Domination in Graphs and Fuzzy Graphs

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By

LEKHA. A

Research Department of Mathematics
St. Mary's College, Thrissur
Kerala, India- 680020

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Dr. Parvathy. K.S
Research Supervisor
P.G.Department and Research
Centre of Mathematics
St.Mary's College, Thrissur
Kerala-680020

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Parvathy

Thrissur
20. 09. 2023

Dr. Parvathy K. S
Research Supervisor

Dr. PARVATHY K. S.
Associate professor
Department of Mathematics
St.Mary's College, Thrissur-20

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St. Mary's College,
Thrissur-680020 Lekha. A

ABSTRACT

Graphs are fundamental mathematical structures used to represent and analyze pairwise relationships between objects. Domination theory is a significant area of study within graph theory, as it addresses several practical problems arising in diverse applications. It provides tools and techniques to solve problems related to network design, optimization, and resilience, making it an essential branch of graph theory. Researchers explore generalizations and variations of domination problems that have applications in different contexts.

In classical domination, a vertex in a graph dominates only the vertices in its immediate neighborhood. But there are situations where a vertex can influence all vertices within a given distance. In this thesis, we made an attempt to study different variations of domination where some distance conditions are imposed on the dominated set.

Dankelmann et al. introduced exponential domination which handles situations in which the influence of a vertex extends to any arbitrary distance but decays exponentially with that distance. Goddard et al. defined disjunctive domination, which keeps the exponential decay of the influence, but only considers distances one and two. Several works have been done on this topic by many researchers. We have explored some properties of disjunctive domination in some classes of graphs and in various graph products.

Efficiency in domination theory often involves reducing redundancy within the dominating sets. We have introduced and studied efficient and nearly efficient disjunctive dominating sets in graphs, which have importance in minimizing waste and maximizing the effectiveness of dominating sets in practical applications.

We introduced a strength-based domination parameter where the dominating strength of a vertex extends to all the other vertices in the graph. This general concept is a generalization of the usual domination and disjunctive domination. Several properties of this new parameter have been investigated. The exact values or bounds of the strength-based domination number are obtained in various classes of graphs.

Domination in fuzzy graphs was first introduced by A. Somasundaram and S. Somasundaram. Many others studied several other variations of domination in fuzzy graphs. We also attempted to study dominating sets in fuzzy graphs as fuzzy subsets of the vertex set.

Key words: domination in graphs; domination number; disjunctive domination number; graph operations; domination in fuzzy graphs.

പഠനസംഗ്രഹം

ഗണിതശാസ്ത്രത്തിലെ വളരെ സജീവവും രസകരവുമായ ഒരു ഗവേഷണ മേഖലയാണ് ഗ്രാഫ് തിയറി അഥവാ ഗ്രാഫ് സിദ്ധാന്തം. ഗ്രാഫുകൾ എന്നത് വ്യക്തികളുടെയോ വസ്തുക്കളുടേയോ പരസ്പര ബന്ധങ്ങളെ ചിത്രീകരിക്കുന്ന ഗണിത ഘടനയാണ്. നിത്യജീവിതത്തിൽ മാനവരാശിക്ക് ആവശ്യമായ സാങ്കേതിക ഉപകരണങ്ങളുടെ ശൃംഖലയെ പ്രതിനിധീകരിക്കാൻ വളരെ വ്യാപകമായി ഗ്രാഫുകൾ ഉപയോഗിച്ചു വരുന്നു. സങ്കീർണ്ണമായ നെറ്റ് വർക്കുകളുടെ രൂപകല്പനക്കും വിശകലനത്തിനും വളരെ സഹായകരമായ സാധ്യതകൾ ഗ്രാഫ് തിയറിയിലുണ്ട്.

ഗ്രാഫ് തിയറി യിലെ വളരെ സജീവമായ ഗവേഷണ മേഖലകളിലൊന്നാണ് ഡോമിനേഷൻ തിയറി അഥവാ ആധിപത്യ സിദ്ധാന്തം. ഒരു പ്രദേശത്തെ വിഭവങ്ങളുടേയും സൗകര്യങ്ങളുടേയും ഉത്തമമായ സ്ഥാനനിർണയം നടത്തുന്നതിന് ഗ്രാഫ് ആധിപത്യ സിദ്ധാന്തം ഉപയോഗിക്കാവുന്നതാണ്. ആശയവിനിമയ നെറ്റ്വർക്കുകളിലും സാമൂഹ്യ നെറ്റ്വർക്കുകളിലും രൂപകല്പന, വിശകലനം തുടങ്ങിയവയ്ക്ക് ഗ്രാഫ് ആധിപത്യ സിദ്ധാന്തം ഉപയോഗിച്ചു വരുന്നു . രൂപകല്പനാ സിദ്ധാന്തം, കോഡിങ് സിദ്ധാന്തം, സ്ഥാനനിർണയം, നിരീക്ഷണ ആശയവിനിമയം, ഒപ്റ്റിമൈസേഷൻ സിദ്ധാന്തം തുടങ്ങി പല മേഖലകളിലുള്ള പ്രായോഗികത ഈ സിദ്ധാന്തത്തിലുള്ള ഗേവഷണം സജീവമായി നിലനിർത്തുന്നു. ആധിപത്യ ആശയങ്ങളുടെ വ്യതിയാനങ്ങളും പൊതു വൽക്കരണങ്ങളും പ്രായോഗിക ജീവിതത്തിലെ വിവിധങ്ങളായ പ്രശ്നങ്ങളെ പരിഹരിക്കാൻ ഗവേഷകർ ഉപയോഗിച്ചു വരുന്നു.

'DOMINATION IN GRAPHS AND FUZZY GRAPHS' എന്ന ഈ പ്രബന്ധത്തിൽ ആറു അദ്ധ്യായങ്ങൾ ആണ് ഉൾപ്പെടുത്തിയിട്ടുള്ളത്. ഈ പ്രബന്ധത്തിലേക്ക് ആവശ്യമുള്ള അടിസ്ഥാന നിർവചനങ്ങളും ആശയങ്ങളുമാണ് ഒന്നാമത്തെ അധ്യായത്തിൽ ഉള്ളത്. പ്രായോഗിക സാധ്യതകളെ മുൻനിർത്തിക്കൊണ്ട് വിവിധതരം ആധിപത്യ സിദ്ധാന്ത ആശയങ്ങളും സാങ്കേതിക വസ്തുതകളും അവയുടെ അവലോകനവുമാണ് രണ്ട് മുതൽ അഞ്ചു വരെയുള്ള അദ്ധ്യായങ്ങളിൽ ചേർത്തിരിക്കുന്നത്. ഈ പഠനത്തിൽ നിന്നുള്ള നിഗമനങ്ങളും ഭാവിയിലേക്കുള്ള സാധ്യതകളുമാണ് അവസാനത്തെ അധ്യായത്തിൽ ഉൾപ്പെടുത്തിയിരിക്കുന്നത്.

ഈ പ്രബന്ധത്തിൽ പ്രതിപാദിച്ചിരിക്കുന്ന വിവിധ ആധിപത്യ സിദ്ധാന്ത വ്യതിയാനങ്ങൾ, നൂതന ആശയങ്ങൾ, അവയുടെ അവലോകനങ്ങൾ എന്നിവ പല സാങ്കേതിക പ്രശ്നങ്ങളുടേയും ശാസ്ത്ര പ്രശ്നങ്ങളുടേയും പരിഹാര സാധ്യതകൾക്ക് ഉതകുമെന്ന് പ്രതീക്ഷിക്കുന്നു.

Contents

CHAPTER 1

Introduction

The subject of graph theory had its beginnings in recreational math problems. The history of graph theory may be specifically traced to 1735, when Leonhard Euler, a Swiss Mathematician, solved the famous ‘Konigsberg bridge problem’. Some puzzles and several problems of practical nature have been instrumental in the development of various topics in graph theory. While the solution of ’Konigsberg problem’ lead to the development of Eulerian graph theory, the challenging Hamiltonian graph theory was developed from the ’Around the World’ game of Sir William Hamilton. The origin of graph theory is well recorded in the historic book by Biggs, Lloyd and Wilson [9].

In the present century graph theory has grown into one of the most interdisciplinary branches in mathematics with a great variety of applications. Graph theory can be used as a mathematical tool for designing and analyzing communication networks, social network systems etc. It has wide range of applications in almost all branches of science, engineering, social sciences and even in linguistics.

Domination is a flourishing area of graph theory. Similar to the development of other areas of graph theory, the game of chess becomes inspirational for the study of dominating sets in graphs. Although the mathematical study of dominating sets in graphs began around 1960 the subject has historical roots dating back to 1862, when C. F De Jaenisch [44] analyzed the problem of determining the minimum number of queens which are necessary to cover an $n \times n$ chessboard mathematically. This problem is said to be the origin of the study of dominating sets in graphs.

In 1958, Claude Berge [6] wrote a book on graph theoy, in which he defined for the first time the concept of domination number of a graph (although he called this number the 'coefficient of external stability'). In 1962, Oystein Ore [49] published his book on graph theory, in which he used, for the first time, the names 'dominating set' and 'domination number'. He used the notation $d(G)$ for the domination number of a graph. In 1977, Cockayne and Hedetniemi [17] published a survey of the few results known at that time about dominating sets in graphs, in which they used the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation.

Domination has applications in facility location problems, in problems involving finding sets of representatives, in land surveying, in monitoring communication or electrical networks, in modeling biological or social networks etc. Part of what motivates so much research into domination is the multitude of varieties of domination. Various types of domination are obtained by imposing additional conditions on the method of domination so as to meet a specific purpose.

This thesis entitled **Domination in Graphs and Fuzzy Graphs** intends to make a small contribution to the vast ocean of domination theory in graphs.

1.1 Basic concepts in graph theory

This section handles the basic notations, terminology and definitions relevant to this work [12, 3, 10].

Definition 1.1.1. [12] A graph $G = (V, E)$ consists of a finite nonempty set V of objects called vertices together with a set E of unordered pairs of vertices of G called edges. The edge $e = \{u, v\}$ is said to join the vertices u and v . We write $e = uv$ and say that u and v are adjacent vertices; u and e are incident, as are v and e. If e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are adjacent edges.

A graph is *trivial* if its vertex set is a singleton and it contains no edges, and *nontrivial* otherwise. An edge with identical end points is called a loop. Edges joining the same pair of vertices are called multiple edges. A graph which has no loops and multiple edges is called a *simple graph*. The graphs considered in this thesis are all simple.

Definition 1.1.2. [12] The number of vertices in G is called the order of G and the number of edges in G is called the size of G.

Definition 1.1.3. [12] The degree of a vertex v in a graph G is defined to be the number of edges incident with v and is denoted by $d(v)$. The minimum of $\{d(v) : v \in V(G)\}$ is denoted by δ and the maximum of $\{d(v) : v \in V(G)\}$ is denoted by Δ.

A vertex of degree zero is an *isolated vertex* and a vertex of degree one is a *pendant vertex* or a *leaf*. The edge incident on a pendant vertex is a *pendant edge*. Any vertex which is adjacent to a pendant vertex is called a *support vertex*.

Definition 1.1.4. [12] If G is a graph of order n, then a vertex of degree $n-1$ is called a universal vertex.

Definition 1.1.5. [12] A vertex u is called a neighbor of a vertex v in G, if uv is an edge of G. The set of all neighbors of v is the open neighborhood of v and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G.

Definition 1.1.6. [12] A graph H is called a subgraph of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A subgraph H of a graph G is a proper subgraph of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A spanning subgraph of G is a subgraph H of G with $V(H) = V(G)$.

For a set S of vertices of G, the subgraph induced by S is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$. Similarly, for a subset E' of $E(G)$, the edge induced subgraph $\langle E' \rangle$ is the subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E'.

Let v be a vertex of a graph G and $|V(G)| \geq 2$. Then the induced subgraph $\langle V(G) - \{v\} \rangle$ is denoted by $G - v$ and it is the subgraph of G obtained by the removal of v and the edges incident with v. If $e \in E(G)$, the spanning subgraph with edge set $E(G) - \{e\}$ is denoted by $G - e$ and it is the subgraph of G obtained by the removal of the edge e.

Definition 1.1.7. [12] A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.

Definition 1.1.8. [12] A graph G is called bipartite if the vertex set $V(G)$ can be partitioned into two subsets X and Y such that each edge of G has one end in X and the other end in Y. (X, Y) is called a partition of G.

Definition 1.1.9. [12] A graph G is said to be complete bipartite if G is simple, bipartite with bipartition (X, Y) and each vertex of X is joined to every vertex of Y. If $|X| = m$ and $|Y| = n$, then G is denoted by $K_{m,n}$. The graph $K_{1,n-1}$ is called a star.

Definition 1.1.10. [12] The complement of a simple graph G, denoted by $\bar{G}$, is a simple graph with vertex set $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if they are non adjacent in G.

Definition 1.1.11. [12] A walk W in a graph G is a finite, non- empty, alternating sequence $u_0, e_1, u_1, ..., u_{n-1}, e_n, u_n$ of vertices and edges of G, beginning and ending with vertices, such that $e_i = u_{i-1}u_i$, for $1 \leq i \leq n$. This walk joins u_0 and u_n, and may also be denoted $(u_0, u_1, u_2, ..., u_{n-1}, u_n)$; it is sometimes called a $u_0 - u_n$ walk. It is closed if $u_0 = u_n$ and is open otherwise. If all the vertices $u_0, u_1, ..., u_n$ are distinct, then W is called a $u_0 - u_n$ path, P. A path on n vertices is denoted by P_n.

Definition 1.1.12. [12] A cycle of length $n \geq 3$ in a graph G is a sequence $(u_0, u_1, u_2, ..., u_{n-1}, u_0)$ of vertices of G such that for $0 \leq i \leq n-2$, the vertices u_i and u_{i+1} are adjacent, u_{n-1} and u_0 are adjacent and $u_0, u_1, u_2, ..., u_{n-1}$ are distinct. A cycle on n vertices is denoted by C_n.

Definition 1.1.13. [12] Two vertices u and v of G are said to be connected if there is a (u, v) path between them. A graph G is said to be connected if every pair of vertices of G are joined by a path. A maximal connected subgraph of G is called a component of G.

Definition 1.1.14. [12] The distance between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path connecting them.

Definition 1.1.15. [12] Let S be a set of vertices in a graph G. The distance of a vertex v in G from S is defined as $d(v, S) = \min\{d(u, v) : u \in S\}$. If $v \in S$, then $d(v, S) = 0$.

Definition 1.1.16. [12] The eccentricity $e(v)$ of a vertex v of a connected graph G is $\max\{d(u,v) : u \in V(G)\}$. That is, $e(v)$ is the distance between v and a vertex farthest from v.

Definition 1.1.17. [12] The radius, $rad(G)$, is the minimum eccentricity among the vertices of G, while the diameter, $diam(G)$, of G is the maximum eccentricity. Consequently, diameter of G is the greatest distance between any two vertices of G.

A graph G has radius 1 if and only if G contains a universal vertex. A vertex v is central vertex if $e(v) = rad(G)$ and the center, $Cen(G)$, is the subgraph of G induced by its central vertices.

Definition 1.1.18. The second neighborhood of a vertex $v \in V$, denoted by $N_2(v)$, is $\{u \in V : d(u,v) = 2\}$.

Definition 1.1.19. [10] The Lexicographic product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1[G_2]$ whose vertex set is $V_1 \times V_2$ in which $((u_1, v_1), (u_2, v_2))$ is an edge if $u_1u_2 \in E_1$ or u_1, u_2 are equal and $v_1v_2 \in E_2$.

Definition 1.1.20. *[27]*Tensor product or Cross Product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2$ whose vertex set is $V_1 \times V_2$ and edge set is $\{((u_1, v_1), (u_2, v_2)) : u_1u_2 \in E_1 \textit{ and } v_1v_2 \in E_2\}$.

Definition 1.1.21. *[27]*The Strong Product or Normal Product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \boxtimes G_2$ whose vertex set is $V_1 \times V_2$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1v_2 \in E_2$ or $u_1u_2 \in E_1$ and $v_1 = v_2$ or $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

Definition 1.1.22. *[27]*The Cartesian Product $G_1 \square G_2$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ in which $(u_1, v_1), (u_2, v_2)$ is an edge if and only if either $u_1 = u_2$ and $v_1v_2 \in E_2$ or $u_1u_2 \in E_1$ and $v_1 = v_2$.

1.2 Basic concepts in fuzzy graph theory

Azriel Rosenfeld [51] introduced the concept of fuzzy graphs in 1975, which is a best tool to handle the real-life uncertainties. He described several fuzzy analogues of graph-theoretic concepts such as paths, cycles, trees and connectedness. Since then several works were done on fuzzy graphs. In this section we review some basic definitions and notations of fuzzy graphs. For terminology in fuzzy graphs we refer to Mordeson and Nair [48].

Definition 1.2.1. [48] Let S be a finite non empty set. A fuzzy subset of S is a mapping $\mu : S \to [0,1]$. If μ, ν are two fuzzy subsets of S, then

1. $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in S$ and
2. $\mu \subset \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in S$ and there exists at least one $x \in S$ such that $\mu(x) < \nu(x)$.

Definition 1.2.2. [48] A fuzzy graph $\mathcal{G} = (V, \mu, \sigma)$ is a non empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\sigma : V \times V :\to [0,1]$ such that for all $u, v \in V$, $\sigma(u,v) \leq \mu(u) \wedge \mu(v)$. μ is called fuzzy vertex set of $\mathcal{G}$ and σ is called the fuzzy edge set of $\mathcal{G}$ respectively.

Definition 1.2.3. [48] The order p and size q of a fuzzy graph $\mathcal{G} = (V, \mu, \sigma)$ are defined to be

$$p = \sum_{x \in V} \mu(x) \quad \text{and} \quad q = \sum_{(x,y) \in V \times V} \sigma(x,y)$$

Definition 1.2.4. Let $\mathcal{G} = (V, \mu, \sigma)$ be a fuzzy graph and $S \subset V$. Then the scalar cardinality of S is defined to be $\sum_{v \in S} \mu(v)$ and it is denoted by $|S|$. Then p denotes the scalar cardinality of V, also called the order of $\mathcal{G}$.

Definition 1.2.5. [48] The degree of a vertex v is defined as $d(v) = \sum_{u \neq v} \sigma(u,v)$. The minimum degree of $\mathcal{G}$ is $\delta(\mathcal{G}) = \wedge\{d(v) : v \in V\}$ and the maximum degree is $\Delta(\mathcal{G}) = \vee\{d(v) : v \in V\}$. A vertex v in a fuzzy graph is called an isolated vertex if $\sigma(u,v) = 0$ for all $u \in V$.

Definition 1.2.6. [48] A path P of length n is a sequence of distinct vertices $u_0, u_1, u_2, ..., u_n$ such that $\sigma(u_{i-1}, u_i) > 0$ for $i = 1, 2, ..., n$. Strength of a path is the degree of membership of the weakest edge in P. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle. The strength of a cycle is the strength of the weakest edge in it.

Definition 1.2.7. [48] The strength of connectedness between two vertices u and v is defined as the maximum of the strengths of all paths between u and v and is denoted by $CONN_{\mathcal{G}}(u,v)$. A fuzzy graph $\mathcal{G} = (\mu, \sigma)$ is connected if for every $u, v \in V$, $CONN_{\mathcal{G}}(u,v) > 0$.

Definition 1.2.8. [48] An edge $e = (u, v)$ of a fuzzy graph is called an effective edge if $\sigma(u, v) = \mu(u) \wedge \mu(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighborhood of u and is denoted by $EN(u)$.

Definition 1.2.9. [48] A fuzzy graph $\mathcal{G}$ is said to be complete if $\sigma(u, v) = \mu(u) \wedge \mu(v)$, for all $u, v \in V$.

1.3 Domination in crisp graphs

The following are some of the fundamental definitions and results pertaining to domination in crisp graphs.

Definition 1.3.1. [32] Let $G = (V, E)$ be a graph. A subset S of V is called a **dominating set** of G if every vertex $v \in V$ is either an element of S or is adjacent to at least one element in S. The minimum cardinality of a dominating set is called the **domination number** of G and is denoted by $\gamma(G)$. A dominating set of minimum cardinality is called a $\gamma(G)$-set of G.

Definition 1.3.2. *[32] A dominating set S is a **minimal dominating set** if no proper subset of it is a dominating set. The maximum cardinality of a minimal dominating set is called the **upper domination number** of G and is denoted by $\Gamma(G)$.*

Definition 1.3.3. [32] A set $S \subseteq V(G)$ is a **total dominating set** of G if every vertex $v \in V$ is adjacent to at least one vertex in S. The minimum cardinality of a total dominating set in a graph G is called its **total domination number**, denoted by $\gamma_t(G)$. A $\gamma_t(G)$-set is a total dominating set in G of cardinality $\gamma_t(G)$.

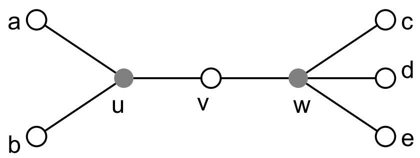

Figure 1.1

For the graph G in Figure 1.1, $\{u, w\}$ is a $\gamma(G)$-set, $\{u, v, w\}$ is a $\gamma_t(G)$-set, $\{a, b, c, d, e\}$ is a minimal dominating set of maximum cardinality, $\gamma(G) = 2$, $\gamma_t(G) = 3$ and $\Gamma(G) = 5$.

Definition 1.3.4. *[5] A dominating set S is called efficient if every vertex of G is dominated by exactly one vertex of S, that is, $|N[v] \cap S| = 1$ for every $v \in V(G)$.*

Theorem 1.3.5. *[32] For any connected graph G,*

$$\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \gamma(G).$$

Definition 1.3.6. *[32]*A subset S of V is called an independent set of G if no two vertices of S are adjacent. A dominating set that is also independent is called an independent dominating set. The minimum cardinality of an independent dominating set is called the independent domination number of G and is denoted by $i(G)$. The maximum cardinality of an independent set in G is called the independence number of G and is denoted by $\beta_0(G)$.

Definition 1.3.7. *[32]*Let S be a set of vertices of a graph G and let $u \in S$. We say that a vertex v is a private neighbor of u (with respect to S) if $N[v] \cap S = \{u\}$. If $N[u] \cap S = \{u\}$, then u is its own private neighbor. A dominating set S is a minimal dominating set if and only if every vertex in S has at least one private neighbor.

Definition 1.3.8. [18, 32] A set S of vertices in G is irredundant if every vertex $v \in S$ has at least one private neighbor. An irredundant set S is called a maximal irredundant set if no proper superset of S is irredundant. The minimum cardinality of a maximal irredundant set in a graph G is called the irredundance number of G and is denoted by $ir(G)$. The maximum cardinality of an irredundant set is called the upper irredundance number of G and is denoted by $IR(G)$.

The six parameters of domination, independence and irredundance are connected by a chain of inequalities as shown in the following theorem which was first observed by Cockayne, Hedetniemi and Miller in 1978.

Theorem 1.3.9. *[18] For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

In 1968, V.G Vizing [55] posed the following conjecture.

Conjecture 1.3.10. *[55] For every pair of finite graphs G and H,*

$$\gamma(G \square H) \geq \gamma(G)\gamma(H).$$

Vizing's conjecture is the most significant unresolved problem in the field of domination theory.

In classical domination a vertex in a graph dominates only the vertices in its immediate neighborhood. But there are situations where a vertex can influence all vertices within a given distance. This kind of situation is considered in distance domination [40].

Definition 1.3.11. *[31]*For an integer $k \geq 1$, a set $S \subseteq V(G)$ is a distance-k dominating set or simply k-dominating set of G if every vertex of $V \setminus S$ is within distance k from some vertex of S. Minimum cardinality among all k-dominating sets of G is called the k-domination number of G and is denoted by $\gamma_k(G)$. It can be observed that $\gamma(G) = \gamma_1(G)$. For the graph given in Figure 1.1, $\{v\}$ is a distance-2 dominating set and $\gamma_2(G) = 1$.

In [21] Dankelmann et al. introduce exponential domination which handles the situations in which the influence of a vertex extends to any arbitrary distance but decays exponentially with that distance. Here it is considered that the 'dominating power' of a vertex decreases exponentially, by the factor $\frac{1}{2}$, with distance. There are two types of exponential domination; porous and non-porous. In non-porous exponential domination, vertices in an exponential domination set block the influence of each other. Whereas in porous exponential domination, the influence of exponential dominating vertices are not blocked.

Definition 1.3.12. *[21]* Let G be a graph and $S \subseteq V(G)$. For each vertex $u \in S$ and for each $v \in V(G) \setminus S$ we define $\bar{d}(u, v) = \bar{d}(v, u)$ to be the length of a shortest path in $\langle V(G) - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise.
For $v \in V(G)$ define

$$w_S(v) = \begin{cases} \sum_{u \in S} \frac{1}{2^{\bar{d}(u,v)-1}} & \text{if } v \notin S \\ 2 & \text{if } v \in S \end{cases}$$

If $w_S(v) \geq 1$ for each $v \in V(G)$, then S is a non- porous exponential dominating set. The smallest cardinality of a non- porous exponential dominating set is the exponential domination number, $\gamma_e(G)$.

Definition 1.3.13. *[21]* Let G be a graph and S a set of vertices of G. For $v \in V(G)$, define $w^*(v) = \sum_{u \in S} \frac{1}{2^{d(u,v)-1}}$. A porous exponential dominating set (or p-exponential dominating set) of G is a set $S \subseteq V(G)$ with $w^*(v) \geq 1$ for all $v \in V$. The minimum cardinality of a porous exponential dominating set is the p-exponential domination number, denoted by $\gamma_e^*(G)$.

Observation 1.3.14. *[21] For every graph G, $\gamma_e^*(G) \leq \gamma_e(G) \leq \gamma(G)$.*

Theorem 1.3.15. [21] For every positive integer n, $\gamma_e^*(P_n) = \gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.

Theorem 1.3.16. *[21] For every integer $n \geq 3$,*

$$\gamma_e(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \lceil \frac{n}{4} \rceil & \text{if } n \neq 4 \end{cases}$$

Motivated by the difficulties to calculate exponential domination number, Goddard et al. defined [26] disjunctive domination, which keeps the exponential decay of the influence, but considers only distances one and two into account.

Definition 1.3.17. [26] For a positive integer b, a set S of vertices in a graph G is a b-disjunctive dominating set, abbreviated bDD-set, in G if every vertex v not in S is adjacent to a vertex of S or has at least b vertices in S at a distance 2 from it in G. The b-disjunctive domination number of G, denoted by $\gamma_b^d(G)$, is the minimum cardinality of a bDD-set in G. The parameter γ_1^d is the distance-2 domination number.

In the special case when $b = 2$, we call a $2DD$-set simply a disjunctive dominating set, abbreviated DD-set, and we call the 2-disjunctive domination number, $\gamma_2^d(G)$, simply the disjunctive domination number. A DD-set of cardinality $\gamma_2^d(G)$ is called a $\gamma_2^d(G)$-set [34]. For the rest of this thesis we look solely at the situation where $b = 2$.

We say that a vertex $v \in V$ is dominated by a set S if v has a neighbor in S. $v \in V$ is disjunctively dominated by a set S if v is at a distance 2 from at least two vertices of S. Further, if v has a neighbor in S, we say S dominates the vertex v, while if v is at a distance 2 from at least two vertices of S, we say S disjunctively dominates the vertex v [36].

Definition 1.3.18. [36] A set of vertices in G is a disjunctive total dominating set, abbreviated DTD-set, of G if every vertex is adjacent to a vertex of S or has at least two vertices in S at a distance 2 from it. The disjunctive total domination number, $\gamma_t^d(G)$, is the minimum cardinality of a DTD-set in G .

Some fundamental results of disjunctive domination which we will require in subsequent chapters of the thesis are given below.

Proposition 1.3.19. *[26]*For any graph G,

1. $\gamma_2^d(G) \leq \gamma(G)$
2. $\gamma_2^d(G) \geq \gamma_e(G)$

Proposition 1.3.20. [26]

1. $\gamma_2^d(G) = n$ if and only if $G = \bar{K_n}$.
2. $\gamma_2^d(G) = 1$ if and only if $\gamma(G) = 1$.
3. $\gamma_2^d(K_{n,m}) = \gamma(K_{n,m}) = 2$ for all $n, m \geq 2$.

Proposition 1.3.21. [26] For any graph G,

1. if $\gamma(G) = 2$, then $\gamma_2^d(G) = 2$
2. if G has diameter at most 2, then $\gamma_2^d(G) \leq 2$

Theorem 1.3.22. [26] For every positive integer n, $\gamma_2^d(P_n) = \left\lceil \frac{n+1}{4} \right\rceil$.

Theorem 1.3.23. *[26] For every integer $n \geq 3$,*

$$\gamma_2^d(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n \neq 4 \end{cases}$$

Theorem 1.3.24. *[26] For a two dimensional grid graph $G_{2,m}$ given by $P_2 \square P_m$, $m \geq 1$,*

$$\gamma_2^d(G_{2,m}) = \left\lceil \frac{m+2}{3} \right\rceil.$$

1.4 Domination in fuzzy graphs

This section deals with the variants of domination in fuzzy graphs.

Definition 1.4.1. *[54]* Let $\mathcal{G} = (V, \mu, \sigma)$ be a fuzzy graph and $u, v \in (V, \mu)$. Then u dominates v in $\mathcal{G}$ if $\sigma(u, v) = \mu(u) \wedge \mu(v)$. Then v dominates u also. A subset S of V is called a dominating set in $\mathcal{G}$ if for every $v \notin S$, there exists $u \in S$ such that u dominates v. The minimum scalar cardinality of a dominating set in $\mathcal{G}$ is called the domination number of $\mathcal{G}$ and is denoted by $\gamma(\mathcal{G})$ or γ.

Bhutani and Rosenfeld [8] have introduced the concept of strong edges of a fuzzy graph . An edge (u, v) is strong if $\sigma(u, v) = CONN(u, v)$. If edge (u, v) is strong, then the vertex v is a strong neighbor vertex u.

A Nagoorgani and V.T Chandrasekaran [25] defined that vertex u dominates v, if (u, v) is a strong edge. A vertex u dominates itself and its strong neighbors. A set S of vertices of $\mathcal{G} = (V, \mu, \sigma)$ is a strong dominating set if every vertex of $V(G) - S$ is a strong neighbor of some vertex in S. A minimum strong dominating set in a fuzzy graph $\mathcal{G}$ is a strong dominating set of minimum scalar cardinality. The scalar cardinality of a minimum strong dominating set is called the strong domination number of $\mathcal{G}$.

O.T Manjusha and M.S Sunitha [46] defined different types of edges, neighborhoods and neighborhood degree of a vertex in a fuzzy graph. In [45] the same authors defined strong domination in fuzzy graphs using membership values of strong edges. The weight of a strong dominating set S is defined as $W(S) = \sum_{u \in S} m(u, v)$, where $m(u, v)$ is the minimum of the membership values of the strong edges incident on u. The strong domination number is the minimum weight of strong dominating set of $\mathcal{G}$.

The above definitions of domination in fuzzy graphs do not consider the fuzzy subsets of the vertex set.

In 1987, Hedetniemi *et al.* [33] introduced the concept of fractional domination and fractional domination number in crisp graphs. Properties of minimal dominating function and fractional domination in graphs was studied by E. J Cockayne *et al.* in [14]. A dominating function (DF) of a graph $G = (V, E)$ is a function $f : V \to [0, 1]$ such that

$$\sum_{x \in N[v]} f(x) \geq 1$$

for all $v \in V$, where $N[v]$ is the closed neighborhood of v. A DF f is called minimal (MDF) if there is no function $g : V \to [0, 1]$ such that $g < f$ and g is a DF. For any DF f, let

$$|f| = \sum_{v \in V} f(v)$$

The fractional domination number $\gamma_f(G)$ is defined by

$$\gamma_f(G) = \min\{|f| : f \text{ is an MDF of } G\}.$$

In [1] the authors defined (r,s)-fuzzy domination in fuzzy graphs as a fuzzy subset of the vertex set. Let $\mathcal{G} = (V, \mu, \sigma)$ be a fuzzy graph. Let $r, s \in [0, 1]$ and $r < s$. A fuzzy subset μ_1 of μ is called an (r, s)-fuzzy dominating set of $\mathcal{G}$ if

$$\Big(\sum_{\sigma(u,v)\geq r} \mu_1(u)\Big) + \mu_1(v) \geq s \text{ for all } v \in V.$$

Several properties of (r, s)-fuzzy domination and related parameters in fuzzy graphs are studied in [1].

1.5 Background of the work

In this section, we shall provide a brief analysis of literature on the domination problem both in crisp graphs and fuzzy graphs.

The well-known concept of domination in graphs and related subset problems is one of the major topics of research in graph theory. It is a is an excellent tool for studying situations that can be represented by networks in which a vertex can exert influence on all vertices in its neighborhood. Haynes *et.al* [32] provide a great resource for the fundamentals of domination in graphs. Surveys of a number of advanced topics in domination are given in [31].

Domination, independence, and irredundance are closely related concepts. There has been a vast amount of work published in this area due to the richness of mathematical theory and the variety of practical applications of these concepts. Dominating sets have been first studied by Berge [6], Ore [49] and Cockayne [17, 15], independent dominating sets have been studied by Berge [7] and Cockayne [16]. The concept of irredundance set was introduced by Cockayne et al. [18]. Several results on domination, independence and irredundance are given in [19, 20, 13]. The domination chain [18], the inequality involving six parameters on domination, independence and irredundance, has become one of the strongest focal point of research in domination theory. In chapter five, we established a similar domination chain in fuzzy graphs.

Another active area of research in domination theory is the Vizing's Conjecture [55] and related problems. Several versions of Vizing's conjecture for various domination-type invariants are studied by many researchers. Vizing-like inequality for other type of graph products also have drawn attention of many mathematicians [11].

Many domination parameters are formed by combining domination with other graph theoretical properties. Some of them are defined by imposing conditions on the dominating set, but conditions can also be placed on the dominated set or the method of domination. In this thesis we consider the cases where conditions are imposed on the dominated set. Different variations of the domination parameters used in this work are all based on the concept of distance of a vertex from the dominating set.

Meir and Moon [47] introduced the concept of a k-packing and a k-dominating set (called a ”k-covering” in [47]) in a graph. In 1976 Slater [53, 39] considered the problem of finding a minimum k-dominating set. Properties of minimum k-dominating sets are studied in [53]. The concept of distance domination in graphs find applications in many situations and structures which give rise to graphs. The concepts of total k-dominating sets and k-independent dominating sets of G are introduced in [39]. In the literature, independent domination has drawn a lot of attention. Properties of distance independent sets are studied in [24]. The concept of k-irredundance was introduced in [29]. Various relations involving distance domination parameters can be found in [39, 28, 38, 37].

In exponential domination [21], the domination power of a vertex can reach any arbitrary distance, but it diminishes exponentially as the distance increases. Exponential domination is the only framework in the literature in which the effect of an exponential dominating vertex is global with respect to other vertices. Even concepts such as distance domination [40] appear basically local because they can be reduced to ordinary domination by considering the proper powers of the underlying graph. There are only few results on exponential domination. As mentioned in Henning et al. [41], the global nature of exponential domination makes it more difficult to study. This might be the reason for the limited number of results on it. A model like this could be used to analyze information dissemination in social networks, where the impact of the information reduces every time it is passed on. In this thesis we introduce another variant of domination that is also global in nature.

Many existing domination-type structures are expensive to implement. Most of the variations on domination and total domination studied in literature tend to focus on adding restrictions which in turn increases their implementation costs. Disjunctive domination, proposed and studied by Goddard et al. [26] is a method for relaxation of the domination number. This motivated us in the study of disjunctive domination in graphs. Several properties of disjunctive and total disjunctive domination number of a graph are given in [35].

Domination in fuzzy graphs was first introduced by A. Somasundaram and S. Somasundaram [54]. They defined domination in fuzzy graphs using the concept of effective edge in a fuzzy graph. A. Nagoorgani and V.T Chandrasekaran [25] defined domination in fuzzy graphs using strong edges. In [45], O.T Manjusha and M.S. Sunitha defined strong domination in fuzzy graphs using membership values of strong edges. All these definitions use either the effective or the strong edges of a fuzzy graph. But there are situations where we need to consider all the non-zero edges into account, even if they are very small in strength. Further these definitions of domination in fuzzy graphs do not consider the fuzzy subsets of the vertex set. But while considering fuzzy graphs and their subset problems it is more apt to consider fuzzy subsets of the vertex set than their crisp subsets. As far as we know, (r, s)-fuzzy domination [1] is the only framework in literature where the authors consider the fuzzy subset of its vertex set. But this definition also does not take into account all the non- zero edges incident at a vertex. Motivated by all these, we defined a fuzzy dominating subset of a fuzzy graph as a fuzzy subset of its vertex set, in which we also considered all the non-zero edges incident at a vertex. Our definition allows the importance of all the edges incident at a vertex. Further, in most of the papers, the authors considered domination as a symmetric relation between two vertices. That is, whenever there is an effective edge (or strong edge) between two vertices each vertex dominates the other irrespective of their strength. But in actual situations it is not always possible that a vertex of weaker potential dominates another with stronger potential. We have taken this also into consideration while defining fuzzy dominating sets in fuzzy graphs.

1.6 Organization of the thesis

This thesis entitled **'Domination in Graphs and Fuzzy Graphs'** deals with different distance related domination parameters both in crisp and fuzzy graphs. It is divided into six chapters. We shall now give a summary of each chapter.

The first chapter is an introduction and contains the basic definitions and terminology both in crisp and fuzzy graph theory. It also includes the literature on different domination parameters in both crisp and fuzzy graphs.

The second chapter deals with Disjunctive Domination in Graphs. We attempted to investigate different properties of disjunctive domination number in Lexicographic, Tensor, Strong and Cartesian products of crisp graphs. Further we have investigated disjunctive domination number of some new graphs derived from given graphs. We also found the disjunctive domination number of some corona related graphs.

In the third chapter we define Efficient Disjunctive Dominating Sets and Nearly Efficient Disjunctive Dominating Sets in graphs. We have examined their existence in several graphs especially in two dimensional grid graphs. We proved the existence of Nearly Efficient Disjunctive Dominating sets in infinite two dimensional grid graphs.

In Chapter 4, we initiate a study of Strength Based Domination or sb-domination in graphs. This chapter is a collaboration work of ours with Dr. S. Arumugam, National Center for Advanced Research in Discrete Mathematics, Kalasalingam Academy of Research and Education, Anand Nagar, Krishnankoil-626126, Tamilnadu, India. We present several fundamental results on the new concept.

The fifth chapter deals with domination in fuzzy graphs. We introduce fuzzy domination in fuzzy graphs and present several basic results of this parameter. Several parameters arising from this concept are also introduced and studied.

The Sixth chapter is a concluding chapter, consisting of summary and scope for further studies.

All the graphs considered in this thesis are finite, un-directed and simple. Some results of this thesis are published in the journals given in page 123 of this thesis.

CHAPTER 2

Disjunctive Domination in Graphs

2.1 Introduction

Dominating sets in graphs suggest good solutions for several optimization problems in graphs. Disjunctive domination introduced by Goddard *et al.*, allows some relaxation in domination and has greater flexibility in modeling networks. By Definition 1.3.17, $S \subseteq V(G)$ is a disjunctive dominating set of G if every $v \in V \setminus S$ is either adjacent to a vertex of S or has at least 2 vertices in S at a distance 2 from it. Since its introduction in 2014, several works were done on this topic by many researchers. Some of the important works can be seen in [35, 34, 36]. Motivated by its advantages we studied this domination parameter in detail. We obtained several results on the disjunctive domination number of various product graphs, corona related graphs and some other classes of graphs. First section of this chapter present properties of disjunctive domination in different product graphs. In the next section we find the disjunctive domination number of certain types of derived graphs.

Some results of this chapter are included in the following papers.

1. Lekha A, Parvathy K. S: Properties of disjunctive domination in product graphs, *Malaya Journal of Matematik (MJM),* vol. 8, no. 1, pp. 37-41, 2020.
2. Lekha A, Parvathy K. S: On disjunctive domination number of corona related graphs, *J. Math. Comput. Sci.,* vol. 11, no. 3, pp. 2538-2550, 2021.

2.2 Disjunctive domination in product graphs

Various graph products clearly model processor connections in multiprocessor systems. In order to boost the performance of such the system one must know the properties of the underlying graph structure. How a graph invariant works on graph products is also an important problem. There are different types of graph products, each with its own set of applications and theoretical interpretations. Domination number in product graphs has been studied for a long time. Among various products, the Cartesian product is the center of study in almost all works in literature. These studies are focused largely on Vizing's conjecture. In this section we attempt to determine the relationship between the disjunctive domination number of different types of graph products and their factors.

Definition 2.2.1. [50] A graph parameter ϕ is super-multiplicative (respectively, sub-multiplicative) with respect to a graph product $*$ if $\phi(G_1 * G_2) \geq \phi(G_1)\phi(G_2)$ (respectively, $\phi(G_1 * G_2) \leq \phi(G_1)\phi(G_2)$) for all pairs of graphs G_1 and G_2.

2.2.1 Disjunctive Domination in Lexicographic Products

The following theorem shows that the disjunctive domination number is sub-multiplicative with respect to Lexicographic product.

Theorem 2.2.2. $\gamma_2^d(G_1[G_2]) \leq \gamma_2^d(G_1)\gamma_2^d(G_2)$ for all graphs G_1 and G_2.

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times S_2$ is a DD-set of $G_1[G_2]$.

claim

Let (u, v) be a vertex in $G_1[G_2]$ which is not in $S_1 \times S_2$.

case (i)

Let $u \in V_1 \setminus S_1$ and $v \in S_2$. If u is adjacent to $u_1 \in S_1$, then (u, v) is adjacent to $(u_1, v) \in S_1 \times S_2$. If u is disjunctively dominated by $u_1, u_2 \in S_1$, then $(u_1, v), (u_2, v) \in S_1 \times S_2$ and $d((u, v), (u_1, v)) = d((u, v), (u_2, v)) = 2$. So (u, v) is disjunctively dominated by $S_1 \times S_2$.

case (ii)

Let $u \in S_1$ and $v \in V_2 \setminus S_2$. If v is adjacent to $v_1 \in S_2$, then (u, v) is adjacent to $(u, v_1) \in S_1 \times S_2$. If v is disjunctively dominated by $v_1, v_2 \in S_2$, then $(u, v_1), (u, v_2) \in$

$S_1 \times S_2$ and $d((u,v),(u,v_1)) = d((u,v),(u,v_2)) = 2$ so that (u,v) is disjunctively dominated by $S_1 \times S_2$.

case (iii)

Let $u \in V_1 \setminus S_1$ and $v \in V_2 \setminus S_2$.
If u is adjacent to $u_1 \in S_1$ and v_1 is any vertex in S_2, then (u,v) is adjacent to $(u_1,v_1) \in S_1 \times S_2$. If u is disjunctively dominated by $u_1,u_2 \in S_1$, then $(u_1,v_1),(u_2,v_1) \in S_1 \times S_2$ and $d((u,v),(u_1,v_1)) = d((u,v),(u_2,v_1)) = 2$ so that (u,v) is disjunctively dominated by $S_1 \times S_2$.

From the above cases it follows that in each case (u,v) is either dominated or disjunctively dominated by elements of $S_1 \times S_2$. Thus $S_1 \times S_2$ is a DD-set in $G_1[G_2]$. Hence $\gamma_2^d(G_1[G_2]) \leq \gamma_2^d(G_1)\gamma_2^d(G_2)$ for all graphs G_1 and G_2. □

Remark 2.2.1. 1. The above bound is sharp. If $G_1 = P_2$ and $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1[G_2]) = 2$ so that $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)\gamma_2^d(G_2)$.

2. Strict inequality may also occur in the above result. For example consider the graphs $G_1 = P_2$ and $G_2 = S_4 \circ K_1$. Then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 4$, $\gamma_2^d(G_1[G_2]) = 2$. Here $\gamma_2^d(G_1[G_2]) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

Theorem 2.2.3. 1. $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)$ if G_2 has a universal vertex. In particular for any positive integer n, $\gamma_2^d(G[K_n]) = \gamma_2^d(G)$.

2. $\gamma_2^d(G_1[G_2]) = 2$, if G_1 has a universal vertex, but G_2 has no such vertex. In particular, if $G_1 = K_n$ and G_2 has no universal vertex, then $\gamma_2^d(G_1[G_2]) = 2$.

3. If both G_1 and G_2 have universal vertices, then $\gamma_2^d(G_1[G_2]) = 1$. In particular if $G_1 = K_n$ and $G_2 = K_m$, where m,n are positive integers, then $\gamma_2^d(G_1[G_2]) = 1$.

Proof. 1. Let v be a universal vertex of G_2 and S_1 be a γ_2^d-set of G_1. Then $S_1 \times v$ disjunctively dominates $G_1[G_2]$. The minimality of $S_1 \times v$ follows from the minimality of the γ_2^d-set S_1 of G_1. Thus, $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)$.

2. Let u be a universal vertex of G_1 and v_1, v_2 are any two vertices in G_2. $\{(u,v_1),(u,v_2)\}$ forms a γ_2^d-set of $G_1[G_2]$, for if (u',v') is an arbitrary vertex in $G_1[G_2] \setminus \{(u,v_1),(u,v_2)\}$, then it is dominated by both (u,v_1) and (u,v_2)

whenever $u \neq u'$ and disjunctively dominated by $\{(u, v_1), (u, v_2)\}$ whenever $u = u'$.

3. Let u and v be universal vertices in G_1 and G_2 respectively. Then (u, v) dominates all the vertices in $G_1[G_2]$. So, $\gamma_2^d(G_1[G_2]) = 1$.

□

Corollary 2.2.4. $\gamma_2^d(G_1[G_2]) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ if G_2 has a universal vertex.

Theorem 2.2.5. Let G_1 be a graph without isolated vertices and G_2 be a non-trivial graph. Then,

$$\gamma_2^d(G_1[G_2]) \leq 2\gamma_2^d(G_1).$$

Proof. Let S be a DD-set of G_1 and x, y are any two distinct vertices in G_2. We can show that $(S \times x) \cup (S \times y)$ is a DD-set of $G_1[G_2]$. Clearly, $S \times x$ dominates or disjunctively dominates all the vertices in $(G_1 \setminus S) \times G_2$. Now, let (u, v) be a vertex in $S \times G_2$. Let u' be a vertex in G_1 which is adjacent to u in G_1. Then (u, v) is adjacent to (u', x) which is adjacent to $(u, x) \in S \times x$ and $(u, y) \in S \times y$ in $G_1[G_2]$. It shows that every vertex in $S \times G_2$ has at least two vertices in $(S \times x) \cup (S \times y)$ at a distance 2 from it in $G_1[G_2]$. Thus $(S \times x) \cup (S \times y)$ is a DD-set in $G_1[G_2]$, proving that $\gamma_2^d(G_1[G_2]) \leq 2\gamma_2^d(G_1)$. □

Remark 2.2.2.

1. If G_1 has a universal vertex, but G_2 has no such vertex, then equality occurs in the above relation.

2. If both G_1 and G_2 have a universal vertex then, strict inequality occurs in the above result.

3. If G_1 has a γ_2^d -set in which a pair of vertices are adjacent or if some vertex in G_1 is dominated by two different vertices in S, then strict inequality occurs in 2.2.1.

Theorem 2.2.6. If G_1 has no isolated vertex, then for all graphs G_2, $\gamma_2^d(G_1[G_2]) \leq \gamma_t^d(G_1)$, where $\gamma_t^d(G_1)$ is the total disjunctive domination number of G_1.

Proof. Let S be a TDD-set of G_1. For any vertex $x \in G_2$, we can show that $S \times x$ is a DD-set in $G_1[G_2]$. It is clear that $S \times x$ dominates or disjunctively dominates $(G_1 \setminus S) \times G_2$. Now let (u, v) be any vertex in $S \times x$. u is either adjacent to $u' \in S$

or has two vertices u_1 and u_2 in S at a distance 2 from it. Then (u, v) is either dominated by $(u', x) \in S \times x$ or disjunctively dominated by $(u_1, x), (u_2, x) \in S \times x$, showing that $S \times x$ is a disjunctive dominating set in $G_1[G_2]$. This proves that, $\gamma_2^d(G_1[G_2]) \leq \gamma_t^d(G_1)$. □

Remark 2.2.3. The bound given in the above theorem is sharp. If G_1 has a universal vertex and G_2 has no such vertex, then $\gamma_2^d(G_1[G_2]) = \gamma_t^d(G_1) = 2$. We may also note that strict inequality in the bound can be achieved. Consider the graphs $G_1 = P_5$, $G_2 = P_2$. Then $\gamma_t^d(G_1) = 3$, $\gamma_2^d(G_1[G_2] = 2$ and hence $\gamma_2^d(G_1[G_2]) < \gamma_t^d(G_1)$.

2.2.2 Disjunctive Domination in Tensor Products

There is no consistent relation between the disjunctive domination number of the tensor product of two graphs and the product of their disjunctive domination numbers. There are graphs in which $\gamma_2^d(G_1 \times G_2) > \gamma_2^d(G_1)\gamma_2^d(G_2)$, $\gamma_2^d(G_1 \times G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ and $\gamma_2^d(G_1 \times G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

Example 2.2.1.

1. $\gamma_2^d(P_5 \times P_3) = 4 > \gamma_2^d(P_5)\gamma_2^d(P_3)$.
2. $\gamma_2^d(C_3 \times C_4) = 2 = \gamma_2^d(C_3)\gamma_2^d(C_4)$.
3. If G_1 is the graph given in Figure 2.1, then $\gamma_2^d(G_1 \times G_1) = 2 < \gamma_2^d(G_1)\gamma_2^d(G_1)$.

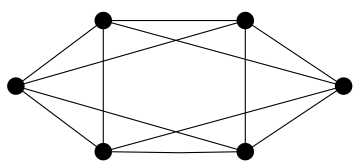

Figure 2.1

Theorem 2.2.7. For any two graphs G_1 and G_2 with at least two vertices and G_2 having no isolated vertices,

$$\gamma_2^d(G_1 \times G_2) \leq min \; \{ \; \gamma_2^d(G_1)|G_2|, \quad \gamma_2^d(G_2)|G_1| \}$$

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs with γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times V_2$ and $V_1 \times S_2$ are both DD-sets in $G_1 \times G_2$.

claim

Let (u, v) be a vertex in $G_1 \times G_2$.

If $u \in S_1$, then $(u, v) \in S_1 \times V_2$. If $u \notin S_1$, then u is either dominated by $x \in S_1$ or disjunctively dominated by two different vertices $x_1, x_2 \in S_1$. If u is dominated by $x \in S_1$, then the vertex (u, v) in $G_1 \times G_2$ is dominated by $(x, v') \in S_1 \times V_2$, where v' is some vertex adjacent to v in G_2. If u is disjunctively dominated by $x_1, x_2 \in S_1$, then the vertices $(x_1, v), (x_2, v) \in S_1 \times V_2$ are such that $d((u, v), (x_1, v)) = d((u, v), (x_2, v)) = 2$. That is, (u, v) has two vertices in $S_1 \times V_2$ at a distance two from it. So, $(u, v) \in G_1 \times G_2$ is disjunctively dominated by $S_1 \times V_2$. Thus $S_1 \times V_2$ is a DD-set of $G_1 \times G_2$. Similarly, $V_1 \times S_2$ is also a DD-set of $G_1 \times G_2$. From these it follows that, $\gamma_2^d(G_1 \times G_2) \leq min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$. □

Remark 2.2.4. 1. The above bound is sharp. For example, if $G_1 = P_2$, $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$ and $\gamma_2^d(G_1 \times G_2) = 4$. In this case, $\gamma_2^d(G_1 \times G_2) = min\ \{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$

2. Strict inequality may occur in the above result.
If $G_1 = P_3$ and $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1 \times G_2) = 5$, $min\{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\} = 6$. Here,
$\gamma_2^d(G_1 \times G_2) < min\{\ \gamma_2^d(G_1)|G_2|,\ \ \gamma_2^d(G_2)|G_1|\}$.

2.2.3 Disjunctive Domination in Strong Products

The following theorem shows that the disjunctive domination number is sub-multiplicative with respect to strong product.

Theorem 2.2.8. For any two non trivial graphs G_1 and G_2,

$$\gamma_2^d(G_1 \boxtimes G_2) \leq \gamma_2^d(G_1)\gamma_2^d(G_2).$$

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times S_2$ is a DD- set of $G_1 \boxtimes G_2$.

claim

Let $(u, v) \notin S_1 \times S_2$ be a vertex in $G_1 \boxtimes G_2$.

case (i)

Let $u \in V_1 \setminus S_1$ and $v \in S_2$. Then either u is dominated by $x \in S_1$ or is disjunctively dominated by two different vertices $x_1, x_2 \in S_1$. If u is dominated by $x \in S_1$, then (u, v) is dominated by $(x, v) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$. If u is disjunctively dominated by two different vertices $x_1, x_2 \in S_1$, then $(x_1, v), (x_2, v) \in S_1 \times S_2$ and $d((u, v), (x_1, v)) = d((u, v), (x_2, v)) = 2$ so that (u, v) is disjunctively dominated by $S_1 \times S_2$ in $G_1 \boxtimes G_2$.

case (ii)

Let $u \in V_1$ and $v \in V_2 \setminus S_2$. Then either v is dominated by $y \in S_2$ or is disjunctively dominated by two different vertices $y_1, y_2 \in S_2$. If v is dominated by $y \in S_2$, (u, v) is dominated by $(u, y) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$. If v is disjunctively dominated by two vertices $y_1, y_2 \in S_2$, then $(u, y_1), (u, y_2) \in S_1 \times S_2$ and $d((u, v), (u, y_1)) = d((u, v), (u, y_2)) = 2$ so that (u, v) is disjunctively dominated by $S_1 \times S_2$ in $G_1 \boxtimes G_2$.

case (iii)

Let $u \in V_1 \setminus S_1$ and $v \in V_2 \setminus S_2$. If u is dominated by $x \in S_1$ and v is dominated by $y \in S_2$, then (u, v) is dominated by $(x, y) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$.

If u is disjunctively dominated by two different vertices $x_1, x_2 \in S_1$ in G_1 and v is dominated by $y \in S_2$ in G_2, then (u, v) is adjacent to (u_1, y) which is again adjacent to $(x_1, y) \in S_1 \times S_2$. Similarly, (u, v) is also adjacent to (u_2, y) which is again adjacent to $(x_2, y) \in S_1 \times S_2$. Thus $d((u, v), (x_1, y)) = d((u, v), (x_2, y)) = 2$. In other words (u, v) is disjunctively dominated by two different vertices $(x_1, y), (x_2, y) \in S_1 \times S_2$. Similarly if u is dominated by $x \in S_1$ in G and v is disjunctively dominated $y_1, y_2 \in S_2$ in G_2, then (u, v) is disjunctively dominated by $(x, y_1), (x.y_2) \in S_1 \times S_2$ in $G_1 \boxtimes G_2$.

If u and v are both disjunctively dominated by S_1 in G_1 and S_2 in G_2 respectively, then there exist $x_1, x_2 \in S_1$ and $y_1, y_2 \in S_2$ such that $d(u, x_1) = d(u, x_2) = 2$ in G_1 and $d(v, y_1) = d(v, y_2) = 2$ in G_2.Then there exist $u_1, u_2 \in V_1 \setminus S_1$ such that u is adjacent to u_1 and u_2 where u_1, u_2 are respectively adjacent to x_1 and x_2 in G. Similarly, there exist $v_1, v_2 \in V_2 \setminus S_2$ such that v is adjacent to v_1 and v_2 where v_1, v_2 are respectively adjacent to y_1 and y_2 in G_2. Thus in $G_1 \boxtimes G_2$, vertex (u, v) is adjacent to (u_1, v_1) and (u_2, v_2) which are respectively adjacent to (x_1, y_1) and (x_2, y_2) in $S_1 \times S_2$. Then, $d((u, v), (x_1, y_1)) = d((u, v), (x_2, y_2)) = 2$, proving that (u, v) is disjunctively dominated by $S_1 \times S_2$.

The above cases show that $S_1 \times S_2$ is a DD-set in $G_1 \boxtimes G_2$.
Thus $\gamma_2^d(G_1 \boxtimes G_2) \leq \gamma_2^d(G_1)\gamma_2^d(G_2)$.

□

Remark 2.2.5.
1. The above bound is sharp. For example if $G_1 = P_2$ and $G_2 = P_7$, then $\gamma_2^d(G_1) = 1, \gamma_2^d(G_2) = 2$, $\gamma_2^d(G_1 \boxtimes G_2) = 2$. So $\gamma_2^d(G_1 \boxtimes G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$.

2. Strict inequality occurs if $G_1 = G_2 = P_4$. Then $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 2$ and $\gamma_2^d(G_1 \boxtimes G_2) = 2$. Hence, $\gamma_2^d(G_1 \boxtimes G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

2.2.4 Disjunctive Domination in Cartesian Products

First we give a general upper bound for the Cartesian product.

Theorem 2.2.9. For any two graphs G_1 and G_2,

$$\gamma_2^d(G_1 \Box G_2) \leq min\ \{\ \gamma_2^d(G_1)|V(G_2)|,\ \ \gamma_2^d(G_2)|V(G_1)|\}$$

Proof. Let G_1 and G_2 are two graphs with γ_2^d-sets S_1 and S_2 respectively. We can show that $S_1 \times V_2$ and $V_1 \times S_2$ are both DD-sets of $G_1 \Box G_2$.

claim

Let (u, v) be a vertex in $G_1 \Box G_2$. If $u \in S_1$, then $(u, v) \in S_1 \times V_2$. If $u \notin S_1$, then u is either dominated by $x \in S_1$ or disjunctively dominated by two different vertices $x_1, x_2 \in S_1$. If u is dominated by $x \in S_1$, then (u, v) is adjacent to $(x, v) \in S_1 \times V_2$. If u is disjunctively dominated by $x_1, x_2 \in S_1$, then the vertices $(x_1, v), (x_2, v) \in S_1 \times V_2$ are such that $d((u, v), (x_1, v)) = d((u, v), (x_2, v)) = 2$. That is, (u, v) has two vertices in $S_1 \times V_2$ at a distance two from it. Thus it is disjunctively dominated by $S_1 \times V_2$. Hence $S_1 \times V_2$ is a DD-set of $G_1 \Box G_2$. Similarly, $V_1 \times S_2$ is also a DD-set of $G_1 \Box G_2$. Thus $\gamma_2^d(G_1 \Box G_2) \leq min\ \{\ \gamma_2^d(G_1)|V(G_2)|,\ \ \gamma_2^d(G_2)|V(G_1)|\}$. □

Remark 2.2.6.
1. The tightness of the above bound can be seen in the example where $G_1 = P_2$ and $G_2 = P_3$.

2. Strict inequality occurs if $G_1 = P_2$ and $G_2 = P_7$.
Here $\gamma_2^d(P_2 \Box P_7) = 3 < min\ \{\ \gamma_2^d(P_2)|V(P_7)|,\ \ \gamma_2^d(P_7)|V(P_2)|\}$

Remark 2.2.7. In general, the Vizing's like inequality $\gamma_2^d(G_1 \Box G_2) \geq \gamma_2^d(G_1)\gamma_2^d(G_2)$ is not true in disjunctive domination. There are graphs in which $\gamma_2^d(G_1 \Box G_2) > \gamma_2^d(G_1)\gamma_2^d(G_2)$, $\gamma_2^d(G_1 \Box G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2)$ and $\gamma_2^d(G_1 \Box G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

For example,

1. If $G_1 = P_7$ and $G_2 = P_2$, then $\gamma_2^d(G_1 \Box G_2) = 3 > \gamma_2^d(G_1)\gamma_2^d(G_2)$.

2. If $G_1 = C_4$ and $G_2 = P_2$, then $\gamma_2^d(G_1 \Box G_2) = \gamma_2^d(G_1)\gamma_2^d(G_2) = 2$.

3. If $G_1 = G_2 = C_4$, then $\gamma_2^d(G_1) = \gamma_2^d(G_2) = 2$ and $\gamma_2^d(G_1 \Box G_2) = 2$.
   In this case, $\gamma_2^d(G_1 \Box G_2) < \gamma_2^d(G_1)\gamma_2^d(G_2)$.

Theorem 2.2.10. For any two graphs G_1 and G_2, where G_1 has a γ- set which is such that the vertices not in this set are twice dominated, $\gamma_2^d(G_1 \Box G_2) \leq \gamma(G_1)\gamma(G_2)$.

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with γ- sets S_1 and S_2 respectively. Let the elements of $V_1 \setminus S_1$ are dominated by two different vertices in S_1. We can show that $S_1 \times S_2$ is a disjunctive dominating set of $G_1 \Box G_2$. Let (u, v) be a vertex in $G_1 \Box G_2$.

case (i)

If $u \in S_1$ and $v \in S_2$, then $(u, v) \in S_1 \times S_2$.

case (ii)

Let $u \in S_1$ and $v \in V_2 \setminus S_2$. If v is dominated by $x \in S_2$ in G_2, then (u, v) is dominated by $(u, x) \in S_1 \times S_2$ in $G_1 \Box G_2$. Similar is the case when $u \in V_1 \setminus S_1$ and $v \in S_2$.

case (iii)

Let $u \in V_1 \setminus S_1$ and $v \in V_2 \setminus S_2$. By hypothesis u is adjacent to two different vertices $x_1, x_2 \in S_1$ in G_1 and v is adjacent to $y \in S_2$ in G_2. Then in $G_1 \Box G_2$, (u, v) is adjacent to (u, y) which is adjacent to (x_1, y) and $(x_2, y) \in S_1 \times S_2$. Thus there are two different vertices (x_1, y), $(x_2, y) \in S_1 \times S_2$ such that $d((u, v), (x_1, y)) = d((u, v), (x_2, y)) = 2$. Hence (u, v) is disjunctively dominated by $S_1 \times S_2$.

The above cases show that $S_1 \times S_2$ is a disjunctive dominating set of $G_1 \Box G_2$. Hence $\gamma_2^d(G_1 \Box G_2) \leq \gamma(G_1)\gamma(G_2)$. □

Theorem 2.2.11. For any two positive integers m, n, $\gamma_2^d(K_m \Box K_n) = 2$.

Proof. Let $(u_1, v_1), (u_2, v_2)$ are two distinct vertices in $K_m \Box K_n$. A vertex $(x, y) \in K_m \Box K_n$ which not dominated by these vertices is such that $d((u_1, v_1), (x, y)) = d((u_2, v_2), (x, y)) = 2$. Hence $\{(u_1, v_1), (u_2, v_2)\}$ is a DD-set in $K_m \Box K_n$ which gives $\gamma_2^d(K_m \Box K_n) \leq 2$. If $u_1 \neq u_2$ and $v_1 \neq v_2$ then (u_1, v_1) and (u_2, v_2) are not adjacent in $K_m \Box K_n$. So there does not exist a universal vertex in $K_m \Box K_n$ which implies that $\gamma_2^d(K_m \Box K_n) \geq 2$. Therefore $\gamma_2^d(K_m \Box K_n) = 2$. □

2.3 Disjunctive domination in corona related graphs

In this section we find the disjunctive domination number of some new graphs derived from two given graphs.

2.3.1 Disjunctive domination in neighborhood corona of graphs

Definition 2.3.1. *[43]*Let G and H be two graphs on n and m vertices respectively. Then the neighborhood corona, $G \star H$ is the graph obtained by taking n copies of H and for each i, making all vertices in the i^{th} copy of H adjacent with the neighbors of $v_i \in G, i = 1, 2, ..., n$.

Notation: H_v denotes the copy of H in $G \star H$ corresponding to $v \in G$.

Definition 2.3.2. *[52]*The splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' , respectively.

The splitting graph was introduced by Sampathkumar and Walikar [52]. In neighborhood corona $G \star H$ if we take $H = K_1$ then it becomes the splitting graph of G.

Observation 2.3.3. Let $v \in G$ and v' be any vertex in the copy of H corresponding to v. Then, for any $u \neq v$ in G, $d(u, v) = d(u, v')$ in $G \star H$. This follows directly from the definition of neighborhood corona of graphs.

Theorem 2.3.4. If S is a disjunctive dominating set of neighborhood corona of any graph, then for any $v \in S$ there exists $u \in S$ such that $d(u, v) \leq 2$.

Proof. Let $G_1 = G \star H$, S is a disjunctive dominating set of G_1 and $v \in S$. Choose vertex $v_1 \in G_1$ such that $d(v, v_1) = 2$. Such a vertex always exists in G_1 because if $v \in G$, v_1 can be any vertex in the copy of H corresponding to v and if $v \in H$ then v_1 can be chosen as the vertex on G corresponding to this $v \in H$. Then for the domination or disjunctive domination of v_1 there must be another vertex u in S such that $d(u, v) \leq 2$. □

Observation 2.3.5. For any two graphs G and H , $\gamma_2^d(G \star H) \geq 2$.

Theorem 2.3.6. If radius of G is less than or equal to 2, then for any graph H, $\gamma_2^d(G \star H) = 2$.

Proof. By Observation 2.3.5, $\gamma_2^d(G \star H) \geq 2$. Radius of $G \star H$ is also 2. Let $u \in C(G)$, where $C(G)$ is the center of G. Then $S = \{u, u'\}$, where u' is any vertex in the copy of u, is a disjunctive dominating set of $G \star H$. So $\gamma_2^d(G \star H) = 2$. □

Corollary 2.3.7. If G has a universal vertex, then for any graph H, $\gamma_2^d(G \star H) = 2$.

Theorem 2.3.8. For any two graphs G and H,

$$\gamma_2(G) \leq \gamma_2^d(G \star H) \leq 2\gamma_2(G)$$

where $\gamma_2(G)$ is the distance -2 domination number of G.

Proof. Let S be a γ_2^d-set of $G \star H$. Let $S' = (S \cap V(G)) \cup \{v \in V(G) : S \cap H_v \neq \phi\}$. Then S' is a distance-2 dominating set of G. Hence $\gamma_2(G) \leq \gamma_2^d(G \star H)$. Now let S be a distance-2 dominating set of G and let S' be a set of vertices formed by taking exactly one vertex from each $H_v, v \in S$. Then $S \cup S'$ is a disjunctive dominating set of $G \star H$. Hence $\gamma_2^d(G \star H) \leq 2\gamma_2(G)$. □

The bounds given in the above theorem are sharp. For example the lower bound is achieved by the family of graphs $\mathcal{G}$ given in Figure 2.2. The upper bound is achieved by the family of graphs $G = S(K_{1,n})$ obtained from $K_{1,n}$ by subdividing each edge once. The case when $n = 2$ and $H = K_1$ is illustrated in Figure 2.3.

Observation 2.3.9. Corresponding to each integer $k \geq 2$ and $i \in \{1, 2, 3, ..., k\}$, there exists a graph G for which $\gamma_2(G) = k$ and $\gamma_2^d(G) = \gamma_2^d(G \star H) = k + i$ for any graph H. This is illustrated in the following example.

Example 2.3.1. For each $k \geq 2$, let G_k^0 be the graph derived from the edge corona $P_k \diamond K_1$ of the path P_k by subdividing each of its edge once and by attaching a

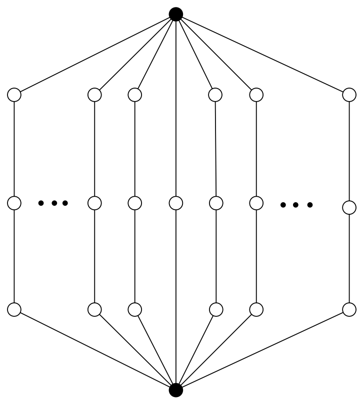

Figure 2.2: *A family of graphs $\mathcal{G}$ for which $\gamma_2(\mathcal{G}) = \gamma_2^d(\mathcal{G} \star H)$*

Figure 2.3: *A graph G and $G \star K_1$ for which $\gamma_2^d(G \star K_1) = 2\gamma_2(G)$*

pendant vertex at each vertex of the path P_k. For $i \in \{1, 2, 3, ..., k\}$, let G_k^i be the graph obtained from G_k^0 by attaching a path P_3 at i distinct vertices of the path P_k of G_k^0. Then, $\gamma_2(G_k^0) = \gamma_2(G_k^1) = ... = \gamma_2(G_k^k) = k$ and $\gamma_2^d(G_k^i) = \gamma_2^d(G_k^i \star H) = k + i$ for $0 \leq i \leq k$. The case when $k = 3$ is depicted in Figure 2.4.

Theorem 2.3.10. For any two graphs G and H,

$$\gamma_2^d(G \star H) \leq 2\gamma_2^d(G).$$

Equality is attained if and only if G has a universal vertex.

Proof. Let S be a γ_2^d-set of G. It can be observed that all vertices in $G \star H$, except the vertices in the copy H_v corresponding to $v \in S$ are dominated or disjunctively dominated by S. Let S' be a set formed by taking exactly one vertex from H_v corresponding to each $v \in S$. Then $S \cup S'$ is a disjunctive dominating set of $G \star H$ and $|S \cup S'| = 2|S|$. Hence $\gamma_2^d(G \star H) \leq 2\gamma_2^d(G)$.

If G has a universal vertex, it follows from Theorem 2.3.6 that

$$\gamma_2^d(G \star H) = 2\gamma_2^d(G) = 2.$$

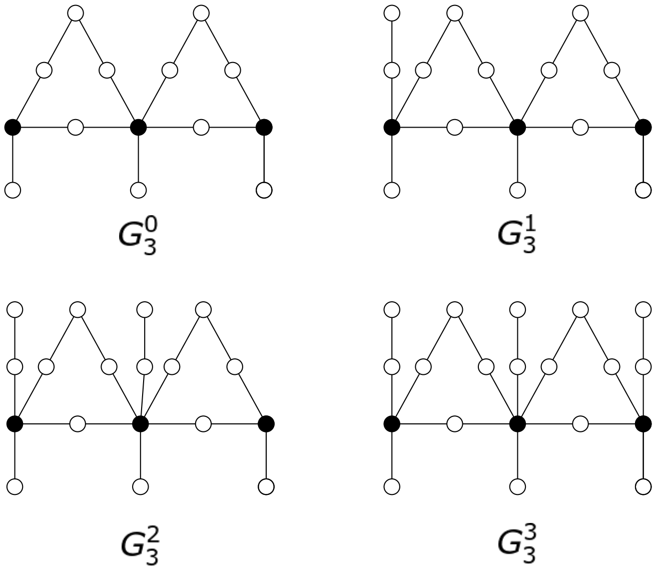

Figure 2.4: *Distance-2 dominating sets of* $G_3^0, G_3^1, G_3^2, G_3^3$. *Here* $\gamma_2(G_3^0) = \gamma_2(G_3^1) = \gamma_2(G_3^2) = \gamma_2(G_3^3) = 3$ *and* $\gamma_2^d(G_3^i) = \gamma_2^d(G_3^i \star H) = 3+i, 0 \leq i \leq 3$

If G has no universal vertex then, every γ_2^d-set of G must contain at least two vertices. Let S be a γ_2^d-set of G and $u \in S$. Then there exist at least one vertex $v \in S$ such that $d(u,v) \leq 4$.

Case (i) $d(u,v) \leq 2$

Let S be a γ_2^d-set of G and S' be a set formed by taking exactly one vertex from each copy of a vertex in $S \setminus \{u,v\}$. Then $S \cup S'$ is a disjunctive dominating set of $G \star H$ and $|S \cup S'| = 2\gamma_2^d(G) - 2$. Hence $\gamma_2^d(G \star H) < 2\gamma_2^d(G)$.

Case (ii) $3 \leq d(u,v) \leq 4$

Let w be a vertex on the uv-path of G such that $d(u,w)$ and $d(v,w)$ are both less than or equal to 2. If S and S' are defined as in *Case (i)*, then $S \cup S' \cup \{w\}$ is a disjunctive dominating set of $G \star H$ of cardinality $2\gamma_2^d(G) - 1$. Hence $\gamma_2^d(G \star H) < 2\gamma_2^d(G)$. □

Remark 2.3.1. There exists no particular relation between the disjunctive domination numbers of a graph and its neighborhood corona. There are graphs for which $\gamma_2^d(G \star H) < \gamma_2^d(G)$, $\gamma_2^d(G \star H) = \gamma_2^d(G)$ and $\gamma_2^d(G \star H) > \gamma_2^d(G)$. Following are some examples for this.

1. Disjunctive domination number of Petersen graph is 2 as realized by any pair of vertices. Disjunctive domination number of neighborhood corona of Petersen

graph and any graph H is also 2. Thus in this case $\gamma_2^d(G) = \gamma_2^d(G \star H)$.

2. Let $G = Q_4$, the hypercube of dimension 4, which is constructed using 2^4 vertices labeled with 4-bit binary numbers in which two vertices (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are adjacent whenever $x_i \neq y_i$ for exactly one $i \in \{1, 2, 3, 4\}$. The set $\{0000, 1111\}$ of its vertices is a disjunctive dominating set of G. Hence $\gamma_2^d(G) = 2$. Let H be any other graph. No two vertices in $\gamma_2^d(G \star H)$ disjunctively dominate all the vertices in it, but the subset $\{0000, 0011, 1111\}$ is one of its disjunctive dominating set. Hence $\gamma_2^d(G \star H) = 3$. In this case, $\gamma_2^d(G) < \gamma_2^d(G \star H)$.

3. Let G be a graph obtained by subdividing each edge of $K_{1,n}$ once where $n > 2$. Then $\gamma_2^d(G) = n > 2$. But $\gamma_2^d(G \star H) = 2$ because the centre vertex together with one vertex in its copy is a disjunctive dominating set of $G \star H$. Hence in this case, $\gamma_2^d(G \star H) < \gamma_2^d(G)$.

Theorem 2.3.11. For any two graphs G and H, $\gamma_2^d(G \star H) \leq \gamma_2^d(G)$ if G has a γ_2^d-set in which corresponding to every $u \in S$ there exists $v \in S$ such that $d(u, v) \leq 2$.

Proof. Let S be a γ_2^d-set of G. All vertices in $G \star H$, except the vertices in the copy H_v corresponding to $v \in S$ are dominated or disjunctively dominated by S. All the vertices in H_v are at a distance 2 from v. These vertices are dominated or disjunctively dominated by S if there is another vertex $u \in S$ such that $d(u, v) \leq 2$. Hence if G has such a γ_2^d-set, then it is a disjunctive dominating set of $G \star H$ as well. Thus, $\gamma_2^d(G \star H) \leq \gamma_2^d(G)$. □

We use the following notations in the next lemma and theorem. Let $v_1, v_2, ..., v_n$ be the vertices on P_n of $G = P_n \star H$. Let H_{v_i} denote the be copy of H corresponding to $v_i \in P_n$ for $i = 1, 2, ..., n$.
Let

$$\begin{aligned} V_i &= \{v_i\} \cup V(H_{v_i}) \quad \text{for } i = 1, 2, ..., n \\ &= \emptyset \quad \text{if } i < 1 \text{ or } i > n \end{aligned}$$

and

$$\begin{aligned} W_i &= V_1 \cup V_2 \cup ... \cup V_i \quad \text{for } i = 1, 2, ..., n \\ &= W_n \quad \text{for } i > n. \end{aligned}$$

Lemma 2.3.12. If D is any disjunctive dominating set of $P_n \star H$, then

(i) $|D \cap W_3| \geq 2$ or $D \cap W_2 \neq \emptyset$ and $|D \cap W_4| \geq 2$.

(ii) If $|D \cap W_4| = 2$, then $|D \cap V_4| \leq 1$.

(iii) $|D \cap (W_n \setminus W_{n-3})| \geq 2$ or $D \cap (W_n \setminus W_{n-2}) \neq \emptyset$ and $|D \cap (W_n \setminus W_{n-4})| \geq 2$.

(iv) If $|D \cap (W_n \setminus W_{n-4})| = 2$, then $|D \cap V_{n-3}| \leq 1$

Proof. (i) and (ii) are easily followed from the observation that the vertices in V_i has no contribution towards the disjunctive domination of V_{i-3} or V_{i+3}. By symmetry we can get (iii) and (iv). □

Lemma 2.3.13. If $n \geq 6$ and D is any disjunctive dominating set of $P_n \star H$, then $|D \cap W_8| \geq 3$ and if $|D \cap W_8| = 3$, then $|D \cap (W_8 \setminus W_4)| \leq 1$.

Proof. From Lemma 2.3.12 (i) and (ii), we see that if $|D \cap W_4| = 2$, then it will contribute at most half towards the disjunctive domination of V_6. Hence D must contain at least one more vertex from its first or second neighborhood. Thus $|D \cap W_8| \geq 3$. If $|D \cap W_8| = 3$, it also follows from Lemma 2.3.12(i) that $|D \cap (W_8 \setminus W_4)| \leq 1$. □

Lemma 2.3.14. If $n \geq 8$ and D is any disjunctive dominating set of $P_n \star H$, then $|D \cap W_{10}| \geq 4$ and if $|D \cap W_{10}| = 4$, then $|D \cap (W_{10} \setminus W_8)| \leq 1$.

Proof. From Lemma 2.3.13, we see that if $|D \cap W_8| = 3$, then it will contribute at most half towards the disjunctive domination of at least one vertex in V_7 or V_8. Hence D must contain at least one more vertex from its first or second neighborhood. Thus $|D \cap W_{10}| \geq 4$. If $|D \cap W_{10}| = 4$, it also shows that $|D \cap (W_{10} \setminus W_8)| \leq 1$. □

Theorem 2.3.15. For any graph H and for $n \geq 2$

$$\gamma_2^d(P_n \star H) = \begin{cases} 2\lceil \frac{n-1}{6} \rceil + 1 & \text{if } n \equiv 0, 1 \ (mod\ 6) \\ 2\lceil \frac{n}{6} \rceil & \text{if } n \equiv 2, 3, 4, 5 \ (mod\ 6) \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices on P_n. Let H_{v_i}, V_i, W_i, $i = 1, 2, ..., n$ be as defined above.

Case (i) $2 \leq n \leq 5$

Singletons in W_2 will not dominate $P_2 \star H$. On the other hand there are two element subsets of W_2 that are disjunctive dominating sets of $P_2 \star H$. Hence $\gamma_2^d(P_2 \star H) = 2$. If $n = 3, 4, 5$ any two element subsets of V_3 containing v_3 form a disjunctive dominating set of $P_n \star H$. Thus $\gamma_2^d(P_n \star H) = 2$ if $2 \leq n \leq 5$. Examples of γ_2^d sets of $P_5 \star K_1$ are depicted in Figure 2.5.

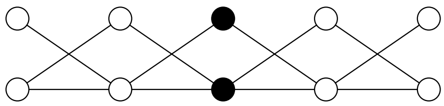
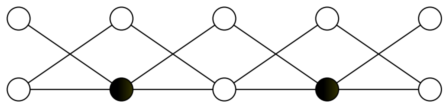

Figure 2.5: γ_2^d *sets of* $P_5 \star K_1$

Case (ii) $n = 6, 7$

$\{2, 4, 6\}$ is a disjunctive dominating set of $P_n \star H$. Hence $\gamma_2^d(P_n \star H) \leq 3$ for $n = 6, 7$. Now from Lemma 2.3.12 (i), we see that $|D \cap W_4| \geq 2$. If $|D \cap W_4| = 2$, then by Lemma 2.3.12(ii), D will contribute at most half towards the disjunctive domination of V_6. Hence D must contain at least one more vertex. Thus $|D| \geq 3$. Therefore $\gamma_2^d(P_n \star H) = 3$ if $n = 6, 7$. γ_2^d set of $P_7 \star K_1$ is depicted in Figure 2.6.
From cases (i) and (ii) we get

$$\gamma_2^d(P_n \star H) = \begin{cases} 2 & \text{if } 2 \leq n \leq 5 \\ 3 & \text{if } n = 6, 7 \end{cases}$$

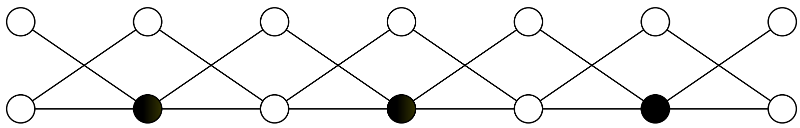

Figure 2.6: γ_2^d *set of* $P_7 \star K_1$

Case (iii) $8 \leq n \leq 11$

From Lemma 2.3.12(i), we see that $|D \cap W_4| \geq 2$. Similarly from Lemma 2.3.12(iii) we get $|D \cap (W_8 \setminus W_4)| \geq 2$. As $W_4 \cap (W_8 \setminus W_4) = \emptyset$, it follows that $|D| \geq 4$. On the other hand $D = \{2, 4, 6, 8\}$ is a disjunctive dominating set of $P_8 \star H$. Hence $\gamma_2^d(P_8 \star H) = 4$.

Now $D = \{2, 4, 8, 10\}$ is a disjunctive set of $P_{11} \star H$. Hence $\gamma_2^d(P_n \star H) \leq 4$ if $n = 9, 10, 11$. As $\gamma_2^d(P_8 \star H) = 4$, we conclude that $\gamma_2^d(P_n \star H) \geq 4$ for $n = 9, 10, 11$. Thus $\gamma_2^d(P_n \star H) = 4$ for $n = 8, 9, 10, 11$.

γ_2^d set of $P_{11} \star K_1$ is depicted in Figure 2.7.

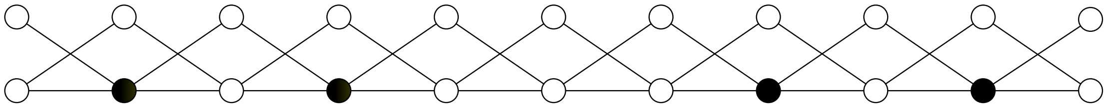

Figure 2.7: γ_2^d *set of* $P_{11} \star K_1$

Case (iv) $n \geq 12$

Let $n \equiv 0, 1 \ (mod\ 6)$

If $n = 6k$, partition the vertices in $P_n \star H$ into three sets W_8, $X = W_n \setminus W_{n-4}$ and $Y = V(P_n \star H) \setminus (W_8 \cup X)$.
Then by Lemma 2.3.13, we get $|D \cap W_8| \geq 3$. By Lemma 2.3.12 (iii) and (iv) we see that $|D \cap X| \geq 2$. Now consider any subset $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$ of Y for any six consecutive indices $i, i+1, i+2, i+3, i+4, i+5$. For the disjunctive domination of $V_{i+2} \cup V_{i+3}$ it is clear that D must contain at least two vertices from $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$. As this is true for any set of six consecutive sets V_i in Y and since $|Y| = 6(k-2)$ we get $|D \cap Y| \geq 2(k-2)$. Thus $|D| \geq 3 + 2(k-2) + 2 = 2k + 1 = 2\lceil \frac{n-1}{6} \rceil + 1$.

On the other hand $D = \{v_{6i+2}, v_{6i+4} : i = 0, 1, 2, ..., k-1\} \cup \{v_{6k}\}$ is a disjunctive dominating set of $P_{6k} \star H$. As the number of vertices in this set is $2k+1$, we get $\gamma_2^d(P_n \star H) \leq 2\lceil \frac{n-1}{6} \rceil + 1$. Thus if $n = 6k$, $\gamma_2^d(P_n \star H) = 2\lceil \frac{n-1}{6} \rceil + 1$.

If $n = 6k+1$, then $\gamma_2^d(P_n \star H) \geq 2k+1$ as $\gamma_2^d(P_{6k} \star H) = 2k+1$. $D = \{v_{6i+2}, v_{6i+4} : i = 0, 1, 2, ..., k-1\} \cup \{v_{6k}\}$ is also a disjunctive dominating set of $P_n \star H$ for $n = 6k+1$. Hence $\gamma_2^d(P_n \star H) = 2k + 1 = 2\lceil \frac{n-1}{6} \rceil + 1$ for $n = 6k+1$.

γ_2^d set of $P_{13} \star K_1$ is depicted in Figure 2.8.

Let $n \equiv 2, 3, 4, 5 \ (mod\ 6)$

Let $n = 6k+2$. Partition vertices in $P_n \star H$ into three sets W_4, $X = W_n \setminus W_{n-4}$ and $Y = V(P_n \star H) \setminus (W_4 \cup X)$. As in the above case we see that $|D \cap W_4| \geq 2$,

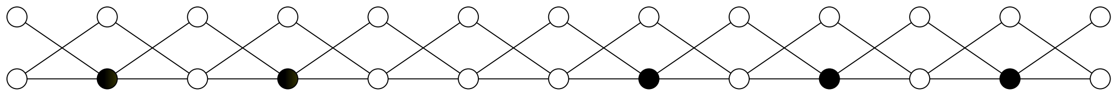

Figure 2.8: γ_2^d *set of* $P_{13} \star K_1$

$|D \cap X| \geq 2$ and $|D \cap Y| \geq 2(k-1)$. Hence $|D| \geq 2 + 2(k-1) + 2 = 2k + 2 = 2\lceil \frac{n}{6} \rceil$. Thus $\gamma_2^d(P_n \star H) \geq 2\lceil \frac{n}{6} \rceil$ if $n \geq 6k + 2$.

Now if $n = 6k + 5$, then $D = \{v_{6i+2}, v_{6i+4} : i = 0, 1, 2, ..., k-1\} \cup \{v_{6k+2}, v_{6k+4}\}$ is a disjunctive dominating set of $P_n \star H$. As the number of vertices in this set is $2k + 2 = 2\lceil \frac{n}{6} \rceil$, we get $\gamma_2^d(P_n \star H) \leq 2\lceil \frac{n}{6} \rceil$ if $n = 6k + 5$. Hence $\gamma_2^d(P_n \star H) \leq 2\lceil \frac{n}{6} \rceil$ if $n \leq 6k + 5$.

Thus $\gamma_2^d(P_n \star H) = 2\lceil \frac{n}{6} \rceil$ if $n = 6k + 2, 6k + 3, 6k + 4$ and $6k + 5$.

By summing up the cases (i), (ii),(iii) and (iv) we see that if $n \geq 2$,

$$\gamma_2^d(P_n \star H) = \begin{cases} 2\lceil \frac{n-1}{6} \rceil + 1 & \text{if } n \equiv 0, 1 \ (mod\ 6) \\ 2\lceil \frac{n}{6} \rceil & \text{if } n \equiv 2, 3, 4, 5 \ (mod\ 6) \end{cases}$$

□

Corollary 2.3.16. For any $n \geq 2$

$$\gamma_2^d(S'(P_n)) = \begin{cases} 2\lceil \frac{n-1}{6} \rceil + 1 & \text{if } n \equiv 0, 1 \ (mod\ 6) \\ 2\lceil \frac{n}{6} \rceil & \text{if } n \equiv 2, 3, 4, 5 \ (mod\ 6) \end{cases}$$

where $S'(P_n)$ is the splitting graph of P_n.

Theorem 2.3.17. For any integer $n \geq 3$ and for any graph H,

$$\gamma_2^d(C_n \star H) = \begin{cases} 2\lceil \frac{n}{6} \rceil - 1 & \text{if } n \equiv 1, 2 \ (mod\ 6) \\ \\ 2\lceil \frac{n}{6} \rceil & \text{otherwise} \end{cases}$$

Proof. Let $\{v_1, v_2, v_3, ..., v_n\}$ be the vertices of C_n. If $n = 3, 4, 5, 6$, then $D = \{v_1, v_3\}$ is a disjunctive dominating set of $C_n \star H$. As there is no universal vertex, we also see that $\gamma_2^d(C_n \star H) \geq 2$. Thus $\gamma_2^d(C_n \star H) = 2$ if $n = 3, 4, 5, 6$. γ_2^d-set of $C_6 \star K_1$ is depicted in Figure 2.9.

Now let $n \geq 7$ and let H_{v_i} be the copy of H corresponding to $v_i \in C_n$, where

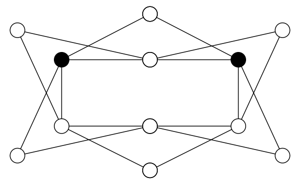

Figure 2.9: γ_2^d *set of* $C_6 \star K_1$

$i \in \{1, 2, ..., n\}$. Let V_i denote $\{v_i\} \cup V(H_{v_i})$ for $i = 1, 2, ..., n$. It can be noted here that a vertex in V_i contributes only half towards the disjunctive domination of a vertex in V_{i+2} and V_{i-2}. It has no contribution towards the disjunctive domination of a vertex in V_{i+3} and V_{i-3}.

It can also be noted here that a vertex $x \in D \cap V_i$ contributes at most half towards the disjunctive domination of least one vertex in V_i. So D must contain one more vertex in the first or second neighborhood it. Hence corresponding to each vertex $x \in D \cap V_i$, there exist a vertex $y \in D$ such that $d(x, y) \leq 2$.

Let $n \equiv 0(mod\ 6)$

If $n = 6k$, then $D = \{v_1, v_3, v_7, v_9, v_{13}, v_{15}, ..., v_{6k-5}, v_{6k-3}\} = \{v_{6j-5}, v_{6j-3} : j = 1, 2, ..., k\}$ is a disjunctive dominating set of $C_{6k} \star H$ of cardinality $2k$. Hence if $n = 6k$, then $\gamma_2^d(C_n \star H) \leq 2k = 2\lceil \frac{n}{6} \rceil$.
The reverse inequality can be seen as follows. Let D be any disjunctive dominating set of $G = C_n \star H$. Consider any subset $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$ of the vertex set of G for any six consecutive indices $i, i+1, i+2, i+3, i+4, i+5$. For the disjunctive domination of $V_{i+2} \cup V_{i+3}$ it is clear that D must contain at least two vertices from $V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3} \cup V_{i+4} \cup V_{i+5}$. As this is true for any set of six consecutive sets V_i in G we get $|D| \geq 2k$. Hence if $n = 6k$, then $\gamma_2^d(C_n \star H) = 2k = 2\lceil \frac{n}{6} \rceil$.
γ_2^d-set of $C_{12} \star K_1$ is depicted in Figure 2.10.

Let $n \geq 7$ **and** $n \equiv 1, 2(mod\ 6)$

Let $n = 6k + 1$ and let D be any disjunctive dominating set of $C_n \star H$. Let $x \in D$. Without loss of generality we can assume that $x \in V_1$. Then there exists another vertex $y \in D$ such that $d(x, y) \leq 2$. Hence either $|D \cap (V_{n-1} \cup V_n \cup V_1)| \geq 2$ or

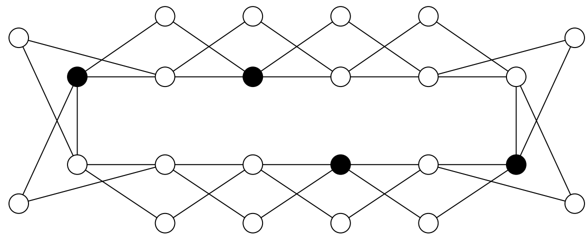

Figure 2.10: γ_2^d *set of* $C_{12} \star K_1$

$|D\cap(V_1\cup V_2\cup V_3)| \geq 2$. Let $|D\cap(V_1\cup V_2\cup V_3)| \geq 2$. If $n \geq 7$ and $|D\cap(V_1\cup V_2\cup V_3)| = 2$, then they will contribute at most half towards the disjunctive domination of vertices in V_5 and V_6. Hence D must contain at least one more vertex from their first or second neighborhood. Thus $|D \cap (V_1 \cup V_2 \cup ... \cup V_7)| \geq 3$. Now from the remaining set of vertices in $C_n \star H$, D must contain be at least two vertices corresponding to every set of six consecutive V_i's in $\{V_8, V_9, ..., V_{6k+1}\}$. Thus $|D| \geq 3+2(k-1) = 2k+1 = 2\lceil \frac{n}{6} \rceil - 1$. Hence $\gamma_2^d(C_n \star H) \geq 2\lceil \frac{n}{6} \rceil - 1$ if $n \geq 6k + 1$.

On the other hand, $D = \{v_{6j-5}, v_{6j-3} : j = 1, 2, ..., k\} \cup \{v_{6k+1}\}$ is a disjunctive dominating set of $C_n \star H$ if $n = 6k + 1$, $6k + 2$. As the number of vertices in this set is $2k + 1$ we get $\gamma_2^d(C_n \star H) \leq 2k + 1 = 2\lceil \frac{n}{6} \rceil - 1$.
Thus $\gamma_2^d(C_n \star H) = 2\lceil \frac{n}{6} \rceil - 1$ if $n = 6k + 1$ or $6k + 2$.
γ_2^d-set of $C_{14} \star K_1$ is depicted in Figure 2.11.

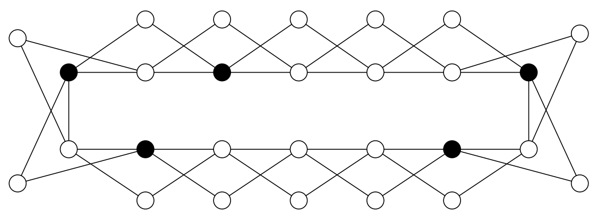

Figure 2.11: γ_2^d *set of* $C_{14} \star K_1$

Let $n \equiv 3, 4, 5 (mod\ 6)$

Let $n = 6k + 3$ and let D be any disjunctive dominating set of $C_n \star H$. For the disjunctive domination of vertices in $V_1 \cup V_2 \cup ... \cup V_{6k}$ at least 2k vertices are needed. Two vertices that contribute to the disjunctive domination of V_{i+2} and V_{i+3} from six consecutive sets $V_i, V_{i+1,...,V_{i+5}}$ provide at most half to the disjunctive domination of a vertex outside this set. Hence for the disjunctive domination of vertices in $V_{2k+1} \cup V_{2k+2} \cup V_{2k+3}$ at least two more vertices are needed in D. Thus $|D| \geq 2k+2 = 2\lceil \frac{n}{6} \rceil$. On the other hand, $D = \{v_{6j-5}, v_{6j-3} : j = 1, 2, ..., k\} \cup \{v_{6k+1}, v_{6k+3}\}$ is a disjunctive dominating set of $C_n \star H$ of cardinality $2k+2$. Thus $\gamma_2^d(C_n \star H) = 2k+2 = 2\lceil \frac{n}{6} \rceil$.

γ_2^d-sets of $C_{15} \star H$ is depicted in Figure 2.12.

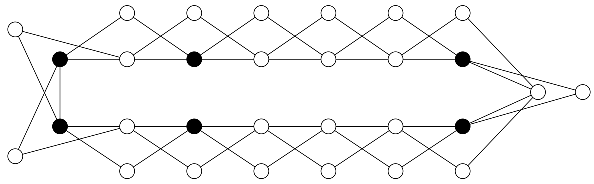

Figure 2.12: *γ_2^d set of $C_{15} \star K_1$*

By summing up all the above results we see that, if $n \geq 3$ then,

$$\gamma_2^d(C_n \star H) = \begin{cases} 2\lceil \frac{n}{6} \rceil - 1 & \text{if } n \equiv 1, 2 \ (mod\ 6) \\ \\ 2\lceil \frac{n}{6} \rceil & \text{otherwise} \end{cases}$$

□

Corollary 2.3.18. For any integer $n \geq 3$,

$$\gamma_2^d(S'(C_n)) = \begin{cases} 2\lceil \frac{n}{6} \rceil - 1 & \text{if } n \equiv 1, 2 \ (mod\ 6) \\ \\ 2\lceil \frac{n}{6} \rceil & \text{otherwise} \end{cases}$$

where $S'(C_n)$ is the splitting graph of C_n.

Theorem 2.3.19. $\gamma_2^d(G \star H) = 2$ for $G \cong K_n,\ K_{1,n},\ W_{1,n}$ where n is a positive integer greater than 3 for $W_{1,n}$.

Proof. Since these graphs have a universal vertex it follows from Theorem 2.3.6. □

Theorem 2.3.20. For all positive integers m, n, $\gamma_2^d(K_{m,n} \star H) = 2$.

Proof. Let $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, .v_2, ..., v_n\}$ be the partition of the vertex set of $K_{m,n}$. Any one vertex in U dominates all the vertices in V and the vertices in their copies. Similarly an arbitrary vertex in V dominates all the vertices in U and the vertices in their copies. Thus a γ_2^d-set contains exactly two vertices. Hence, $\gamma_2^d(K_{m,n} \star H) = 2$. □

2.3.2 Disjunctive domination in edge corona of graphs

Definition 2.3.21. *[42]Let G and H be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges respectively. The edge corona $G \diamond H$ of G and H is the graph obtained by taking one copy of G and m_1 copies of H and then joining two end-vertices of the i^{th} edge of G to every vertex in the i^{th} copy of H.*

Theorem 2.3.22. For any nontrivial graph G and a graph H,

$$\gamma_2^d(G) \leq \gamma_2^d(G \diamond H).$$

Proof. Let S be a γ_2^d-set of $G \diamond H$. Let H_e denote the copy of H corresponding to an edge $e \in E(G)$. Let the set S' be formed such that it contains one of the incident vertices of each edge $e \in E(G)$ for which $|S \cap H_e| = 1$ and both the incident vertices if $|S \cap H_e| \geq 2$. Let $D = (S \cap V(G)) \cup S'$. Then $D \subset V(G)$. S is a disjunctive dominating set of $G \diamond H$ and $d(w, D) \leq d(w, S)$ for any vertex $w \in V(G)$ shows that D is a disjunctive dominating set of G. Hence $\gamma_2^d(G) \leq \gamma_2^d(G \diamond H)$. □

Note

It may be noted that $\gamma_2^d(G \diamond H)$ can be much larger than $\gamma_2^d(G)$. For example if G is the friendship graph F_n, which is constructed by joining n copies of C_3 with a common vertex, then $\gamma_2^d(G) = 1$ whereas $\gamma_2^d(G \diamond H) = n$. The case when $n = 4$ is depicted in Figure 3.3.

Theorem 2.3.23. For every positive integer $n > 1$, $\gamma_2^d(P_n \diamond H) = \lceil \frac{n}{3} \rceil$

Proof. Let $v_1, v_2, v_3, ...v_n$ be the vertices and $e_1, e_2, e_3, ..., e_{n-1}$ be the edges of P_n. Let $H_1, H_2, ...H_{n-1}$ be the copies of H corresponding to the edges of P_n.

Let S be any disjunctive dominating set of $P_n \diamond H$. Let the set S' be formed such that it contains one of the incident vertices of each edge e_i for which $|S \cap H_i| = 1$

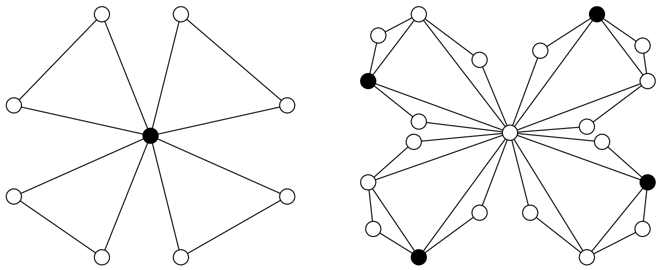

Figure 2.13: $\gamma_2^d(F_4) = 1$ *but* $\gamma_2^d(F_4 \diamond K_1) = 4$

and both the incident vertices if $|S \cap H_i| \geq 2$. Let $D = (S \cap V(G)) \cup S'$. Then D is a disjunctive dominating set of $G \diamond H$ with $|D| \leq |S|$. Hence we can assume with out loss of generality that, if D is any γ_2^d-set of $P_n \diamond H$, then $D \cap H_i = \emptyset$ for all $i = 1, 2, 3, ..., n-1$.

case (i) $n \equiv 0$ $(mod\ 3)$

Let $n = 3k$. The set $D = \{v_2, v_5, v_8, ...v_{n-1}\}$ is a disjunctive dominating set of $P_n \diamond H$ of cardinality k. Hence $\gamma_2^d(P_n \diamond H) \leq k = \lceil \frac{n}{3} \rceil$.

To prove the reverse inequality let us suppose that D is a disjunctive dominating set of $P_n \diamond H$ such that $D \cap H_i = \emptyset$ for $i = 1, 2, ..., n-1$. For the disjunctive domination of v_1 and vertices in H_1, it is clear either $v_1 \in D$ or $v_2 \in D$. Now consider any three consecutive vertices v_i, v_{i+1}, v_{i+2}, $i \geq 2$ on P_n. If any of these vertices are not in D, then both v_{i-1} and v_{i+3} must be in D. But these two vertices contributes at most half towards the disjunctive domination of vertices in H_i and H_{i+1}, which is a contradiction to our assumption that D is a disjunctive dominating set. Hence $D \cap \{v_i, v_{i+1}, v_{i+2}\} \neq \emptyset$. Thus corresponding to any three consecutive vertices on P_n, there must be at least one vertex in D. Hence $|D| \geq k = \lceil \frac{n}{3} \rceil$.

Thus when $n = 3k$, $\gamma_2^d(P_n \diamond H) = k = \lceil \frac{n}{3} \rceil$.

case (ii) $n \equiv 1, 2$ $(mod\ 3)$

Let $n = 3k + 1$ *or* $3k + 2$. In this case $\{v_2, v_5, v_8, ...v_{3k-1}, v_{3k+1}\}$ is a disjunctive dominating set of cardinality $k + 1$. Hence $\gamma_2^d(P_n \diamond H) \leq k + 1 = \lceil \frac{n}{3} \rceil$.

To prove the reverse inequality let $n = 3k + 1$ and let D be any disjunctive dominating set as in case (i). As before it can be seen that at least one vertex is required from every set of three consecutive vertices on P_n. Also for the disjunctive domination of v_1 and vertices in H_1 either v_1 or v_2 must be in D. Hence k vertices are needed for the disjunctive domination of first $3k$ vertices on P_n and the vertices in the copy of H corresponding to the edges between them and one of them is v_1 or v_2. These k vertices in D contribute at most half towards the disjunctive domination of v_{k+1}. Hence $|D| \geq k+1$ when $n = 3k+1$. So when $n = 3k+1$ or $3k+2$, $\gamma_2^d(P_n \diamond H) \geq k+1 = \lceil \frac{n}{3} \rceil$

Thus $\gamma_2^d(P_n \diamond H) = \lceil \frac{n}{3} \rceil$ if $n = 3k+1$, $3k+2$. By summing up the cases (i) and (ii) we get the theorem. □

Theorem 2.3.24. For every positive integer $n > 3$,

$$\gamma_2^d(C_n \diamond H) = \left\lceil \frac{n}{3} \right\rceil$$

.

Proof. The proof is similar to the proof of $\gamma_2^d(P_n \diamond H)$. □

Theorem 2.3.25. For every positive integer $n \geq 3$,

$$\gamma_2^d(K_n \diamond H) = 2.$$

Proof. Let u be an arbitrary vertex in K_n. It dominates all the vertices in K_n and the copies of H corresponding to the edges incident with u. Let e be an edge which is not incident with u and let H_e be the copy of H corresponding to e. Vertices in H_e are at a distance 2 from u. Let $v \neq u$ be any other vertex in K_n. Then H_e is dominated or disjunctively dominated by $\{u, v\}$, i.e, it is a γ_2^d-set of $K_n \diamond H$. Hence, $\gamma_2^d(K_n \diamond H) = 2$. □

Theorem 2.3.26. For $m, n \geq 2$,

$$\gamma_2^d(K_{m,n} \diamond H) = 2.$$

Proof. Any two vertices in $K_{m,n}$ dominates or disjunctively dominates all the vertices in $K_{m,n}$ as well as the vertices in the copies of H corresponding to its edges. It is also obvious that a single vertex cannot dominate all the vertices. Hence, $\gamma_2^d(K_{m,n} \diamond H) = 2$. □

Theorem 2.3.27. If $W_{1,n}$ is the wheel graph on $n+1$ vertices, then

$$\gamma_2^d(W_{1,n} \diamond H) = \left\lceil \frac{n}{4} \right\rceil + 1$$

Proof. Let u be the center of the wheel. It dominates all the vertices in $W_{1,n}$ and the vertices in all the copies of H corresponding to the edges of $W_{1,n}$ incident at u. Let V' denote the vertices in the copies of H corresponding to the edges not incident at the center u. All the vertices in V' are at a distance 2 from u. For the disjunctive domination of these vertices, each vertex in V' needs at least one more vertex at a distance 2 from it. Since at least $\lceil \frac{n}{4} \rceil + 1$ vertices are required for this $\gamma_2^d(W_{1,n} \diamond H) \geq \lceil \frac{n}{4} \rceil + 1$. Since $S = \{v_1, v_5, v_9, ..., v_{4k+1}\}$ of the vertices on the rim of the wheel is such a set, we get $\{u\} \cup S$ is a disjunctive dominating set of $W_{1,n} \diamond H$. Hence $\gamma_2^d(W_{1,n} \diamond H) \leq \lceil \frac{n}{4} \rceil + 1$. Thus,

$$\gamma_2^d(W_{1,n} \diamond H) = \left\lceil \frac{n}{4} \right\rceil + 1$$

□

Theorem 2.3.28. $\gamma_2^d(G \diamond H) = 1$ if and only if $G = K_{1,n}$.

Proof. Let $G = K_{1,n}$. The center vertex of $K_{1,n}$ dominates all the vertices in $K_{1,n} \diamond H$. Hence, $\gamma_2^d(K_{1,n} \diamond H) = 1$. Conversely let $\gamma_2^d(G \diamond H) = 1$. This is possible if and only if all the edges of G are incident at a single vertex, i.e, $G = K_{1,n}$. □

2.4 Conclusion

In this chapter we have made an attempt to study the properties of disjunctive domination in different graph products and in two types of corona related graphs. We found the disjunctive domination number of neighborhood and edge corona of some standard classes of graphs. It will be interesting to study the impact of other graph operations on disjunctive domination.

CHAPTER 3

Efficient Disjunctive Dominating Sets in Graphs

3.1 Introduction

Efficiency in domination theory often involves reducing redundancy within the dominating sets. This means avoiding situations where multiple dominating vertices cover the same portion of the graph, thus minimizing waste. Bange, Barkauskas and Slater [4, 5] introduced the notion of efficient dominating sets in graphs. They defined a dominating set D of G as an efficient dominating set of G, if every vertex in V has exactly one vertex in D in its closed neighborhood. Equivalently, the distance between any two vertices in D is at least three. All graphs do not have efficient dominating sets. For example, the Peterson graph cannot be efficiently dominated. The efficient domination number of a graph, denoted by $F(G)$, is the maximum number of vertices that can be dominated by a set $D \subset V$, that dominates each vertex at most once. A graph G of order $n = |V(G)|$ has an efficient dominating set if and only if $F(G) = n$.

In this chapter we introduce Efficient Disjunctive Dominating sets(EDD-sets)

Some results of this chapter are included in the following paper.
Lekha A, Parvathy K. S: Efficient disjunctive dominating sets in graphs, *Advances in Mathematics: Scientific Journal,* vol. 10, no. 3, pp. 1215-1226, 2021.

in graphs. We examine the existence of EDD-sets in some graphs, characterize all paths,cycles, two dimensional grid graphs having EDD-sets. We also introduce the notion of Nearly Efficient Disjunctive Dominating sets(NEDD-sets) in graphs and provide a proof to show the existence of an NEDD-set in an infinite two dimensional grid graph. We focus mainly on two dimensional grid graphs, i.e, the Cartesian product of two paths. Grid graphs have importance in computer architecture as they model parallel processor networks and they have applications in various fields like sensor networks, coding theory and robotics. Hence the study of graph theoretic properties of these graphs is a significant problem.

3.2 Efficient disjunctive dominating sets

Definition 3.2.1. Let $D \subset V$. Define a function $f_D : V \to R$ by

$$f_D(u) = |N[u] \cap D| + \frac{1}{2}|N_2(u) \cap D|.$$

D is called an **Efficient Disjunctive Dominating set** or **EDD-set** if $f_D(u) = 1$ for all $u \in V$. In other words, D is an efficient disjunctive dominating set if each vertex of V is either dominated by exactly one vertex in D or disjunctively dominated by exactly two vertices in D.

An EDD-set is a disjunctive dominating set for which the total amount of domination and disjunctive domination done by it is minimum. Hence cardinality of an EDD-set is $\gamma_2^d(G)$.

Definition 3.2.2. The efficient disjunctive domination number $F_2^d(G)$ of a graph is the maximum number of vertices in the graphs for which $f_D(v) = 1$ among all subsets D of V.

A graph G of order $n = |V(G)|$ has an efficient disjunctive dominating set if and only if $F_2^d(G) = n$. Most graphs do not have an efficient disjunctive dominating set. If a graph G has an efficient disjunctive dominating set, we say that G is efficiently disjunctive dominatable graph or EDD-graph. Some examples of EDD-graphs are given in Figure 3.1.

Example 3.2.1. Petersen graph is not an EDD-graph. Disjunctive domination number of this graph is 2, as realized by any two vertices. But any pair of vertices

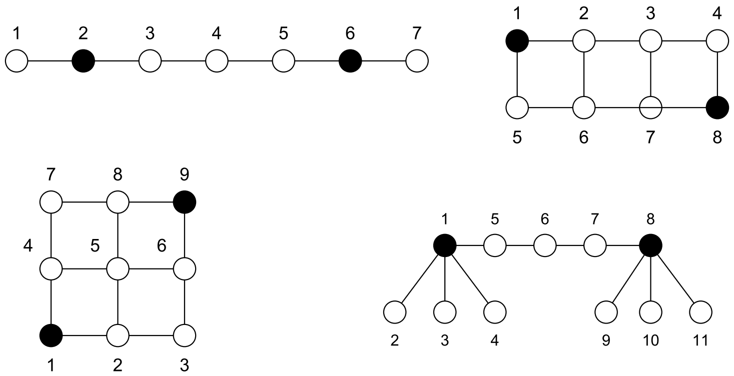

Figure 3.1: *EDD-graphs*

of this graph lie on a C_5 shows that $f_D(u) > 1$ for all $D \subset V$ with $|D| = 2$. Hence it has no EDD-set.

Observation

All graphs having a universal vertex are EDD-graphs. In particular complete graphs, star graphs and wheel graphs are EDD-graphs.

Lemma 3.2.3. If D is an EDD-set, then $d(u, v) \geq 4$ for every pair of vertices $u, v \in D$.

Proof. Let $u, v \in D$ and $d(u, v) < 4$. Then there exist at least one vertex w on the uv-path such that $f_D(w) > 1$ which is a contradiction to the definition of EDD-sets. □

Theorem 3.2.4. For $n \geq 3$, C_n is an EDD-graph if and only if $n = 3$ or $n \equiv 0 \ (mod\ 4)$, but $n \neq 4$.

Proof. By Theorem 1.3.23, if $n \geq 3$,

$$\gamma_2^d(C_n) = \begin{cases} 2 & \text{if n=4} \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

It can be verified easily that C_3 is an EDD-graph, but C_4 is not an EDD-graph. Let $n \geq 5$. An EDD-set in C_n, if it exists, has cardinality $\lceil \frac{n}{4} \rceil$. Let $\{1, 2, 3, ..., n\}$ be the vertices of C_n.

Case (i) $n = 4k,\ k > 1$

$\gamma_2^d(C_n) = k$ and $D = \{1, 5, 9, ..., 4k-3\}$ of cardinality k is an EDD-set of C_n if $n = 4k$, $k > 1$. Hence if $n \equiv 0\ (mod\ 4)$, but $n \neq 4$, C_n is an EDD-graph.

Case (ii) $n \equiv 1, 2, 3\ (mod\ 4)$

If $n = 4k+1$ or $4k+2$ or $4k+3$, then $\gamma_2^d(C_n) = k+1$. Hence in any disjunctive dominating set there exist at least one pair of vertices u, v such that $d(u, v) < 4$. Hence it follows from Lemma 3.2.3 that C_n is not an EDD- graph in this case. □

Theorem 3.2.5. For every positive integer n, P_n is an EDD-graph unless $n \equiv 0\ (mod\ 4)$.

Proof. By Theorem 1.3.22, $\gamma_2^d(P_n) = \lceil \frac{n+1}{4} \rceil$ for all n . Hence an EDD-set in P_n, if it exists, has cardinality $\lceil \frac{n+1}{4} \rceil$. Let $\{1, 2, 3, ..., n\}$ be the vertices of P_n.

Case (i) $n \equiv 1, 2\ (mod\ 4)$

Let $n = 4k+1$ or $4k+2$. Then $D = \{1, 5, 9, ..., 4k+1\}$ is an EDD-set of P_n. Hence P_n is an EDD-graph if $n \equiv 1, 2\ (mod\ 4)$.

Case (ii) $n \equiv 3\ (mod\ 4)$

Let $n = 4k+3$. Then $D = \{2, 6, 10, ..., 4k+2\}$ is an EDD-set of P_n.
Hence P_n, $n \equiv 3\ (mod\ 4)$ is an EDD-graph.

Case (iii) $n \equiv 0\ (mod\ 4)$

Let $n = 4k$. We may note that any γ_2^d-set D of P_n has $k+1$ vertices and hence there must be at least one pair of vertices $u, v \in D$ with $d(u, v) < 4$, which is a contradiction to the definition of an EDD-set. Hence P_n if $n \equiv 0\ (mod\ 4)$ is not an EDD-graph. □

Theorem 3.2.6. $G_{2,m} = P_2 \Box P_m$ is an EDD-graph if and only if $m \equiv 1\ (mod\ 3)$.

Proof. By Theorem 1.3.24, $\gamma_2^d(G_{2,m}) = \lceil \frac{m+2}{3} \rceil$ and the proof of this theorem [26] shows that a γ_2^d-set of $G_{2,m}$ contains one vertex from every third column, taken from alternating rows, together with a vertex in the last column if no vertex is already taken from there.
Let $V(P_2) = \{1, 2\}$ and $V(P_m) = \{1, 2, ..., m\}$. Then,
$V(G_{2,m}) = \{(1,1), (1,2), ...(1,m), (2,1), (2,2)..., (2,m)\}$

Case (i) $m \equiv 1$ $(mod\ 3)$

Let $m = 3k+1$. Then $D = \{(1,1),(2,4),(1,7),(2,10),...,(1,3k+1)\}$ or $\{(1,1),(2,4),(1,7),(2,10),...,(2,3k+1)\}$ is an EDD-set depending on whether k is even or odd. The case when $m = 10$ is illustrated in Figure 3.2.

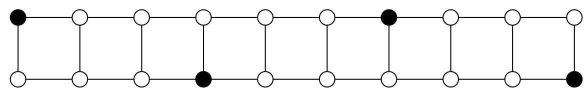

Figure 3.2: *An EDD-set of* $G_{2,10}$

Case (ii) $m \equiv 0, 2$ $(mod\ 3)$

Let $m = 3k$ or $3k+2$. Then the construction of a γ_2^d-set of $G_{2,m}$ given in [26] shows that, any γ_2^d-set of $G_{2,m}$ contains two vertices within a distance less than 4 between them. Hence it follows from Lemma 3.2.3 that these are not EDD-graphs. □

Theorem 3.2.7. $G = P_3 \Box P_3$ is an EDD-graph.

Proof. Let $V(P_3) = \{1,2,3\}$. Then,

$$V(P_3 \Box P_3) = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}.$$

It can be verified that $D = \{(1,1),(3,3)\}$ is an EDD-set of G. Hence $P_3 \Box P_3$ is an EDD-graph. □

Theorem 3.2.8. $G = P_4 \Box P_3$ is not an EDD-graph.

Proof. Consider the graph $G = P_4 \Box P_3$ given in Figure 3.3. If possible let D be an EDD-set of G. It is clear that D must contain at least two vertices in G. Also from Lemma 3.2.3 it follows that there cannot be 3 vertices in D. Hence an EDD-set of G, if it exists, must be of order 2. If $(2,2)$ or $(3,2)$ is in D, Lemma 3.2.3 shows that D cannot contain a second vertex because all the other vertices of G are within a distance of 3 from these two vertices. Hence there are only two different possibilities for the set D. Without loss of generality we can assume the two different cases as $D = \{(1,1),(3,3)\}$ or $D = \{(1,1),(4,2)\}$.

Case(i) $D = \{(1,1),(3,3)\}$

Consider the vertex $(4,1)$. It is at a distance 3 from both the vertices in D. So it is neither dominated nor disjunctively dominated by D. This is a contradiction to the

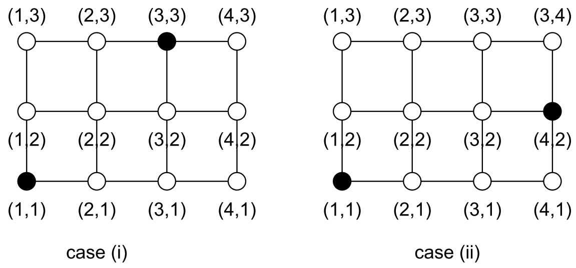

Figure 3.3: $P_4 \Box P_3$

choice of D.

Case(ii) $D = \{(1,1),(4,2)\}$

In this case the vertex $(2,3)$ is at a distance 3 from both the vertices in D. Hence it is neither dominated nor disjunctively dominated by D which is again a contradiction to the choice of D.

Thus in both cases we arrive at a contradiction to the assumption that D is an EDD-set of G. So we conclude that $G = P_4 \Box P_3$ has no EDD-set. $\square$

Theorem 3.2.9. A two dimensional grid graph $G = P_n \Box P_m$ has no EDD-set if $n \geq 4$ and $m \geq 3$.

Proof. Let $V(P_n) = \{u_1, u_2, ..., u_n\}$ and $V(P_m) = \{v_1, v_2, ..., v_m\}$. If possible let D be an EDD-set of G. It is clear that D must contain at least two vertices. Choose some vertex $(u_i, v_j) \in D$. Since G contains $P_4 \Box P_3$ as a subgraph, with out loss of generality we can assume that there exist a path P_3 in G having vertices $(u_i, v_j), (u_{i+1}, v_j), (u_{i+2}, v_j)$. Relabel the vertices of G as $(u_i, v_j) = (0,0)$, $(u_{i\pm x}, v_{j\pm y}) = (\pm x, \pm y)$.

Vertex $(0,0) \in D$ dominates $(1,0), (-1,0), (0,1)$ and $(0,-1)$. Vertex $(2,0)$ is at a distance 2 from $(0,0) \in D$. For the disjunctive domination of this vertex, D must contain another vertex which is also at a distance 2 from $(2,0)$. Lemma 3.2.3 shows that vertices $(1,1)$ and $(1,-1)$ cannot be in D. Hence D must contain one vertex from the set $\{(4,0),(2,2),(2,-2),(3,1),(3,-1)\}$ whichever exists in G. Due to symmetry of G, we need to consider only one vertex from the vertices $\{(2,-2),(2,2)\}$ and one vertex from the vertices $\{(3,1),(3,-1)\}$. Hence without loss of generality we can assume that there are only three possible cases: $(i)(4,0) \in D$, $(2,2) \in D$ or $(3,1) \in D$.

Case(i) $(4, 0) \in D$

Since G contains $P_4 \Box P_3$ as sub-graph, vertex $(2, 1)$ or $(2, -1)$ will be in G. Both these vertices are at a distance three from $(0, 0)$ and $(4, 0)$. Due to symmetry of G we can assume, with out loss of generality, that $(2, 1) \in G$. For the domination or disjunctive domination of this vertex, D must contain one vertex from its closed neighborhood

$$N[(2, 1)] = \{(2, 1), (1, 1), (3, 1), (2, 0), (2, 2)\}$$

or two vertices from its second neighborhood

$$N_2((2, 1)) = \{(1, 0), (3, 0), (2, -1), (0, 1), (4, 1), (1, 2), (3, 2), (2, 3)\}.$$

If D contains a vertex from $N[(2, 1)]$, then $f_D(2, 0)$ will be greater than one, which contradicts the definition of D. For disjunctive domination of $(2, 1)$, if two vertices from its second neighborhood are chosen in D, these vertices together with the already chosen vertices $(0, 0)$ and $(4, 0)$ in D do not satisfy Lemma 3.2.3. Thus $(2, 1)$ is neither dominated nor disjunctively dominated by D, which contradicts the definition of D.

Case(ii) $(2, 2) \in D$

Since G contains $P_4 \Box P_3$ as sub-graph, vertex $(3, 0)$ or $(-1, 2)$ will be in G. Due to symmetry of G we can assume that $(3, 0) \in G$. For the domination of this vertex, D must contain one vertex from its closed neighbor set $N[(3, 0)] = \{(3, 0), (2, 0), (4, 0), (3, 1), (3, -1)\}$. But any of these vertices in D makes $f_D(2, 0) > 1$. Now for the disjunctive domination of $(3, 0)$, there must be two vertices in D from its second neighborhood,

$$N_2((3, 0)) = \{(4, -1), (4, 1), (3, -2), (3, 2), (2, -1), (2, 1), (1, 0), (5, 0)\}.$$

But Lemma 3.2.3 shows that the only possible case is $(5, 0), (3, -2) \in D$. Suppose these two vertices are in D. Then $(3, 0)$ is disjunctively dominated by D. Now consider the vertex $(3, 1) \in G$. This vertex is at a distance 2 from $(2, 2) \in D$ and at a distance 3 from other vertices chosen in D. For the disjunctive domination of this vertex, D must contain another vertex from its second neighborhood. But all the vertices in its second neighborhood are at a distance less than 4 from the already chosen vertices, which is a contradiction to the choice of D.

Case(iii) $(3,1) \in D$

Vertex $(1,2)$ or $(2,-1)$ will be in G. We can assume, without loss of generality, that $(1,2) \in G$. For the domination of this vertex, D must contain one vertex from its closed neighbor set $N[(1,2)] = \{(1,2),(0,2),(2,2),(1,3),(1,1)\}$. But any of these vertices in D makes $f_D(1,1) > 1$. Now for the disjunctive domination of $(1,2)$, there must be two vertices in D from its second neighborhood,

$$N_2((1,2)) = \{(-1,2),(0,1),(0,3),(1,0),(1,4),(2,1),(2,3),(3,2)\}.$$

All these vertices except $(1,4)$ are at a distance less than 4 from the already chosen vertices in D. Hence disjunctive domination of $(1,2)$ is also not possible, contradicting our hypothesis on D.

From the above cases we can conclude that an EDD-set cannot exist in a grid graph which has an induced sub-graph isomorphic to $P_4 \Box P_3$. $\square$

Theorem 3.2.10. $G = P_n \Box P_m$ is an EDD-graph if and only if

(i) n= 2, m= 3k+1

(ii) n= m= 3

Proof. It follows from Theorems 3.2.6, 3.2.7, 3.2.8 and 3.2.9. $\square$

3.3 Nearly efficient disjunctive dominating sets

From the above theorem we can see that an infinite grid graph has no EDD-set, but it has a disjunctive dominating set with the following property.

Theorem 3.3.1. An infinite grid graph G has a disjunctive dominating set D such that for each vertex $u \in V$ in G, $1 \leq f_D(u) < 2$.

Proof. Let $\mathbb{Z}$ denote the additive group of integers, $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ the product of $\mathbb{Z}$ with itself and $\mathbb{Z}_8$ the group of integers modulo 8. Let

$$f : \mathbb{Z}^2 \to \mathbb{Z}_8$$

be the homomorphism given by

$$f(x, y) = x + 3y \; for \; (x, y) \in \mathbb{Z}^2.$$

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then $f(e_1) = 1$ and $f(e_2) = 3$.
For all $u = (x, y) \in \mathbb{Z}^2$,
$f(u \pm e_1) = f(u) \pm 1 = f(u) + 1 \; or f(u) + 7$ in $\mathbb{Z}_8$,
$f(u \pm e_2) = f(u) \pm 3 = f(u) + 3$ or $f(u) + 5$,
$f(u \pm 2e_1) = f(u) \pm 2 = f(u) + 2$ or $f(u) + 6$,
$f(u \pm 2e_2) = f(u) \pm 6 = f(u) + 6$ or $f(u) + 2$,
$f(u + e_1 + e_2) = f(u) + 4$,
$f(u + e_1 - e_2) = f(u) - 2 = f(u) + 6$,
$f(u + e_2 - e_1) = f(u) + 2$,
$f(u - e_1 - e_2) = f(u) - 4 = f(u) + 4$.
The unit ball $B(u)$ about a vertex $u \in \mathbb{Z}^2$ is defined as the set

$$B(u) = \{v : d(u, v) \leq 1\}.$$

The ball $B^2(u)$ about $u \in \mathbb{Z}^2$ and of radius 2 is defined as the set

$$B^2(u) = \{v : d(v, u) \leq 2\}.$$

So

$$B(u) = \{u, u \pm e_1, u \pm e_2\}$$

and

$$B^2(u) = \{u, u \pm e_1, u \pm e_2, u \pm 2e_1, u \pm 2e_2, u + e_1 + e_2, u + e_1 - e_2, u + e_2 - e_1, u - e_1 - e_2\}.$$

Hence

$$f(B(u)) = \{f(u), f(u) + 1, f(u) + 3, f(u) + 5, f(u) + 7\}$$

and

$$f(B^2(u)) = \{f(u), f(u) + 1, f(u) + 2, f(u) + 3, f(u) + 4, f(u) + 5, f(u) + 6, f(u) + 7\}.$$

Thus

$$f(B^2(u)) = f(u) + \mathbb{Z}_8 = \mathbb{Z}_8.$$

So f restricted to $B(u)$ is a bijection to a subset of $\mathbb{Z}_8$ and its restriction to $B^2(u)$ is an onto map from $B^2(u)$ to $\mathbb{Z}_8$.

Consider the subset $D = f^{-1}(0)$ of V. It is easy to see that,

$$B(u) \cap B^2(v) = \phi \quad if\ u \neq v\ and\ u, v \in f^{-1}(0).$$

We can show that D is a disjunctive dominating set of G such that each vertex in V is either dominated by exactly one vertex in D or disjunctively dominated by 2 or 3 vertices in D so that $f_D(u) = 1\ or\ \frac{3}{2}$ for all $u \in V$.

Let $u = (x, y)$ be any element of V. Following are the different possibilities for u.

(i) If $f(u) = 0$, then $u \in D$ and $f_D(u) = 1$.

(ii) If $f(u) = 1$, then
$f(u - e_1) = f(x - 1, y) = 0$ and so $u - e_1 \in D$. Also $u - e_1$ is at a distance one from u. Hence u is dominated by $u - e_1 \in D$ and $f_D(u) = 1$.

(iii) If $f(u) = 2$, then
$f(u - 2e_1) = f(x - 2, y) = 0$,
$f(u + 2e_2) = f(x, y + 2) = 0\ in\ \mathbb{Z}_8$ and
$f(u + e_1 - e_2) = f(x + 1, y - 1) = 0$.
Hence $u - 2e_1, u + 2e_2, u + e_1 - e_2 \in D$. These vertices are at a distance 2 from u. So u is disjunctively dominated by these three vertices in D and so $f_D(u) = \frac{3}{2} < 2$.

(iv) If $f(u) = 3$, then
$f(u - e_2) = f(x, y - 1) = 0$ and so $u - e_2 \in D$. Then u is dominated by $u - e_2 \in D$ and $f_D(u) = 1$.

(v) If $f(u) = 4$, then
$f(u + e_1 + e_2) = f(x + 1, y + 1) = 0$ in $\mathbb{Z}_8$ and
$f(u - e_1 - e_2) = f(x - 1, y - 1) = 0$
So $u + e_1 + e_2, u - e_1 - e_2 \in D$. Thus u is at a distance of 2 from two different

vertices in D or it is disjunctively dominated by two vertices in D and so $f_D(u) = 1$.

(vi) If $f(u) = 5$, then
$f(u + e_2) = f(x, y + 1) = 0$ in $\mathbb{Z}_8$ and so $u + e_2 \in D$. Then u is dominated by $u + e_2 \in D$ and $f_D(u) = 1$.

(vii) If $f(u) = 6$, then
$f(u + 2e_1) = f(x + 2, y) = 0$ in $\mathbb{Z}_8$
$f(u - 2e_2) = f(x, y - 2) = 0$ and
$f(u - e_1 + e_2) = f(x - 1, y + 1) = 0$ in $\mathbb{Z}_8$.
Hence $u + 2e_1, u - 2e_2, u - e_1 + e_2 \in D$. Thus u is at a distance of 2 from three different vertices in D. Hence it is disjunctively dominated by three vertices in D and so $f_D(u) = \frac{3}{2}$.

(viii) If $f(u) = 7$, then
$f(u + e_1) = f(x + 1, y) = 0$ in $\mathbb{Z}_8$ and so $u + e_1 \in D$. Then u is dominated by $u + e_1 \in D$ and $f_D(u) = 1$.

Thus in all the cases $u \in V$ is either dominated exactly once or disjunctively dominated by 2 or 3 vertices in D and so $1 \leq f_D(u) \leq \frac{3}{2} < 2$ for all $u \in V$. □

The above theorem motivated us to define a **nearly efficient disjunctive dominating set** in a graph.

Definition 3.3.2. Let $G = (V, E)$. A subset D of V for which $B(u) \cap B^2(v) = \phi$ *for every* $u, v \in D$ and $1 \leq f_D(u) < 2$ for every $u \in V$ is called a **nearly efficient disjunctive dominating set** or **NEDD**-set. A graph having an NEDD-set is called an **NEDD-graph.**

Example 3.3.1. NEDD-sets of $P_5 \square P_3$, $P_5 \square P_5$ and $P_5 \square P_2 \square P_2$ are shown in Figure 3.4.

Even though an infinite two dimensional grid graph has an NEDD-set, it can be observed that all finite two dimensional grid graphs are not NEDD-graphs.

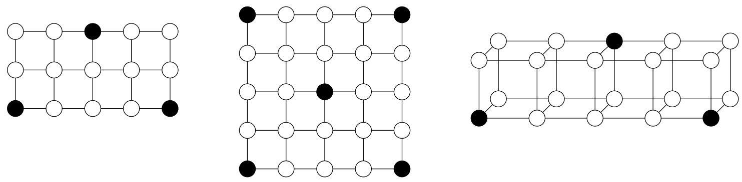

Figure 3.4: *NEDD-graphs*

3.4 Conclusion

In this chapter we have introduced and studied efficient and nearly efficient disjunctive dominating sets (EDD-sets and NEDD-sets) in graphs which have importance in minimizing waste and maximizing the effectiveness of dominating sets in practical applications. We examined the existence of EDD-sets and NEDD-sets in some graphs, especially in two dimensional grid graphs. Existence of EDD-sets and NEDD-sets in three dimensional grid graphs are interesting topics for further study. Study of existence of these sets in other graphs also have further scope for research.

CHAPTER 4

Strength Based Domination in Graphs

4.1 Introduction

Several models of domination have been investigated during the past four decades. Independent domination, total domination, connected domination, paired domination and restrained domination are some of the domination models which are based on considering restrictions on the induced subgraph $G[S]$ or $G[V \setminus S]$. For a dominating set S of a graph G, if $v \in V \setminus S$, then $|N[v] \cap S| \geq 1$. Restrictions based on what we allow for $N[v] \cap S$, domination models such as k-domination, locating domination and perfect domination have been investigated. Some of the domination models based on the concept of distance are distance-k-domination, k-step domination and (k, r)-domination.

In several real life situations such as social networks, communication networks and biological networks, the influence of the vertex extends beyond its neighborhood but decreases with distance. To address this problem Dankelmann et al. [21] considered the case where the dominating power of a vertex decreases exponentially

This chapter is a collaboration work of ours with Dr. S. Arumugam, National Centre for Advanced Research in Discrete Mathematics, Kalasalingam Academy of Research and Education, Anand Nagar, Krishnankoil-626126, India (s.arumugam.klu@gmail.com) by the factor $\frac{1}{2}$ with distance. In Disjunctive domination [26], two vertices v_1 and v_2 with $d(v, v_1) = d(v, v_2) = 2$ dominate the vertex v. Exponential domination is a global concept with respect to the vertices. In this chapter we introduce a new concept in domination, called strength based domination which is also global with respect to the graphs. This general concept is a generalization of the usual domination and disjunctive domination.

4.2 Strength based domination in graphs

Definition 4.2.1. Let $G = (V, E)$ be a connected graph. Let $u, v \in V$. The dominating strength between u and v is defined as $s(u, v) = \frac{1}{d(u, v)}$.

Definition 4.2.2. Let $A \subseteq V$ and $v \in V \setminus A$. Then,

$$s(v, A) = \sum_{u \in A} s(u, v) = \sum_{u \in A} \frac{1}{d(u, v)}$$

is defined as the dominating strength of A on v.

Definition 4.2.3. Let $G = (V, E)$ be a connected graph. A subset D of V is called a strength based dominating set or a sb-dominating set of G if for every $v \in V \setminus D$, there exists a subset D_1 of D such that $s(v, D_1) \geq 1$. Minimum cardinality of a sb-dominating set of G is called the sb-domination number of G and is denoted by $\gamma_{sb}(G)$. Any sb-dominating set of cardinality γ_{sb} is called a γ_{sb}-set of G.

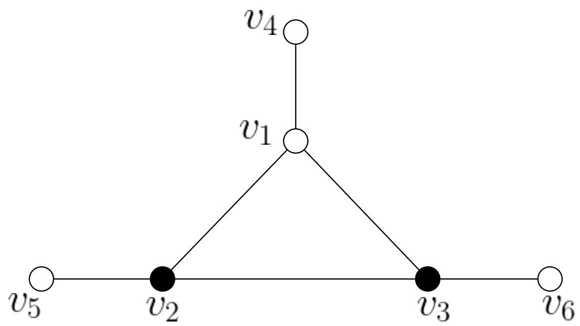

Figure 4.1: A graph with $\gamma_{sb}(G) = 2$ and $\gamma(G) = 3$

Observation 4.2.4. Let G be a connected graph. Clearly $\gamma_{sb}(G) = 1$ if and only if $\gamma(G) = 1$. Also, if $\gamma(G) = 2$, then $\gamma_{sb}(G) = 2$. However the converse is not true.

For the graph G given in Figure 4.1 , $S = \{v_2, v_3\}$ is a sb-dominating set. Hence $\gamma_{sb}(G) = 2$. However $\gamma(G) = 3$.

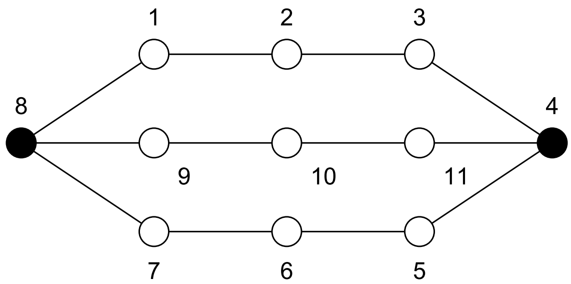

Figure 4.2: A graph with $\gamma_{sb}(G) = 2$

Example 4.2.1. For the graph G given in Figure 4.2, $S = \{4, 8\}$ is a sb-dominating set of G. Hence $\gamma_{sb}(G) \leq 2$. Since $\gamma(G) \neq 1$, it follows that $\gamma_{sb}(G) \geq 2$. Hence $\gamma_{sb}(G) = 2$.

Example 4.2.2. Consider the graph G given in Figure 4.3. Let $P_1 = \{2, 5, 8\}$, $P_2 = \{3, 6, 9\}$ and $P_3 = \{4, 7, 10\}$. $S = \{8, 9, 10\}$ is a sb-dominating set of G and hence $\gamma_{sb}(G) \leq 3$. Now let D be any sb-dominating set of G. If $|D| = 2$, then $D \cap P_i = \emptyset$ for at least one i. Let $D \cap P_1 = \emptyset$. Then $s(8, D) \leq \frac{1}{4} + \frac{1}{4} < 1$, which is a contradiction. Hence $\gamma_{sb}(G) \geq 3$. Thus $\gamma_{sb}(G) = 3$.

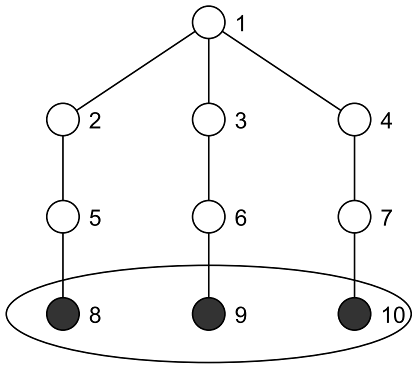

Figure 4.3: A graph with $\gamma_{sb}(G)$=3

Example 4.2.3. For the Petersen graph G given in Figure 4.4, $S = \{1, 2\}$ is a sb-dominating set of G. Hence $\gamma_{sb}(G) = 2$.

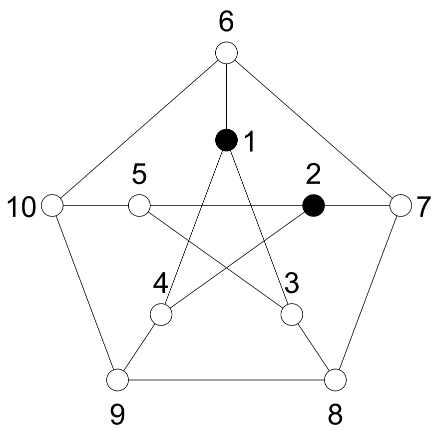

Figure 4.4: γ_{sb}-set of Petersen Graph

Theorem 4.2.5. *For any graph G, $\gamma_{sb}(G) \leq \gamma_2^d(G) \leq \gamma(G)$,where $\gamma_2^d(G)$ is the disjunctive domination number of G.*

Proof. Every dominating set of G is a disjunctive dominating set as well as a sb-dominating set. Every disjunctive dominating set of G is a sb-dominating set. Hence $\gamma_{sb}(G) \leq \gamma_2^d(G) \leq \gamma(G)$. □

Remark 4.2.1. *There are graphs for which all these three are equal. For example $\gamma_{sb}(P_n) = \gamma_2^d(P_n) = \gamma(P_n)$ for $1 \leq n \leq 6$.*
There are graphs for which all these are different. For example $\gamma_{sb}(C_{13}) = 3$, $\gamma_2^d(C_{13}) = 4$ and $\gamma(C_{13}) = 5$.

Observation 4.2.6. *1. $\gamma_{sb}(G) = n$ if and only if $G = \bar{K_n}$.*

2. $\gamma_{sb}(G) = 1$ if and only if G has a universal vertex.

3. If $\gamma_2^d(G) \leq 3$, then $\gamma_{sb}(G) = \gamma_2^d(G)$.

Theorem 4.2.7. If $\gamma_{sb}(G) = 2$, then $diam(G) \leq 6$.

Proof. Let $S = \{u, v\}$ be a sb-dominating set of G. If S is a dominating set of G, then $\gamma(G) = 2$. Then by Theorem 1.3.5, $diam(G) \leq 5$.
Suppose S is not a dominating set. Let $S_1 = N[u] \cup N[v]$ and $S_2 = V \setminus S_1$.
Clearly $S_2 \neq \emptyset$. Let $x, y \in V$.

Case (i) $x, y \in S_2$

Then $d(x, u) = d(y, u) = 2$. Hence $d(x, y) \leq 4$.

Case (ii) $x \in S_2$ **and** $y \in S_1$

Since $y \in S_1$, we may assume without loss of generality that $y \in N(u)$. Now $d(x,u) = 2$ and $d(u,y) = 1$. Hence $d(x,y) \leq 3$.

Case (iii)$x, y \in S_1$

If x and y have a common neighbor, then $d(x,y) \leq 2$. Suppose $N(x) \cap N(u) = \emptyset$.
Let $x \in N(u)$ and $y \in N(v)$. If u and v are adjacent, then $d(x,y) = 3$.
Suppose u and v are nonadjacent. Let $w \in S_2$.
Then $d(x,y) \leq d(x,u) + d(u,w) + d(w,v) + d(v,y) \leq 1 + 2 + 2 + 1$.
Hence $d(x,y) \leq 6$.
Thus diam $(G) \leq 6$. □

Theorem 4.2.8. *Let k be a positive integer with $2 \leq k \leq 6$. Then there exists a graph G such that $\gamma_{sb}(G) = 2$, $\gamma(G) > 2$ and diam $(G) = k$.*

Proof. We consider five cases.

Case (i) $k = 2$

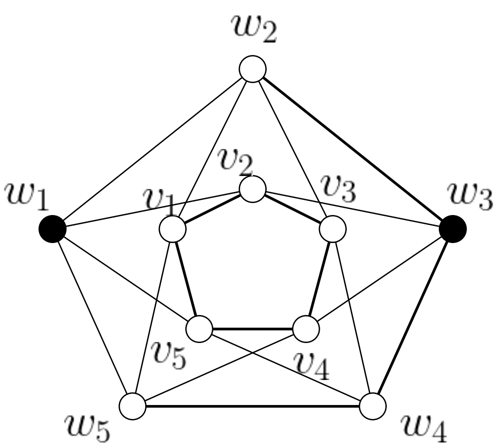

Figure 4.5: *A graph G with $\gamma(G) > 2$, $\gamma_{sb}(G) = 2$ and diam$(G) = 2$*

Let G be the graph obtained from two copies $G_1 = (v_1, v_2, v_3, v_4, v_5, v_1)$ and $G_2 = (w_1, w_2, w_3, w_4, w_5, w_1)$ of C_5 such that $N(w_i) \cap V(G_1) = \{v_{i-1}, v_{i+1}\}$. The graph G is given in Figure 4.5.
Let $S = \{w_1, w_3\}$, $S_1 = \{w_2, w_4, w_5, v_2, v_4, v_5\}$ and $S_2 = \{v_1, v_3\}$.
Clearly v_1 and v_3 are sb-dominatined by S. Hence $\gamma_{sb}(G) = 2$.
Also any two non-adjacent vertices of G have a common neighbor and hence

$diam(G) = 2$.

Now let $x, y \in V(G)$. Since $N(x) \cap N(y) \neq \emptyset$ and $|N(x)| = |N(y)| = 4$ we have $|N[x] \cup N[y]| \leq 9$. Hence $\{x, y\}$ is not a dominating set of G. Thus $\gamma(G) > 2$.

Case (ii) $k = 3$

Let G be the graph obtained from $K_4 \circ K_1$ by removing one pendent vertex.
Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and let w_i be the pendent vertex adjacent to $v_i, 1 \leq i \leq 3$.
Clearly $\gamma(G) = 3$. Also $S = \{v_1, v_2\}$ is a sb-dominating set of G and hence $\gamma_{sb}(G) = 2$.
Since $d(w_1, w_3) = 3$, it follows that $diam(G) = 3$.

Case (iii) $k = 4$

Let $P_3 = (v_1, v_2, v_3)$. Let $G = P_3 \circ K_1$ and let w_i be the pendent vertex adjacent to v_i, $1 \leq i \leq 3$. Then $S = \{v_1, v_3\}$ is a sb-dominating set of G. Hence $\gamma_{sb}(G) = 2$.
Also $\gamma(G) = 3$. Since $d(w_1, w_3) = 4$, it follows that $diam(G) = 4$.

Case (iv) $k = 5$

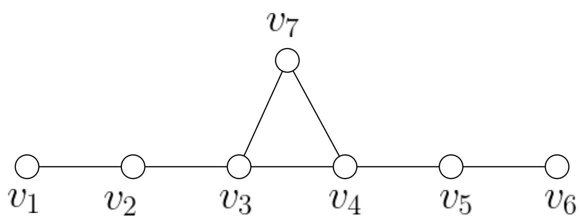

Figure 4.6: *A graph G with* $\gamma(G) = 3$, $\gamma_{sb} = 2$ *and diam(G)=5*

Let G be the graph given in Figure 4.6. Clearly $S = \{v_2, v_5\}$ is a sb-dominating set of G and hence $\gamma_{sb}(G) = 2$. Also $\gamma(G) = 3$. Since $d(v_1, v_6) = 5$, it follows that $diam(G) = 5$.

Case (v) $k = 6$

Let $P_7 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$. Clearly $\gamma(G) = 3$ and $diam(G) = 3$. Also $S = \{v_2, v_6\}$ is a sb-dominating set of G and hence $\gamma_{sb}(G) = 2$.

□

Definition 4.2.9. Let $G = (V, E)$ be a connected graph and let $v \in V$. Then the dominating strength of v, denoted by $ds(v)$, is defined by $ds(v) = \sum_{u \neq v} \frac{1}{d(u,v)}$. The sequence $ds(v_1) \geq ds(v_2) \geq ... \geq ds(v_n)$) is called the dominating strength sequence of G.

Clearly, $ds(v) \geq deg(v)$.

Let $\Delta_{sb} = max\{ds(v) : v \in V\}$ and $\delta_{sb} = min\{ds(v) : v \in V\}$.

Definition 4.2.10. A graph is called $sb-regular$ if $ds(v)$ is a constant for all $v \in V$. K_n, $K_{m,m}$, C_n, Q_3 and Petersen graph are some examples of sb-regular graphs.

Example 4.2.4. For the graph in Figure 4.6, $ds(v_1) = ds(v_6) = \dfrac{157}{60}$, $ds(v_2) = ds(v_5) = \dfrac{43}{12}$, $ds(v_3) = ds(v_4) = \dfrac{13}{3}$ and $ds(v_7) = \dfrac{11}{3}$.
For this graph $\Delta_{sb} = ds(v_3) = ds(v_4) = \dfrac{13}{3}$ and $\delta_{sb} = ds(v_1) = ds(v_6) = \dfrac{157}{60}$.

Theorem 4.2.11. Let G be the graph obtained from $K_{1,k}, k \geq 3$, by sub- dividing each edge $k-2$ times. Then $\gamma_{sb}(G) = k$.

Proof. Let $V(K_{1,k}) = \{v, v_1, v_2, ..., v_k\}$ where $deg(v) = k$. Let v_{i_1}, v_{i_2}, $v_{i_{(k-2)}}$ be the vertices sub-dividing the edge vv_i. Clearly $S = \{v_{1_1}, v_{2_1}, ..., v_{k_1}\}$ is a sb-dominating of G and hence $\gamma_{sb}(G) \leq k$. Now let D be any γ_{sb}-set of G. Let P_i denote the path $(v, v_{i_1}, v_{i_2}, ...v_{i_{(k-2)}}, v_i)$. If $D \cap V(P_i) = \emptyset$, then $s(v_i, D) \leq \frac{k-1}{k} < 1$ which is a contradiction. Hence $D \cap P_i \neq \emptyset$ for all i, $1 \leq i \leq k$. Hence $\gamma_{sb}(G) = |D| \geq k$. Thus $\gamma sb(G) = k$. □

Corollary 4.2.12. For any positive integer k, there exists a graph G with $\gamma_{sb}(G) = k$.

Proof. If $k = 1$ or 2, any graph with $\gamma(G) = 1$ or 2 has sb-domination numbers $1, 2$ respectively. For $k \geq 3$, the graph given in Theorem 4.2.11 has $\gamma_{sb}(G) = k$. The cases $k = 3, 4$ are depicted in Figure 4.7.

□

4.3 On sb-domination number of paths

We now proceed to determine the sb-domination number of paths.

Theorem 4.3.1. Let $P_n = (1, 2, 3, ..., n)$ be a path of order n. Then $\gamma_{sb}(P_n) \leq \lceil \frac{n+1}{4} \rceil$ and equality holds if and only if $1 \leq n \leq 15$.

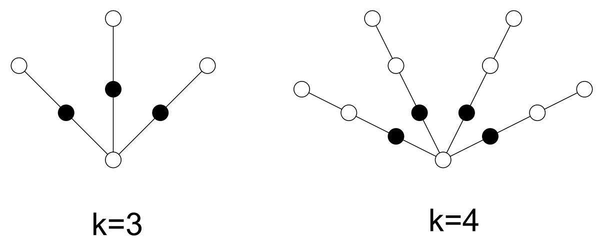

Figure 4.7: Examples of graphs with $\gamma_{sb} = k$ for $k = 3, 4$

Proof. It follows from Theorem 1.3.22 that $\gamma_2^d(P_n) = \lceil \frac{n+1}{4} \rceil$. Also $\gamma_{sb}(P_n) \leq \gamma_2^d(P_n)$ and hence $\gamma_{sb}(P_n) \leq \lceil \frac{n+1}{4} \rceil$.

Now $\gamma_2^d(P_n) = \begin{cases} 1 & \text{if } 1 \leq n \leq 3 \\ 2 & \text{if } 4 \leq n \leq 7 \\ 3 & \text{if } 8 \leq n \leq 11 \end{cases}$

Further, for any graph G, if $\gamma_2^d(G) \leq 3$, then $\gamma_{sb}(G) = \gamma_2^d(G)$. Hence it follows that $\gamma_{sb}(P_n) = \gamma_2^d(P_n) = \lceil \frac{n+1}{4} \rceil$ if $1 \leq n \leq 11$.

Now let $n = 12$.

Then $\gamma_{sb}(P_{12}) \leq \lceil \frac{12+1}{4} \rceil = 4$.

Now, suppose P_{12} has a sb-dominating set D with $|D| = 3$. Let $D = \{i_1, i_2, i_3\}$ where $1 \leq i_1 < i_2 < i_3 \leq 12$.

If $i_3 \leq 6$, then $s(12, D) \leq \frac{1}{2}$, which is a contradiction. Hence $i_3 \geq 7$. By a similar argument we have $i_1 \leq 6$. Hence we may assume without loss of generality that $i_1 \leq 6$ and $7 \leq i_2 < i_3$. Let $i_2 = 6 + a$ and $i_3 = 6 + b$ where $1 \leq a < b$.

Now, $s(1, i_2) + s(1, i_3) < \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Hence $s(1, i_1) > \frac{2}{3}$.

Therefore, $\dfrac{1}{i_1 - 1} > \dfrac{2}{3}$, which implies that $i_1 \leq 2$.

Now, $s(4, D) = \dfrac{1}{4 - i_1} + \dfrac{1}{2 + a} + \dfrac{1}{2 + b} \geq 1$.

Since $a < b$, we have

$$\frac{1}{4 - i_1} + \frac{2}{2 + a} > 1 \tag{4.3.1}$$

If $i_1 = 1$, then (4.3.1) implies that $\dfrac{2}{2 + a} > 1 - \dfrac{1}{3} = \dfrac{2}{3}$.

Hence $2 + a < 3$, which is a contradiction.

If $i = 2$, then (4.3.1) implies that $\dfrac{2}{2 + a} > \dfrac{1}{2}$. Hence $2 + a < 4$, so that $a = 1$.

Thus $i_1 = 6 + a = 7$.

Now, $s(4,D) = \dfrac{1}{4-i_1} + \dfrac{1}{2+a} + \dfrac{1}{2+b} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{2+b} \geq 1.$
Hence $\dfrac{1}{2+b} \geq \dfrac{1}{6}$, so that $b \leq 4$.
Thus $i_3 = 6 + b \leq 10$.
Now, $s(12,D) = \dfrac{1}{10} + \dfrac{1}{5} + \dfrac{1}{12-i_3} \geq 1.$
Hence $\dfrac{1}{12-i_3} \geq 1 - \dfrac{1}{10} - \dfrac{1}{5} = \dfrac{7}{10}.$
Therefore $84 - 7i_3 \geq 10$.

Thus $7i_3 \geq 74$, which is a contradiction, since $i_3 \leq 10$.

Thus D is not a sb-dominating set of P_{12}.

Hence $\gamma_{sb}(P_{12}) \geq 4$ and therefore $\gamma_{sb}(P_{12}) = 4 = \lceil \frac{n+1}{4} \rceil$.

Now let $13 \leq n \leq 15$.

Since $\gamma_{sb}(P_{12}) = 4$, it follows that $\gamma_{sb}(P_n) \geq 4$ if $13 \leq n \leq 15$.

Also $D = \{2, 6, 10, n-1\}$ is a sb-dominating set of P_n for $13 \leq n \leq 15$ and so $\gamma_{sb}(P_n) = 4$ if $13 \leq n \leq 15$.

Thus $\gamma_{sb}(P_n) = \lceil \frac{n+1}{4} \rceil$ if $1 \leq n \leq 15$.

Suppose $n = 16$. Then $\lceil \frac{n+1}{4} \rceil = 5$. We claim that $\gamma_{sb}(P_{16}) = 4$.

Let $D = \{2, 7, 11, 15\}$. The vertices in $\{1, 3, 6, 8, 10, 12, 14, 16\}$ are dominated by D. Also $s(4,D) = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{11} > 1$, $s(5,D) = \dfrac{1}{3} + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{10} > s(4,D) > 1$, $s(9,D) > \dfrac{1}{2} + \dfrac{1}{2} = 1$ and $s(13,D) > \dfrac{1}{2} + \dfrac{1}{2} = 1$.
Thus D is a sb-dominating set of P_{16}. Hence $\gamma_{sb}(P_{16}) \leq 4$.

Also $\gamma_{sb}(P_{16}) \geq \gamma_{sb}(P_{15}) = 4$.

Thus $\gamma_{sb}(P_{16}) = 4 < \lceil \frac{16+1}{4} \rceil$.

Hence $\gamma_{sb}(P_n) = \lceil \frac{n+1}{4} \rceil$ if and only if $1 \leq n \leq 15$. □

Theorem 4.3.2. Let $P_n = (1, 2, 3, ..., n)$ be a path of order n. Then $\gamma_{sb}(P_n) = \lceil \frac{n+1}{5} \rceil$ if and only if $16 \leq n \leq 29$ and $n \neq 19$ and 24. Also $\gamma_{sb}(P_n) = \lceil \frac{n+1}{5} \rceil + 1$ if $n = 19$ or 24.

Proof. It follows from Theorem 4.3.1 that $\gamma_{sb}(P_{16}) = 4 = \lceil \frac{16+1}{5} \rceil$.

Now let $n = 17$ or 18.

Let $D = \{2, 7, 12, 17\}$. Then the vertices in $\{1, 3, 6, 8, 11, 13, 16, 17, 18\}$ are dominated by D.

Now $s(4, D) = s(15, D) = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{13} > 1$, $s(5, D) = s(14, D) = \frac{1}{3} + \frac{1}{2} + \frac{1}{7} + \frac{1}{12} > s(4, D) > 1$, $s(9, D) = \frac{1}{7} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} > s(5, D) > 1$ and $s(10, D) = \frac{1}{8} + \frac{1}{3} + \frac{1}{2} + \frac{1}{7} > s(5, D) > 1$. Hence D is a sb-dominating set of P_{17} and P_{18}.
Thus $\gamma_{sb}(P_n) \leq 4$ if $n = 17$ or 18.
Also $\gamma_{sb}(P_{17}) \geq \gamma_{sb}(P_{16}) = 4$ and $\gamma_{sb}(P_{18}) \geq \gamma_{sb}(P_{16}) = 4$.
Thus $\gamma_{sb}(P_n) = 4 = \lceil \frac{n+1}{5} \rceil$ if $n = 17$ or 18.

Since $\{2, 7, 12, 17\}$ is a sb-dominating set of P_{18}, it follows that $D = \{2, 7, 12, 17, 18\}$ is a sb-dominating set of P_{19}. Hence $\gamma_{sb}(P_{19}) \leq 5$.
We now claim that any set $D = \{i_1, i_2, i_3, i_4\}$ where $1 \leq i_1 < i_2 < i_3 < i_4 \leq 19$ is not a sb-dominating set of P_{19}.
Let $S_1 = \{1, 2, 3, ..., 10\}$ and $S_2 = \{11, 12, 13, ..., 19\}$.
Suppose $|D \cap S_1| = 1$. Let $D \cap S_1 = \{i_1\}$.

If $i_1 \leq 2$, then $s(4, D) \leq \frac{1}{2} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} < 1$.

If $i_1 > 2$, then $s(1, D) \leq \frac{1}{2} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} < 1$.
Hence $|D \cap S_1| = 1$ is not possible. Therefore $|D \cap S_1| \geq 2$.
Similarly, $|D \cap S_2| \geq 2$.
Since $|D| = 4$, it follows that $|D \cap S_1| = |D \cap S_2| = 2.$, that is, $i_1, i_2 \in S_1$ and $i_3, i_4 \in S_2$.

If $i_1 \geq 4$, then $s(1, D) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{10} + \frac{1}{11} < 1$. Hence $i_1 \leq 3$. Similarly, $i_4 \geq 17$.
If $i_1 = 3$, then $s(1, D) \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{16} < 1$.
Hence $i_1 \leq 2$. Similarly $i_4 \geq 18$.

If $i_2 \geq 8$, then $s(4, D) \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} < 1$.
Hence $i_1 \leq 2$, $3 \leq i_2 \leq 7$.
Similarly $13 \leq i_3 \leq 17$, $i_4 \geq 18$.
Now, $s(10, D) \leq \frac{1}{8} + \frac{1}{3} + \frac{1}{3} + \frac{1}{8} < 1$
Thus 10 is not sb-dominated by D. Thus if $|D| = 4$, then D is not a sb-dominating set of P_{19}.

Hence $\gamma_{sb}(P_{19}) \geq 5$ and so $\gamma_{sb}(P_{19}) = 5 = \lceil \frac{n+1}{5} \rceil + 1$.

Now $D = \{2, 7, 12, 17, (n-1)\}$ is a sb-dominating set of P_n if $20 \leq n \leq 23$. Hence it follows that $\gamma_{sb}(P_n) = 5 = \lceil \frac{n+1}{5} \rceil$ if $20 \leq n \leq 23$.

Now let $n = 24$. It can be verified that $D = \{2, 7, 12, 17, 22, 23\}$ is a sb-dominating set of P_{24}. Hence $\gamma_{sb}(P_{24}) \leq 6$.

We now claim that any subset $D = \{i_1, i_2, i_3, i_4, i_5\}$ where $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq 24$ is not a sb-dominating set of P_{24}.
Let $S_1 = \{1, 2, 3, ..., 12\}$ and $S_2 = \{13, 14, 15, ..., 24\}$.
Suppose $|D \cap S_1|$=1. Let $D \cap S_1 = \{i_1\}$.

If $i_1 \leq 2$, then $s(4, D) \leq \dfrac{1}{2} + \dfrac{1}{9} + \dfrac{1}{10} + \dfrac{1}{11} + \dfrac{1}{12} < \dfrac{1}{2} + \dfrac{4}{9} < 1$.

If $i_1 > 2$, then $s(1, D) \leq \dfrac{1}{2} + \dfrac{1}{12} + \dfrac{1}{13} + \dfrac{1}{14} + \dfrac{1}{15} < \dfrac{1}{2} + \dfrac{1}{3} < 1$.

Therefore $|D \cap S_1| \geq 2$. Without loss of generality let $|D \cap S_1| = 2$ and $|D \cap S_2| = 3$. Hence $i_1, i_2 \in S_1$ and $i_3, i_4, i_5 \in S_2$.

If $i_1 \geq 4$, then $s(1, D) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} < 1$. Hence $i_1 \leq 3$.
Now if $i_5 \leq 21$, then $s(24, D) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{12} + \frac{1}{21} < 1$. Hence $i_5 \geq 22$.

If $i_1 = 3$ and $i_2 \geq 5$, then $s(1, D) \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} < 1$. Hence if $i_1 = 3$, then $i_2 \leq 4$. But then, $s(9, D) < \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} < 1$.
Therefore $i_1 \neq 3$. Hence $i_1 \leq 2$.

Since $i_1 \leq 2$ and $i_5 \geq 22$, if $i_2 \geq 8$, then $s(5, D) \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{8} + \frac{1}{9} + \frac{1}{17} < 1$.
Hence $i_2 \leq 7$.

Now let $i_5 = 22$. Since $i_1 \leq 2$ and $i_2 \leq 7$, if $i_4 \leq 19$, then $s(24, D) \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{17} + \frac{1}{22} < 1$. Hence if $i_5 = 22$, then $i_4 \geq 20$.

Now $s(10, D) \leq \frac{1}{8} + \frac{1}{3} + \frac{1}{10} + \frac{1}{12} + \frac{1}{i_3 - 10}$.
If $s(10, D) \geq 1$, then $\frac{1}{i_3 - 10} \geq 1 - \left(\frac{1}{8} + \frac{1}{3} + \frac{1}{10} + \frac{1}{12}\right) = \frac{43}{120}$.
Hence $i_3 - 10 \leq \frac{120}{43}$, which is a contradiction since $i_3 \geq 13$. Therefore $i_5 \neq 22$.

Hence $i_5 \geq 23$.

Now since $i_1 \leq 2$, $i_2 \leq 7$ and $i_5 \geq 23$, if $i_3 \geq 14$, then $s(10, D) \leq \frac{1}{8} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{13} < 1$. Hence $i_3 \leq 13$. Since $i_3 \in S_2$, it follows that $i_3 = 13$.

Now if $i_4 \geq 18$, then $s(10, D) \leq \frac{1}{8} + \frac{1}{3} + \frac{1}{3} + \frac{1}{8} + \frac{1}{13} < 1$. Hence if $i_4 \leq 17$. But then $s(20, D) < 1$ which is a contradiction. Thus if $|D| = 5$, then D is not a sb-dominating set of P_{24}. Hence $\gamma_{sb}(P_{24}) \geq 6$ and so $\gamma_{sb}(P_{24}) = 6 = \lceil \frac{24+1}{5} \rceil + 1$.

Now let $25 \leq n \leq 28$ and let $D = \{2, 7, 12, 17, 22, (n-1)\}$. The set of vertices dominated by D is $\{1, 3, 6, 8, 11, 13, 16, 18, 21, 23, n-2, n\}$.

Also, $s(4, D) > \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{13} + \dfrac{1}{18} > 1$, $s(5, D) > s(4, D) > 1$. Similarly, $s(9, D)$, $s(10, D)$, $s(14, D)$, $s(15, D)$, $s(19, D)$, $s(20, D)$, $s(24, D)$ and $s(25, D)$ are all greater than $s(4, D) > 1$. Hence D is a sb-dominating set of P_n if $25 \leq n \leq 28$. Hence $\gamma_{sb}(P_n) \leq 6$. Also $\gamma_{sb}(P_n) \geq \gamma_{sb}(P_{24}) = 6$ for $n \geq 25$.
Thus $\gamma_{sb}(P_n) = 6 = \lceil \frac{24+1}{5} \rceil$ if $25 \leq n \leq 28$.

Now it can be verified that $D = \{2, 7, 12, 18, 23, 28\}$ is a sb-dominating set of P_{29} and hence $\gamma_{sb}(P_{29}) \leq 6$. Also $\gamma_{sb}(P_{29}) \geq \gamma_{sb}(P_{28}) = 6$. Hence $\gamma_{sb}(P_{29}) = 6 = \lceil \frac{n+1}{5} \rceil$.

Now we claim that $\gamma_{sb}(P_n) < \lceil \frac{n+1}{5} \rceil + 1$ if $n > 29$.
It can be verified that $D = \{2, 7, 13, 19, 25, 30\}$ is a sb-dominating set of P_{30} and P_{31}. Hence $\gamma_{sb}(P_n) = 6 < \lceil \frac{n+1}{5} \rceil$ if $n = 30, 31$. Thus, $\gamma_{sb}(P_n) = \lceil \frac{n+1}{5} \rceil$ if and only if $16 \leq n \leq 29$ and $n \neq 19$ and 24 and $\gamma_{sb}(P_n) = \lceil \frac{n+1}{5} \rceil + 1$ if $n = 19$ or 24. □

Corollary 4.3.3. $\gamma_{sb}(P_n) = 6$ *if* $24 \leq n \leq 31$.

Lemma 4.3.4. There exists a sb-dominating set $D = \{i_1, i_2, ... i_k\}$ for the path P_n where $1 \leq i_1 < i_2 < ... < i_k \leq n$ and $d(i_j, i_{j+1}) \geq 3$ for each i.

Proof. Suppose $d(i_j, i_{j+1}) = 1$ and let $(j, j+1)$ be the first such pair.
Let $D' = (D - \{i_j, i_{j+1}\}) \cup \{i_j - 1, i_{j+1} + 1\}$. We claim that D' is a γ_{sb}- set of P_n. Clearly the vertices $i_j - 2, i_j, i_{j+1}, i_{j+1} + 2$ are dominated by D'. Let u be any other vertex and let $u \leq i_j - 3$. Let $d(u, i_j) = d$. Now,

$$\begin{aligned} s(u,\{i_j-1,i_{j+1}+1\})-s(u,\{i_j,i_{j+1}\}) &= \Big(\frac{1}{d-1}+\frac{1}{d+2}\Big)-\Big(\frac{1}{d}+\frac{1}{d+1}\Big) \\ &= \Big(\frac{1}{d-1}-\frac{1}{d}\Big)-\Big(\frac{1}{d+1}-\frac{1}{d+2}\Big) \\ &= \frac{1}{(d-1)d}-\frac{1}{(d+1)(d+2)}\Big) \\ &> 0. \end{aligned}$$

Hence $s(u,D') > s(u,D) \geq 1$. Similarly, if $u \geq i_{j+1}+3$, then also $s(u,D') > s(u,D) \geq 1$. Thus D' is a sb-dominating set.

By repeating this process we may assume with out loss of generality that $D = \{i_1, i_2, ...i_k\}$ with $d(i_j, i_{j+1}) \geq 2$.

Now, suppose that there exists a pair of vertices i_j, i_{j+1} in D such that $d(i_j, i_{j+1}) = 2$ and let $(j, j+1)$ be the first such pair. If $i_j = i_1 = 1$, then replace i_2 by 4 or 5. If $i_j \geq 2$, let $D' = (D - \{i_j, i_{j+1}\}) \cup \{i_j - 1, i_{j+1}+1\}$. Clearly, the vertices $i_j-2, i_j, i_{j+1}, i_{j+1}+2$ are dominated by D'. Let u be any other vertex.

If $u = i_j + 1$ then $s(u,D') \geq \dfrac{1}{d(u,i_j-1)} + \dfrac{1}{d(u,i_{j+1}+1)} = \frac{1}{2}+\frac{1}{2} = 1$.

Let $u \leq i_j - 3$ and let $d(u,i_j) = d$. Then $d \geq 3$.

Now,

$$\begin{aligned} s(u,\{i_j-1,i_{j+1}+1\})-s(u,\{i_j,i_{j+1}\}) &= \Big(\frac{1}{d-1}+\frac{1}{d+3}\Big)-\Big(\frac{1}{d}+\frac{1}{d+2}\Big) \\ &= \Big(\frac{1}{d-1}-\frac{1}{d}\Big)-\Big(\frac{1}{d+2}-\frac{1}{d+3}\Big) \\ &= \frac{1}{(d-1)d}-\frac{1}{(d+2)(d+3)}\Big) \\ &> 0. \end{aligned}$$

Hence $s(u,D') > s(u,D) \geq 1$. Similarly, if $u \geq i_{j+1}+3$, then also $s(u,D') > s(u,D) \geq 1$. Thus D' is a sb-dominating set. By repeating this process we obtain a sb-dominating set D with distance between any two consecutive vertices of D at least 3. □

Theorem 4.3.5. *$\gamma_{sb}(P_n) = 7$ if $32 \leq n \leq 37$.*

Proof. Let $n = 32$. Then $D = \{2, 7, 12, 17, 22, 27, 32\}$ is a sb-dominating set of P_{32}. Hence $\gamma_{sb}(P_{32}) \leq 7$.

Suppose $D = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ where $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq 32$ is a sb-dominating set of P_{32} such that $d(i_j, i_{j+1}) \geq 3$ for $j = 1, 2, 3, 4, 5$.

Let $S_1 = \{1,2,3...,16\}$ and $S_2 = \{17,18,...,32\}$. We claim that D is not a sb-dominating set.
Suppose that $i_1, i_2 \in S_1$ and $i_3, i_4, i_5, i_6 \in S_2$.
If $1 \leq i_1 < i_2 \leq 8$, then $s(8,12) < 1$ and if $9 \leq i_1 < i_2 \leq 16$, then $s(1,D) < 1$. Now, suppose that $1 \leq i_1 \leq 8$ and $9 \leq i_2 \leq 16$. In this case if $i_1 \leq 2$, then $s(4,D) < 1$ and if $i_1 > 2$, then $s(1,D) < 1$. Thus D is not an sb-dominating set, which is a contradiction. Therefore $|D \cap S_1| \geq 3$. Similarly $|D \cap S_2| \geq 3$. Since $|D| = 6$, it follows that $|D \cap S_1| = |D \cap S_2| = 3$.

Hence we may assume that $i_1, i_2, i_3 \in S_1$ and $i_4, i_5, i_6 \in S_2$. Let $i_1 > 2$. Then $s(1,D) \leq \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{16}+\frac{1}{19}+\frac{1}{22} < 1$ which is a contradiction. Hence $i_1 \leq 2$. Similarly, $i_6 \geq 31$. Let there exists a pair of vertices i_j, i_{j+1} in D such that $d(i_j, i_{j+1}) \geq 8$. Then, $s(i_j+4, D) \leq \frac{2}{4}+\frac{2}{7}+\frac{2}{10} < 1$ which is again a contradiction. Hence $d(i_j, i_{j+1}) \leq 7$.

Now, we claim that $d(i_j, i_{j+1}) \geq 4$ for each $j = 1,2,3,4,5$. Suppose there exists a pair of vertices i_j, i_{j+1} in D such that $d(i_j, i_{j+1}) = 3$ and let $(j, j+1)$ be the first such pair.
If $i_j = i_1 = 1$, then replace i_2 by 6. Then vertices $2,5,7$ are dominated by $\{i_1, i_2\}$. Now, $s(3,D) \geq \frac{1}{2}+\frac{1}{3}+\frac{1}{10}+\frac{1}{17}+\frac{1}{24}+\frac{1}{31} > 1$ and $s(4,D) > s(3,D) > 1$. Also if $u \geq 8$, then $s(u,D') > s(u,D) > 1$. Thus D' is also a sb-dominating set.

If $i_j \geq 2$, let $D' = (D - \{i_j, i_{j+1}\}) \cup \{i_j - 1, i_{j+1}+1\}$. Clearly, the vertices $i_j - 2, i_j, i_{j+1}, i_{j+1}+2$ are dominated by D'. Let u be any other vertex.
If $u = i_j + 1$ then $s(u,D') \geq \frac{1}{2}+\frac{1}{3}+\frac{1}{10}+\frac{1}{17}+\frac{1}{24}+\frac{1}{31} > 1$.
If $u = i_j + 2$ then $s(u,D') \geq s(i_j+1, D') > 1$.
Now let $u \leq i_j - 3$ and let $d(u, i_j) = d$. Then $d \geq 3$.

Now,

$$\begin{aligned} s(u,\{i_j-1, i_{j+1}+1\}) - s(u,\{i_j, i_{j+1}\}) &= \Big(\frac{1}{d-1}+\frac{1}{d+4}\Big) - \Big(\frac{1}{d}+\frac{1}{d+3}\Big) \\ &= \Big(\frac{1}{d-1}-\frac{1}{d}\Big) - \Big(\frac{1}{d+3}-\frac{1}{d+4}\Big) \\ &= \frac{1}{(d-1)d} - \frac{1}{(d+3)(d+4)}\Big) \\ &> 0. \end{aligned}$$

Hence $s(u, D') > s(u, D) \geq 1$. Similarly, if $u \geq i_{j+1} + 3$, then also $s(u, D') > s(u, D) \geq 1$. Thus D' is a sb-dominating set.

By repeating this process we obtain a sb-dominating set D with distance between any two consecutive vertices of D at least 4. Hence we may assume with out loss of generality that $d(i_j, i_{j+1}) \geq 4$ in D.

Suppose there exists a pair of vertices i_j, i_{j+1} such that $d(i_j, i_{j+1}) = 7$. As $i_1 \leq 2$, $i_6 \geq 31$ and $4 \leq d(i_{j'}, i_{j'+1}) \leq 7$ for all other $j' \in \{1, 2, 3, 4, 5\}$, we see that $s(i_j + 4, D) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{7} + \frac{1}{8} + \frac{1}{14} + \frac{1}{15} < 1$ which is a contradiction. Hence $d(i_j, i_{j+1}) < 7$ for all $j = 1, 2, 3, 4, 5$.

Now $n = 32$, $|D| = 6$, $i_1 \leq 2$, $i_6 \geq 31$, $d(i_j, i_{j+1}) \leq 6$ shows that $d(i_j, i_{j+1}) \neq 4$ for any $j = 1, 2, 3, 4, 5$.

Hence distance between consecutive vertices in D is either 5 or 6. Among all such possible subsets D, it is clear that either $d(i_1, i_2)$ or $d(i_5, i_6)$ is 6. If $d(i_1, i_2) = 6$, then vertex $i_1 + 3$ of P_{32} is not sb-dominated. If $d(i_5, i_6) = 6$, then vertex $i_5 + 3$ of P_{32} is not sb-dominated. Thus 6 vertices are not enough to sb-dominate P_{32}.

Therefore $\gamma_{sb}(P_{32}) = 7$.

For $33 \leq n \leq 37$, $\gamma_{sb}(P_n) \geq \gamma_{sb}(P_{32}) = 7$. Also it can be verified that $D = \{2, 7, 13, 19, 25, 31, (n-1)\}$ is a sb-dominating set with $|D| = 7$. Hence, $\gamma_{sb}(P_n) = 7$ if $32 \leq n \leq 37$. □

Theorem 4.3.6. $\gamma_{sb}(P_n) = 8$ *if* $38 \leq n \leq 45$.

Proof. Let $n = 38$. Suppose there exists a sb-dominating set $D = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$ of P_{38} where $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i_7 \leq 38$. We claim that D is not a sb-dominating set.

Let $S_1 = \{1, 2, 3..., 19\}$ and $S_2 = \{20, 18, ..., 38\}$. We may assume that $i_1, i_2, i_3 \in S_1$, $i_4, i_5, i_6, i_7 \in S_2$ and $d(i_j, i_{j+1}) \geq 3$. As in Theorem 4.3.5, we can prove here also that $d(i_j, i_{j+1}) \geq 4$ for all $j = 1, 2, 3, 4, 5, 6$.

Let $i_1 > 2$. Then $s(1, D) \leq (\frac{1}{2} + \frac{1}{6} + \frac{1}{10}) + (\frac{1}{19} + \frac{1}{23} + \frac{1}{27} + \frac{1}{31}) < 1$ which is a contradiction. Hence $i_2 \leq 2$.

Also, if $i_7 \leq 36$, then $s(38, D) \leq (\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14}) + (\frac{1}{19} + \frac{1}{23} + \frac{1}{27}) < 1$, which is again a contradiction. Hence $i_7 \geq 37$. If $d(i_j, i_{j+1}) = 8$ for some pair i_j, i_{j+1}, then $s(i_j + 4, D) < 1$. Hence $d(i_j, i_{j+1}) \leq 7$ for all $j = 1, 2, 3, 4, 5, 6$.

Let $d(i_j, i_{j+1}) = 7$ for some pair i_j, i_{j+1}. It can be verified then that if $j = 1, 2, 5$ or 6, then $s(i_j + 4, D) < 1$. Thus if $d(i_j, i_{j+1}) = 7$, then $j = 3$ or 4. If both

$d(i_3, i_4) = d(i_4, i_5) = 7$, then $s(i_3 + 4, D) < 1$. Hence there can be at most one pair i_j, i_{j+1} such that $d(i_j, i_{j+1}) = 7$ which is either $d(i_3, i_4)$ or $d(i_4, i_5)$. But in this case there must exist another pair $(i_{j'}, i_{j'+1})$ such that $d(i_{j'}, i_{j'+1}) = 6$. It can be verified now that $s(i_{j'} + 3, D) < 1$. Thus $d(i_j, i_{j+1}) \leq 6$ for all $j = 1, 2, 3, 4, 5, 6$.
If $d(i_j, i_{j+1}) = 6$ for all consecutive pairs of i_j and i_{j+1}, then vertices $i_1 + 3$ and $i_6 + 3$ are not sb-dominated.
Hence $d(i_j, i_{j+1}) \leq 5$ for at least one pair i_j, i_{j+1}. As $n = 38$, $|D| = 7$ and $d(i_j, i_{j+1}) \leq 6$ for all j, it follows that the only possible case left is $d(i_j, i_{j+1}) = 5$ for exactly one pair of consecutive vertices and is 6 for all other $d(i_j, i_{j+1})$.
Now if $d(i_j, i_{j+1}) = 5$ for $i_j \leq 19$, then vertex $i_7 - 3$ is not sb-dominated. If $d(i_j, i_{j+1}) = 5$ for $i_j > 19$, then vertex $i_1 + 3$ is not sb-dominated. Hence 7 vertices are not enough to sb-dominate P_{38}.
Therefore $\gamma_{sb}(P_{38}) \geq 8$.
On the other hand $D = \{2, 7, 12, 17, 22, 27, 32.37\}$ is a sb-dominating set of P_{38}.
Hence $\gamma_{sb}(P_{38}) = 8$.
For $39 \leq n \leq 45$, $\gamma_{sb}(P_n) \geq \gamma_{sb}(P_{38}) = 8$.
Also $D = \{2, 7, 13, 19, 25, 31, 37.(n-1)\}$ is a sb-dominating set of P_n for $39 \leq n \leq 43$, $D = \{2, 7, 13, 19, 26, 32, 38, 43\}$ is a sb-dominating set of P_{44} and $D = \{2, 7, 13, 20, 26, 33.39, 44\}$ is a sb-dominating set of P_{45}. Hence $\gamma_{sb}(P_n) = 8$ for $38 \leq n \leq 45$. □

In a similar way we can prove that

$$\begin{aligned} \gamma_{sb}(P_n) &= 9 \quad \text{for } 46 \leq n \leq 55 \\ &= 10 \quad \text{for } 56 \leq n \leq 62 \\ &= 11 \quad \text{for } 63 \leq n \leq 69 \\ &= 12 \quad \text{for } 70 \leq n \leq 78 \end{aligned}$$

We get the following theorems from the above discussions of $\gamma_{sb}(P_n)$ for $n \geq 30$.

Theorem 4.3.7.

$$\begin{aligned} \gamma_{sb}(P_n) &= \left\lceil \frac{n}{5} \right\rceil \quad \textit{if } 30 \leq n \leq 40 \textit{ and } n \neq 31, 36, 37 \\ &= \left\lceil \frac{n}{5} \right\rceil - 1 \quad \textit{if } n = 31, 36, 37 \textit{ or } 41 \leq n \leq 50 \end{aligned}$$

n	$\gamma_{sb}(P_n)$
$1 \le n \le 3$	1
$4 \le n \le 7$	2
$8 \le n \le 11$	3
$12 \le n \le 18$	4
$19 \le n \le 23$	5
$24 \le n \le 31$	6
$32 \le n \le 37$	7
$38 \le n \le 45$	8
$46 \le n \le 55$	9
$56 \le n \le 62$	10
$63 \le n \le 69$	11
$70 \le n \le 78$	12

Table 4.1: Table showing $\gamma_{sb}(P_n)$ for $1 \le n \le 78$

Theorem 4.3.8.

$$\begin{aligned}\gamma_{sb}(P_n) &= \left\lceil \frac{n}{6} \right\rceil \quad \textit{if } 51 \le n \le 72 \textit{ and } n \ne 55, 61, 62, 67, 68, 69 \\ &= \left\lceil \frac{n}{6} \right\rceil - 1 \quad \textit{if } n = 55, 61, 62, 67, 68, 69\end{aligned}$$

Theorem 4.3.9. $\gamma_{sb}(P_n) < \lceil \frac{n}{6} \rceil$ *if* $n \ge 73$.

Table 4.2 given below shows that the difference $\gamma(P_n) - \gamma_{sb}(P_n)$ becomes large as n increases.

Remark 4.3.1. Because of the divergence of the series $\sum \frac{1}{n}$, it can be observed that $\gamma_{sb}(P_n)$ can be much smaller than $\lceil \frac{n}{6} \rceil$ for large values of n.

4.4 On sb-domination number of cycles

In this section we determine $\gamma_{sb}(C_n)$ for $3 \le n \le 48$. Throughout this section we denote the cycle C_n by $C_n = (1, 2, 3, ..., n, 1)$.

Theorem 4.4.1.

$$\begin{aligned}\gamma_{sb}(C_n) &= 2 \quad \textit{if } n = 4 \\ &\le \left\lceil \frac{n}{4} \right\rceil \quad \textit{otherwise}\end{aligned}$$

Proof. From Theorem 1.3.23 we get, for every integer $n \ge 3$,

$$\gamma_2^d(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n \ne 4 \end{cases}$$

n	$\gamma(P_n)$	$\gamma_{sb}(P_n)$	$\gamma(P_n) - \gamma_{sb}(P_n)$
$1 \leq n \leq 3$	1	1	0
$4 \leq n \leq 6$	2	2	0
7	3	2	1
10	4	3	1
15	5	4	1
16	6	4	2
23	8	5	3
30	10	6	4
31	11	6	5
37	13	7	6
45	15	8	7
55	19	9	10
61	21	10	11
76	26	12	14

Table 4.2: As n increases, the difference between $\gamma(P_n)$ and $\gamma_{sb}(P_n)$ becomes more significant.

Now theorem follows from the observation that

$$\begin{aligned} \gamma_{sb}(G) &= \gamma_2^d(G) \quad \text{if } \gamma_2^d(G) \leq 3 \\ &\leq \gamma_2^d(G) \quad \text{otherwise} \end{aligned}$$

□

Theorem 4.4.2. $\gamma_{sb}(C_n) = \lceil \frac{n}{4} \rceil$ *for* $5 \leq n \leq 12$.

Proof. We know that $\gamma_{sb}(G) = \gamma_2^d(G)$ if $\gamma_2^d(G) \leq 3$
and $\gamma_2^d(C_n) = \lceil \frac{n}{4} \rceil$ if $5 \leq n \leq 12$.
Hence $\gamma_{sb}(C_n) = \lceil \frac{n}{4} \rceil$ for $5 \leq n \leq 12$. □

Theorem 4.4.3. $\gamma_{sb}(C_n) = 3$ *for* $9 \leq n \leq 13$.

Proof. From Theorem 4.4.2, $\gamma_{sb}(C_n) = 3$ for $9 \leq n \leq 12$.

Now for $n = 13$,

$$\gamma_{sb}(C_{13}) \geq 3 \quad \text{as } \gamma_{sb}(C_{12}) = 3$$

Also $D = \{1, 6, 10\}$ is a sb-dominating set of C_{13}. Hence, $\gamma_{sb}(C_{13}) = 3$.

□

Theorem 4.4.4. $\gamma_{sb}(C_n) = 4$ *if* $14 \leq n \leq 20$.

Proof. $D = \{1, 5, 9, 13\}$ is a sb-dominating set of C_n for $n = 14, 15, 16$ and $D = \{1, 6, 11, 16\}$ is a sb-dominating set of C_n for $n = 17, 18, 19, 20$.
Therefore

$$\gamma_{sb}(C_n) \leq 4 \quad \text{if } 14 \leq n \leq 20. \tag{4.4.1}$$

Now let $n = 14$. Let $D = \{i_1, i_2, i_3\}$ where $1 \leq i_1 < i_2 < i_3 \leq 14$ be a sb-dominating set of C_{14}.
If $d(i_j, i_{j+1}) = 6$, then $s(i_j + 3, D) < 1$.
Hence $d(i_j, i_{j+1})$ is 4 or 5. We assume without loss of generality that
$d(i_1, i_2) = d(i_2, i_3) = 5$ and $d(i_3, i_1) = 4$.
Then, $s(i_1 + 3, D) < 1$ which is a contradiction. Therefore, $\gamma_{sb}(C_{14}) \geq 4$
Thus, $\gamma_{sb}(C_{14}) = 4$.
Now $\gamma_{sb}(C_n) \geq \gamma_{sb}(C_{14}) = 4$ if $n \geq 14$.
Hence we get from (4.4.1) that $\gamma_{sb}(C_n) = 4$ if $14 \leq n \leq 20$. □

Theorem 4.4.5. $\gamma_{sb}(C_n) = 5$ *if* $21 \leq n \leq 27$.

Proof. $D = \{1, 5, 9, 13, 17\}$ is a sb-dominating set of C_{21}, $D = \{1, 6, 11, 16, 21\}$ is a sb-dominating set of C_n for $n = 22, 23, 24, 25$ and $D = \{1, 7, 12, 18, 23\}$ is a sb-dominating set of C_n for $n = 26, 27$.
Therefore

$$\gamma_{sb}(C_n) \leq 5 \quad \text{if } 21 \leq n \leq 27. \tag{4.4.2}$$

Let $n = 21$ and let 4 vertices are enough to sb-dominate C_{21}
Let $D = \{i_1, i_2, i_3, i_4\}$ where $1 = i_1 < i_2 < i_3 < i_4 \leq 21$ is a sb-dominating set of C_{21}.
First we show that distance between two consecutive vertices in D cannot be ≥ 6.

Claim

Let $d(i_1, i_2) = 6$.
As we have fixed $i_1 = 1$, we get $i_2 = 7$.
Now sb-domination of vertex 4 is possible only if $i_3 \leq 10$ and $i_4 \geq 19$.
But then vertex 14 is not sb-dominated.
Thus if D is a sb-dominating set, distance between two consecutive vertices in $D \leq 5$.
This is not possible as $|D| = 4$ and $n = 21$.

Hence,

$$\gamma_{sb}(C_{21}) \geq 5 \tag{4.4.3}$$

From (4.4.2) and (4.4.3) we get, $\gamma_{sb}(C_{21}) = 5$ if $21 \leq n \leq 27$. □

Theorem 4.4.6. $\gamma_{sb}(C_n) = 6$ *if* $28 \leq n \leq 36$.

Proof. $D = \{1, 6, 11, 16, 21, 26\}$ is a sb-dominating set of C_n if $28 \leq n \leq 30$ and $D = \{1, 7, 13, 19, 25, 31\}$ is a sb-dominating set of C_n for $31 \leq n \leq 36$ Therefore

$$\gamma_{sb}(C_n) \leq 6 \quad \text{if } 28 \leq n \leq 36. \tag{4.4.4}$$

Let $n = 28$ and $D = \{i_1, i_2, i_3, i_4, i_5\}$ where $1 = i_1 < i_2 < i_3 < i_4 < i_5 \leq 28$ is a sb-dominating set of C_{28}.
Distance between any two consecutive vertices in D cannot be ≥ 8.

Claim

Let $d(i_1, i_2) = 8$
Then $s(i_1 + 4, D) \leq (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{7} + \frac{1}{7}) + \frac{1}{10} < 1$ shows that distance between any two consecutive vertices in D cannot be ≥ 8.

We can also see that minimum distance between any pair of consecutive vertices in D is ≥ 4.

Claim

Let $d(i_1, i_2) = 3$. As $n = 28$ and $|D| = 5$ at most one such pair can only occur.
Now replace i_1 by $i_1' = i_1 - 1$ and i_2 by $i_2' = i_2 + 1$ in D, so that $d(i_1', i_2') = 5$
Now $s(i_1' + 2, D) \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{9} > 1$
Similarly, $s(i_1' + 3, D) > 1$
Therefore $i_1' + 2, i_1' + 3$ are sb-dominated. It is obvious that strengths received by vertices outside (i_1', i_2') increases by this replacement. Hence all vertices are sb-dominated by the set D obtained after this replacement.
Thus, we can assume without loss of generality that minimum distance between two consecutive vertices in $D \geq 4$.
Now if $d(i_1, i_2) = 7$, then $s(i_1 + 4, D) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{7} + \frac{1}{8} + \frac{1}{11} < 1$

Hence, $d(i_1, i_2) \leq 6$.
This shows that distance between any two pairs of consecutive vertices in D is at most 6.
If $d(i_1, i_2) = 4$, then distance between all other pairs of consecutive vertices must be 6, as $n = 28$.
But then $S(i_3 + 3, D) = (\frac{1}{3} + \frac{1}{3}) + (\frac{1}{9} + \frac{1}{9}) + \frac{1}{13}) < 1$
Therefore the distance between consecutive pairs of vertices in D is either 5 or 6. As $n = 28$, in this case, we see that distance between three pairs of consecutive vertices is 6 and two pairs of consecutive vertices is 5. Now the inequality $\frac{1}{3}+\frac{1}{3}+\frac{1}{9}+\frac{1}{8}+\frac{1}{14} < 1$ shows that here also, all vertices are not sb-dominated. Thus five vertices are not enough to sb-dominate C_{28}. Hence $\gamma_{sb}(C_{28}) = 6$.
Now from (4.4.4), we conclude that, $\gamma_{sb}(C_n) = 6$ for $28 \leq n \leq 36$ □

Theorem 4.4.7. $\gamma_{sb}(C_n) = 7$ *for* $37 \leq n \leq 42$.

Proof. Let $D = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ where $1 = i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < 37$ is a sb-dominating set of C_{37}.
Since minimum distance between two consecutive vertices in D is 3 and since $\frac{1}{4} + \frac{1}{4} + \frac{2}{7} + \frac{2}{10} < 1$, we can assume that distance between two consecutive vertices in D is ≤ 7.
Let $d(i_1, i_2) = 3$. Atmost one such pair can occur as $n = 37, |D| = 6$ and $d(i_j, i_{j+1}) \leq 7$. We can replace the vertices in D to get another sb-dominating set $D' = \{i_1 - 1, i_2 + 1, i_3, i_4, i_5, i_6\}$ where $d(i_1 - 1, i_2 + 1) = 5$, $d(i_1 - 1, i_6) \leq 6$ and $d(i_2 + 1, i_3) \leq 6$.
Claim

$$s(i_1 + 1, D') \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{9} > 1.069$$
$$> 1$$

Similarly, $s((i_1 + 2, D') > 1$
Other vertices are also sb-dominated by D'. Hence we can assume that distance between two consecutive vertices in D is at least four.
Now we prove that distance between two consecutive vertices in D is greater than four. Let $d(i_1, i_2) = 4$. If we replacet i_1 by $i_1 - 1$ and i_2 by $i_2 + 1$, the resulting set D' is also a sb- dominating set.

Claim

$$s(i_1+1,D') \geq \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{10}+\frac{1}{15}+\frac{1}{17} > 1$$
$$s(i_1+2,D') \geq \frac{1}{3}+\frac{1}{3}+\frac{1}{9}+\frac{1}{9}+\frac{1}{16}+\frac{1}{16} > 1$$

and similarly $s(i_1+3,D') > 1$.

Also i_1, i_2 are dominated by D'.

Strength received by vertices outside (i_1, i_2) with respect to D' is more than that received from D. Hence D' is also a sb-dominating set.

From these we can assume that minimum distance between two consecutive vertices in D is five.

As $n = 37$ and $|D| = 6$, distance between at least one pair of consecutive vertices in D is seven. Let $d(i_1, i_2) = 7$. But then, $s(i_1+3, D) \leq \frac{1}{3}+\frac{1}{4}+\frac{1}{8}+\frac{1}{9}+\frac{1}{13}+\frac{1}{14} < 1$. Hence six vertices are not enough to sb-dominate C_{37}.

Now $D = \{1, 7, 13, 19, 25, 31, 37\}$ is a sb-dominating set of C_{37} shows that $\gamma_{sb}(C_{37}) = 7$.

For $n \geq 37$, $\gamma_{sb}(C_n) \geq 7$. Also, $D = \{1, 7, 13, 19.25, 31, 37\}$ ia a sb-dominating set of C_n for $37 \leq n \leq 42$.

Therefore $\gamma_{sb}(C_n) = 7$ if $37 \leq n \leq 42$. □

Theorem 4.4.8. $\gamma_{sb}(C_n) = 8$ *for* $43 \leq n \leq 52$.

Proof. Let $n = 43$.

$D = \{1, 7, 13, 19, 25, 31, 37, 43\}$ is a sb-dominating set of order 8 for C_{43}. Hence $\gamma_{sb}(C_{43}) \leq 8$. Assume that 7 vertices are enough to sb-dominate C_{43} and let $D = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$ where $1 = i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i_7 \leq 43$ is a sb-dominating set of C_{43}.

As $n = 43$ and $|D| = 7$, as in the previous theorem we can assume that distance between two consecutive vertices in D is at least five. Let i_j, i_{j+1} are two consecutive vertices in D with $d(i_j, i_{j+1}) = 8$.

Now, $s(i_j+4, D) \leq \frac{2}{4}+\frac{2}{9}+\frac{2}{14}+\frac{1}{9} < 1$, shows that distance between two consecutive vertices in D cannot be eight or more. Hence it must be is $5, 6$ or 7. Now following are the different possible cases.

Case (i)

Let distance between one pair of consecutive vertices in D is 7 and distance between all other pairs are 6.

Let $d(i_1, i_2) = 7$ and $d(i_j, i_{j+1}) = 6$ for $j \neq 1$.

Now $s(i_1 + 4, D) = \frac{1}{4} + \frac{1}{3} + \frac{1}{10} + \frac{1}{9} + \frac{1}{16} + \frac{1}{15} + \frac{1}{21} < 1$ proves that this case is not possible.

Case (ii)

Distance between two pairs of consecutive vertices in D are 7, one pair is 5 and rest are 6.

Case (iii)

Distance between three pairs of consecutive vertices in D are 7, two pairs are 5 and the remaining are 6.

Case (iv)

Distance between four pairs of consecutive vertices in D are 7 and three pairs are 5. Simple calculations as in case (i) show that these cases are also not possible, if D is a sb- dominating set of C_{43}. Hence seven vertices are not enough to sb-dominate C_{43}. Now $D = \{1, 7, 13, 19, 25, 31, 37, 43\}$ is a sb-dominating set of C_{43} proves that $\gamma_{sb}(C_{43}) = 8$.

For $n > 43$, $\gamma_{sb}(C_n) \geq 8$.

Also $D = \{1, 7, 13, 19, 25, 31, 37, 43\}$ is a sb-dominating set of C_n for $43 \leq n \leq 48$ and $D = \{1, 7, 14, 20, 27, 33, 40, 46\}$ is a sb-dominating set of C_n for $46 \leq n \leq 52$.

Hence $\gamma_{sb}(C_n) = 8$ if $43 \leq n \leq 52$. □

Proceeding similarly, we can find $\gamma_{sb}(C_n)$ for $n \geq 53$. The above theorems are summarized in the following theorem.

Theorem 4.4.9.

$$\begin{aligned}
\gamma_{sb}(C_n) &= 1 \quad \text{for } n = 3 \\
&= 2 \quad \text{for } n = 4 \\
&= \left\lceil \frac{n}{4} \right\rceil \quad \text{for } 5 \leq n \leq 16 \text{ and } n \neq 13 \\
&= \left\lceil \frac{n}{4} \right\rceil - 1 \quad \text{for } n = 13
\end{aligned}$$

$$\begin{aligned}
\gamma_{sb}(C_n) &= \left\lceil \frac{n}{5} \right\rceil \quad \text{for } 17 \leq n \leq 30 \text{ and } n \neq 26, 27 \\
&= \left\lceil \frac{n}{5} \right\rceil - 1 \quad \text{if } n = 26, 27 \\
&= \left\lceil \frac{n}{6} \right\rceil \quad \text{for } 31 \leq n \leq 48
\end{aligned}$$

For $n \geq 49$, $\gamma_{sb}(C_n) \leq \lceil \frac{n}{6} \rceil$.

Because of the divergence of the series $\sum \frac{1}{n}$, it can be observed here also that, $\gamma_{sb}(C_n)$ can be much smaller than $\lceil \frac{n}{6} \rceil$ for large value of n. This can be seen in Corollary 4.7.7.

4.5 Some bounds for sb-domination number

The following theorem gives a lower bound for $\gamma_{sb}(G)$ in terms of the number of vertices n and Δ_{sb}.

Theorem 4.5.1. *Let $G = (V, E)$ be a connected graph of order n. Then,*

$$\gamma_{sb}(G) \geq \lceil \frac{n}{1 + \Delta_{sb}} \rceil$$

.

Proof. Let D be any γ_{sb}-set of G. Each $v \in D$ has $ds(v) \leq \Delta_{sb}$. Then

$$\Delta_{sb}|D| + |D| \geq \sum_{v \in D} ds(v) + |D| \geq n$$

. Hence $\gamma_{sb}(G) = |D| \geq \lceil \frac{n}{1+\Delta_{sb}} \rceil$.

□

Note: The bound given in the above theorem is sharp. For example, the graph given in Figure 4.2 achieves this bound. Here $\Delta_{sb} = 5.75, n = 11$ *and* $\lceil \frac{n}{1+\Delta_{sb}} \rceil = \gamma_{sb}(G) = 2$.

Another stronger and sharp lower bound for $\gamma_{sb}(G)$ is given in the following theorem.

Theorem 4.5.2. *Let G be a graph with ds-sequence $(ds(v_1), ds(v_2), ..., ds(v_n))$. Let $t = min\{k : k + ds(v_1) + ds(v_2) + ... + ds(v_k) \geq n\}$. Then $\gamma_{sb}(G) \geq t$.*

Proof. Let $S \subset V$. If $|S| < t$, then $|S| + \sum_{v \in S} ds(v) < n$. Hence S is not an sb-dominating set of G. Thus $\gamma_{sb}(G) \geq t$. □

This is also a tight bound. For the graph in Figure 4.2, $t = 2 = \gamma_{sb}(G)$. The following theorem gives a sharp upper bound for $\gamma_{sb}(G)$ in terms of the number of vertices n and Δ_{sb}.

Theorem 4.5.3. *Let $G = (V, E)$ be a connected graph of order n. Then,*

$$\gamma_{sb}(G) \leq n - \lfloor \Delta_{sb} \rfloor$$

.

Proof. Let $\lfloor \Delta_{sb} \rfloor = n - i$. Then,

$$n - i \leq \Delta_{sb} < n - i + 1.$$

Let $v \in V(G)$ has $ds(v) = \Delta_{sb}$.

$$So, \quad n - i \leq ds(v) < n - i + 1.$$

From this it follows that

$$n - i \leq deg(v) + \frac{n - 1 - deg(v)}{2}$$

$$Hence, \quad deg(v) \geq n - 2i + 1.$$

$$Therefore, \quad n - 2i + 1 \leq deg(v) \leq n - i$$

$$Thus, \quad deg(v) = n - 2i + j \quad where \quad 1 \leq j \leq i$$

Consider the graph $G_1 = G \setminus N[v]$.
Then $|G_1| = 2i - j - 1$.

Suppose there are k isolated vertices in G_1. If $k = 0$, then $\lfloor |G_1|/2 \rfloor$ vertices are only required to dominate it. Since $|G_1| \leq 2i - 2$, we can see that at most $i - 1$ vertices are enough to dominate G_1. These together with v will give a sb-dominating set of order i.

If $k > 0$, let A be the set of isolated vertices in G_1. Then, $G \setminus N[v] \setminus A$ has a dominating set S such that

$$|S| \leq \lfloor \frac{2i - 2 - k}{2} \rfloor = i - 1 - \lceil \frac{k}{2} \rceil.$$

If $k \neq 2$, let

$$S_1 = S \cup \{v\} \cup T$$

where $T \subset A$ and $|T| = 1$ *or* 2 where

$$|T| = \begin{cases} 1 & \text{if } k = 1 \\ 2 & \text{otherwise} \end{cases}$$

If $k = 2$, let

$$S_1 = S \cup \{v_1, v_2\}$$

where $\{v_1, v_2\} \subset N(v)$ and it dominates A.
In all the cases we get $|S_1| \leq i = n - \lfloor \Delta_{sb} \rfloor$

□

Theorem 4.5.4. Let $G = (V, E)$ be a connected graph. Let $v \in V(G)$ has $ds(v) = \Delta_{sb}$ and let $G_1 = G \setminus N[v]$. Then $\gamma_{sb}(G) = \lceil n - \Delta_{sb} \rceil$ if and only if one of the following conditions hold.

(a) $|G_1 \cap N_2(v)| = 1$ or 2.

(b) $|G_1| = 3$ and $G_1 \cong 3K_1$ or $K_1 \cup K_2$ where the vertices of G are such that corresponding to any pair of vertices in G there exist a vertex which is not adjacent to these two and is at a distance at least 3 from one of them.

(c) $|G_1| = 4$ and $G_1 \cong 4K_1$ or $2K_1 \cup K_2$ or $2K_2$ where the vertices of G are such that there is no vertex from $N_3(v)$ and corresponding to any pair of vertices in G there exist a vertex which is not adjacent to these two and is at a distance at least 3 from one of them.

Proof. According to the construction of the sb-dominating set in the above theorem equality holds if and only if the components of $G_1 =< G - N[v] >$ are isolated vertices or C_4 or corona of a graph.

Case (i)

Suppose G_1 has a C_4 component. Let $u \in C_4 \cap N_2(v)$. Let $u' \in C_4$ be such that $d(u, u') = 2$. If $u' \in S$ then u' dominates two of its adjacent vertices in C_4. The vertex $u \in C_4$ u gets a strength $\frac{1}{2}$ from u' and $\frac{1}{2}$ from v. Thus vertices in C_4 are sb-dominated by one single vertex in C_4 together with v instead of two vertices taken in the constuction of S in the theorem. Thus strict inequality occurs in this case.

Case (ii)

Suppose G_1 contains a corona on more than 3 vertices. Then it contains a corona on P_3 as its sub-graph. Let u_1, u_2, u_3 be the vertices of P_3. Then u_1 and u_3 are enough to sb-dominate the corona of this P_3. But according to the construction 3 vertices are taken in S. Thus equality does not occur in this case.

Case (iii)

Suppose G_1 has corona on two vertices as its sub-graph, that is, G_1 contains a P_4 as its sub-graph.
Here there are two cases.
Subcase (i) at least one pendent vertex of P_4 is in $N_2(v)$.
Let this pendent vertex be u. Choose $u' \in P_4$ which is such that $d(u, u') = 2$. u' together with v sb-dominates P_4. Hence in this case too equality does not occur.
Subcase (ii) both pendent vertices of P_4 are in in $N_3(v)$.
Let v_1 and v_2 be the vertices in $N_3(v)$ and let their neighbors in $N_2(v)$ are u_1 and u_2 respectively. According to the construction of sb-dominating set in the theorem two vertices from this component are included in S. But contribution to $n - \lfloor \Delta_{sb} \rfloor$ from these 4 vertices is $4 - 1 = 3 > 2$. Hence equality does not occur.

Thus $\gamma_{sb}(G) = n - \lfloor \Delta_{sb} \rfloor$ if and only if G_1 contains only isolated vertices and $K_2's$. We can show that at most 4 vertices can be there in G_1. It is clear that 2 vertices from $N_2(v)$ together with v are enough to sb-dominate all vertices in $N_2(v)$. Hence equality does not occur if there are more than two vertices in $N_2(v)$.

As in the proof given in subcase (ii) of case (iii) we can prove that if there are are more than one vertex from $N_3(v)$ equality does not occur.

Suppose there are 4 vertices in $N_2(v)$ and a vertex in $N_3(v)$. Two vertices from this part together with v are enough to sb-dominate G. So $\gamma_{sb}(G) \leq 3$ but $n - \lfloor \Delta_{sb} \rfloor = 4$. Hence strict inequality occurs in the theorem.

Thus there are at most 4 vertices in G_1 if equality occurs.

- If there is only one vertex in G_1 it must be from $N_2(v)$. Here $\lceil n - \Delta_{sb} \rceil = 2$ and $\gamma_{sb} = 2$. Thus equality occurs.

- If there are two vertices in G_1 either one is in $N_2(v)$ and other in $N_3(v)$ or both are in $N_2(v)$. If one vertex is in $N_3(v)$, then $\gamma_{sb}(G) = 2 \leq \lceil n - \Delta_{sb} \rceil = 3$. Thus equality does not occur in this case. If both vertices are in $N_2(v)$, then $\gamma_{sb} = n - \Delta_{sb} = 2$.

- If G_1 contains 3 vertices either all the vertices are in $N_3(v)$ or two of them are in $N_2(v)$ and one is in $N_3(v)$. In both the cases $n - \lfloor \Delta_{sb} \rfloor = 3$. Then equality occurs if and only if $\gamma_{sb} = 3$. Hence corresponding to any two vertices in G there must exist a vertex v_1 which is not adjacent to these vertices and is such that it is at a distance 3 from one of these.

- If there are four vertices in G_1 either 3 of them are in $N_2(v)$ and one is in $N_3(v)$ or all the four vertices are in $N_2(v)$. In the former case $\gamma_{sb} = 2 \leq \lceil n - \Delta_{sb} \rceil = 3$. Hence equality does not occur. In the latter case equality occurs if $\gamma_{sb} = 3$. Hence corresponding to any two vertices in G there must exist a vertex v_1 which is not sb-dominated by these vertices.

□

Theorem 4.5.5. If G has a universal vertex, then $\gamma_{sb}(G) = 1 = n - \Delta_{sb}$.

Proof. Let $u \in G$ be a universal vertex. Then $\{u\}$ is a γ_{sb}- set of G and $\Delta_{sb} = n - 1$. Therefore $\gamma_{sb}(G) = 1 = n - \Delta_{sb}$. □

4.6 On sb-domination number and diameter of a graph

Theorem 4.6.1. Let G be a connected graph, then $\gamma_{sb}(G) \leq diam(G)$, where $diam(G)$ is the diameter of G and this bound is sharp.

Proof. Let $diam(G) = d$. Let $D = \{v_1, v_2, ..., v_d\}$ be any subset of V with cardinality d. Since $d(u, v) \leq d$, for all $u \in V \setminus D$, it follows that $s(u, D) \geq 1$. Hence D is a sb-dominating set of G and thus $\gamma_{sb}(G) \leq diam(G)$. For any complete bipartite graph $K_{m,n}$, where $m, n \geq 2$, we have $\gamma_{sb}(G) = diam(G)$ □

Corollary 4.6.2. *If G is a self centered graph, then $\gamma_{sb}(G) \leq r$, where r is the radius of G.*

Remark 4.6.1. 1. If $diam(G) = 1$, then $\gamma_{sb}(G) = 1$.

2. If $diam(G) = 2$, then

$$\begin{aligned}\gamma_{sb}(G) &= 1 \quad \text{if } G \text{ has a universal vertex}\\ &= 2 \quad \text{otherwise}\end{aligned}$$

3. If diam(G)=3, then $\gamma_{sb}(G) = 2$ *or* 3. It is obvious that $\gamma_{sb}(G) = 2$ if and only if $\gamma_2^d(G) = 2$ and $\gamma_{sb}(G) = 3$ if and only if $\gamma_2^d(G) = 3$.

Theorem 4.6.3. *Given any integer n, there exists a graph G for which $diam(G) = \gamma_{sb}(G) = n$.*

Proof. Let $G = K_n \square K_n \square ... \square K_n$, the cartesian product of n copies of K_n. Then $diam(G) = n$. Hence $\gamma_{sb}(G) \leq n$. We can show that no subset of $V(G)$ of order less than n can sb-dominate G. Let $D = \{u_1, u_2, ..., u_{n-1}\} \subset V(G)$ where $u_1 = (u_1^1, u_1^2, ..., u_1^n)$, $u_2 = (u_2^1, u_2^2, ..., u_2^n)$, ..., $u_{n-1} = (u_{n-1}^1, u_{n-1}^2, ..., u_{n-1}^n)$., Let $x = (x_1, x_2, ..., x_n) \in V(G)$ be such that $x_1 \neq u_1^1, u_2^1, ..., u_{n-1}^1$, $x_2 \neq u_1^2, u_2^2, ..., u_{n-1}^2$,..., $x_n \neq u_1^n, u_2^n, ..., u_{n-1}^n$. Then $d(x, u_1) = n$, $d(x, u_2) = n$,..., $d(x, u_{n-1} = n)$. So,

$$\begin{aligned}s(x, D) &= \frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}, (n-1) times\\ &= \frac{n-1}{n}\\ &< 1\end{aligned}$$

Hence D with $|D| \leq n-1$ cannot be a sb-dominating set of G. Thus $\gamma_{sb}(G) = n$. □

Observation 4.6.4. *Let r be the radius of G. Let $C_0, C_1, C_2, ..., C_r$ represent vertices G with eccentricity $r, r+1, r+2, ...2r$ respectively. Then*

1. *Center of G is a sb-dominating set if it contains r vertices.*
2. *If $|C_0| \leq r$, but $|C_0 \cup C_1| \geq r+1$, then $\gamma_{sb}(G) \leq r+1$.*
3. *If $|C_0 \cup C_1 \cup ... \cup C_i| \geq r+i$, then $\gamma_{sb}(G) \leq r+i$.*

Remark 4.6.2. *Let G be a graph, $u \in V(G)$ with $e(u) = t$ and let $G_1 = G - N_t(u)$ where $N_t(u) = \{v \in V : d(u, v) = t\}$. If D_1 is a sb-dominating set of G_1, then $s(v_t, D_1) \geq \frac{1}{2}$ for all $v_t \in N_t(u)$.*

Proof. Let $v_{t-1} \in N_{t-1}(u)$ and let $v_t v_{t-1} \in E(G)$. Then

$$s(v_{t-1}, D_1) = \sum_{v \in D_1} \frac{1}{d(v, v_{t-1})} \geq 1$$

Now,

$$s(v_t, D_1) \geq \sum_{v \in D_1} \frac{1}{d(v, v_{t-1}) + 1}$$

$$\geq \sum_{v \in D_1} \frac{1}{2d(v, v_{t-1})} \geq \frac{1}{2}$$

□

4.7 Bounds for sb-domination number of paths and cycles for large value of n

In this section, we provide a method to find upper bounds for sb-domination number of paths and cycles for large values n, by making use of the divergence property of the series $\sum \frac{1}{n}$.

Definition 4.7.1. For each $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be defined as the integer satisfying the conditions, $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2n_k - 1} \geq k$, but $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2n_k - 3} < k$.

Example 4.7.1. (i) When $k = 2$, $n_k = 8$

Proof. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{13} < 2$
and $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{13} + \frac{1}{15} > 2$
Therefore, $2n_k - 1 = 15, n_k = 8$ □

(ii) When $k = 3$, $n_k = 57$

Proof. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{111} < 3$
and $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{113} > 3$
Therefore, $2n_k - 1 = 113, n_k = 57$ □

Values of n_k for some $k \in \mathbb{N}$ are shown the Table 4.3.

Similarly we can also define n_k for $k \in \mathbb{R}$.

If $k \in \mathbb{R}$, let $n_k \in \mathbb{N}$ be such that $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2n_k - 1} \geq k$, but $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2n_k - 3} < k$. Examples of n_k for some $k \in \mathbb{R}$ are given in Table 4.4.

k	1	2	3	4	5
n_k	1	8	57	419	3092

Table 4.3: Table showing n_k for $k \in \mathbb{N}$

k	1.25	1.5	1.75	2.25	2.5	2.75	3.25	3.5
n_k	2	3	5	13	21	35	94	254

Table 4.4: Table showing n_k for some values of $k \in \mathbb{R}$

These definitions of n_k for $k \in \mathbb{N}$ and $k \in \mathbb{R}$ are useful for defining upper bounds for sb-domination number of paths and cycles.

4.7.1 Bounds in terms of n_k for sb-domination number of paths

Theorem 4.7.2. *For $2k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be as defined above and let $m = 2kn_k + 1$. Then for any path P_r with $r \leq m$, $\gamma_{sb}(P_r) \leq n_k$.*

Proof. Let $m = 2kn_k + 1$ where $2k \in \mathbb{N}$ and n_k as defined above. Consider the path P_m where $m = 2kn_k + 1$ having vertices given by $\{1, 2, 3, ..., 2kn_k + 1\}$.
Let $D \subset V(P_m)$ be given by $D = \{k+1, 3k+1, 5k+1, ..., (2n_k - 1)k+1\}$ having n_k vertices. We can show that D is a sb-dominating set of P_m. The domination strength received by the vertex 1 from the set D is given by $s(1, D) = \frac{1}{k}+\frac{1}{3k}+\frac{1}{5k}...+\frac{1}{(2n_k-1)k} \geq 1$ from definition of n_k.
Hence vertex 1 is sb-dominated by D. Now let j be an arbitrary vertex on P_m. With out of generality we may assume that $j \leq \frac{m}{2}$.
Let $i \in \mathbb{N}$ be such that $(2i - 1)k + 1, (2i + 1)k + 1 \in D$ and j lies between these two vertices in D. Since the vertices in D are equally spaced, we see that, $s(j, D) = \Big(\frac{1}{d((2i-1)k+1, j)} + \frac{1}{d(j, (2i+1)k+1)}\Big) + \Big(\frac{1}{d((2i-3)k+1, j)} + \frac{1}{d(j, (2i+3)k+1)}\Big) + ... + \Big(\frac{1}{d(k+1, j)} + \frac{1}{d(j, (4i-1)k+1)}\Big) + \frac{1}{d(j, (4i+1)k+1)} + ... + \frac{1}{d(j, (2n_k-1)k+1)} \geq \frac{1}{k} + \frac{1}{3k} + \frac{1}{5k} ... + \frac{1}{(2n_k-1)k} \geq 1.$

That is the strength received by any vertex j from D is greater than that received by the vertex 1 from D. Hence D is a sb-dominating set of P_m. Therefore $\gamma_{sb}(P_m) \leq |D| = n_k$.
Now for $r \leq m$, $\gamma_{sb}(P_r) \leq \gamma_{sb}(P_m)$.

Hence we get, $\gamma_{sb}(P_r) \leq n_k$ for all $r \leq m$ where $m = 2kn_k + 1$. □

Corollary 4.7.3. *If $m = 2kn_k + 1$, then*

$$\begin{aligned}\gamma_{sb}(p_n) &\leq \left\lceil \frac{n-1}{2k} \right\rceil \quad \text{if } n \geq m \\ &\leq \left\lceil \frac{n-1}{2(k-1)} \right\rceil \quad \text{if } n < m\end{aligned}$$

Proof. Let $n \geq m$. Then from the proof of the above theorem we see that $D = \{k+1, 3k+1, ..., (2(\frac{n-1}{2k}) - 1)k + 1\}$ or $D = \{k+1, 3k+1, ..., (2\lfloor \frac{n-1}{2k} \rfloor - 1)k + 1, n\}$ is a sb-dominating set of P_n of order $\lceil \frac{n-1}{2k} \rceil \geq n_k$ according as $n - 1$ is divisible exactly by $2k$ or not.
Hence, $\gamma_{sb}(p_n) \leq \left\lceil \frac{n-1}{2k} \right\rceil$ if $n \geq m$.

Similarly if $n \geq 2(k-1)n_{k-1} + 1$, then $\gamma_{sb}(p_n) \leq \left\lceil \frac{n-1}{2(k-1)} \right\rceil$.
Thus we get

$$\begin{aligned}\gamma_{sb}(p_n) &\leq \left\lceil \frac{n-1}{2k} \right\rceil \quad \text{if } n \geq m \\ &\leq \left\lceil \frac{n-1}{2(k-1)} \right\rceil \quad \text{if } n < m\end{aligned}$$

□

Note 4.7.1. The upper bounds given in Theorem 4.7.2 and Corollary 4.7.3 are much larger than the actual values of $\gamma_{sb}(P_n)$ for $n \geq 2$. But the following theorems give better upper bounds of $\gamma_{sb}(P_n)$ in terms of n_k. For each value of k, we can find several values of n for which $\gamma_{sb}(P_n)$ attains these bounds.

The following theorem provides a good upper bound for $\gamma_{sb}(P_m)$, $m \geq 2kn_{k-1}+1$.

Theorem 4.7.4. If $2kn_{k-1} + 1 \leq m < 2kn_k + 1$, then $\gamma_{sb}(P_m) \leq \lceil \frac{m-1}{2k} + 1 \rceil$.

Proof. By definition of n_{k-1} we have $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2n_{k-1}-1} \geq k - 1$. Therefore

$$\frac{1}{k} + \frac{1}{3k} + \frac{1}{5k} + \frac{1}{7k} + ... + \frac{1}{(2n_{k-1} - 1)k} \geq \frac{k-1}{k} = 1 - \frac{1}{k}$$

Consider the path $P_m = \{1, 2, 3, ..., m\}$ where m is such that

$$2kn_{k-1} + 1 \leq m < 2kn_k + 1$$

Let $D \subset V(P_m)$ be given by $D = D_1 \cup D_2$ where $D_1 = \{k + 1, 3k + 1, 5k +$

$1, ..., (2\lfloor\frac{m-1}{2k}\rfloor - 1)k + 1\}$ and $D_2 = \{k, m-k+1\}$

$$\begin{aligned}
s(1, D_1) &= \frac{1}{k} + \frac{1}{3k} + \frac{1}{5k} + \frac{1}{7k} + ... + \frac{1}{(2\lfloor\frac{m-1}{2k}\rfloor - 1)k} \\
&\geq \frac{1}{k} + \frac{1}{3k} + \frac{1}{5k} + \frac{1}{7k} + ... + \frac{1}{(2n_{k-1} - 1)k} \quad (\text{as } m \geq 2kn_{k-1} + 1) \\
&\geq 1 - \frac{1}{k} \\
S(1, D_2) &\geq \frac{1}{k-1} + \frac{1}{m-k} > \frac{1}{k}
\end{aligned}$$

Therefore,

$$\begin{aligned}
s(1, D) &= s(1, D_1) + s(1, D_2) \\
&> 1 - \frac{1}{k} + \frac{1}{k} \\
&> 1
\end{aligned}$$

Hence vertex 1 is sb-dominated by D. Now consider any vertex j between 1 and $k+1$.

Then $s(j, D) > s(1, D)$, for all $1 < j < k+1$, ie., all vertices between 1 and $k+1$ are sb-dominated by D. Similarly vertices between $m-k+1$ and m also get sb-dominated by D.

Let j be any other vertex on P_m such that $k+1 < j \leq \frac{m}{2}$. Then we can find an i such that j lies between $(2i-1)k+1$ and $(2i+1)k+1$. Its distance from one of it is less than k and other one is less than $3k$. We can see that,

$$\begin{aligned}
s(j, D) &\geq \frac{1}{k} + \frac{1}{3k} + \frac{1}{5k} + \frac{1}{7k} + ... + \frac{1}{(2n_{k-1} - 1)k} \\
&\geq 1
\end{aligned}$$

Thus any vertex $j \leq \frac{m}{2}$ is sb-dominated by D. By the same argument we see that vertices $j > \frac{m}{2}$ are also sb-dominated by D.

Thus $\gamma_{sb}(P_m) \leq \lceil\frac{m-1}{2k} + 1\rceil$. □

$\gamma_{sb}(P_m)$ attains the bound given in the above theorem for many values of m.

Theorem 4.7.5. *If* $m \geq (2k-1)n_{k-1} + 1$, *then* $\gamma_{sb}(P_m) \leq \left\lceil\frac{m-1}{2k-1} + 1\right\rceil$.

Proof. By definition of n_{k-1},

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{2n_{k-1}-1}\geq k-1$$

Therefore,

$$\begin{aligned}\frac{1}{k-\frac{1}{2}}+\frac{1}{3(k-\frac{1}{2})}+...+\frac{1}{(2n_{k-1}-1)(k-\frac{1}{2})} &\geq \frac{k-1}{k-\frac{1}{2}}\\ &\geq \frac{2k-2}{2k-1}\\ &\geq 1-\frac{1}{2k-1}\\ ie., \frac{1}{k-\frac{1}{2}}+\frac{1}{3k-\frac{3}{2}}+\frac{1}{5k-\frac{5}{2}}...+\frac{1}{(2n_{k-1}-1)k-\frac{2n_{k-1}-1}{2}} &\geq 1-\frac{1}{2k-1}\end{aligned}$$

$$\text{Let}\quad D=D_1\cup D_2$$

where $D_1=\{k,3k-1,5k-2,...,k+(\lfloor\frac{m-1}{2k-1}-1\rfloor)(2k-1),m-(k-1)\}$ and $D_2=\{k-1,m-(k-2)\}$.
Then $|D|=\lceil\frac{m-1}{2k-1}+2\rceil$. Now

$$\begin{aligned}s(1,D_1) &= \frac{1}{k-1}+\frac{1}{3k-2}+\frac{1}{5k-3}+...+\frac{1}{m-k}\\ &\geq \frac{1}{k-\frac{1}{2}}+\frac{1}{3k-\frac{3}{2}}+\frac{1}{5k-\frac{5}{2}}+...+\frac{1}{(2n_{k-1}-1)k-\frac{2n_{k-1}-1}{2}}\\ &\geq 1-\frac{1}{2k-1}\\ S(1,D_2) &\geq \frac{1}{k-2}+\frac{1}{m-k+1}\\ &\geq \frac{1}{2k-1}\end{aligned}$$

Therefore,

$$\begin{aligned}s(1,D) &= s(1,D_1)+S(1,D_2)\\ &> 1-\frac{1}{2k-1}+\frac{1}{2k-1}\\ &> 1\end{aligned}$$

Hence vertex 1 is sb-dominated by D. Now, we see as in the previous theorem that, $s(j,D)\geq s(1,D)$ for all $j\in P_m$. ie., the strength received by any other vertex $j\in P_m$ is greater than that received by vertex 1. Hence D is a sb-dominating set of

P_m.

Therefore, $\gamma_{sb}(P_m) \leq \left\lceil \frac{m-1}{2k-1} + 2 \right\rceil$. □

Even though finding exact values of sb-domination number of P_n for large values of n is difficult, above theorems are useful for finding approximate values of $\gamma_{sb}(P_n)$ when n is large.

4.7.2 Bounds in terms of n_k for sb-domination number of cycles

Here we give some upper bounds for sb-domination number of cycles in terms of n_k where k need not be an integer. Here we consider only the cases where $2k \in \mathbb{N}$.

Theorem 4.7.6. *For any $2k \in \mathbb{N}$ with $2k \geq 4$, $\gamma_{sb}(C_m) \leq 2n_{\frac{k}{2}}$ for $m \leq 4kn_{\frac{k}{2}}$.*

Proof. By defenition of $n_{\frac{k}{2}}$,

$$1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n_{\frac{k}{2}} - 1} \geq \frac{k}{2}$$

Therefore,

$$\frac{2}{k} + \frac{2}{3k} + \frac{2}{5k} + ... + \frac{2}{k(2n_{\frac{k}{2}} - 1)} \geq 1$$

Consider the cycle C_m where $m = 4kn_{\frac{k}{2}}$. Let $V(C_m) = \{1, 2, 3, ..., m\}$.
Let $D \subset V(C_m)$ be given by $D = \{1, 2k + 1, 4k + 1, ..., 2kn_{\frac{k}{2}} + 1, 2k(n_{\frac{k}{2}} + 1) + 1, ..., 4kn_{\frac{k}{2}} - 2k + 1\}$, where the distance between each pair of consecutive vertices in D is $2k$.

Consider any vertex $v = 2rk + j + 1$ on C_m which is between two consecutive vertices $v_r = 2rk + 1$ and $v_{r+1} = 2(r + 1)k + 1$ in D. As $\frac{1}{k-1} - \frac{1}{k} > \frac{1}{k} - \frac{1}{k+1}$ we observe that $\frac{1}{d(v_r, v)} - \frac{1}{d(v, v_{r+1})} = \frac{1}{j} - \frac{1}{2k - j} > \frac{2}{k}$, $\frac{1}{d(v_{r-1}, v)} - \frac{1}{d(v, v_{r+2})} > \frac{2}{3k}$ and so on. Therefore,

$$\begin{aligned} s(v, D) &\geq \frac{2}{k} + \frac{2}{3k} + \frac{2}{5k} + ... + \frac{2}{(2n_{\frac{k}{2}} - 1)k} \\ &\geq 1 \end{aligned}$$

Hence any vertex $v \in C_m$ is sb-dominated by D. Thus D is a sb-dominating set of C_m with $|D| = 2n_{\frac{k}{2}}$. Hence, $\gamma_{sb}(C_m) \leq 2n_{\frac{k}{2}}$ for $m = 4kn_{\frac{k}{2}}$. It also proves that for

k	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
$n_{\frac{k}{2}}$	1	2	3	5	8	13	21	35	57	94	254
$2kn_{\frac{k}{2}}$	2	4	6	10	16	26	42	70	114	188	508
$4kn_{\frac{k}{2}}$	8	20	36	70	128	234	420	770	1368	2444	7112

Table 4.5: Table showing $n_{\frac{k}{2}}$, $2n_{\frac{k}{2}}$, $4n_{\frac{k}{2}}$ for some values of k

$m \leq 4kn_{\frac{k}{2}}$, $\gamma_{sb}(C_m) \leq 2n_{\frac{k}{2}}$ □

Remark 4.7.1. *The upper bound given in the above theorem for $\gamma_{sb}(C_m)$ is sharp for many values of m. From the above theorem and the values given in Table 4.5 we get the following.*

1. $\gamma_{sb}(C_n) \leq 2$, *if $n \leq 8$. We see from Theorem 4.4.1 and Theorem 4.4.2 that this bound is attained for $n = 4, 5, 6, 7, 8$.*

2. $\gamma_{sb}(C_n) \leq 4$ *if $n \leq 20$. From Theorem 4.4.4 we see that this bound is attained for $14 \leq n \leq 20$.*

3. $\gamma_{sb}(C_n) \leq 6$ *if $n \leq 36$. From Theorem 4.4.6 we see that this bound is attained for $26 \leq n \leq 36$.*

4. $\gamma_{sb}(C_n) \leq 10$ *if $n \leq 70$.*

5. $\gamma_{sb}(C_n) \leq 16$ *if $n \leq 128$ and so on.*

From Theorem 4.4.9 we get the exact values of $\gamma_{sb}(C_n)$ for $n \leq 48$. The upper and lower bounds for larger values of n are given in Corollary4.7.7.

Corollary 4.7.7. 1. *If $49 \leq n \leq 70$, then $\lceil \frac{n}{7} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{6} \rceil$*

2. *If $71 \leq n \leq 128$, then $\lceil \frac{n}{8} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{7} \rceil$*

3. *If $129 \leq n \leq 234$, then $\lceil \frac{n}{9} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{8} \rceil$*

4. *If $235 \leq n \leq 420$, then $\lceil \frac{n}{10} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{9} \rceil$*

5. *If $421 \leq n \leq 770$, then $\lceil \frac{n}{11} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{10} \rceil$*

6. *If $771 \leq n \leq 1368$, then $\lceil \frac{n}{12} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{11} \rceil$*

7. *If* $1369 \leq n \leq 2444$, *then* $\lceil \frac{n}{13} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{12} \rceil$

8. *If* $2445 \leq n \leq 7620$, *then* $\lceil \frac{n}{14} \rceil \leq \gamma_{sb}(C_n) \leq \lceil \frac{n}{13} \rceil$

Proof. These results follow easily from Theorem 4.7.6 and the values of $n_{\frac{k}{2}}$ given in Table 4.5 □

Corollary 4.7.8. Let $k \in \mathbb{R}$ be such that $2k \in \mathbb{N}$ and $2k \geq 4$ and let $m = 4kn_{\frac{k}{2}}$. Then for any $n \geq m$, $\gamma_{sb}(C_n) \leq \left\lceil \dfrac{n}{2k} \right\rceil$ if $n \geq m$.

Proof. As $n \geq 4kn_{\frac{k}{2}}$, from the proof of the above theorem we see that $D = \{1, 2k+1, 4k+1, ..., (\lfloor \frac{n-1}{2k} \rfloor)2k+1\}$ is a sb-dominating set of C_n of order $\lceil \frac{n}{2k} \rceil$. Hence $\gamma_{sb}(C_n) \leq \left\lceil \dfrac{n}{2k} \right\rceil$ if $n \geq m$. □

Corollary 4.7.9. *If* $m \geq 2(2k-1)n_{\frac{2k-1}{4}}$, *then* $\gamma_{sb}(C_n) \leq \left\lceil \dfrac{n}{2k-1} \right\rceil$

Proof. By replacing k by $k - \frac{1}{2}$ in Corollary 4.7.8 we get the result. □

Theorem 4.7.10. Let $k \in \mathbb{R}$ be such that $2k \in \mathbb{N}$ and $2k \geq 4$. Then for any $n \in \mathbb{N}$ with $2(2k-1))n_{\frac{2k-1}{4}} < n \leq 4kn_{\frac{k}{2}}$, then

$$\gamma_{sb}(C_m) \leq \min\left\{ \left\lceil \frac{n}{2k-1} \right\rceil, 2n_{\frac{k}{2}} \right\}$$

Proof. From Theorem 4.7.6, we get $\gamma_{sb}(C_m) \leq 2n_{\frac{k}{2}}$ if $n \leq 4kn_{\frac{k}{2}}$.
From Corollary 4.7.9 , we get $\gamma_{sb}(C_m) \leq \left\lceil \dfrac{n}{2k-1} \right\rceil$ if $m \geq 2(2k-1)n_{\frac{2k-1}{4}}$. By combining two results we see that if $2(2k-1))n_{\frac{2k-1}{4}} < n \leq 4kn_{\frac{k}{2}}$, then $\gamma_{sb}(C_m) \leq \min\left\{ \left\lceil \dfrac{n}{2k-1} \right\rceil, 2n_{\frac{k}{2}} \right\}$. □

Remark 4.7.2. The bound given in Theorem 4.7.10 is attained for many values of n.

1. If $k = 2.5$, then $2(2k-1)n_{\frac{2k-1}{4}} = 8$ and $4kn_{\frac{k}{2}} = 20$.
   Hence Theorem 4.7.10 says that if $8 < n \leq 20$, then
   $\gamma_{sb}(C_m) \leq \min\left\{ \left\lceil \dfrac{n}{4} \right\rceil, 4 \right\}$.
   This bound is attained for all values of n in $8 < n \leq 20$ except $n = 13$ as we know from Theorem 4.4.3 and Theorem 4.4.4 that

$$\begin{aligned} \gamma_{sb}(C_m) &= 3 \quad \text{for } n = 9, 10, 11, 12, 13 \\ &= 4 \quad \text{for } 14 \leq n \leq 20 \end{aligned}$$

For $n = 13$, $\gamma_{sb}(C_m) = 3 < \min\left\{\left\lceil \frac{13}{4} \right\rceil, 4\right\}$.

2. If $k = 3$, then $2(2k-1)n_{\frac{2k-1}{4}} = 20$ and $4kn_{\frac{k}{2}} = 36$.
   Hence Theorem 4.7.10 says that if $20 < n \leq 36$, then $\gamma_{sb}(C_m) \leq \min\left\{\left\lceil \frac{n}{5} \right\rceil, 6\right\}$. This bound is also attained for all values of n in $20 < n \leq 36$ except $n = 26, 27$ as we know from Theorem 4.4.5 and Theorem 4.4.6 that

$$\begin{aligned} \gamma_{sb}(C_m) &= 5 \quad \text{for } 21 \leq n \leq 27 \\ &= 6 \quad \text{for } 28 \leq n \leq 36 \end{aligned}$$

   For $n = 26, 27$, $\gamma_{sb}(C_m) = 5 < \min\left\{\left\lceil \frac{n}{5} \right\rceil, 6\right\}$.

4.8 Conclusion

In this chapter we have introduced the concepts of dominating strength of a vertex and strength based domination number of graphs and have presented several basic results of this parameter. Even though it is hard to find exact values of the sb-domination number of graphs having a large number of vertices, one can always attempt to find good bounds for such graphs. We found exact values of the sb-domination number of paths and cycles for some values of n and have provided several bounds of it for large values of n. Such studies in other classes of graphs and graph products will be interesting. Other important properties and algorithmic aspects of this parameter can also be explored further.

CHAPTER 5

Domination in Fuzzy Graphs

5.1 Introduction

Research in crisp graph theory focuses on several subset problems of the vertex set. Dominating sets, independent sets, vertex cover, vertex cut, neighborhood of a vertex are all crisp subsets of the vertex set of a graph. In studies of subset problems in fuzzy graph theory, it is more apt to consider the fuzzy subsets of the vertex set. In the present literature, the fuzzy subsets of the vertex sets are not much studied. For example, the neighborhood and the closed neighborhood[48] of a vertex v in a fuzzy graph are defined as $N(v) = \{u \in V : \sigma(u, v) > 0\}$ and $N[v] = N(v) \cup \{v\}$. But it is obvious that the influence of a neighbor vertex u on v depends on the intensity of relation between u and v, i.e., it depends on $\sigma(u, v)$. This motivate us to redefine the neighborhood of a vertex $v \in V$ as a fuzzy subset ν_v of μ given by $\nu_v : V \to [0, 1]$ and

$$\begin{aligned} \nu_v(u) &= 0 && \text{if } u = v \\ &= \sigma(u, v) && \text{if } u \neq v \end{aligned}$$

Some results of this chapter are included in the following papers.

1. Lekha A, Parvathy K. S: Fuzzy domination in fuzzy graphs, *Journal of Intelligent & Fuzzy Systems,* Vol. 44, No. 2, pp. 3205–3212, 2023.
2. Lekha A, Parvathy K. S: On fz-domination number of fuzzy graphs, *Ratio Mathematica Journal of Mathematics, Statistics and Applications,* Vol. 46, 2023.

The closed neighborhood of vertex v can be defined as the fuzzy subset $\bar{\nu}_v : V \to [0,1]$ given by

$$\begin{aligned} \bar{\nu}_v(u) &= \mu(u) \quad && \text{if } u = v \\ &= \sigma(u,v) \quad && \text{if } u \neq v \end{aligned}$$

For example, in the fuzzy graph $\mathcal{G}$ in Figure 5.1,

$$\nu_u = \{(u,0),(v,0.3),(w,0.5)\}$$

and

$$\bar{\nu}_u = \{(u,0.9),(v,0.3),(w,0.5)\}$$

In a similar way, we can define other fuzzy subsets of the vertex set. In this chapter

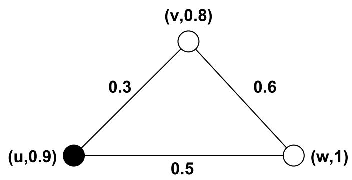

Figure 5.1: *Fuzzy Graph $\mathcal{G}$*

we define a dominating set of a fuzzy graph as a fuzzy subset of its vertex set and study some of its properties.

5.2 Fuzzy dominating sets in fuzzy graphs

Definition 5.2.1. *Let $\mathcal{G} = (V, \mu, \sigma)$ be a fuzzy graph on a finite set V. A fuzzy subset μ' of μ is called a fuzzy dominating set or fz-dominating set of $\mathcal{G}$ if for every $v \in V$,*

$$\mu'(v) + \sum_{x \in V} \Big(\sigma(x,v) \wedge \mu'(x)\Big) \geq \mu(v)$$

A fuzzy subset μ' is a minimal fz-dominating set, if $\mu'' \subset \mu'$ is not an fz-dominating set.

Definition 5.2.2. *Fuzzy domination number or fz-domination number of a fuzzy graph $\mathcal{G}$, denoted by $\gamma_{fz}(\mathcal{G})$, is defined as*

$$\gamma_{fz}(\mathcal{G}) = min\ \{|\mu'| : \mu' \text{ is a minimal fz-dominating set of } \mathcal{G}\}$$

Remark 5.2.1. *In the case of a crisp graph, the fz-domination number and the fractional domination number are equal.*

Example 5.2.1. *Consider the fuzzy graph $\mathcal{H} = (\mu, \sigma)$ given in Figure 5.2.*

The fuzzy subsets of μ given by

$$\mu_1 = \{(u, 0.5), (v, 0), (w, 0.5)\}$$

$$\mu_2 = \{(u, 0.2), (v, 0.3), (w, 0.2)\}$$

$$\mu_3 = \{(u, 0.4), (v, 0.1), (w, 0.4)\}$$

$$\mu_4 = \{(u, 0.1), (v, 0.4), (w, 0.2)\}$$

are all minimal fz-dominating sets of $\mathcal{H}$.

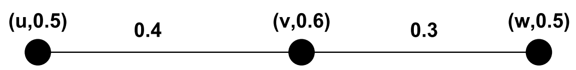

Figure 5.2: *Fuzzy graph, $\mathcal{H}$*

Remark 5.2.2. *The fz-domination in fuzzy graphs is a generalization of the concept of domination in Crisp graphs. A crisp graph $G = (V, E)$ can be viewed as a fuzzy graph $G = (V, \mu, \sigma)$ where μ and σ are the characteristic functions χ_V and χ_E of V and E respectively. The characteristic function of any dominating set S of G is a fuzzy subset μ' of μ such that*

1. *if $v \in S$, then $\mu'(v) = \mu(v) = 1$*
2. *if $v \notin S$, then $\mu'(v) = 0$, but there exists $u \in S$ such that $(u, v) \in E$. Hence, $\mu'(u) = \sigma(u, v) = 1$.*

In both the cases,

$$\mu'(v) + \sum_{x \in V} \Big(\sigma(x, v) \wedge \mu'(x)\Big) \geq \mu(v) = 1.$$

Definition 5.2.3. *For an fz-dominating set μ' of $\mathcal{G}$, the boundary of μ', denoted by $B_{\mu'}$, is defined as*

$$B_{\mu'} = \Big\{v \in V : \mu'(v) + \sum_{x\in V} \big(\sigma(x,v) \wedge \mu'(x)\big) = \mu(v)\Big\}$$

and the positive set of μ', denoted by $P_{\mu'}$, is defined as

$$P_{\mu'} = \{v \in V : \mu'(v) > 0\}$$

Theorem 5.2.4. *An fz-dominating set μ' of a fuzzy graph $\mathcal{G}$ is a minimal fz-dominating set if and only if each $v \in P_{\mu'}$ is either in $B_{\mu'}$ or there exists a vertex $u \in B_{\mu'}$ such that $\sigma(u,v) \geq \mu'(v)$.*

Proof. Let μ' be a minimal fz-dominating set of $\mathcal{G}$ and let $v \in P_{\mu'}$ but $v \notin B_{\mu'}$. Let there exist no $u \in B_{\mu'}$ such that $\sigma(u,v) > 0$.
Let

$$r = \wedge\Big\{\Big(\mu'(x) + \sum_{y\in V}\sigma(x,y) \wedge \mu'(y)\Big) - \mu(x) : x \in N[v]\Big\}$$

Then the fuzzy subset $\mu'' \subset \mu'$ defined by

$$\begin{aligned} \mu''(x) &= \mu'(x) - r \quad \text{if } x = v \\ &= \mu'(x) \qquad\quad \text{otherwise} \end{aligned}$$

is also an fz-dominating set of $\mathcal{G}$, which is a contradiction to the minimality of μ'. Hence there must exist at least one $u \in B_{\mu'} \cap N[v]$ such that $\sigma(u,v) > 0$. For each $u \in B_{\mu'} \cap N[v]$, let $\mu'(v) > \sigma(u,v)$.
Now let

$$\begin{aligned} r &= \wedge\Big\{\Big(\mu'(x) + \sum_{y\in V}\sigma(x,y) \wedge \mu'(y)\Big) - \mu(x) \\ &\qquad\qquad : x \in N[v] \setminus B_{\mu'}\Big\} \\ s &= \wedge\{\mu'(v) - \sigma(u,v) : u \in B_{\mu'} \cap N[v]\} \\ \text{and } t &= \wedge\{r, s\} \end{aligned}$$

Then μ'' defined by

$$\begin{aligned} \mu''(x) &= \mu'(x) - t \quad \text{if } x = v \\ &= \mu'(x) \qquad\quad \text{otherwise} \end{aligned}$$

is an fz-dominating subset of $\mathcal{G}$, such that $\mu'' \subset \mu'$, which is again a contradiction to the choice of μ'. Thus we see that if μ' is a minimal fz-dominating set, then each $v \in P_{\mu'}$ is either in $B_{\mu'}$ or there exists a vertex $u \in B_{\mu'}$ such that $\sigma(u,v) \geq \mu'(v)$.

Conversely let μ' be an fz-dominating subset of $\mathcal{G}$ such that each $v \in P_{\mu'}$ is either in $B_{\mu'}$ or there exists a vertex $u \in B_{\mu'}$ such that $\mu'(v) \leq \sigma(u,v)$. Consider a fuzzy subset $\mu'' \subset \mu'$ such that $\mu'' \neq \mu'$. Then there exists some $u \in V$ such that $\mu''(u) < \mu'(u)$.
If $u \in B_{\mu'}$, then

$$\mu''(u) + \sum_{x \in V} \Big(\sigma(u,x) \wedge \mu''(x)\Big) < \mu'(u) + \sum_{x \in V} \Big(\sigma(u,x) \wedge \mu'(x)\Big) = \mu(u)$$

If $u \notin B_{\mu'}$, then by our assumption, there exists a vertex $x \in B_{\mu'}$ such that $\mu'(u) \leq \sigma(u,x)$. Hence $\mu''(u) < \sigma(u,x)$.
This implies that,

$$\mu''(x) + \sum_{v \in V} \Big(\sigma(x,v) \wedge \mu''(v)\Big) < \mu'(x) + \sum_{v \in V} \Big(\sigma(x,v) \wedge \mu'(v)\Big) = \mu(x)$$

Thus in both the cases μ'' cannot be an fz-dominating subset of μ. Hence μ' is a minimal fz-dominating set. □

Theorem 5.2.5. *For any fuzzy graph $\mathcal{G}$, $\gamma_{fz}(\mathcal{G}) \geq M$, where $M = \max_{v \in V} \mu(v)$.*

Proof. Let $x \in V$ be such that $\mu(x) = M$. Let μ' be an fz-dominating subset of μ. Then,

$$\mu'(x) + \sum_{v \in V} (\mu'(v) \wedge \sigma(x,v)) \geq \mu(x) = M$$

Hence, $|\mu'| = \sum_{v \in V} \mu'(v) \geq M$.

Since this is true for all fz-dominating subset μ' of μ, we get $\gamma_{fz}(\mathcal{G}) \geq M$. □

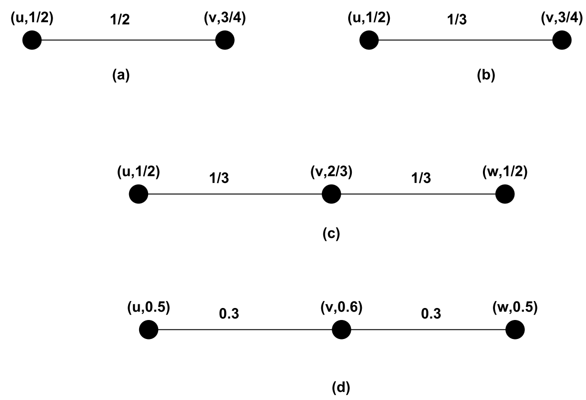

Figure 5.3: *Fuzzy graphs* $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$

Example 5.2.2. (a) Consider the fuzzy graph $\mathcal{G}_1 = (\mu_1, \sigma_1)$ given in Figure 5.3(a).
$\mu_1' = \{(v, \frac{3}{4})\}$ is a minimum fz-dominating set of $\mathcal{G}_1$ and $\gamma_{fz}(\mathcal{G}_1) = \dfrac{3}{4}$.

(b) Consider the fuzzy graph $\mathcal{G}_2 = (\mu_2, \sigma_2)$ given in Figure 5.3(b).
Let $\mu_2' = \{(u, \frac{1}{6}), (v, \frac{7}{12})\}$. Then,

$$\mu_2'(u) + \Big(\sigma_2(u, v) \wedge \mu_2'(v)\Big) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} = \mu_2(u)$$

and

$$\mu_2'(v) + \Big(\sigma_2(u, v) \wedge \mu_2'(u)\Big) = \frac{7}{12} + \frac{1}{6} = \frac{3}{4} = \mu_2(v)$$

show that μ_2' is an fz-dominating set of $\mathcal{G}_2$. Hence

$$\gamma_{fz}(\mathcal{G}_2) \leq \frac{1}{6} + \frac{7}{12} = \frac{3}{4}$$

It follows from Theorem (5.2.5) that, $\gamma_{fz}(\mathcal{G}_2) \geq \dfrac{3}{4}$.
So, $\gamma_{fz}(\mathcal{G}_2) = \dfrac{3}{4}$.

(c) Consider the fuzzy graph $\mathcal{G}_3 = (\mu_3, \sigma_3)$ given in Figure 5.3(c).

$\mu_3' = \{(u, \frac{1}{6}), (v, \frac{1}{3}), (w, \frac{1}{6})\}$ is an fz-dominating set of $\mathcal{G}_3$.
Hence, $\gamma_{fz}(\mathcal{G}_3) \leq \dfrac{1}{6} + \dfrac{1}{3} + \dfrac{1}{6} = \dfrac{2}{3}$.
Also, Theorem (5.2.5) shows that $\gamma_{fz}(\mathcal{G}_3) \geq \dfrac{2}{3}$. Therefore $\gamma_{fz}(\mathcal{G}_3) = \dfrac{2}{3}$.

(d) Consider the fuzzy graph $\mathcal{G}_4 = (\mu_4, \sigma_4)$ given in Figure 5.3(d).
$\mu_4' = \{(u, 0.2), (v, 0.3), (w, 0.2)\}$ is a fuzzy dominating set of $\mathcal{G}_4$.
Hence, $\gamma(\mathcal{G}_4) \leq 0.2 + 0.3 + 0.2 = 0.7$.
Let μ_4'' be another fz-dominating set of $\mathcal{G}_4$. It is obvious that $\mu_4''(u)$ and $\mu_4''(w)$ must be at least 0.2. Let $\mu_4''(u) = x$, where $x \geq 0.2$.
Then

$$\begin{aligned} |\mu_4''| &\geq x + 0.5 - x + \mu_4''(w) \\ &\geq 0.5 + \mu_4''(w) \\ &\geq 0.7 \end{aligned}$$

Hence, $\gamma_{fz}(\mathcal{G}_4) \geq 0.7$
Therefore, $\gamma_{fz}(\mathcal{G}_4) = 0.7$.

Theorem 5.2.6. *$\gamma_{fz}(\mathcal{G}) \leq p$ where $p = \sum_{u \in V} \mu(u)$. Also $\gamma_{fz}(\mathcal{G}) = p$ if and only if all the vertices are isolated.*

Proof. The vertex set (V, μ) itself is an fz-dominating set of $\mathcal{G}$. Hence $\gamma_{fz}(\mathcal{G}) \leq p$.
If all the vertices are isolated then, μ is the only fz-dominating set of the graph.
Hence in this case $\gamma_{fz}(\mathcal{G}) = p$.
Suppose $u \in V$ is not an isolated vertex. Then there exist a vertex v such that $\sigma(u, v) \neq 0$.
Let $\mu(u) \wedge \sigma(u, v) = q$.
Now consider μ' defined by,

$$\mu'(x) = \begin{cases} \mu(x) & \text{if } x \neq u \\ \mu(x) - q & \text{if } x = u \end{cases}$$

Then μ' is an fz-dominating set of $\mathcal{G}$ and

$$|\mu'| = \sum_{v \in V} \mu'(v) = p - q$$

Hence

$$\gamma_{fz}(\mathcal{G}) \leq p - q.$$

Since $q \neq 0$, we get $\gamma_{fz}(\mathcal{G}) < p$. □

Definition 5.2.7. *A vertex v in a fuzzy graph is said to be fuzzy isolated if*

$$\mu(v) > \sum_{x \in V} \sigma(v, x)$$

The vertex u in Figure 5.1 is a fuzzy isolated vertex.

Theorem 5.2.8. *If u is not a fuzzy isolated vertex of $\mathcal{G}$, then*

$$\gamma_{fz}(\mathcal{G}) \leq p - \mu(u).$$

In particular, if $M = max\{\mu(v) : v$ is not a fuzzy isolated vertex of $\mathcal{G}\}$, then $\gamma_{fz}(\mathcal{G}) \leq p - M$.

Proof. Suppose $u \in V$ is not a fuzzy isolated vertex of $\mathcal{G}$ and let μ' be defined by,

$$\mu'(v) = \begin{cases} \mu(v) & \text{if } v \neq u \\ 0 & \text{if } v = u \end{cases}$$

Then $\mu' \subset \mu$ is an fz-dominating set of $\mathcal{G}$ and $|\mu'| = p - \mu(u)$. Hence

$$\gamma_{fz}(\mathcal{G}) \leq p - \mu(u).$$

If $M = \max\{\mu(v) : v$ is not a fuzzy isolated vertex of $\mathcal{G}\}$, then it follows that $\gamma_{fz}(\mathcal{G}) \leq p - M$. □

Theorem 5.2.9. *For any fuzzy graph $\mathcal{G}$,*

$$\gamma_{fz}(G) \leq p - \Delta$$

where $\Delta = \vee\{d(v) : v \in V\}$

Proof. Let $u \in V$ be such that $d(u) = \Delta$. Let μ' be defined by,

$$\begin{aligned} \mu'(v) &= \mu(v) && \text{if } v = u \\ &= \mu(v) - \sigma(u, v) && \text{otherwise} \end{aligned}$$

Then μ' is an fz-dominating set of $\mathcal{G}$ and $|\mu'| = p - \Delta$.
Hence $\gamma_{fz}(\mathcal{G}) \leq p - \Delta$. □

Theorem 5.2.10. *For any fuzzy graph $\mathcal{G}$,*

$$\gamma_{fz}(\mathcal{G}) \geq p - 2q.$$

Proof. Let μ' be a minimum fz-dominating set of a fuzzy graph $\mathcal{G}$. Then,

$$\gamma_{fz}(\mathcal{G}) = \sum_{v \in V} \mu'(v)$$

Also for each $v \in V$,

$$\mu'(v) + \Big(\sum_{u \in V} \mu'(u) \wedge \sigma(v, u) \Big) \geq \mu(v)$$

By taking sum of all these inequalities, we get

$$\sum_{v \in V} \Big(\mu'(v) + \sum_{u \in V} \sigma(u, v) \wedge \mu'(u) \Big) \geq \sum_{v \in V} \mu(v)$$

This shows that

$$\gamma_{fz}(\mathcal{G}) + 2q \geq p$$

Hence $\gamma_{fz}(\mathcal{G}) \geq p - 2q$ □

Theorem 5.2.11. *Let $\mathcal{G}$ be a complete fuzzy graph. Then $\gamma_{fz}(\mathcal{G}) = M$ where $M = \max_{v \in V} \mu(v)$.*

Proof. $\mathcal{G}$ is a complete fuzzy graph implies that $\sigma(u, v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$. Let $M = \mu(x)$, $x \in V$. Let μ' be defined by,

$$\begin{aligned} \mu'(v) &= M \quad \text{if } v = x \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Consider $v \in V$ where $v \neq x$. Then,

$$\begin{aligned} \mu(v) = \quad & \mu(v) \wedge \mu(x) \\ = \quad & \sigma(x, v) \\ = \quad & \mu'(x) \wedge \sigma(x, v) \\ \leq \quad & \mu'(v) + \sum_{u \in V} \big(\mu'(u) \wedge \sigma(u, v) \big) \end{aligned}$$

Thus μ' is an fz-dominating set of $\mathcal{G}$. Now $|\mu'| = M$ shows that $\gamma_{fz}(\mathcal{G}) \leq M$. On the other hand, by Theorem 5.2.5, $\gamma_{fz}(\mathcal{G}) \geq M$. Thus $\gamma_{fz}(\mathcal{G}) = M$. □

5.3 The fz-domination and graph operations

In this section, we study some properties of fz- domination number under the action of different graph operations in fuzzy graphs. Bounds for the fz- domination number of some graph products are obtained and the conditions for the sharpness of these bounds are examined.

5.3.1 Union of two fuzzy graphs and fz- domination

Union of the fuzzy graphs $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ is the fuzzy graph $\mathcal{G} \cup \mathcal{H} = (V, \mu, \sigma)$
where

$$V = V_1 \cup V_2$$

$$\begin{aligned} \mu(u) &= \mu_1(u) \quad \text{if } u \in V_1 \setminus V_2 \\ &= \mu_2(u) \quad \text{if } u \in V_2 \setminus V_1 \\ &= \mu_1(u) \vee \mu_2(u) \quad \text{if } u \in V_1 \cap V_2 \end{aligned}$$

and

$$\begin{aligned} \sigma(u,v) &= \sigma_1(u,v) \quad \text{if } u \in V_1 \setminus V_2, v \in V_1 \\ &= \sigma_2(u,v) \quad \text{if } u \in V_2 \setminus V_1, v \in V_2 \\ &= \sigma_1(u,v) \vee \sigma_2(u,v) \quad \text{if } u, v \in V_1 \cap V_2 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The following theorem gives a general upper bound for the fz-domination number of union of two fuzzy graphs.

Theorem 5.3.1. *For any two non- trivial fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$,*

$$\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$$

Proof. Consider the fuzzy graphs $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$. Let μ_1' and μ_2' be the minimum fz-dominating sets of $\mathcal{G}$ and $\mathcal{H}$ respectively. Let the fuzzy subset

μ' of V be defined by

$$\begin{aligned} \mu'(u) &= \mu'_1(u) \quad \text{if } u \in V_1 \setminus V_2 \\ &= \mu'_2(u) \quad \text{if } u \in V_2 \setminus V_1 \\ &= \mu'_1(u) \vee \mu'_2(u) \quad \text{if } u \in V_1 \cap V_2 \end{aligned}$$

Now let $v \in V$.

Case (i)

If $v \in V_1 \setminus V_2$, then

$$\begin{aligned} \mu(v) &= \mu_1(v) \\ &\leq \Big(\mu'_1(v) + \sum_{x \in V_1} \sigma_1(x,v) \wedge \mu'_1(x)\Big) \\ &= \mu'(v) + \sum_{x \in V} \sigma(x,v) \wedge \mu'(x). \end{aligned}$$

Case (ii)

If $v \in V_2 \setminus V_1$, then

$$\begin{aligned} \mu(v) &= \mu_2(v) \\ &\leq \Big(\mu'_2(v) + \sum_{x \in V_2} \sigma_2(x,v) \wedge \mu'_2(x)\Big) \\ &= \mu'(v) + \sum_{x \in V} \sigma(x,v) \wedge \mu'(x). \end{aligned}$$

Case (iii)

If $v \in V_1 \cap V_2$

$$\begin{aligned} \mu(v) &= \mu_1(v) \vee \mu_2(v) \\ &\leq \Big(\mu'_1(v) + \sum_{x \in V_1} \sigma_1(x,v) \wedge \mu'_1(x)\Big) \vee \Big(\mu'_2(v) + \sum_{x \in V_2} \sigma_2(x,v) \wedge \mu'_2(x)\Big) \\ &\leq (\mu'_1(v) \vee \mu'_2(v)) + \Big(\sum_{x \in V_1 \setminus V_2} (\sigma_1(x,v) \wedge \mu'_1(x)) + \sum_{x \in V_2 \setminus V_1} (\sigma_2(x,v) \wedge \mu'_2(x)) \\ &+ \sum_{x \in V_1 \cap V_2} (\sigma_1(x,v) \vee \sigma_2(x,v)) \wedge (\mu'_1(x) \vee \mu'_2(x))\Big) \\ &\leq \mu'(v) + \sum_{x \in \mathcal{V}} \sigma(x,v) \wedge \mu'(x). \end{aligned}$$

Thus μ' is an fz-dominating set of $\mathcal{G} \cup \mathcal{H}$ and $\mu'(v) \leq \mu_1'(v) + \mu_2'(v)$.
Hence $|\mu'| \leq |\mu_1'| + |\mu_2'|$.
Thus, $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$. □

Remark 5.3.1. *Bound given in the above theorem is tight. Obviously equality holds if the vertex sets of $\mathcal{G}$ and $\mathcal{H}$ are disjoint. The following example shows that equality may hold even if they are not disjoint. For the graphs in Figure 5.4, $\gamma_{fz}(\mathcal{G}) = 0.5$, $\gamma_{fz}(\mathcal{H}) = 0.6$ and $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) = 1.1$ so that $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) = \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$.*

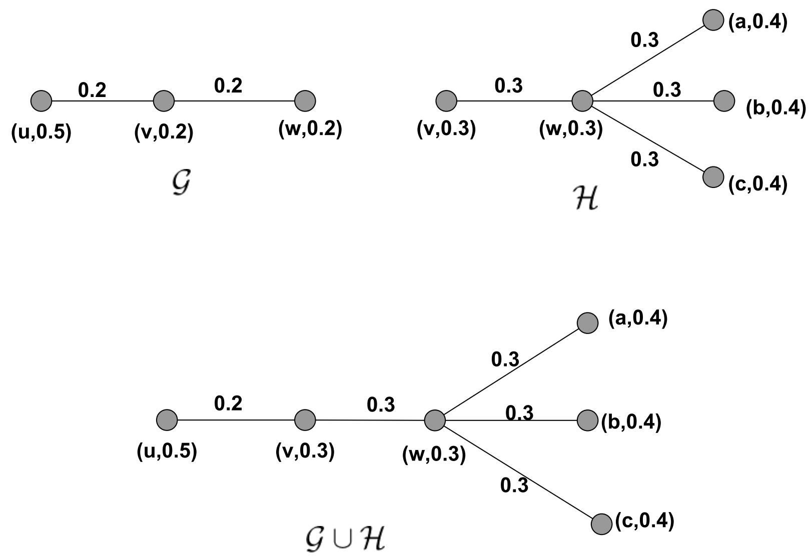

Figure 5.4: *Fuzzy Graphs $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$*

5.3.2 Join of two fuzzy graphs and fz- domination

For any two fuzzy graphs $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ whose vertex sets are disjoint, the join $\mathcal{G} + \mathcal{H}$ is the fuzzy graph defined by
$\mathcal{G} + \mathcal{H} = (V, \mu, \sigma)$ where $V = V_1 \cup V_2$,

$$\begin{aligned} \mu(u) &= \mu_1(u) \quad \text{if } u \in V_1 \\ &= \mu_2(u) \quad \text{if } u \in V_2 \end{aligned}$$

and

$$\begin{aligned} \sigma(u, v) &= \sigma_1(u, v) \quad \text{if } u, v \in V_1 \\ &= \sigma_2(u, v) \quad \text{if } u, v \in V_2 \\ &= \mu_1(u) \wedge \mu_2(v) \text{ if } u \in V_1 \text{ and } v \in V_2 \end{aligned}$$

Theorem 5.3.2. *For any two non- trivial fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$ whose vertex sets are disjoint,*

$$\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq max\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

Proof. Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \phi$. Let $\gamma_{fz}(\mathcal{G}) \geq \gamma_{fz}(\mathcal{H})$ and let μ_1' be a minimum fz-dominating set of $\mathcal{G}$. Define $\mu' \subset \mu$ by

$$\begin{aligned} \mu'(u) &= \mu_1'(u) \quad \text{if } u \in \mathcal{G} \\ &= 0 \quad \text{if } u \in \mathcal{H} \end{aligned}$$

Let m be such that $m = max\{\mu_2(u); u \in \mathcal{H}\}$.
Now $m \leq \gamma_{fz}(\mathcal{H}) \leq \gamma_{fz}(\mathcal{G})$ implies that μ' is an fz-dominating set of $\mathcal{G}+\mathcal{H}$.
Hence,

$$\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq max\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

□

In the following discussion M, m_1 and m_2 denote the maximum membership value of a vertex in $\mathcal{G}+\mathcal{H}$, $\mathcal{G}$ and $\mathcal{H}$ respectively.

Observation 5.3.3. *It is possible that*

$$\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq min\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

For example, if $M \leq \gamma_{fz}(\mathcal{H}) \leq \gamma_{fz}(\mathcal{G})$, then $\mu' \subset \mu$ defined by

$$\begin{aligned} \mu'(u) &= \mu_2'(u) \quad \textit{if } u \in \mathcal{H} \\ &= 0 \quad \textit{if } u \in \mathcal{G} \end{aligned}$$

is an fz-dominating set of $\mathcal{G}+\mathcal{H}$ and hence $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq \gamma_{fz}(\mathcal{H})$. Here equality occurs if $M = \gamma_{fz}(\mathcal{H})$. The following example shows that strict inequality can also occur in this relation.

Example 5.3.1. *Consider the fuzzy graphs $\mathcal{G}_1$, $\mathcal{G}_1$ and $\mathcal{G}_1+\mathcal{G}_2$ given in Figure 5.5. Here $\gamma_{fz}(\mathcal{G}_1) = 1.6$, $\gamma_{fz}(\mathcal{G}_2) = 1$, $M = 0.8$, $\gamma_{fz}(\mathcal{G}_1+\mathcal{G}_2) = 0.9$ so that*

$$\gamma_{fz}(\mathcal{G}_1+\mathcal{G}_2) < \gamma_{fz}(\mathcal{G}_2)$$

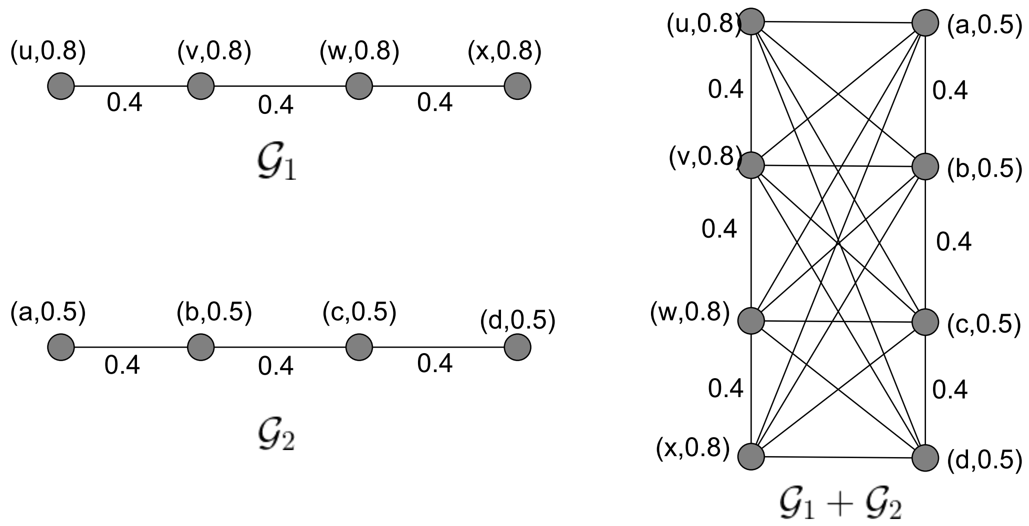

Figure 5.5: *Fuzzy Graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_1 + \mathcal{G}_2$*

Observation 5.3.4. *If $\gamma_{fz}(\mathcal{H}) \leq M \leq |\mu_2|$, then $\gamma_{fz}(\mathcal{G}+\mathcal{H}) = M$.*

Claim: *Define $\mu_2'' \supset \mu_2'$ in $\mathcal{H}$ such that $|\mu_2''| = M$. Then, μ_2'' is an fz-dominating set of $\mathcal{G}+\mathcal{H}$. Hence $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq M$. Also, since there is a vertex of membership value M in $\mathcal{G}+\mathcal{H}$, we get $\gamma_{fz}(\mathcal{G}+\mathcal{H}) = M$.*

Observation 5.3.5. *If $m_1 \leq |\mu_2|$ and $m_2 \leq |\mu_1|$, then $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq m_1 + m_2$.*

Claim: *Define $\mu_1' \subset \mu_1$ in $\mathcal{G}$ such that $|\mu_1'| = m_2$ and $\mu_2' \subset \mu_2$ in $\mathcal{H}$ such that $|\mu_2'| = m_1$. Now μ' defined by*

$$\begin{aligned} \mu'(u) &= \mu_1'(u) \quad \textit{if } u \in \mathcal{G} \\ &= \mu_2'(u) \quad \textit{if } u \in \mathcal{H} \end{aligned}$$

is an fz- dominating set in $\mathcal{G}+\mathcal{H}$. Hence $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq m_1 + m_2$.

Observation 5.3.6. *If $|\mu_2| \leq M$, then $\mu' \subset \mu$ in $\mathcal{G}+\mathcal{H}$ defined by*

$$\begin{aligned} \mu'(u) &= \mu_2(u) \quad \textit{if } u \in \mathcal{H} \\ &= max\{0, \mu_1'(u) - |\mu_2|\} \quad \textit{if } u \in \mathcal{G} \end{aligned}$$

is an fz- dominating set in $\mathcal{G}+\mathcal{H}$. Then, $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) - n|\mu_2|$ where n is the number of vertices $u \in \mathcal{G}$ having $\mu_1(u) \geq |\mu_2|$.

Observation 5.3.7. *If $\gamma_{fz}(\mathcal{H}) \leq M \leq \gamma_{fz}(\mathcal{G})$, then $\mu' \subset \mu$ in $\mathcal{G}+\mathcal{H}$ defined by*

$$\begin{aligned} \mu'(u) &= \mu_2'(u) \quad \text{if } u \in \mathcal{H} \\ &= max\{0, \mu_1'(u) - \gamma_{fz}(\mathcal{H})\} \quad \text{if } u \in \mathcal{G} \end{aligned}$$

is an fz- dominating set in $\mathcal{G}+\mathcal{H}$. Then, $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) - n\gamma_{fz}(\mathcal{H})$ where n is the number of vertices $u \in \mathcal{G}$ having $\mu_1(u) \geq \gamma_{fz}(\mathcal{H})$.

5.3.3 Corona of two fuzzy graphs and fz- domination

Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and let $\mathcal{K}_1 = (\{u\}, \mu_2(u))$ be the trivial fuzzy graph. The corona of $\mathcal{G}$ and $\mathcal{K}_1$ is the fuzzy graph $\mathcal{G} \circ \mathcal{K}_1$ obtained by attaching a copy of $\mathcal{K}_1$ to each vertex $v_i \in V_1$ such that $\sigma(v_i, u_i) = \mu_1(v_i) \wedge \mu_2(u_i)$ where u_i represents the vertex in the copy of $\mathcal{K}_1$ corresponding to $v_i \in V_1$.

Observation 5.3.8. *The following two results are obvious.*

1. $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq \gamma_{fz}(\mathcal{G})$
2. $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$

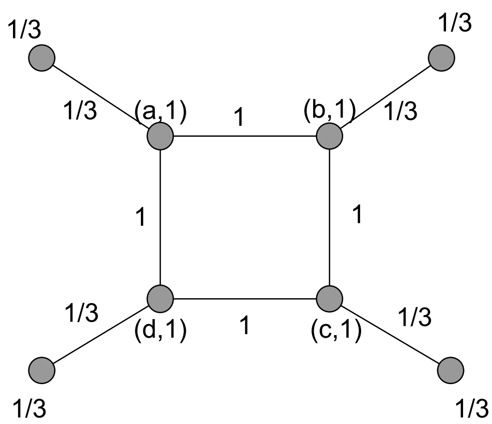

Figure 5.6: *Fuzzy graph $\mathcal{G} \circ \mathcal{K}_1$*

Remark 5.3.2. *The following example shows that equality may occur in Observation 5.3.8(a).*
Consider the fuzzy graph $\mathcal{G} \circ \mathcal{K}_1$ given in Figure 5.6. $\mu' = \{(a, \frac{1}{3}), (b, \frac{1}{3}), (c, \frac{1}{3}), (d, \frac{1}{3})\}$ is a minimum fuzzy dominating set of $\mathcal{G}$ and $\gamma_{fz}(\mathcal{G}) = \frac{4}{3}$. μ' is also a fuzzy dominating set of $\mathcal{G} \circ \mathcal{K}_1$. Therefore, $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \frac{4}{3} = \gamma_{fz}(\mathcal{G})$.

On the other hand from Observation 5.3.8(a), $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq \gamma_{fz}(\mathcal{G})$.
Thus we get $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = \gamma_{fz}(\mathcal{G})$.

Theorem 5.3.9. $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \gamma_{fz}(\mathcal{G}) + n\mu_2(u)$ *where* n *is the number of vertices in* V_1.

Proof. Let μ be the fuzzy subset of $\mathcal{G} \circ \mathcal{K}_1$ and μ'_1 be a minimum fuzzy dominating set of $\mathcal{G}$. Let μ' be a fuzzy subset of μ defined by

$$\begin{aligned} \mu'(v) &= \mu'_1(v) \quad \text{if } v \in V_1 \\ &= \mu_2(v) \quad \text{otherwise} \end{aligned}$$

Then μ' is a fuzzy dominating set of $\mathcal{G} \circ \mathcal{K}_1$ and

$$\begin{aligned} |\mu'| &= |\mu'_1| + n|\mu_2| \\ &= \gamma_{fz}(\mathcal{G}) + n\mu_2(u) \end{aligned}$$

Therefore $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \gamma_{fz}(\mathcal{G}) + n\mu_2(u)$ □

Theorem 5.3.10. *If* $\mu_2(u) \geq \mu_1(v)$ *for all* $v \in V_1$*, then*

$$\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$$

Proof. It is clear that $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$
Now let μ' be defined by

$$\begin{aligned} \mu'(v) &= o \quad \text{if } v \in V_1 \\ &= \mu_2(v) \quad \text{if } v = u \end{aligned}$$

Then, μ' is a fuzzy dominating set of $\mathcal{G} \circ \mathcal{K}_1$.
Therefore, $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq |\mu'| = n\mu_2(u)$.
Hence $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$ □

Remark 5.3.3. *The condition in Theorem 5.3.10 is not necessary. For example, consider the corona of the fuzzy graph* $\mathcal{G}'$ *given in Figure 5.7.*
Here, $\mu_2(u) < \mu_1(v)$ *for all* $v \in V_1$*. Now* $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$ *implies that* $\gamma_{fz}(\mathcal{G}' \circ \mathcal{K}'_1) \geq 2$.
Also $\mu' = \{(a, \frac{1}{2}), (b, \frac{1}{2}), (c, \frac{1}{2}), (d, \frac{1}{2})\}$ *is a fuzzy dominating set of* $\mathcal{G}' \circ \mathcal{K}'_1$.
Hence $\gamma_{fz}(\mathcal{G}' \circ \mathcal{K}'_1) = 2 = n\mu_2(u)$

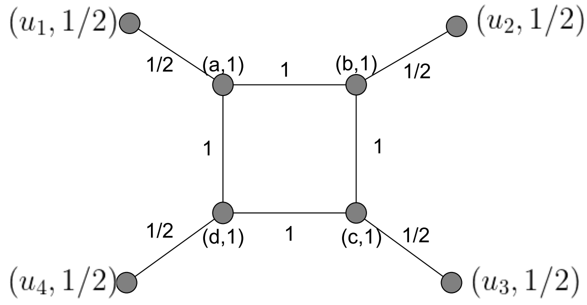

Figure 5.7: $\mathcal{G}' \circ \mathcal{K}'_1$

5.3.4 Cartesian product of two fuzzy graphs and fz- domination

Cartesian product of two fuzzy graphs is defined as follows. For any two fuzzy graphs $\mathcal{G}_1 = (V_1, \mu_1, \sigma_1)$ and $\mathcal{G}_2 = (V_2, \mu_2, \sigma_2)$, the Cartesian product is the fuzzy graph $\mathcal{G}_1 \Box \mathcal{G}_2 = (V, \mu_1 \times \mu_2, \sigma_1 \times \sigma_2)$ where $V = V_1 \times V_2$,

$$(\mu_1 \times \mu_2)(a, b) = \mu_1(a) \wedge \mu_2(b)$$

and

$$\begin{aligned}(\sigma_1 \times \sigma_2)\Big((a_1, b_2), (a_2, b_2)\Big) &= \mu_1(a_1) \wedge \sigma_2(b_1, b_2) \quad \text{if } a_1 = a_2 \\ &= \sigma_1(a_1, a_2) \wedge \mu_2(b_1) \quad \text{if } b_1 = b_2 \\ &= 0 \quad \text{otherwise}\end{aligned}$$

We first give a general upperbound for the fz- domination number of Cartesian product of two fuzzy graphs.

Theorem 5.3.11. *For any two nontrivial fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$,*

$$\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) \leq min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\}$$

where m and n are the number of vertices with nonzero membership values in $\mathcal{G}$ and $\mathcal{H}$ respectively.

Proof. Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ where

$$V_1 = \{(u_1, \mu_1(u_1)), (u_2, \mu_1(u_2)), ..., (u_m, \mu_1(u_m))\}$$

and

$$V_2 = \{(v_1, \mu_2(v_1)), (v_2, \mu_2(v_2)), ..., (v_n, \mu_2(v_n))\}$$

$\mathcal{G}\square\mathcal{H} = (V, \mu, \sigma)$ where $V = V_1 \times V_2$, $\mu(u, v) = \mu_1(u) \wedge \mu_2(v)$ and

$$\begin{aligned}\sigma\Big((u_i, v_j), (u_i', v_j')\Big) &= \sigma_1(u_i, u_i') \quad \text{if } v_j = v_j' \\ &= \sigma_2(v_j, v_j') \quad \text{if } u_i = u_i' \\ &= 0 \quad \text{otherwise}\end{aligned}$$

Let $\mathcal{G}_j$ denotes the fuzzy sub-graph of $\mathcal{G}\square\mathcal{H}$ induced by $V_1 \times v_j \subset V_1 \times V_2$. Then,

$$V(\mathcal{G}_j) = \{(u_1, v_j), (u_2, v_j), ..., (u_m, v_j)\}$$

$$\mu(u_i, v_j) = \mu_1(u_i) \wedge \mu_2(v_j) \leq \mu_1(u_i)$$

and

$$\sigma\Big((u_i, v_j), (u_i', v_j)\Big) = min\{\sigma_1(u_i, u_i'), \mu_2(v_j)\} \leq \sigma(u_i, u_i')$$

Claim: $\gamma_{fz}(\mathcal{G}_j) \leq \gamma_{fz}(\mathcal{G})$. Define μ_j' on $\mathcal{G}_j$ as $\mu_j'(u_i, v_j) = \mu_1'(u_i) \wedge \mu_2(v_j)$
Consider $(u_i, v_j) \in \mathcal{G}_j$. μ_1' is an fz-dominatng set of $\mathcal{G}$ implies that
$\mu_1(u_i) \leq \mu_1'(u_i) + \sum_{u_k \in \mathcal{G}} \sigma_1(u_k, u_i) \wedge \mu_1'(u_k)$.
Hence,

$$\begin{aligned}\mu_1(u_i) \wedge \mu_2(v_j) &\leq \mu_1'(u_i) \wedge \mu_2(v_j) + \sum_{u_k \in \mathcal{G}} \sigma_1(u_k, u_i) \wedge \mu_1'(u_k) \wedge \mu_2(v_j) \\ &\leq \mu_j'(u_i, v_j) + \sum_{u_k \in \mathcal{G}} \sigma((u_k, v_j), (u_i, v_j)) \wedge \mu_j'(u_k, v_j)\end{aligned}$$

That is,

$$\mu(u_i, v_j) \leq \mu_j'(u_i, v_j) + \sum_{u_k \in \mathcal{G}} (\sigma((u_k, v_j), (u_i, v_j)) \wedge \mu_j'(u_k, v_j)$$

Thus we get μ_j' is an fz- dominating set of $\mathcal{G}_j$ for $j = 1, 2, ..., n$
Also $|\mu_j'| \leq |\mu_1'|$ shows that $\gamma_{fz}(\mathcal{G}_j) \leq \gamma_{fz}(\mathcal{G})$ for $j = 1.2...., n$
Hence

$$\gamma_{fz}(\mathcal{G}\square\mathcal{H}) \leq n\gamma_{fz}(\mathcal{G})$$

Similarly,

$$\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) \leq m\gamma_{fz}(\mathcal{H})$$

Thus we get,

$$\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) \leq min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\}$$

□

Equality may hold in the above theorem. For example for the graphs $\mathcal{G}$,$\mathcal{H}$ and $\mathcal{G}\Box\mathcal{H}$ given in Figure 5.8, $\gamma_{fz}(\mathcal{G}) = 0.2$, $\gamma_{fz}(\mathcal{H}) = 0.2$ and $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) = 0.4$ so that $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) = min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\}$

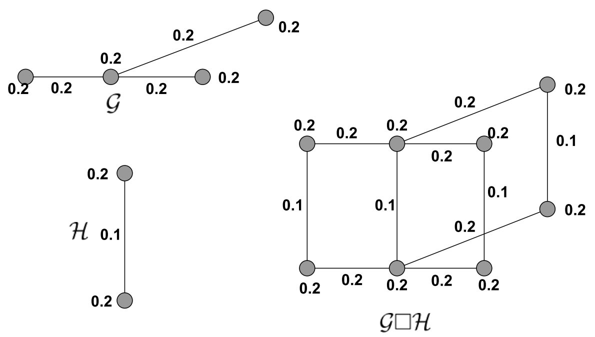

Figure 5.8: *Fuzzy graph $\mathcal{G}$,$\mathcal{H}$ and $\mathcal{G}\Box\mathcal{H}$*

The following conjecture on Cartesian product of crisp was made by V. G. Vizing in 1968.
For every pair of finite crisp graphs G and H,

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H)$$

Vizing's conjecture is arguably the main open problem in the area of domination theory. Here we examine whether Vizing's like inequality holds in the case of fz-domination of fuzzy graphs. A fuzzy graph $\mathcal{G}$ is said to satisfy Vizing's conjecture if $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) \geq \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$ for every fuzzy graph $\mathcal{H}$.

Definition 5.3.12. *A fuzzy graph $\mathcal{H} = (V_1, \mu_1, \sigma_1)$ is called a partial fuzzy subgraph of $\mathcal{G} = (V, \mu, \sigma)$ induced by V_1 if $V_1 \subset V$, $\mu_1(u) = \mu(u)$ if $u \in V_1$, 0 otherwise and $\sigma_1(u, v) = \sigma(u, v) \wedge \mu(u) \wedge \mu(v)$ for all $u, v \in V$.*

Definition 5.3.13. *The spanning fuzzy subgraph of $\mathcal{G} = (V, \mu, \sigma)$ is the partial fuzzy subgraph $\mathcal{G}' = (V_1, \mu', \sigma')$ where $V = V_1$ and $\mu = \mu'$*

If $\mathcal{G}'$ is a spanning fuzzy subgraph of the fuzzy graph $\mathcal{G}$, then $\gamma_{fz}(\mathcal{G}') \geq \gamma_{fz}(\mathcal{G})$).

Theorem 5.3.14. *If $\mathcal{G}$ satisfies Vizing's Conjecture and $\mathcal{G}'$ is a spanning fuzzy subgraph of $\mathcal{G}$ such that $\gamma_{fz}(\mathcal{G}') = \gamma_{fz}(\mathcal{G})$, then $\mathcal{G}'$ also satisfies Vizing's Conjecture.*

Proof. $\mathcal{G}'\Box\mathcal{H}$ is a spanning fuzzy sub- graph of $\mathcal{G}\Box\mathcal{H}$ for every fuzzy graph $\mathcal{H}$. Hence

$$\begin{aligned}\gamma_{fz}(\mathcal{G}'\Box\mathcal{H}) &\geq \gamma_{fz}(\mathcal{G}\Box\mathcal{H})\\ &\geq \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H}) = \gamma_{fz}(\mathcal{G}')\gamma_{fz}(\mathcal{H})\end{aligned}$$

□

The following example shows that in general this inequality does not hold for fz-domination in fuzzy graphs.

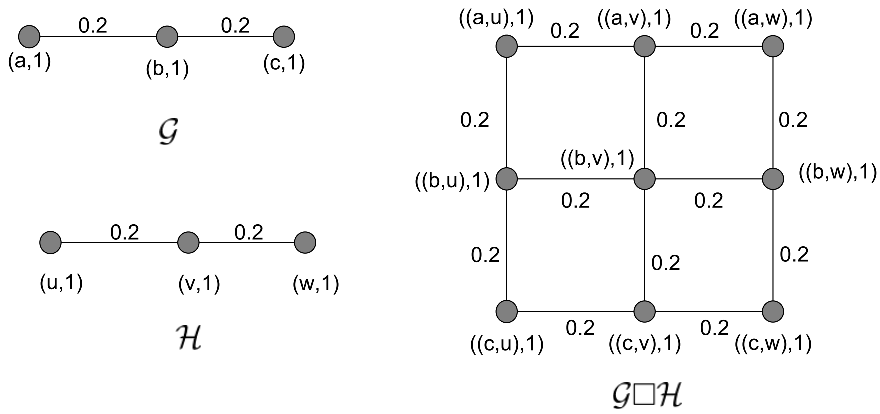

Figure 5.9: *Fuzzy Graphs $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{G}\Box\mathcal{H}$*

Example 5.3.2. *Consider the fuzzy graphs $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ given in Figure 5.9. For $\mathcal{G}$, $V_1 = \{(a,1),(b,1),(c,1)\}$, $\sigma_1(a,b) = \sigma_1(b,c) = 1, \sigma_1(a,c) = 0$. For $\mathcal{H}$, $V_2 = \{(u,1),(v,1),(w,1)\}$, $\sigma_2(u,v) = \sigma_2(v,w) = 1, \sigma_2(u,w) = 0$. $\mu_1' = \{(a,0.8),(b,0.6),(c,0.8)\}$ is a minimum fuzzy dominating set of $\mathcal{G}$.*
Hence $\gamma_{fz}(\mathcal{G}) = 2.2$. Similarly $\gamma_{fz}(\mathcal{H}) = 2.2$
Now $\mu' = \{((a,u),0.6),((a,v),0.4),((a,w),0.6)\},((b,u),0.4),((b,v),0.2),$
$((b,w),0.4),((c,u),0.6),((c,v),0.4),((c,w),0.6)$ is a minimum fuzzy dominating set of $\mathcal{G}\Box\mathcal{H}$. Hence $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) = 4.2$
Here

$$\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$$

There are fuzzy graphs for which

1. $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$

2. $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) = \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$

3. $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) > \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$

5.4 Fuzzy irredundance and fuzzy independence

In this section we introduce the concept of fuzzy irredundance and fuzzy independence corresponding to fz-domination and prove the fz-domination chain in fuzzy graphs.

5.4.1 Fuzzy Irredundant sets of a fuzzy graph

Definition 5.4.1. *Let $\mathcal{G} = (V, \mu, \sigma)$ be a fuzzy graph. A fuzzy subset μ' of μ is called a fuzzy irredundant or fz-irredundant set of $\mathcal{G}$ if each $v \in V$ with $\mu(v) > 0$ is either in $B_{\mu'}$ or there exists $u \in B_{\mu'}$ with $\mu'(v) \leq \sigma(u, v)$.*

Remark 5.4.1. *It follows from from Theorem 5.2.4 that any minimal fz-dominating set of a fuzzy graph is fz-irredundant.*

Converse of the above remark is not true,that is, an fz-irredundant set of a fuzzy graph need not be an fz-dominating set. The property of being an fz-irredundant set of $\mathcal{G}$ is a hereditary property. Hence we can define a maximal fz-irredundant set.

Definition 5.4.2. *A fuzzy subset μ' is called a maximal fz-irredundant set if $\mu'' \supset \mu'$ is not an fz-irredundant set.*

Definition 5.4.3. *The fz-irredundance number $ir_{fz}(\mathcal{G})$ and upper fz-irredundance number $IR_{fz}(\mathcal{G})$ are defined as*

$$ir_{fz}(\mathcal{G}) = \wedge\{|\mu'| : \mu' \textit{is a maximal fz-irredundant set of } \mathcal{G}\}$$

and

$$IR_{fz}(\mathcal{G}) = \vee\{|\mu'| : \mu' \textit{is a maximal fz-irredundant set of } \mathcal{G}\}$$

Theorem 5.4.4. *Let $\mathcal{G} = (\mu, \sigma)$ be a fuzzy graph. An fz-dominating set μ' of $\mathcal{G}$ is minimal fz-dominating set if and only if it is fz-irredundant set.*

Proof. Let μ' be an fz-dominating set that is also an fz-irredundant set.
Consider a fuzzy subset μ'' of μ with $\mu''(x) < \mu'(x)$ for at least one $x \in V$.
Clearly, $\mu'(x) > 0$ and since μ' is an fz-irredundant set, there exists $w \in B_{\mu'}$ such that either $x = w$ or $\mu'(x) \leq \sigma(x, w)$.

Case (i)

If $x = w$, then $\mu''(x) < \mu'(x)$ implies that $\mu''(w) + \sum_{v \in V} \mu''(v) \wedge \sigma(w, v) < \mu(w)$.

Case (ii)

If $x \neq w$, then $\mu'(x) \leq \sigma(x, w)$ implies that $\mu''(x) < \sigma(x, w)$ and so $\mu''(w) + \sum_{v \in V} \mu''(v) \wedge \sigma(w, v) < \mu(w)$.
Thus in both cases μ'' is not fz-dominating. It proves that μ' is a minimal fz-dominating set of $\mathcal{G}$.
The converse follows from Remark 5.4.1. □

Theorem 5.4.5. *Every minimal fz- dominating set μ' in a fuzzy graph $\mathcal{G} = (\mu, \sigma)$ is a maximal fz-irredundant set of $\mathcal{G}$.*

Proof. Let μ' be a minimal fz-dominating set of $\mathcal{G}$. It follows from Theorem 5.4.4 that μ' is an fz-irredundant set of $\mathcal{G}$. Consider $\mu'' \supset \mu'$ with $\mu''(x) > \mu'(x)$ for some $x \in V$. Obviously $\mu''(x) > 0$. Let us assume that μ'' is also an fz- irredundant set. Then by definition either $x \in B_{\mu''}$ or there exists $w \in B_{\mu''}$ such that $\mu''(x) \leq \sigma(w, x)$.

If $x \in B_{\mu''}$, then $\mu''(x) + \sum_{v \in V} \Big(\mu''(v) \wedge \sigma(v, x)\Big) = \mu(x)$. As $\mu''(x) > \mu'(x)$, it follows that $\mu'(x) + \sum_{v \in V} \Big(\mu'(v) \wedge \sigma(v, x)\Big) < \mu''(x) + \sum_{v \in V} \Big(\mu''(v) \wedge \sigma(v, x)\Big) = \mu(x)$, so that μ' is not an fz-dominating set, a contradiction.

Now consider the second case where $w \in B_{\mu''}$ with $\mu''(x) \leq \sigma(w, x)$. As $w \in B_{\mu''}$, we get $\mu''(w) + \sum_{v \in V} \Big(\mu''(v) \wedge \sigma(v, w)\Big) = \mu(w)$. Since $\mu''(x) \leq \sigma(w, x)$ and $\mu''(x) > \mu'(x)$, we see that $\mu'(x) < \sigma(w, x)$. Now the inequalities $\mu'(x) < \sigma(w, x)$, $\mu''(x) \leq \sigma(w, x)$ and $\mu'(x) < \mu''(x)$ show that $\mu'(w) + \sum_{v \in V} \Big(\mu'(v) \wedge \sigma(v, w)\Big) < \mu''(w) + \sum_{v \in V} \Big(\mu''(v) \wedge \sigma(v, w)\Big) = \mu(w)$, so that μ' is not an fz-dominating set, a contradiction again. Hence μ' is a maximal fz-irredundant set. □

We can define the upper fz- domination number as follows.

Definition 5.4.6. *The upper fz-domination number of a fuzzy graph $\mathcal{G}$, denoted by*

$\Gamma_{fz}(G)$, *is defined as*

$$\Gamma_{fz}(\mathcal{G}) = max \, \{|\mu'| : \mu' \text{ is a minimal fz-dominating set of } \mathcal{G}\}$$

From Theorem 5.4.5 and definitions of $ir_{fz}(\mathcal{G})$, $IR_{fz}(\mathcal{G})$, $\gamma_{fz}(\mathcal{G})$ and $\Gamma_{fz}(\mathcal{G})$ we can get the following theorem.

Theorem 5.4.7. *For any fuzzy graph* $\mathcal{G}$,

$$ir_{fz}(\mathcal{G}) \leq \gamma_{fz}(\mathcal{G}) \leq \Gamma_{fz}(\mathcal{G}) \leq IR_{fz}(\mathcal{G})$$

5.4.2 Fuzzy independent sets of a fuzzy graph

Hedetniemi et al. [33] introduced the concept of dominating function, fractional domination, fractional domination number γ_f and upper fractional domination number Γ_f in graphs. Domke et al.[22] introduced the fractional irredundance numbers ir_f and IR_f.

Domke at al. in chapter 3 of Haynes et al.([30], page 85) raised the following question:
Can we define the concept of a fractional independent function as a function $g : V \to [0, 1]$ in such a way that

(i) the characteristic function of every independent set of vertices is an independent function,

(ii) every maximal independent set of vertices corresponds to a maximal independent function,

(iii) every maximal independent function is also a minimal dominating function?

S. Arumugam et al.[2] tried to answer this. They observed that if $S \subset V$ is an independent set of a crisp graph $G = (V, E)$ and if $f : V \to [0, 1]$ is characteristic function of S, then $f(v) = 1$ if $v \in S$ and since no neighbor of v is in S, $\sum_{u \in N[v]} f(u) = 1$. Motivated by this observation, they defined fractional independent functions as follows.

Definition 5.4.8. *[2] Let* $G = (V, E)$ *be a graph. A function* $f : V \to [0, 1]$ *is called an independent function if for every vertex* v *with* $f(v) > 0$, $\sum_{u \in N[v]} f(u) = 1$. *An* *independent function f is called a maximal independent function(MIF) if for every $v \in V$ with $f(v) = 0$, we have $\sum_{u \in N[v]} f(u) > 1$.*

Clearly if S is an independent set in G, then $f = \chi_S$ is an independent function. If further S is maximal, then χ_S is an MIF. We have already seen that fz-domination in fuzzy graphs coincides with the fractional domination in crisp graphs. Hence we must define fz- independence in fuzzy graphs in such a way that it coincides with fractional independence of crisp graphs. We now define fz- independence in fuzzy graph as follows.

Definition 5.4.9. *Let $\mathcal{G} = (\mu, \sigma)$ be a fuzzy graph. A fuzzy subset μ' of μ is fz-independent if for every vertex $v \in V$ with $\mu'(v) > 0$,*

$$\mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x) = \mu(v)$$

μ' is called maximal fz-independent if for all $v \in V$ with $\mu'(v) = 0$, we have

$$\mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x) \geq \mu(v)$$

Observation 5.4.10. *Let $G = (V, E)$ be a crisp graph and $S \subset V$ is a maximal independent set in G. Then χ_S is a maximal fz-independent set of the fuzzy graph $\mathcal{G} = (\chi_V, \chi_E)$ because*

(i) if $v \in S$, then $\chi_S(v) = 1$ and $\chi_S(x) = 0$ for $x \in N[v]$.
Therefore,

$$\chi_S(v) + \sum_{x \in V} \chi_E(x, v) \wedge \chi_S(x) = 1 = \chi_V(v)$$

(ii) if $v \notin S$, then $\chi_S(v) = 0$, but there exists at least one $u \in N[v]$, such that $\chi_S(u) = 1$, so that

$$\chi_S(v) + \sum_{x \in V} \chi_E(x, v) \wedge \chi_S(x) \geq 1 = \chi_V(v)$$

Observation 5.4.11. *If μ' is an fz-independent set of μ, then $P_{\mu'} \subseteq B_{\mu'}$.*

Theorem 5.4.12. *Let $\mathcal{G} = (\mu, \sigma)$ be a fuzzy graph. An fz-independent set μ' of $\mathcal{G}$ is maximal fz-independent if and only if it is fz-dominating.*

If μ' is a maximal fz-independent set, then by definition it is fz-dominating. Conversely, let μ' be fz- independent and fz- dominating. Let $\mu'' \supset \mu'$ be such that $\mu''(x) > \mu'(x)$ for some $x \in V$. Then clearly $\mu''(x) > 0$.
Now $\mu''(x) + \sum_{v \in V} \mu''(v) \wedge \sigma(v,x) > \mu'(x) + \sum_{v \in V} \mu'(v) \wedge \sigma(v,x) \geq \mu(x)$. Thus μ'' is not fz-independent. Therefore μ' is a maximal fz-independent set of $\mathcal{G}$.

Theorem 5.4.13. *Every maximal fz-independent subset is minimal fz-dominating.*

Proof. Let μ' be a maximal fz-independent subset of μ. Then it follows from definition that μ' is fz-dominating.

Let $\mu'' \subset \mu'$ be such that $\mu''(x) < \mu'(x)$ for some $x \in V$. Clearly $\mu'(x) \neq 0$.
Since μ' is fz- independent we get, $\mu'(x) + \sum_{v \in V} \mu'(v) \wedge \sigma(v,x) = \mu(x)$.
Therefore $\mu''(x) + \sum_{v \in V} \mu''(v) \wedge \sigma(v,x) < \mu'(x) + \sum_{v \in V} \mu'(v) \wedge \sigma(v,x) = \mu(x)$. Thus μ'' is not fz-dominating. Hence μ' is minimal fz-dominating set. □

Definition 5.4.14. *The fuzzy independence number $\beta_{0_{fz}}(\mathcal{G})$ and fuzzy independent domination number $i_{fz}(\mathcal{G})$ are defined as*

$$\beta_{0_{fz}}(\mathcal{G}) = \vee\{|\mu'| : \mu' \textit{is a maximal fz-independent set of } \mathcal{G}\}$$

and

$$i_{fz}(\mathcal{G}) = \wedge\{|\mu'| : \mu' \textit{is a maximal fz-independent set of } \mathcal{G}\}$$

From Theorem 5.4.13 and definitions of $\gamma_{fz}(\mathcal{G})$, $\Gamma_{fz}(\mathcal{G})$, $\imath_{fz}(\mathcal{G})$ and $\beta_{0_{fz}}(\mathcal{G})$ we can get the following theorem.

Theorem 5.4.15. *For any fuzzy graph $\mathcal{G}$*

$$\gamma_{fz}(G) \leq i_{fz}(G) \leq \beta_{0_{fz}}(G) \leq \Gamma_{fz}(G)$$

Combining Theorem 5.4.7 and Theorem 5.4.15 we get the following fz-domination chain.

Theorem 5.4.16. *For any fuzzy graph $\mathcal{G} = (\mu, \sigma)$,*

$$ir_{fz}(\mathcal{G}) \leq \gamma_{fz}(\mathcal{G}) \leq i_{fz}(\mathcal{G}) \leq \beta_{0_{fz}}(\mathcal{G}) \leq \Gamma_{fz}(\mathcal{G}) \leq IR_{fz}(\mathcal{G})$$

5.5 Strength based domination as the fz-domination

Rosenfeld in his classic paper on fuzzy graphs [51] defined distance in fuzzy graphs as follows. For any path $P = x_1, x_2, ..., x_n$ in a fuzzy graph $\mathcal{G} = (\mu, \sigma)$, the σ-length of P is defined as the sum of the reciprocals of P's edge weights, i.e., $l(P) = \sum_1^n \frac{1}{\sigma(x_{i-1}, x_i)}$

For any two vertices u and v, their σ-distance, $\delta(u, v)$ is defined as the smallest σ-length of any path from u to v.

In view of the above definition of distance in fuzzy graphs, we can see sb-domination in crisp graph as an fz-domination in a fuzzy graph. In sb-domination the domination strength of a vertex u on another vertex v is $s(u, v) = \frac{1}{d(u, v)}$. In fz-domination we have seen this as $\mu(u) \wedge \sigma(u, v)$. In the special case where $\mu(u) = 1$ this becomes $\sigma(u, v) = \frac{1}{\delta(u, v)}$, where $\delta(u, v)$ is the σ-distance between the vertices u and v in the fuzzy graph. Thus corresponding to any crisp graph G we can always construct a fuzzy graph $\mathcal{G}$ such that $\gamma_{sb}(G) = \gamma_{fz}(\mathcal{G})$ as given below.
Let $G = (V, E)$ be a crisp graph. Let the fuzzy graph $\mathcal{G} = (V, \mu, \sigma)$ be defined by $\mu(u) = 1$ for all $u \in V$ and $\sigma(u, v) = \frac{1}{d_G(u, v)}$ for all $u, v \in V$, where $d_G(u, v)$ represents the distance between u and v in the crisp graph G. Then from the definitions of sb-domination and fz-domination we see that $\gamma_{sb}(G) = \gamma_{fz}(\mathcal{G})$. We call the fuzzy graph $\mathcal{G}$ corresponding to the crisp graph G, the **sb-completion fuzzy graph** of G.

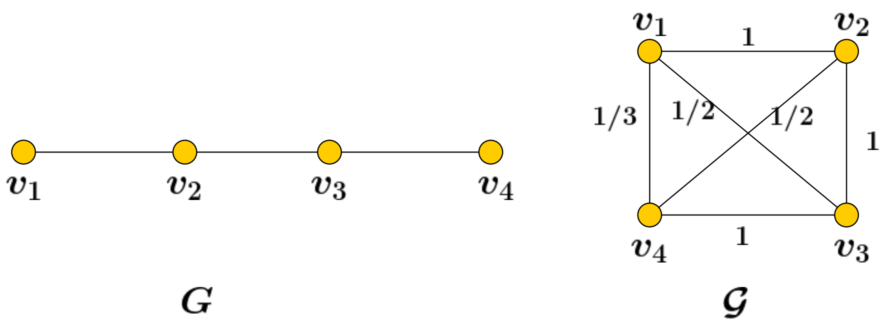

Figure 5.10: *Graph $G = P_4$ and its sb-completion fuzzy graph $\mathcal{G}$*

Illustration 5.5.1. Consider the graph $G = P_4$. The corresponding sb-completion fuzzy graph $\mathcal{G} = (\mu, \sigma)$ is constructed as follows.
$\mu(v_1) = \mu(v_2) = \mu(v_3) = \mu(v_4) = 1$, $\sigma(v_1, v_2) = d_G(v_1, v_2) = 1$
Similarly, $\sigma(v_2, v_3) = \sigma(v_3, v_4) = 1$, $\sigma(v_1, v_3) = \frac{1}{d_G(v_1, v_3)} = \frac{1}{2}$, $\sigma(v_2, v_4) = \frac{1}{2}$,

$\sigma(v_1, v_4) = \dfrac{1}{d_G(v_1, v_4)} = \dfrac{1}{3}$. It can be observed easily that $\gamma_{sb}(G) = \gamma_{fz}(\mathcal{G})$. P_4 and its sb-completion fuzzy graph are depicted in Figure 5.10.

5.6 Conclusion

In this chapter, we have introduced the concept of fz-domination in fuzzy graphs and presented several basic results of this parameter. Other related parameters, fz-independence and fz-irredundance, are also introduced, and the relation between these parameters is established. Other important features of these parameters can be explored further. Investigating the properties of sb-completion graphs will be another interesting area of research.

CHAPTER 6

Conclusion and Further Scope of Research

6.1 Introduction

The theory of domination has advanced remarkably over the past decades as a result of its wide range of applications in optimization, and computational problems. In this thesis we have studied about some distance related domination parameters. The different variations of domination parameters that we discussed in this thesis have applications in facility location problems, in problems involving finding sets of representatives, in monitoring communication or electrical networks etc. Different applications of variations of domination in graphs can be found in [23]

6.2 Summary of the thesis

In this thesis we studied three types of domination parameters namely disjunctive domination and strength based domination in crisp graphs and fz- domination in fuzzy graphs. The application of disjunctive domination arises in facility location problems when the maximum distance to a facility is fixed and one seeks to minimize the number of facilities necessary so that everyone is serviced. Disjunctive domination allows some relaxation in domination and thus reduces the implementation costs. We investigated the properties of disjunctive domination in different type of graph products. The following are some of the results which we have obtained.

- Disjunctive domination number is sub-multiplicative with respect to Lexicographic product and Strong product but not with respect to Tensor and Cartesian products.
- Let G_1 and G_2 be any two graphs. If G_1 has a γ-set which is such that the vertices not in this set are twice dominated, then $\gamma_2^d(G_1 \square G_2) \leq \gamma(G_1)\gamma(G_2)$.
- Evaluated the disjunctive domination number of neighborhood and edge corona of some classes of graphs.
- Introduced the notion of Efficient disjunctive dominating sets in graphs and investigated their existence in some classes of graphs.
- Introduced the notion of nearly efficient disjunctive dominating sets in graphs and established their existence in an infinite two dimensional grid graph.

While most of the domination parameters are based on local conditions, sb-domination introduced by us in this thesis has global nature where the influence of a sb-dominating vertex is global with respect to other vertices. This type of domination are useful in the analysis of dissemination of information in social networks, where the impact of the information decreases every time it is passed on. The following are some of the results which we have obtained in sb-domination.

- Investigated some basic properties of sb-domination in graphs.
- Found the sb-domination number of some standard classes of graphs.
- Established some bounds for the sb-domination number of graphs.

Dominating set helps in analyzing the effect or impact of changes happening in a network. In a fuzzy graph, membership value of a vertex denotes its strength whereas the membership value of an edge denotes the influence of one vertex on the other. It is possible that the strength or needs required at a vertex can be fulfilled by a set of dominating vertices in the graph. The strength of domination that a vertex can contribute to another vertex depends on its membership value as well as the membership value of the edge between them. All these are taken into consideration when we introduced the definition of fuzzy domination in a fuzzy graph. The following are some of the results which we have obtained in fz-domination in fuzzy graphs

- Studied different properties and bounds of fz-domination number of fuzzy graphs.
- Characterized minimal fz-dominating sets in fuzzy graphs.
- Studied the impact of different graph operations on fz-domination.
- Introduced fz-independence, fz-irredundance and established the fz-domination chain in fuzzy graphs.

6.3 Further scope of research

A large number of problems are open for study in disjunctive domination and sb-domination in crisp graphs and fz- domination in fuzzy graphs. Some of them that we found interesting are given below.

Problem 6.3.1. Characterize all graphs having $\gamma(G) = \gamma_2^d(G)$.

Problem 6.3.2. Find a relation between $\gamma_2^d(G)$, order n and size m of G.

Problem 6.3.3. Find $\gamma_2^d(P_m \Box P_n)$ for $n \geq 4$.

Problem 6.3.4. Determine $\gamma_2^d(G)$ of regular graphs.

Problem 6.3.5. Characterize the graphs for which $\gamma_2^d(G \Box H) < \gamma_2^d(G)\gamma_2^d(H)$.

Problem 6.3.6. Characterize the graphs for which $\gamma_2^d(G)\gamma_2^d(H) \leq \gamma_2^d(G \Box H) < \gamma(G)\gamma(H)$.

Problem 6.3.7. Characterize all graphs having $\gamma_e(G) = \gamma_2^d(G) = \gamma_{sb}(G)$.

Problem 6.3.8. Find the class of graphs for which the Vizing's like inequality is true with respect to disjunctive domination.

Problem 6.3.9. Characterize the minimal sb-dominating set of a graph.

Problem 6.3.10. Investigate the impact of different graph operations on sb-domination number.

Problem 6.3.11. Find the class of graphs for which the Vizing's like inequality is true or not true with respect to sb-domination.

Problem 6.3.12. Find bounds for $\gamma_{sb}(G)$ in terms of other different graph parameters.

Problem 6.3.13. Define sb-independence and sb-irredundance suitably and establish the sb-domination chain.

Problem 6.3.14. Investigate sb-domination critical graphs.

Problem 6.3.15. Characterize the class of graphs for which $\gamma_{sb}(G) = \gamma_2^d(G)$.

Problem 6.3.16. Given three positive integers $a < b < c$, does there exist a graph G for which $\gamma_{sb}(G) = a, \gamma_2^d(G) = b$ and $\gamma(G) = c$?

Problem 6.3.17. Find $\gamma_{sb}(G)$ for different classes of graphs.

The following are some interesting problems in fz-domination.

Problem 6.3.18. Given a real number sequence, $0 < a \leq b \leq c \leq d \leq e \leq f$, does there exist a fuzzy graph $\mathcal{G}$ for which $1 \leq ir_{fz}(\mathcal{G}) = a \leq \gamma_{fz}(\mathcal{G}) = b \leq i_{fz}(\mathcal{G}) = c \leq \beta_{0_{fz}}(\mathcal{G}) = d \leq \Gamma_{fz}(\mathcal{G}) = e \leq IR_{fz}(\mathcal{G}) = f$?

Problem 6.3.19. Under what conditions are any of the parameters of the fz-domination chain equal?

Problem 6.3.20. Find bounds for $\gamma_{fz}(\mathcal{G})$ in terms of other different graph parameters.

Problem 6.3.21. Find an algorithm to find γ_{sb} of a graph by finding γ_{fz} of corresponding sb-completion fuzzy graph.

Problem 6.3.22. If G satisfies Vizing's conjecture, is it possible to get a fuzzy graph $\mathcal{G}$, by giving proper membership values to edges in G such that $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$ for all fuzzy graphs $\mathcal{H}$.

Many more interesting questions can be posed on sb-domination and fz- domination. One can attempt to study sb-domination in crisp graphs as fz-domination in the corresponding fuzzy graph and can try to explore more properties of these parameters. It would be interesting to explore the sb-domination number under the impact of different graph operations. Finding sharp upper and lower bounds for sb-domination number of different graph classes is particularly interesting. We conclude the thesis with a positive hope that future research on sb-domination and fz-domination may result in some fascinating findings.

Publications in Journals and Presentations

Publications:

1. Lekha A and Parvathy K. S, “Nearly dominating sets,” *Bulletin of Kerala Mathematics Association,* vol. 14, no. 2, pp. 219-228, 2017.

2. Lekha A and Parvathy K. S, “Properties of disjunctive domination in product graphs,” *Malaya Journal of Matematik (MJM),* vol. 8, no. 1, pp. 37-41, 2020.

3. Lekha A and Parvathy K. S, “Efficient disjunctive dominating sets in graphs,” *Advances in Mathematics: Scientific Journal,* vol. 10, no. 3, pp. 1215-1226, 2021.

4. Lekha A and Parvathy K. S, “On disjunctive domination number of corona related graphs,” *J. Math. Comput. Sci.,* vol. 11, no. 3, pp. 2538-2550, 2021.

5. Lekha A and Parvathy K. S, “Fuzzy domination in fuzzy graphs,” *Journal of Intelligent & Fuzzy Systems,* Vol. 44, No. 2, pp. 3205–3212 , 2023.

6. Lekha A and Parvathy K. S, “On fz- domination number of fuzzy graphs,” *Ratio Mathematica Journal of Mathematics, Statistics and Applications,* Vol. 46, 2023.

Presentations:

1. Lekha A, *Nearly Dominating Sets*, International conference on Science, Technology, Women Studies, Business & Social Sciences 2016 at IMRF Goa during 03-05 of November 2016.

2. Lekha A, *Study of Near Domination in Product Graphs*, National Workshop on Graph Domination and Labelling at St Mary's College, Thrissur during November 23-25, 2017.

3. Lekha A, *Perfect Near Domination in Grid Graphs*, International Conference on Recent Trends in Graph Theory and Combinatorics, ICRTGC-2018 at Cochin University of Science and Technology during April 26-29, 2018.

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