \{(u, v_j), (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$, where $\{u, u_1, u_2, u_3, \ldots, u_n\}$ and $\{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, and u be the central vertex, $u'_i s$ are pendant vertices in S_n. Also let

$V_2 = \{(u', v'_j), (u'_i, v'_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$ be the corresponding vertex set of second copy of B_n. Then $V(G) = V_1 \cup V_2$.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u, v_j) = \begin{cases} b & \text{if} \quad j = 1 \\ c & \text{if} \quad j = 2 \end{cases} \qquad \text{and} \qquad f(u_i, v_j) = \begin{cases} b & \text{if} \quad j = 1 \text{ and } 1 \leq i \leq n \\ c & \text{if} \quad j = 2 \text{ and } 1 \leq i \leq n \end{cases}$$

$$f(u', v'_j) = a \text{ for } 1 \leq j \leq 2 \qquad \text{and} \qquad f(u'_i, v'_j) = a \text{ for } 1 \leq i \leq n, \ 1 \leq j \leq 2$$

Clearly, f is an a-neighbourhood V_4-magic labeling of $Sh(B_n)$. □

Theorem 5.3.47. *$Sh(B_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof. By labeling all the vertices by a, we get $Sh(B_n) \in \Omega_0$. □

Corollary 5.3.48. *$Sh(B_n) \in \Omega_{a,0}$ for all $n \equiv 1 (\text{mod } 2)$.*

Proof. Proof follows from Theorems 5.3.46 and 5.3.47. □

Theorem 5.3.49. *$Sh(G_n) \in \Omega_0$ for all n.*

Proof. Degree of vertices in $Sh(G_n)$ is either 4 or 6 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(G_n))$. □

Theorem 5.3.50. *$Sh(G_n) \in \Omega_a$ for all $n \equiv 2 (\text{mod } 4)$.*

Proof. Let G be the shadow graph $Sh(G_n)$ with $n \equiv 2(\text{ mod } 4)$. Let $V_1 = \{u, u_i : 1 \leq i \leq 2n\}$ and $E_1 = \{uu_{2i-1} : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq 2n-1\} \cup \{u_{2n} u_1\}$ be the vertex and edge set of first copy of G_n. Let $V_2 = \{u', u'_i : 1 \leq i \leq 2n\}$ be the corresponding vertex set of second copy of G_n. Then $V(G) = V_1 \cup V_2$.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u) = b$$

$$f(u') = c$$

$$f(u_i) = a \quad \text{for} \quad 1 \leq i \leq 2n.$$

$$f(u'_i) = \begin{cases} a & \text{if} \quad i \equiv 0 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 1 (\text{mod } 4) \\ a & \text{if} \quad i \equiv 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 3 (\text{mod } 4) \end{cases}$$

Then f is an a-neighbourhood V_4-magic labeling for $Sh(G_n)$. □

Corollary 5.3.51. *$Sh(G_n) \in \Omega_{a,0}$ for all $n \equiv 2(\text{mod } 4)$.*

Proof. Proof directly follows from Theorem 5.3.49 and Theorem 5.3.50. □

5.4 Middle graphs

Theorem 5.4.1. *$M(C_n) \in \Omega_a$ if and only if $n \equiv 0(\text{mod } 2)$.*

Proof. Consider $M(C_n)$ with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ labeled as in figure 5.4. Suppose that $M(C_n) \in \Omega_a$ with a labeling f. Then $N_f^+(u_2) = a$ implies that $f(v_1) + f(v_2) = a$, which implies that either $f(v_1) = b$ or $f(v_1) = c$. Without loss of generality we can assume that $f(v_1) = b$. Then $f(v_2) = c$, $f(v_3) = b$,

$f(v_4) = c$, etc. and so on. Now $N_f^+(u_1) = a$ implies that $f(v_1) + f(v_n) = a$, which again implies that $f(v_n) = c$. Hence $n \equiv 0 (\text{mod } 2)$. Conversely, assume that $n \equiv 0 (\text{mod } 2)$. Then define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0 (\text{mod } 2) \\ c & \text{if} \quad i \equiv 1 (\text{mod } 2) \end{cases}$$

Obviously, f is an a-neighbourhood V_4-magic labeling of $M(C_n)$. This completes the proof of the theorem.

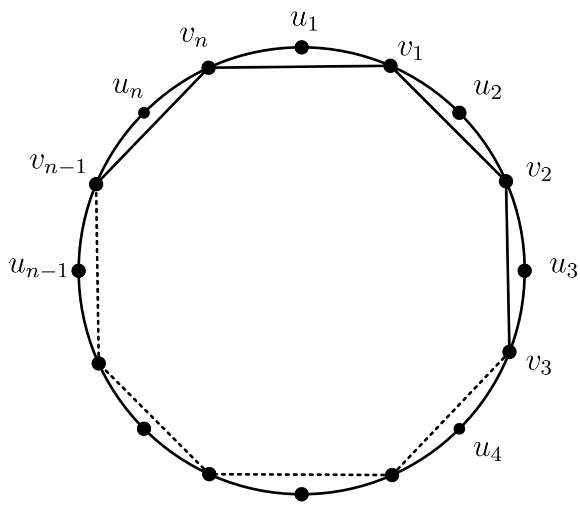

Figure 5.4: The middle graph $M(C_n)$

□

Theorem 5.4.2. *$M(C_n) \in \Omega_0$ for all $n \geq 3$.*

Proof. By labeling all the vertices by a, we get $M(C_n) \in \Omega_0$. □

Corollary 5.4.3. *$M(C_n) \in \Omega_{a,0}$ if and only if $n \equiv 0(\mathrm{mod}\ 2)$.*

Proof. Proof directly follows from Theorem 5.4.1 and Theorem 5.4.2. □

Theorem 5.4.4. *$M(P_n) \notin \Omega_a$ for any n.*

Proof. Consider $M(P_n)$ with vertex set $V = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and edge set $E = \{u_i v_i : 1 \leq i \leq n-1\} \cup \{v_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\}$. Assume that $M(P_n) \in \Omega_a$ for some n with a labeling f. Then $N_f^+(u_1) = a$ implies that $f(v_1) = a$. Now $N_f^+(u_2) = a$ implies that $f(v_1)+f(v_2) = a$. Hence $f(v_2) = 0$, a contradiction. Hence the theorem is proved. □

Theorem 5.4.5. *$M(P_n) \notin \Omega_0$ for any n.*

Proof. Proof is obvious, since $M(P_n)$ has pendant vertex in it. □

Corollary 5.4.6. *$M(P_n) \notin \Omega_{a,0}$ for any n.*

Proof. It directly follows from Theorem 5.4.4. □

Theorem 5.4.7. *$M(K_{1,n}) \in \Omega_a$ if and only if $n \equiv 1(\mathrm{mod}\ 2)$.*

Proof. Consider $M(K_{1,n})$ with vertex set $V = \{u, u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{uv_i, u_i v_i : 1 \leq i \leq n\} \cup \{v_i v_j : 1 \leq i, j \leq n, i \neq j\}$. Assume that $M(K_{1,n}) \in \Omega_a$ with a labeling f. Then, $N_f^+(u_i) = a$ implies that $f(v_i) = a$ for all $1 \leq i \leq n$. Consequently, $N_f^+(u) = a$ gives $na = a$, which implies that $n \equiv 1(\mathrm{mod}\ 2)$. Conversely, assume that $n \equiv 1(\mathrm{mod}\ 2)$. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$

$$f(v_i) = a \quad \text{if} \quad i = 1, 2, 3, \ldots, n$$

$$f(u) = c$$

Then f is an a-neighbourhood V_4-magic labeling of $M(K_{1,n})$. □

Theorem 5.4.8. *$M(K_{1,n}) \notin \Omega_0$ for any n.*

Proof. Proof is obvious due to the presence of pendant vertex in $M(K_{1,n})$. □

Corollary 5.4.9. *$M(K_{1,n}) \notin \Omega_{a,0}$ for any n.*

Proof. It directly follows from Theorem 5.4.8. □

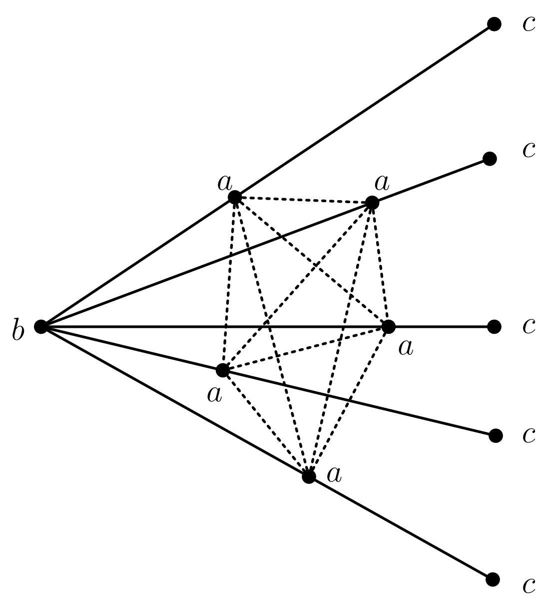

Figure 5.5: An a-neighbourhood V_4-magic labeling of $M(K_{1,5})$

Theorem 5.4.10. *$M(F_m) \in \Omega_0$ for all m.*

Proof. Note that degree of each vertex in $M(F_m)$ is even. By labeling all the vertices by a, we get $M(F_m) \in \Omega_0$ for all m. □

Theorem 5.4.11. *$M(F_m) \notin \Omega_a$ for any m.*

Proof. Consider $M(F_m)$ with vertex set $V = \{w, u_i, v_i, w'_i, u'_i, v'_i : 1 \leq i \leq m\}$ labeled as in figure 5.6. Suppose that $M(F_m) \in \Omega_a$ with a labeling f. Then for each $1 \leq i \leq m$, $N_f^+(u_i) = a = N_f^+(v_i)$ implies that $f(u'_i) = (v'_i)$. Hence $N_f^+(w) = \sum f(u'_i) + \sum f(v'_i) = 0$, which is a contradiction. Hence the theorem is proved.

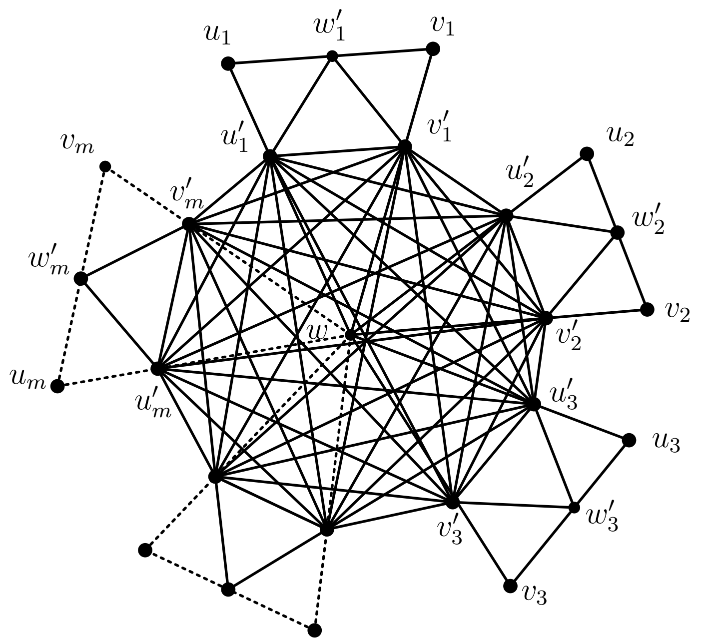

Figure 5.6: The middle graph $M(F_m)$

□

Corollary 5.4.12. *$M(F_m) \notin \Omega_{a,0}$ for any m.*

Proof. It directly follows from Theorem 5.4.11. □

Theorem 5.4.13. *$M(B_{m,n}) \notin \Omega_0$ for any m and n.*

Proof. Proof is obvious, since $M(B_{m,n})$ has pendant vertex in it. □

Theorem 5.4.14. *$M(B_{m,n}) \in \Omega_a$ if and only if m and n are both even.*

Proof. Consider $M(B_{m,n})$ with vertex set $V = \{u, v, w, u_i, v_j, u'_i, v'_j : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E = \{u_i u'_i : 1 \le i \le m\} \cup \{v_j v'_j : 1 \le j \le n\} \cup \{u'_i u'_j : 1 \le i, j \le m, i \ne j\} \cup \{v'_i v'_j : 1 \le i, j \le n, i \ne j\} \cup \{uu'_i : 1 \le i \le m\} \cup \{vv'_j : 1 \le j \le n\} \cup \{wu'_i : 1 \le i \le m\} \cup \{wv'_j : 1 \le j \le n\} \cup \{wu, wv\}$. Assume that m and n are both even. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = c \quad \text{if} \quad i = 1, 2, 3, \dots, m$$

$$f(u'_i) = a \quad \text{if} \quad i = 1, 2, 3, \dots, m$$

$$f(v_j) = b \quad \text{if} \quad j = 1, 2, 3, \dots, n$$

$$f(v'_j) = a \quad \text{if} \quad j = 1, 2, 3, \dots, n$$

$$f(u) = b$$

$$f(v) = c$$

$$f(w) = a$$

Then, f is an a-neighbourhood V_4-magic labeling of $M(B_{m,n})$. Conversely, assume that m and n are not both even. Without loss of generality assume that m is odd. If $M(B_{m,n}) \in \Omega_a$, then we have $f(u_i') = a$ for $i = 1, 2, 3, \ldots, m$. Now $N_f^+(u) = \sum f(u_i) + f(w) = a$ implies that $na + f(w) = a$, which again implies that $f(w) = 0$, a contradiction. Therefore, $M(B_{m,n}) \notin \Omega_a$. □

Corollary 5.4.15. *$M(B_{m,n}) \notin \Omega_{a,0}$ for any n.*

Proof. It directly follows from Theorem 5.4.13. □

Chapter 6

Neighbourhood Barycentric V_4-magic Graphs

This chapter introduces the concept of Neighbourhood barycentric V_4-magic labeling in Graphs. The first section gives the definition of neighbourhood barycentric V_4-magic labeling in graphs and the second section investigates graphs which are a-neighbourhood barycentric V_4-magic or 0-neighbourhood barycentric V_4-magic.

6.1 Introduction

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0. For any graph $G = (V(G), E(G))$, a mapping $f : V(G) \to V_4 \backslash \{0\}$ is said to be Neighbourhood barycentric V_4-magic labeling if the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(u) = \sum_{v \in N(u)} f(v)$ satisfies the following conditions:

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(a) N_f^+ is a constant map, and

(b) For each $u \in V(G)$, $N_f^+(u) = deg(u)f(v_u)$ for some vertex $v_u \in N(u)$

If this constant is p, where p is any non zero element in V_4, we say that f is a p-neighbourhood barycentric V_4-magic labeling of G, and G is said to be a p-neighbourhood barycentric V_4-magic graph. If this constant is 0, we say that f is a 0-neighbourhood barycentric V_4-magic labeling of G, and G a 0-neighbourhood barycentric V_4-magic graph. Through out this chapter we use the following notations.

(i) Λ_a := the class of all a-neighbourhood barycentric V_4-magic graphs, and

(ii) Λ_0 := the class of all 0-neighbourhood barycentric V_4-magic graphs.

6.2 Neighbourhood barycentric V_4-magic graphs

Theorem 6.2.1. *The cycle $C_n \in \Lambda_0$ for all $n \geq 3$.*

Proof. By labeling all the vertices of C_n by a, we get $C_n \in \Lambda_0$. □

Theorem 6.2.2. *$C_n \notin \Lambda_a$ for any $n \geq 3$.*

Proof. Consider C_n with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Now for any $v_i \in V$ and any $u \in N(v_i)$, we have $deg(v_i)f(u) = 0$. Thus condition(ii) of the definition violates. Hence $C_n \notin \Lambda_a$. □

Theorem 6.2.3. *The path $P_n \notin \Lambda_0$ for any $n > 1$.*

Proof. Let P_n be any path with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Suppose that $P_n \in \Lambda_0$ for some $n > 1$ with a labeling f. Then $N_f^+(v_1) = 0$ implies that $f(v_2) = 0$, which is a contradiction. Hence $P_n \notin \Lambda_0$ for all $n > 1$. □

Theorem 6.2.4. *$P_n \notin \Lambda_a$ for any $n > 1$.*

Proof. Proof is obvious, since P_n has vertex of degree 2. □

Theorem 6.2.5. *The complete graph $K_n \in \Lambda_a$ if and only if n is even.*

Proof. Suppose that $K_n \in \Lambda_a$. Then there exists a vertex labeling f such that for any vertex $u \in V(K_n)$, $N_f^+(u) = a = deg(u)f(v_u)$ for some $v_u \in N(u)$. Therefore $a = (n-1)f(v_u)$, implying that n is even. Conversely, suppose that n is even. By labeling all the vertices by a, we get $K_n \in \Lambda_a$. □

Theorem 6.2.6. *$K_n \in \Lambda_0$ if and only if n is odd.*

Proof. Suppose that $K_n \in \Lambda_0$. Then there exists a vertex labeling f such that for any vertex $u \in V(K_n)$, $N_f^+(u) = 0 = deg(u)f(v_u)$ for some $v_u \in N(u)$. Therefore $0 = (n-1)f(v_u)$. Hence n is odd. Conversely, suppose that n is odd. By labeling all the vertices by a, we get $K_n \in \Lambda_0$. This completes the proof of the theorem. □

Theorem 6.2.7. *The graph $K_{m,n} \in \Lambda_a$ if and only if both m and n are odd.*

Proof. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. Assume that $K_{m,n} \in \Lambda_a$ with a labeling f. Then for each $1 \leq i \leq m$, $N_f^+(u_i) = a = deg(u_i)f(v_j)$ for some $1 \leq j \leq n$. Then $nf(v_j) = a$, which implies that n is odd. Similarly for each $1 \leq j \leq n$, $N_f^+(v_j) = a = deg(v_j)f(u_k)$ for some

$1 \leq k \leq m$, implies that m is odd. Conversely, assume that both m and n are odd. Labeling all the vertices by a, we get $K_{m,n} \in \Lambda_a$. □

Theorem 6.2.8. *$K_{m,n} \in \Lambda_0$ if and only if both m and n are even.*

Proof. Let $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. Assume that $K_{m,n} \in \Lambda_0$ with a labeling f. Then for each $1 \leq i \leq m$, $N_f^+(u_i) = 0 = deg(u_i)f(v_j)$ for some $1 \leq j \leq n$. Therefore, $nf(v_j) = 0$, hence n is even. Similarly for each $1 \leq j \leq n$, $N_f^+(v_j) = 0 = deg(v_j)f(u_k)$ for some $1 \leq k \leq m$, implies that m is even. Conversely, assume that both m and n are even. Labeling all the vertices by a, we get $K_{m,n} \in \Lambda_0$. □

Theorem 6.2.9. *The star graph $K_{1,n} \in \Lambda_a$ if and only if n is odd.*

Proof. It directly follows from Theorem 6.2.7. □

Theorem 6.2.10. *$K_{1,n} \notin \Lambda_0$ for any $n \in \mathbb{N}$.*

Proof. It is obvious from Theorem 6.2.8. □

Theorem 6.2.11. *The bistar $B_{m,n} \in \Lambda_a$ if and only if both m and n are even.*

Proof. Consider bistar $B_{m,n}$ with vertex set $V = \{u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ where $u_i (1 \leq i \leq m)$ and $v_j (1 \leq j \leq n)$ are pendant vertices adjacent to u and v respectively. Assume that $B_{m,n} \in \Lambda_a$ with a labeling f. Then $N_f^+(u) = a = deg(u)f(w_u)$ for some $w_u \in N(u)$. Therefore,$(m+1)f(w_u) = a$, implies that m is even. Similarly, $N_f^+(v) = a = deg(v)f(w_v)$ for some $w_v \in N(v)$. Therefore,$(n+1)f(w_v) = a$, implies that n is even. Conversely, assume that both m and n are even. If we label all the vertices by a, we get $B_{m,n} \in \Lambda_a$. □

Theorem 6.2.12. *$B_{m,n} \notin \Lambda_0$ for any m and n.*

Proof. Proof is obvious due to the presence of pendant vertices in $B_{m,n}$. □

Theorem 6.2.13. *The friendship graph $F_m \in \Lambda_0$ for all $m \in \mathbb{N}$.*

Proof. If we label all the vertices of F_m by a, we will get $F_m \in \Lambda_0$. □

Theorem 6.2.14. *$F_m \notin \Lambda_a$ for any $m \in \mathbb{N}$.*

Proof. Proof is obvious due to the presence of even vertices in F_m. □

Theorem 6.2.15. *$C_n \odot K_2 \in \Lambda_0$ for any $n \geq 3$.*

Proof. Degree of vertices in $C_n \odot K_2$ are either 2 or 4. By labeling all the vertices by a, we get $C_n \odot K_2 \in \Lambda_0$. □

Theorem 6.2.16. *$C_n \odot K_2 \notin \Lambda_a$ for any $n \geq 3$.*

Proof. Proof is obvious due to the presence of even vertices in $C_n \odot K_2$. □

Theorem 6.2.17. *$C_n \odot \overline{K}_m \in \Lambda_a$ if and only if m is odd.*

Proof. Suppose that $C_n \odot \overline{K}_m \in \Lambda_a$. Then there exists a labeling f such that, for each $u \in V(C_n \odot \overline{K}_m)$, we have $N_f^+(u) = a = deg(u)f(v_u)$ for some vertex $v_u \in N(u)$. Let $\{u_1, u_2, u_3, \ldots, u_n\}$ be the vertices of C_n and $\{u_{k_1}, u_{k_2}, u_{k_3}, \ldots, u_{k_n}\}$ be the vertex set of k^{th} copy of $\overline{K}_m$. Then $N_f^+(u_1) = a = deg(u_1)f(v)$ for some vertex $v \in N(u_1)$ implies that $a = (m+2)f(v)$, hence m is odd. Conversely, suppose that m is odd. Labeling all the vertices by a, we get $C_n \odot \overline{K}_m \in \Lambda_a$. □

Theorem 6.2.18. *$C_n \odot \overline{K}_m \notin \Lambda_0$ for any m and n.*

Proof. Proof is obvious due to the presence of pendant vertices in $C_n \odot \overline{K}_m$. □

Theorem 6.2.19. *The wheel $W_n \in \Lambda_a$ if and only if n is odd.*

Proof. Consider the wheel graph W_n with vertex set $V = \{u, u_1, u_2, u_3, \ldots, u_n\}$ where u be the central vertex. Assume that $W_n \in \Lambda_a$ for some n with a labeling f. Therefore, $N_f^+(u) = a = deg(u)f(u_k)$ for some $1 \leq k \leq n$. Implies that $nf(u_k) = a$, hence n is odd. Conversely, assume that n is odd. Then labeling all the vertices by a, we get $W_n \in \Lambda_a$. This completes the proof of the theorem. □

Theorem 6.2.20. *$W_n \notin \Lambda_0$ for any n.*

Proof. Let $V = \{u, u_1, u_2, u_3, \ldots, u_n\}$ be the vertex set of W_n with central vertex u. Assume that $W_n \in \Lambda_a$ with a labeling f. Then $N_f^+(u_1) = 0 = deg(u_1)f(v)$ for some $v \in N(u_1)$. Therefore $f(v) = 0$, a contradiction. Hence the proof. □

Theorem 6.2.21. *The helm graph $H_n \notin \Lambda_0$ for any n.*

Proof. Proof is obvious due the presence pendant vertices in H_n. □

Theorem 6.2.22. *$H_n \notin \Lambda_a$ for any n.*

Proof. Proof is obvious due the presence even degree vertices in H_n. □

Theorem 6.2.23. *The flower graph $Fl_n \in \Lambda_0$ for all n.*

Proof. In the flower graph Fl_n, the degree of vertices are either 2 or 4 or $2n$. By labeling all the vertices Fl_n by a, we get $Fl_n \in \Lambda_0$. □

Theorem 6.2.24. *$Fl_n \notin \Lambda_a$ for any n.*

Proof. Proof is obvious due the presence even degree vertices in Fl_n. □

Theorem 6.2.25. *The sunflower graph $SF_n \notin \Lambda_0$ for any n.*

Proof. Since SF_n has odd vertices, $SF_n \notin \Lambda_0$ for all n. □

Theorem 6.2.26. *$SF_n \notin \Lambda_a$ for any n.*

Proof. Proof is obvious due the presence even degree vertices in SF_n. □

Theorem 6.2.27. *Let G be a k-regular graph, then $G \in \Lambda_a$ if and only if k is odd.*

Proof. Let G be any k-regular graph. Suppose that $G \in \Lambda_a$. Then there exists a vertex labeling f such that each vertex $u \in V(G)$, $N_f^+(u) = a = deg(u)f(v_u)$ for some $v_u \in N(u)$. Therefore $kf(v_u) = a$, implies that k is odd. Conversely, suppose that k is odd. By labeling all the vertices by a, we get $G \in \Lambda_a$. □

Theorem 6.2.28. *Let G be a k-regular graph, then $G \in \Lambda_0$ if and only if k is even.*

Proof. Let G be any k-regular graph. Suppose that $G \in \Lambda_0$. Then there exists a vertex labeling f such that each vertex $u \in V(G)$, $N_f^+(u) = 0 = deg(u)f(v_u)$ for some $v_u \in N(u)$. Therefore $kf(v_u) = 0$, implies that k is even. Conversely, suppose that k is even. By labeling all the vertices by a, we get $G \in \Lambda_0$. Completes the proof of the theorem. □

Theorem 6.2.29. *The graph $J(m, n) \in \Lambda_a$ if and only if both m and n are odd.*

Proof. Let G be the graph $J(m,n)$. Then G has $(m+n+4)$ vertices and $(m+n+5)$ edges. Also let $V(G) = V_1 \cup V_2$ where $V_1 = \{w_1, w_2, w_3, w_4\}$, $V_2 = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = E_1 \cup E_2$ where $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\}$, $E_2 = \{w_2u_i, w_4v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Assume that $J(m,n) \in \Lambda_a$ with a labeling f. Then $N_f^+(w_2) = a = deg(w_2)f(w)$ for some vertex $w \in N(w_2)$. Which implies that $(m+2)f(w) = a$, hence m is odd. Also $N_f^+(w_4) = a = deg(w_4)f(w)$ for some vertex $w \in N(w_4)$. Which implies that $(n+2)f(w) = a$, hence n is odd. Conversely, assume that both m and n are odd. Labeling all the vertices by a, we get $J(m,n) \in \Lambda_a$. □

Theorem 6.2.30. *$J(m,n) \notin \Lambda_0$ for any m and n.*

Proof. Proof is obvious, since $J(m,n)$ has pendant vertices in it. □

Theorem 6.2.31. *$< K_{1,n} : m > \in \Lambda_a$ if and only if m is odd and n is even.*

Proof. Let G be the graph $< K_{1,n} : m >$, let $V_i = \{u_i, u_{ij} : 1 \leq j \leq n\}$ be the vertex set of i^{th} copy of $K_{1,n}$ with apex u_i and let u be the unique vertex adjacent to the central vertices $u_i (1 \leq i \leq m)$. Then $V(G) = V_1 \cup V_2 \cup \ldots \cup V_m \cup \{u\}$. Suppose that $< K_{1,n} : m > \in \Lambda_a$ with a labeling f. Then $N_f^+(u) = a = deg(u)f(u_k)$ for some $1 \leq k \leq m$. Therefore $mf(u_k) = a$, implies that m is odd. Similarly $N_f^+(u_i) = a = deg(u_i)f(v)$ for some $v \in N(u_i)$, implies that $(n+1)f(v) = a$ and hence n is even. Conversely, assume that m is odd and n is even. Labeling all the vertices by a, we get $< K_{1,n} : m > \in \Lambda_a$. □

Theorem 6.2.32. *$< K_{1,n} : m > \notin \Lambda_0$ for all m and n.*

Proof. It is obvious due to the presence of pendant vertex in $< K_{1,n} : m >$. □

Theorem 6.2.33. *The Ladder L_n is neither a-neighbourhood barycentric V_4-magic nor 0-neighbourhood barycentric V_4-magic for any n.*

Proof. Proof is obvious since L_n has vertices of degree 2 and 3. □

Theorem 6.2.34. *The Ladder $L_{n+2} \in \Lambda_a$ for all n.*

Proof. If we label all the vertices of L_{n+2} by a, we get $L_{n+2} \in \Lambda_a$. □

Theorem 6.2.35. *$L_{n+2} \notin \Lambda_0$ for any n.*

Proof. Proof is obvious due to the presence of pendant vertices in L_{n+2}. □

Theorem 6.2.36. *The quadrilateral snake $QS_n \in \Lambda_0$ for all n.*

Proof. Note that in QS_n, degree of each vertex is either 2 or 4. By labeling all the vertices of QS_n by a, we get $QS_n \in \Lambda_0$ for all n. □

Theorem 6.2.37. *$QS_n \notin \Lambda_a$ for any n.*

Proof. Proof is obvious due to the presence of even vertices in QS_n. □

Theorem 6.2.38. *The graph $P_2 \square C_n \in \Lambda_a$ for all $n \geq 3$.*

Proof. Consider $P_2 \square C_n$ with vertex set $V = \{(u_i, v_j) : 1 \leq i \leq 2, 1 \leq j \leq n\}$. Then degree of each vertex is 3. Labeling all the vertices by a, we get $P_2 \square C_n \in \Lambda_a$ for all $n \geq 3$. □

Theorem 6.2.39. *$P_2 \square C_n \notin \Lambda_0$ for any $n \geq 3$.*

Proof. Proof is obvious, since $P_2 \square C_n$ has odd vertices. □

Theorem 6.2.40. *The crown graph* $C_n^* \in \Lambda_a$ *for all* $n \geqslant 3$.

Proof. The degree of vertices of a crown graph are either 1 or 3. By labeling all the vertices by a, we get $C_n^* \in \Lambda_a$. □

Theorem 6.2.41. $C_n^* \notin \Lambda_0$ *for any* $n \geq 3$.

Proof. Proof is obvious due to the presence of pendant vertices in C_n^*. □

Theorem 6.2.42. $P_2 \Box C_n^* \in \Lambda_0$ *for all* $n \geq 3$.

Proof. Consider the graph $P_2 \Box C_n^*$. Then degree of each vertex is either 2 or 4. Labeling all the vertices by a, we get $P_2 \Box C_n^* \in \Lambda_0$. □

Theorem 6.2.43. $P_2 \Box C_n^* \notin \Lambda_a$ *for any* $n \geq 3$.

Proof. Proof directly follows since $P_2 \Box C_n^*$ has even vertices. □

Theorem 6.2.44. $B_n = S_n \Box P_2 \in \Lambda_0$ *if and only if* n *is odd.*

Proof. Let $V_1 = \{u, u_1, u_2, u_3, \ldots, u_n\}$ and $V_2 = \{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, where u be the central vertex and $u_i's$ are pendant vertices in S_n. Then $V(B_n) = \{(u, v_j), (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$. Assume that $B_n \in \Lambda_a$ with a labeling f. Then $N_f^+[(u, v_1)] = 0 = deg[(u, v_1)]f(w)$ for some $w \in N[(u, v_1)]$. Therefore $(n+1)f(w) = 0$, implies that n is odd. Conversely, assume that n is odd. By labeling all the vertices by a, we get $B_n \in \Lambda_a$. □

Theorem 6.2.45. $B_n \notin \Lambda_a$ *for any* n.

Proof. Proof is obvious, since B_n has even vertices. □

Theorem 6.2.46. *A tree $T \in \Lambda_a$ if and only if all the vertices of T are odd.*

Proof. Let T be any (p, q) tree with vertex set $V = \{v_1, v_2, v_3, \ldots, v_p\}$ where $deg(v_i) = k_i$ for $i = 1, 2, 3, \ldots, p$. Assume that $T \in \Lambda_a$ with a labeling f. Then for each $1 \leq i \leq p$, $N_f^+(v_i) = a = deg(v_i)f(v_j)$ for some $v_j \in N(v_i)$, which implies that $k_i f(v_j) = a$. Hence k_i is odd. Conversely, assume that for each $1 \leq i \leq p$, k_i is odd. By labeling all the vertices of T by a, we get $T \in \Lambda_a$. □

Theorem 6.2.47. *Let T be any (p, q) tree, then $T \notin \Lambda_0$.*

Proof. Proof is obvious due to the presence of pendant vertices in T. □

We now describe an important property of a-neighbourhood and 0-neighbourhood barycentric V_4-magic graphs. That is, there is no graph which belong to both Λ_a and Λ_0.

Theorem 6.2.48. *The class Λ_a and Λ_0 are disjoint. ie, $\Lambda_a \cap \Lambda_0 = \Phi$.*

Proof. Suppose that there is a graph G such that $G \in \Lambda_a \cap \Lambda_0$. Since $G \in \Lambda_a$, there exists a vertex labeling f such that for any $u \in V(G)$, $N_f^+(u) = a = deg(u)f(v_u)$ for some vertex $v_u \in N(u)$. Again since $G \in \Lambda_0$, there exists a vertex labeling g such that for the same $u \in V(G)$, $N_g^+(u) = 0 = deg(u)g(w_u)$ for some vertex $w_u \in N(u)$. Now we consider the following cases:

Case 1: $deg(u)$ is even.

Then, $N_f^+(u) = a = deg(u)f(v_u)$ implies that $a = 0$, which is a contradiction.

Case 2: $deg(u)$ is odd.

$N_g^+(u) = 0 = deg(u)g(w_u)$ implies that $g(w_u) = 0$, which is also a contradiction.

Therefore, $\Lambda_a \cap \Lambda_0 = \Phi$. □

Chapter 7

Star V_4-magic Labeling of Graphs

This chapter introduces a new type of labeling called star V_4-magic labeling of graphs. The first section of this chapter gives the definition of star V_4-magic labeling in graphs. The second section investigates class of graphs which are a-star V_4-magic or 0-star V_4-magic and both a-star and 0-star V_4-magic.

7.1 Introduction

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0. We say that, a graph $G = (V(G), E(G))$, star V_4-magic if there exists a labeling $f : V(G) \to V_4\backslash\{0\}$ such that the induced mapping $V_f^+ : V(G) \to V_4$ defined by

$$V_f^+(v) = \textstyle\sum_{u\in N(v)} f^*(uv), \text{ where } f^*(uv) = f(u) + f(v)$$

is a constant map. If this constant is p, where p is any non zero element in V_4, then we say that f is a p-star V_4-magic labeling of G and G is said to be a p-star

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V_4-magic graph. If this constant is 0, then we say that f is a 0-star V_4-magic labeling of G and G is said to be a 0-star V_4-magic graph. Through out this chapter we use the following notations:

(i) $\Psi_a :=$ the class of all a-star V_4-magic graphs,

(ii) $\Psi_0 :=$ the class of all 0-star V_4-magic graphs, and

(iii) $\Psi_{a,0} := \Psi_a \cap \Psi_0$.

7.2 Star V_4-magic labeling of graphs

Lemma 7.2.1. *Let G be any graph and $f : V(G) \to V_4\backslash\{0\}$ is any labeling of G, then $\sum_{v\in V} V_f^+(v) = 0$.*

Proof. Let G be the graph with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $f : V(G) \to V_4\backslash\{0\}$ is any labeling of G. Then,

$$\sum_{i=1}^{n} V_f^+(v_i) = 2\sum deg(v_i)f(v_i) = 0.$$

This completes the proof. □

Lemma 7.2.2. *Let G be any (p, q) graph, then $G \in \Psi_0$.*

Proof. Let G be the graph with vertex set $V = \{v_1, v_2, v_3, \ldots, v_p\}$. Labeling all the vertices v_i by a, we get $f^* \equiv 0$. Then $V_f^+(v_i) = 0$ for all $v_i \in V$. This completes the proof of the Lemma. □

Theorem 7.2.3. *$C_n \in \Psi_a$ if and only if $n \equiv 0(\text{mod } 4)$.*

Proof. Assume that $C_n \in \Psi_a$ with a labeling f. Then by Lemma 7.2.1, we have $na = 0$. Therefore $n \equiv 0(\text{mod } 2)$. Then either $n \equiv 0(\text{mod } 4)$ or $n \equiv 2(\text{mod } 4)$. We prove that the case where $n \equiv 2(\text{mod } 4)$ is impossible. For if $n \equiv 2(\text{mod } 4)$, then $n = 4k + 2$ for some positive integer k. Let $v_1, v_2, v_3, \ldots, v_{4k}, v_{4k+1}, v_{4k+2}$ be the vertices of C_n in order. Now $V_f^+(v_2) = a$ implies that $f(v_1) + f(v_3) = a$, which implies that $f(v_1)$ is either b or c. Without loss of generality, we assume that $f(v_1) = b$. If $f(v_1) = b$, then $f(v_3) = c$, $f(v_5) = b$, $f(v_7) = c$, $f(v_9) = b$, $f(v_{11}) = c$. Proceeding like this we get $f(v_{4k+1}) = b$. Then, $V_f^+(v_{4k+2}) = f(v_1) + f(v_{4k+1}) = b + b = 0$, a contradiction. Therefore, $n \equiv 2(\text{mod } 4)$ is impossible. Hence $n \equiv 0(\text{mod } 4)$. Conversely, assume that $n \equiv 0(\text{mod } 4)$. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n in order. Define $f : V(C_n) \to V_4 \backslash \{0\}$ as :

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 3(\text{mod } 4) \\ c & \text{if} \quad i \equiv 1, 2(\text{mod } 4) \end{cases}$$

Then, $V_f^+(v_i) = a \quad$ for $1 \leq i \leq n$. This completes the proof of the theorem. □

Corollary 7.2.4. *$C_n \in \Psi_{a,0}$ if and only if $n \equiv 0(\text{mod } 4)$.*

Proof. Proof follows from Lemma 7.2.2 and Theorem 7.2.3. □

Theorem 7.2.5. *$P_n \in \Psi_a$ if and only if $n \equiv 0(\text{mod } 2)$.*

Proof. Let P_n be a path with vertices $v_1, v_2, v_3, \ldots, v_n$ in order. Assume that $P_n \in \Psi_a$ with a labeling f. Then by Lemma 7.2.1, we have $\sum_{i=1}^{n} V_f^+(v_i) = 0$, implies that $na = 0$. Hence $n \equiv 0(\text{mod } 2)$. Conversely, assume that $n \equiv 0(\text{mod } 2)$. We define $f : V(P_n) \to V_4 \backslash \{0\}$ as:

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \end{cases}$$

Then, $V_f^+(v_i) = a$ for all $v_i \in V(P_n)$. Hence the result. □

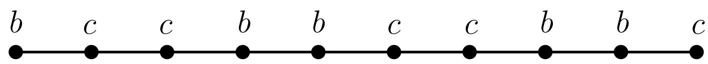

Figure 7.1: An a-star V_4-magic labeling of P_{10}

Corollary 7.2.6. *$P_n \in \Psi_{a,0}$ if and only if $n \equiv 0 (\text{mod } 2)$.*

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 7.2.5. □

Theorem 7.2.7. *The complete graph $K_n \in \Psi_a$ if and only if $n \equiv 0 (\text{mod } 2)$.*

Proof. Suppose that $K_n \in \Psi_a$. Then by Lemma 7.2.1 we have $na = 0$, therefore $n \equiv 0(\text{ mod } 2)$. Conversely, suppose that $n \equiv 0(\text{ mod } 2)$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of K_n. Define $f : V \to V_4 \backslash \{0\}$ as :

$$f(v_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i > 1 \end{cases}$$

Then,

$$V_f^+(v_i) = \begin{cases} (n-1).a = a \quad \text{for} \quad i = 1 \\ a + (n-2).0 = a \quad \text{for} \quad i > 1 \end{cases}$$

Hence the proof is complete. □

Corollary 7.2.8. *$K_n \in \Psi_{a,0}$ if and only if $n \equiv 0 (\text{mod } 2)$.*

Proof. Proof follows from Lemma 7.2.2 and Theorem 7.2.7. □

Theorem 7.2.9. *$K_{1,n} \in \Psi_a$ if and only if n is odd.*

Proof. Consider $K_{1,n}$ with vertex set $V = \{v_i : 0 \leq i \leq n\}$ where v_0 is the apex. Suppose that $K_{1,n} \in \Psi_a$ with a labeling f. Then by Lemma 7.2.1, we have

$$\sum_{i=0}^{n} V_f^+(v_i) = 0$$

Implying that $(n+1)a = 0$, hence n is odd. Conversely, suppose that n is odd. We define $f : V \to V_4 \backslash \{0\}$ as :

$$f(v_i) = \begin{cases} b \quad \text{if} \quad i = 0 \\ c \quad \text{if} \quad i \geq 1 \end{cases}$$

Then f is an a-star V_4-magic labeling of $K_{1,n}$. □

Corollary 7.2.10. *$K_{1,n} \in \Psi_{a,0}$ if and only if n is odd.*

Proof. Proof follows from Lemma 7.2.2 and Theorem 7.2.9. □

Theorem 7.2.11. *$K_{m,n} \in \Psi_a$ if and only if $m + n \equiv 0 (\text{mod } 2)$.*

Proof. Consider $K_{m,n}$ with bipartition $X = \{u_1, u_2, u_3, \ldots, u_m\}$ and $Y = \{v_1, v_2, v_3, v_4, \ldots, v_n\}$. Assume that $K_{m,n} \in \Psi_a$, then by Lemma 7.2.1 we have $(m+n)a = 0$, implies that $m + n \equiv 0 (\text{mod } 2)$. Conversely, assume that $m + n \equiv 0 (\text{mod } 2)$. Then both m and n are odd or m and n are even.

Case 1: m and n are odd.

Define $f : V(K_{m,n}) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, m$$

$$f(v_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Case 2: m and n are even.

Define $f : V(K_{m,n}) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i \geq 2 \end{cases} \qquad f(v_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i \geq 2 \end{cases}$$

In either case, f is an a-star V_4-magic labeling of $K_{m,n}$. □

Corollary 7.2.12. *$K_{m,n} \in \Psi_{a,0}$ if and only if $m + n \equiv 0 (\text{mod } 2)$.*

Proof. Proof follows from Lemma 7.2.2 and Theorem 7.2.11. □

Theorem 7.2.13. *The bistar $B_{m,n} \in \Psi_a$ if and only if $m + n \equiv 0 (\text{mod } 2)$.*

Proof. Consider the bistar $B_{m,n}$ with vertex set $V = \{u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$, where $u_i (1 \leq i \leq m)$ and $v_j (1 \leq j \leq n)$ are pendant vertices adjacent to u and v respectively. Assume that $B_{m,n} \in \Psi_a$ with a labeling f. Therefore, $V_f^+(u) = a$ for all $u \in V$. Then by Lemma 7.2.1, we have $\sum_{u \in V} V_f^+(u) = 0$. Implies that $(m + n + 2)a = 0$, which again implies that $m + n \equiv 0(\text{mod } 2)$. Conversely, assume that $m + n \equiv 0(\text{mod } 2)$. Then we have both m and n are even or m and n are odd.

Case 1: m and n are even

Define $f : V \to V_4 \backslash \{0\}$ as :

$$f(v) = f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, m$$

$$f(u) = f(v_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Case 2: m and n are odd

Define $f : V \to V_4 \backslash \{0\}$ as :

$$f(u) = f(v) = b$$

$$f(u_i) = f(v_j) = c \quad \text{for } i = 1, 2, 3, \ldots, m \text{ and } j = 1, 2, 3, \ldots, n$$

In either case, f is an a-star V_4-magic labeling of $B_{m,n}$. Hence the theorem is proved. □

Corollary 7.2.14. *$B_{m,n} \in \Psi_{a,0}$ if and only if $m + n \equiv 0(\text{mod } 2)$.*

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 7.2.13. □

Theorem 7.2.15. *The wheel graph $W_n \in \Psi_a$ if and only if n is odd.*

Proof. Assume that $W_n \in \Psi_a$. Then by Lemma 7.2.1, we have $(n + 1)a = 0$. Hence n is odd. Conversely, assume that n is odd. Let $V = \{u_i : 0 \leq i \leq n\}$ be the vertex set of W_n, where u_0 be the central vertex. Define $f : V \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

Then, $V_f^+(u_i) = 0$ for all i. This completes the proof. □

Corollary 7.2.16. *$W_n \in \Psi_{a,0}$ if and only if n is odd.*

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 7.2.15. □

Theorem 7.2.17. *$H_n \notin \Psi_a$ for all n.*

Proof. Suppose that $H_n \in \Psi_a$ for some n. Then by Lemma 7.2.1, we have $(2n + 1).a = 0$, implies that $a = 0$, a contradiction. Hence $H_n \notin \Psi_a$ for all n. □

Corollary 7.2.18. *$H_n \notin \Psi_{a,0}$ for all n.*

Proof. It follows from Theorem 7.2.17. □

Theorem 7.2.19. *The Jelly fish $J(m, n) \in \Psi_a$ if and only if $m+n \equiv 0(\bmod\ 2)$.*

Proof. Assume that $J(m, n) \in \Psi_a$. Then by Lemma 7.2.1, we have $(m+n+4)a = 0$. Hence $m + n \equiv 0(\text{mod } 2)$. Conversely, assume that $m + n \equiv 0(\text{mod } 2)$. Then either m and n are even or m and n are odd. Let $V(J(m, n)) = V_1 \cup V_2$ where $V_1 = \{w_1, w_2, w_3, w_4\}$, $V_2 = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = E_1 \cup E_2$, where $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\}$, $E_2 = \{w_2u_i, w_4v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Case 1: Both m and n are even.

Define $f : V(J(m,n)) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i \geq 2 \end{cases}$$

$$f(u_i) = b \ \text{ for } \ 1 \leq i \leq m$$

$$f(v_j) = b \ \text{ for } \ 1 \leq j \leq n$$

Case 2: Both m and n are odd

Define $f : V(J(m,n)) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = \begin{cases} c & \text{if} \quad i = 1,2,3 \\ b & \text{if} \quad i = 4 \end{cases}$$

$$f(u_i) = b \ \text{ for } \ 1 \leq i \leq m$$

$$f(v_j) = c \ \text{ for } \ 1 \leq j \leq n$$

In either case, f is an a-star V_4-magic labeling of $J(m,n)$. □

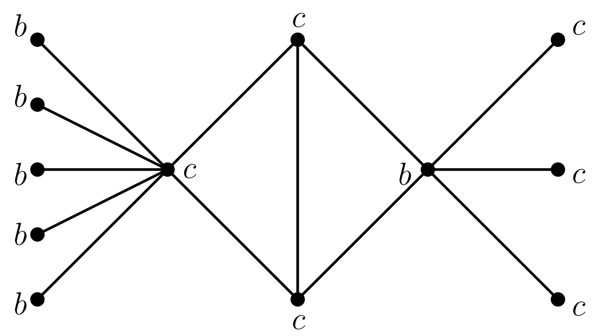

Figure 7.2: An a-star V_4-magic labeling of $J(5,3)$

Theorem 7.2.20. *$J(m, n) \in \Psi_{a,0}$ if and only if $m + n \equiv 0 (\text{mod } 2)$.*

Proof. Proof is obvious from Lemma 7.2.2 and Theorem 7.2.19. □

Theorem 7.2.21. *The crown $C_n^* \in \Psi_a$ for all $n \geq 3$.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the rim vertices and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices adjacent to $u_1, u_2, u_3, \ldots, u_n$ respectively in C_n^*. We define $f : V(C_n^*) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

$$f(v_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Then, $V_f^+(u_i) = V_f^+(v_i) = a$ for $i = 1, 2, 3, \ldots, n$. This completes the proof. □

Corollary 7.2.22. *$C_n^* \in \Psi_{a,0}$ for all $n \geq 3$.*

Proof. Proof obviously follows from Lemma 7.2.2 and Theorem 7.2.21. □

Theorem 7.2.23. *The flower graph $Fl_n \notin \Psi_a$ for all n.*

Proof. Suppose that $Fl_n \in \Psi_a$. Then by Lemma 7.2.1, we have $(2n + 1)a = 0$. Implies that $a = 0$, a contradiction. Hence $Fl_n \notin \Psi_a$ for all n. □

Corollary 7.2.24. *$Fl_n \notin \Psi_{a,0}$ for all n.*

Proof. Proof is obvious from Theorem 7.2.23. □

Theorem 7.2.25. *The friendship graph $F_m \notin \Psi_a$ for all m.*

Proof. Suppose that $F_m \in \Psi_a$. Then by Lemma 7.2.1, we have $(2m+1)a = 0$. Implies that $a = 0$, a contradiction. Hence $F_m \notin \Psi_a$ for all m. □

Corollary 7.2.26. *$F_m \notin \Psi_{a,0}$ for all m.*

Proof. Proof follows from Theorem 7.2.25. □

Theorem 7.2.27. *The book graph $B_n \in \Psi_a$ for all n.*

Proof. Let $V_1 = \{u, u_1, u_2, u_3, \dots, u_n\}$ and $V_2 = \{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, where u be the central vertex and $u_i's$ are pendant vertices in S_n. Then $V(B_n) = \{(u, v_j), (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$. We define $f : V(B_n) \to V_4 \backslash \{0\}$ as :

$$f(u, v_j) = f(u_i, v_j) = b \quad \text{for} \quad 1 \leq i \leq n \ \text{ and } j = 1$$

$$f(u, v_j) = f(u_i, v_j) = c \quad \text{for} \quad 1 \leq i \leq n \ \text{ and } j = 2$$

Then, $V_f^+ \equiv a$. Hence the proof. □

Corollary 7.2.28. *$B_n \in \Psi_{a,0}$ for all n.*

Proof. Proof follows from Lemma 7.2.2 and Theorem 7.2.27. □

Chapter 8

Star V_4-magic Labeling of Some Special Graphs

This chapter investigates star V_4-magic labeling of fan related graphs and some more graphs. The first section provides definitions of fan, fan related graphs and some other graphs. Second section discusses star V_4-magic labeling of fan and fan related graphs. The last section of the chapter investigates star V_4-magic labeling of few more graphs.

8.1 Introduction

The graph $\mathbb{F}_n = P_n \vee K_1$ is called a fan where $P_n : u_1u_2 \ldots u_n$ be a path and $V(K_1) = u$. The Umbrella $U_{n,m}(m > 1)$ is obtained from a fan $\mathbb{F}_n$ by appending a path $P_m : v_1v_2 \ldots v_m$ to the central vertex of the fan $\mathbb{F}_n$ [17]. An extended umbrella graph $U_{n,m,k}$ is a graph obtained by identifying the pendant vertex of the umbrella $U_{n,m}$ with the apex of the star $K_{1,k}$. The Jahangir graph $J_{n,m}$ for

$m \geq 3$ is a graph consisting of a cycle C_{nm} with one additional vertex called the central vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm} [15]. The web graph $W(2,n)$ is the graph obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a single pendant edge to each vertex of the outer cycle [8]. The Jewel Graph J_n is a graph with vertex set $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$ [20].

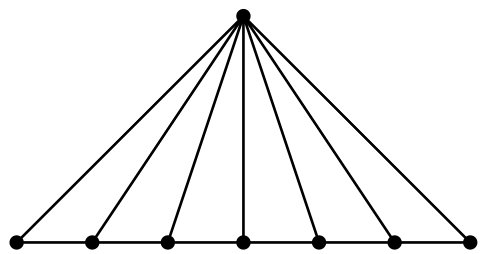

Figure 8.1: Fan graph $\mathbb{F}_7$

8.2 Fan related graphs

Theorem 8.2.1. *The fan* $\mathbb{F}_n \in \Psi_a$ *if and only if* $n \equiv 1(\text{mod } 2)$.

Proof. Assume that $\mathbb{F}_n \in \Psi_a$ with a labeling f. Then by Lemma 7.2.1, we have $(n+1).a = 0$, which implies that $n \equiv 1(\text{mod } 2)$. Conversely, assume that $n \equiv 1(\text{mod } 2)$. Let $\mathbb{F}_n$ be the fan with apex u_0 and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the path P_n in order. Define $f : V(\mathbb{F}_n) \to V_4 \backslash \{0\}$ as:

$$f(u_j) = \begin{cases} b & \text{if} \quad j = 0 \\ c & \text{if} \quad j \geq 1 \end{cases}$$

Then, $V_f^+(u_j) = a$ for all j. This completes the proof of the theorem. □

Corollary 8.2.2. *$\mathbb{F}_n \in \Psi_{a,0}$ if and only if $n \equiv 1 (\text{mod } 2)$.*

Proof. Proof follows from Lemma 7.2.2 and Theorem 8.2.1 □

Theorem 8.2.3. *The umbrella $U_{n,m} \in \Psi_a$ if and only if $(n+m) \equiv 0 (\text{mod } 2)$.*

Proof. Suppose that $U_{n,m} \in \Psi_a$ with a labeling f. Then by Lemma 7.2.1, we have $(n+m).a = 0$. Hence $(n+m) \equiv 0 (\text{mod } 2)$. Conversely, suppose that $(n+m) \equiv 0 (\text{mod } 2)$. Let the vertex set of $U_{n,m}$ be $V = \{u_1, u_2, u_3, \ldots, u_n, u_0 = v_1, v_2, v_3, \ldots, v_m\}$ and edge set $E = \{u_0u_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq m-1\}$. We consider the following cases:

Case 1: $n \equiv 0 (\text{mod } 2)$ and $m \equiv 0 (\text{mod } 2)$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \end{cases}$$

Case 2: $n \equiv 1(\text{mod } 2)$ and $m \equiv 1(\text{mod } 2)$.

Define $f : V \rightarrow V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3(\text{mod } 4) \end{cases}$$

In either case, we have $V_f^+ \equiv a$, completes the proof. □

Corollary 8.2.4. *$U_{n,m} \in \Psi_{a,0}$ if and only if $(n + m) \equiv 0(\text{mod } 2)$.*

Proof. Proof is obvious from Lemma 7.2.2 and Theorem 8.2.3. □

Theorem 8.2.5. *The graph $U_{n,m,k} \in \Psi_a$ if and only if $(n+m+k) \equiv 0(\text{ mod } 2)$.*

Proof. Let the vertex set of $U_{m,n,k}$ be $V = \{u_1, u_2, \ldots, u_n, u_0 = v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_k\}$ and edge set be $E = \{u_0u_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq m-1\} \cup \{v_mw_i : 1 \leq i \leq k\}$. Assume that $U_{n,m,k} \in \Psi_a$ with a labeling f. Then by Lemma 7.2.1, we have $(n+m+k).a = 0$, which implies that $(n+m+k) \equiv 0(\text{ mod } 2)$. Conversely, assume that $(n+m+k) \equiv 0(\text{ mod } 2)$. We consider the following cases:

Case 1: $n \equiv 0(\text{mod } 2)$, $m \equiv 0(\text{mod } 2)$ and $k \equiv 0(\text{mod } 2)$.

Subcase 1: $n \equiv 0(\text{mod } 2)$, $m \equiv 0(\text{mod } 4)$ and $k \equiv 0(\text{mod } 2)$.

Define $f : V \rightarrow V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = c \quad \text{for all} \quad 1 \leq i \leq k.$$

Subcase 2: $n \equiv 0(\text{mod } 2)$, $m \equiv 2(\text{mod } 4)$ and $k \equiv 0(\text{mod } 2)$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = b \quad \text{for all} \quad 1 \leq i \leq k.$$

Case 2: $n \equiv 1(\text{mod } 2)$, $m \equiv 1(\text{mod } 2)$ and $k \equiv 0(\text{mod } 2)$.

Subcase 1: $n \equiv 1(\text{mod } 2)$, $m \equiv 1(\text{mod } 4)$ and $k \equiv 0(\text{mod } 2)$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = c \quad \text{for all} \quad 1 \leq i \leq k.$$

Subcase 2: $n \equiv 1(\text{mod } 2)$, $m \equiv 3(\text{mod } 4)$ and $k \equiv 0(\text{mod } 2)$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = b \quad \text{for all} \quad 1 \leq i \leq k.$$

Case 3: $n \equiv 1(\text{mod } 2)$, $m \equiv 0(\text{mod } 2)$ and $k \equiv 1(\text{mod } 2)$.

Subcase 1: $n \equiv 1(\text{mod } 2)$, $m \equiv 0(\text{mod } 4)$ and $k \equiv 1(\text{mod } 2)$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1,2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0,3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = b \quad \text{for all} \quad 1 \leq i \leq k.$$

Subcase 2: $n \equiv 1 (\text{mod } 2)$, $m \equiv 2 (\text{mod } 4)$ and $k \equiv 1 (\text{mod } 2)$.

Define $f : V \rightarrow V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = c \quad \text{for all} \quad 1 \leq i \leq k.$$

Case 4: $n \equiv 0 (\text{mod } 2)$, $m \equiv 1 (\text{mod } 2)$ and $k \equiv 1 (\text{mod } 2)$.

Subcase 1: $n \equiv 0 (\text{mod } 2)$, $m \equiv 1 (\text{mod } 4)$ and $k \equiv 1 (\text{mod } 2)$.

Define $f : V \rightarrow V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = c \quad \text{for all} \quad 1 \leq i \leq k.$$

Subcase 2: $n \equiv 0 (\text{mod } 2)$, $m \equiv 3 (\text{mod } 4)$ and $k \equiv 1 (\text{mod } 2)$.

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0 \\ c & \text{if} \quad i \geq 1 \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 0, 1(\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3(\text{mod } 4) \end{cases}$$

$$f(w_i) = b \quad \text{for all} \quad 1 \leq i \leq k.$$

In each of the above cases, f gives a-star V_4-magic labeling of $U_{m,n,k}$. □

Corollary 8.2.6. *The graph* $U_{n,m,k} \in \Psi_{a,0}$ *if and only if* $(n+m+k) \equiv 0(\text{ mod } 2)$.

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 8.2.5. □

8.3 Some more graphs

Theorem 8.3.1. *The Jahangir graph* $J_{n,m} \in \Psi_a$ *if and only if* $n \equiv 1(\text{mod } 2)$ *and* $m \equiv 1(\text{mod } 2)$.

Proof. Consider the Jahangir graph $J_{n,m}$ with vertex set $V = \{w, w_i, w_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and edge set $E = \{ww_i : 1 \leq i \leq m\} \cup \{w_{i,j}w_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-2\} \cup \{w_iw_{i,1} : 1 \leq i \leq m\} \cup \{w_{i,n-1}w_{i+1} : 1 \leq i \leq m-1\} \cup \{w_{m,n-1}w_1\}$. Assume that $J_{n,m} \in \Psi_a$. Then by Lemma 7.2.1, we have $(nm+1).a = 0$, implies that $nm \equiv 1(\text{mod } 2)$, which again implies that $n \equiv 1(\text{mod } 2)$ and $m \equiv 1(\text{mod } 2)$. Conversely, assume that $n \equiv 1(\text{mod } 2)$ and $m \equiv 1(\text{mod } 2)$. Here we consider the following two cases:

Case 1: $n \equiv 1 (\text{mod } 4)$ and $m \equiv 1 (\text{mod } 2)$

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(w) = b$$

$$f(w_i) = c \quad \text{for} \quad i = 1, 2, 3, \dots, m$$

$$f(w_{i,j}) = \begin{cases} c & \text{if} \quad j \equiv 0, 1 (\text{mod } 4) \\ b & \text{if} \quad j \equiv 2, 3 (\text{mod } 4) \end{cases}$$

Case 2: $n \equiv 3 (\text{mod } 4)$ and $m \equiv 1 (\text{mod } 2)$

Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(w) = b$$

$$f(w_i) = c \quad \text{for} \quad i = 1, 2, 3, \dots, m$$

$$f(w_{i,j}) = \begin{cases} b & \text{if} \quad j \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad j \equiv 0, 3 (\text{mod } 4) \end{cases}$$

In either case, f gives an a-star V_4-magic labeling of $J_{n,m}$. □

Corollary 8.3.2. *$J_{n,m} \in \Psi_{a,0}$ if and only if $n \equiv 1 (\text{mod } 2)$ and $m \equiv 1 (\text{mod } 2)$.*

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 8.3.1. □

Theorem 8.3.3. *The graph $Bt(n, k) \in \Psi_a$ if and only if $n \equiv 1 (\text{mod } 2)$ and $k \equiv 0 (\text{mod } 2)$.*

Proof. Assume that $Bt(n,k) \in \Psi_a$. Then by Lemma 7.2.1, we have $(nk+n+1).a = 0$, implies that $n(k+1)$ is odd, which again implies that $n \equiv 1(\text{mod } 2)$ and $k \equiv 0(\text{mod } 2)$. Conversely, assume that $n \equiv 1(\text{mod } 2)$ and $k \equiv 0(\text{mod } 2)$. Let $V = \{u, u_i, u_{ij} : 1 \leq i \leq n\ , 1 \leq j \leq k\}$ be the vertex set and $E = \{uu_i, u_iu_{i1}, u_{i1}u_{ij} : 1 \leq i \leq n, 2 \leq j \leq k\}$ edge set of $Bt(n,k)$. We define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u) = b$$

$$f(u_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

$$f(u_{ij}) = \begin{cases} c & \text{if} \quad j = 1 \ \text{ and } \ 1 \leq i \leq n \\ b & \text{otherwise} \end{cases}$$

Clearly, f is an a-star V_4-magic labeling of $Bt(n,k)$. Completes the proof. □

Corollary 8.3.4. *The graph* $Bt(n,k) \in \Psi_{a,0}$ *if and only if* $n \equiv 1(\text{mod } 2)$ *and* $k \equiv 0(\text{mod } 2)$.

Proof. It directly follows from Lemma 7.2.2 and Theorem 8.3.3. □

Theorem 8.3.5. *The graph* $< K_{1,n} : m > \in \Psi_a$ *if and only if* $n \equiv 0(\text{ mod } 2)$ *and* $m \equiv 1(\text{mod } 2)$.

Proof. Let G be the graph $< K_{1,n} : m >$ and let $V_i = \{u_i, u_{ij} : 1 \leq j \leq n\}$ be the vertex set of i^{th} copy of $K_{1,n}$ with apex u_i and let u be the unique vertex adjacent to the central vertices $u_i (1 \leq i \leq m)$ in G. Then $V(G) = V_1 \cup V_2 \cup \ldots \cup V_m \cup \{u\}$. Assume that $G \in \Psi_a$. Then by Lemma 7.2.1, we have $(mn+m+1).a = 0$. Hence $n \equiv 0(\text{mod } 2)$ and $m \equiv 1(\text{mod } 2)$. Conversely, assume that $n \equiv 0(\text{mod } 2)$ and

$m \equiv 1 \pmod 2$. Then $f : V(G) \to V_4 \backslash \{0\}$ defined by:

$$f(u_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, m$$

$$f(u) = f(u_{i,j}) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, m \quad \text{and} \quad j = 1, 2, 3, \ldots, n.$$

gives an a-star V_4-magic labeling of G. Hence the theorem is proved. □

Corollary 8.3.6. *The graph* $< K_{1,n} : m > \in \Psi_{a,0}$ *if and only if* $n \equiv 0 \pmod 2$ *and* $m \equiv 1 \pmod 2$.

Proof. Proof is obvious from Lemma 7.2.2 and Theorem 8.3.5. □

Theorem 8.3.7. *The ladder* $L_n \in \Psi_a$ *for all* n.

Proof. Consider the ladder L_n with vertex set $V = \{u_i, v_i : 1 \le i \le n\}$ and edge set $E = \{u_i v_i : 1 \le i \le n\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n-1\}$. For each $i = 1, 2, 3, \ldots, n$, defining $f(u_i) = b$ and $f(v_i) = c$, we get $L_n \in \Psi_a$. □

Corollary 8.3.8. $L_n \in \Psi_{a,0}$ *for all* n.

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 8.3.7. □

Theorem 8.3.9. *The comb* $CB_n \in \Psi_a$ *for all* n.

Proof. Consider the comb CB_n with vertex set $V = \{u_i, v_i : 1 \le i \le n\}$ and edge set $E = \{u_i v_i : 1 \le i \le n\} \cup \{u_i u_{i+1} : 1 \le i \le n-1\}$. For each $i = 1, 2, 3, \ldots, n$, defining $f(u_i) = b$ and $f(v_i) = c$, we get $CB_n \in \Psi_a$. □

Corollary 8.3.10. $CB_n \in \Psi_{a,0}$ *for all* n.

Proof. Proof directly follows from Lemma 7.2.2 and Theorem 8.3.9. □

Theorem 8.3.11. *The gear graph $G_n \notin \Psi_a$ for any n.*

Proof. Suppose that $G_n \in \Psi_a$. Then by Lemma 7.2.1, we have $(2n+1)a = 0$, which implies that $a = 0$, a contradiction. □

Corollary 8.3.12. *$G_n \notin \Psi_{a,0}$ for any n.*

Theorem 8.3.13. *The web graph $W(2,n) \in \Psi_a$ if and only if $n \equiv 1 (\text{mod } 2)$.*

Proof. Assume that $W(2,n) \in \Psi_a$. Then by Lemma 7.2.1, we have $(3n+1)a = 0$, implies that $n \equiv 1(\text{mod } 2)$. Conversely, assume that $n \equiv 1(\text{mod } 2)$. Let u be the central vertex, let $u_1, u_2, u_3, \ldots, u_n$ be vertices of inner circle, $v_1, v_2, v_3, \ldots, v_n$ be the vertices of outer circle and $w_1, w_2, w_3, \ldots, w_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$ respectively in $W(2,n)$. We define $f : V(W(2,n)) \to V_4 \backslash \{0\}$ by:

$$f(u) = b$$

$$f(u_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

$$f(v_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

$$f(w_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Clearly, f is an a-star V_4-magic labeling of $W(2,n)$. Hence the theorem is proved. □

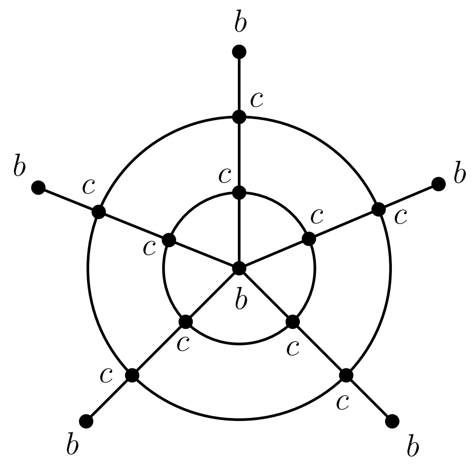

Figure 8.2: An a-star V_4-magic labeling of $W(2,5)$

Corollary 8.3.14. *$W(2,n) \in \Psi_{a,0}$ if and only if $n \equiv 1(\text{mod } 2)$.*

Theorem 8.3.15. *The Jewel graph $J_n \in \Psi_a$ if and only if $n \equiv 0(\text{mod } 2)$.*

Proof. Consider the jewel graph J_n with vertex $V(J_n) = \{u, x, v, y, v_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$. Suppose that $J_n \in \Psi_a$. Then by Lemma 7.2.1, we have $(n+4)a = 0$, hence $n \equiv 0(\text{ mod } 2)$. Conversely, suppose that $n \equiv 0(\text{mod } 2)$. Define $f : V(J_n)) \to V_4 \backslash \{0\}$ as:

$$f(v) = c$$

$$f(u) = f(x) = f(y) = b$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i = 1 \\ c & \text{if} \quad i > 1 \end{cases}$$

Obviously, f is an a-star V_4-magic labeling of J_n. □

Corollary 8.3.16. *$J_n \in \Psi_{a,0}$ if and only if $n \equiv 0 (\text{mod } 2)$.*

Theorem 8.3.17. *$C_n \odot K_2$ admits a-star V_4-magic labeling for $n \equiv 0 (\text{mod } 4)$.*

Proof. Let C_n be the cycle with vertices $u_1, u_2, u_3, \ldots, u_n$ and let v_k and w_k be the vertices of k^{th} copy of K_2. Define $f : V(C_n \odot K_2) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

$$f(w_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0, 3 (\text{mod } 4) \end{cases}$$

Then, $V_f^+(u_i) = V_f^+(v_i) = V_f^+(w_i) = a$. This completes the proof. □

Corollary 8.3.18. *$C_n \odot K_2 \in \Psi_{a,0}$ for $n \equiv 0 (\text{mod } 4)$.*

Theorem 8.3.19. *$P_n \odot \overline{K}_2 \in \Psi_a$ if and only if $n \equiv 0 (\text{mod } 2)$.*

Proof. Let P_n be the cycle with vertices $u_1, u_2, u_3, \ldots, u_n$ and let v_k and w_k be the vertices of k^{th} copy of $\overline{K}_2$. Assume that $P_n \odot \overline{K}_2 \in \Psi_a$. Then by Lemma 7.2.2, we have $3na = 0$, hence $n \equiv 0 (\text{mod } 2)$. Conversely, assume that $n \equiv 0 (\text{mod } 2)$. We define $f : V(C_n \odot K_2) \to V_4 \backslash \{0\}$ as :

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 2,3 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0,1 (\text{mod } 4) \end{cases}$$

$$f(w_i) = \begin{cases} c & \text{if} \quad i \equiv 2,3 (\text{mod } 4) \\ b & \text{if} \quad i \equiv 0,1 (\text{mod } 4) \end{cases}$$

Clearly, f is an a-star V_4-magic labeling of $P_n \odot \overline{K}_2$. □

Corollary 8.3.20. *$P_n \odot \overline{K}_2 \in \Psi_{a,0}$ if an only if $n \equiv 0(\text{mod } 2)$.*

Theorem 8.3.21. *The planar grid $P_m \square P_n \in \Psi_a$ if and only if $mn \equiv 0(\text{ mod } 2)$.*

Proof. Let $\{u_1, u_2, u_3, \ldots, u_m\}$ and $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex sets of P_m and P_n respectively. Assume that $P_m \square P_n \in \Psi_a$. Then by Lemma 7.2.1, we have $mna = 0$, which implies that $mn \equiv 0(\text{mod } 2)$. Conversely, assume that $mn \equiv 0(\text{mod } 2)$. We consider the following cases:

Case 1: $m \equiv 0(\text{mod } 2)$ and $n \equiv 0(\text{mod } 2)$

Define $f : V(P_m \square P_n) \to V_4 \backslash \{0\}$ as:

$$f(u_i, v_j) = \begin{cases} b & \text{if} \quad j \equiv 0,1 (\text{mod } 4) \\ c & \text{if} \quad j \equiv 2,3 (\text{mod } 4) \end{cases}$$

Case 2: $m \equiv 0(\text{mod } 2)$ and $n \equiv 1(\text{mod } 2)$

Define $f : V(P_m \square P_n) \to V_4 \backslash \{0\}$ as:

$$f(u_i, v_j) = \begin{cases} b & \text{if} \quad i \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad i \equiv 2, 3 (\text{mod } 4) \end{cases}$$

Case 3: $m \equiv 1(\text{mod } 2)$ and $n \equiv 0(\text{mod } 2)$

Define $f : V(P_m \square P_n) \to V_4 \backslash \{0\}$ as:

$$f(u_i, v_j) = \begin{cases} b & \text{if} \quad j \equiv 0, 1 (\text{mod } 4) \\ c & \text{if} \quad j \equiv 2, 3 (\text{mod } 4) \end{cases}$$

In each of the above cases f is an a-star V_4-magic labeling of $P_m \square P_n$. □

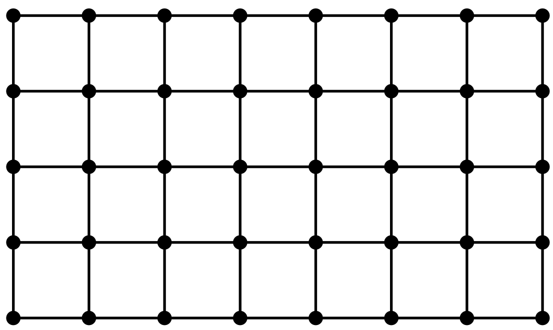

Figure 8.3: The Planar grid $P_8 \square P_5$

Corollary 8.3.22. $P_m \square P_n \in \Psi_{a,0}$ *if and only if* $mn \equiv 0(\text{mod } 2)$.

Chapter 9

Conclusion and Further Scope of Research

This chapter includes a summary of the thesis and some directions in which one can go for further research in this area.

9.1 Summary of the thesis

In this thesis, new types of labelings such as Neighbourhood V_4-magic labeling, Neighbourhood barycentric V_4-magic labeling and Star V_4-magic labeling were introduced. As a first stage, we studied Neighbourhood V_4-magic labeling of cycle, star and path related graphs. Further, Neighbourhood V_4-magic labeling of complete bipartite graphs and regular graphs are discussed, followed by Neighbourhood V_4-magic labeling of splitting, shadow and middle of some special graphs are discussed. The thesis introduced Neighbourhood barycentric V_4-magic labeling of some special graphs like cycle, path, the complete graph K_n, the complete bipartite graph $K_{m,n}$, wheel graph, book graph, trees and some more graphs.

The Star V_4-magic labeling of graphs such as cycle, path, complete bipartite graph, wheel graph, jellyfish graph are studied. Star V_4-magic labeling of special graph like fan graph, umbrella graph and Jewel graph are discussed.

9.2 Further scope of research

(i) Examine Necessary and Sufficient conditions of Neighbourhood V_4-magic labeling of some more graphs.

(ii) Examine Necessary and Sufficient conditions of Neighbourhood V_4-magic labeling of Cartesian product and Lexico graphical product of graphs.

(iii) Investigate Neighbourhood V_4-magic labeling of middle graph of some more graphs.

(iv) Identify Neighbourhood barycentric V_4-magic labeling of some more graphs.

(v) Examine Necessary and Sufficient conditions of Star V_4-magic labeling of some more graphs.

(vi) Find Necessary and Sufficient conditions of Star V_4-magic labeling of Cartesian product and Lexico graphical product of graphs.

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APPENDIX I

List of publications

1. K. P. Vineesh and V. Anil Kumar, *Neighbourhood V_4-magic labeling of some cycle related graphs*, Far East Journal of Mathematical Sciences (FJMS), Volume 111, Number 2, 2019, Pages 263-272.

2. K. P. Vineesh and V. Anil Kumar, *Neighbourhood V_4-magic labeling of star and path related graphs*, Journal of Discrete Mathematical Sciences & Cryptography, Volume 22 (2019), Number 6, Pages 1067–1076.

3. K. P. Vineesh and V. Anil Kumar, *Neighbourhood V_4-magic labeling of some graphs*, Far East Journal of Mathematical Sciences (FJMS), Volume 113, Number 1, 2019, Pages 47-64.

4. K. P. Vineesh and V. Anil Kumar, *Neighbourhood V_4-magic labeling of some splitting graphs*, International Journal research in Advent technology, Volume 7, Number 1, January 2019, Pages 530-535.

5. K. P. Vineesh and V. Anil Kumar, *Neighbourhood barycentric V_4-magic labelings of some graphs*, International Journal of Research and Analytical Reviews, Volume 6, Issue 1, February 2019, Pages 1350-1360.

6. K. P. Vineesh and V. Anil Kumar, *Star magic labeling of some graphs*, Far East Journal of Mathematical Sciences (FJMS), Volume 113, Number 2, 2019, Pages 209-219.

7. K. P. Vineesh and V. Anil Kumar, *Neighbourhood V_4-magic labeling of some shadow graphs*, International Journal of Mathematical Combinatorics, Volume 2 (2019), Pages 86-98.

8. K. P. Vineesh and V. Anil Kumar, *Neighbourhood V_4-magic labeling of some middle graphs*, Malaya Journal of Matematik, Volume 8, Number 2 (2020), Pages 499-501.

9. K. P. Vineesh and V. Anil Kumar, *Star magic labeling of some special graphs*, Communicated to Journal of Discrete Mathematical Sciences & Cryptography, June 2019.

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