

## Ph.D. ENTRANCE EXAMINATION, APRIL 2021

## MATHEMATICS

Time : Two Hours

Maximum : 100 Marks

## Part A

*Answer all questions.**Each question carries 2 marks.*

1. What are the contrapositive of the conditional statement "I come to class whenever there is going to be a test" ?

- a) "If I come to class, then there will be a test."
- b) "If I do not come to class, then there will not be a test."
- c) "If there is not going to be a test, then I don't come to class."
- d) "If there is going to be a test, then I don't come to class."

2.  $\neg(A \vee q) \wedge (A \wedge q)$  is a :

- a) Tautology.
- b) Contradiction.
- c) Contingency.
- d) None of these.

3. General solution of the differential equation

$$y''' + 2y'' - y' - 2y = 0 \text{ is :}$$

- a)  $y(x) = C_1 e^{-2x} + C_2 e^x + C_3 e^{-x}$ .
- b)  $y(x) = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{-2x}$ .
- c)  $y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{-2x}$ .
- d)  $y(x) = C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$ .

4. The Wronskian of two solutions to the differential equation

$$t^4 y'' - t^3 y' - t^8 y \text{ is :}$$

- a)  $ct^2$ .
- b)  $ct^2 + bt$ .
- c)  $\frac{c}{t}$ .
- d)  $\frac{c}{t^2}$ .

**Turn over**



9. If  $x$  and  $y$  are any *two* vectors in an inner product space  $X$ , then :

$$| \langle x, y \rangle | \leq \| x \| \cdot \| y \|.$$

The above inequality is known as :

- (a) Parallelogram law.
- (b) Cauchy-Schwarz's inequality.
- (c) Polarisation identity.
- (d) Bessel's inequality.

10. With respect to the partial ordering  $m \leq n$  whenever  $m$  divides  $n$ , defined on the set  $\{2, 3, 4, 8\}$  :

- a) 8 and 3 are maximal.
- b) All are maximal.
- c) 3 only is maximal.
- d) 8 only is maximal.

11. Which among the following is not true with respect to the permutation group  $S_3$  ?

- a) 3 elements of  $S_3$  are odd.
- b) Non-abelian.
- c) Cyclic.
- d) 3 elements of  $S_3$  are even.

12. The radius of convergence of the power series :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} \text{ is}$$

- a) 1.
- b) 2.
- c)  $\frac{1}{2}$ .
- d) 0.

13. Which among the following is not true ?

- a)  $\frac{1}{x}$  is not uniformly continuous on  $(0, 1]$ .
- b)  $x^2$  is not uniformly continuous on  $[0, \infty)$ .
- c)  $\sqrt{x}$  is uniformly continuous on  $[0, \infty)$ .
- d) None of the above.

14. The value of  $\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx}$  is :

a)  $\frac{1}{k} \ln \frac{a^m}{b^n}$ .

b)  $\frac{1}{k} \ln \left( \frac{a}{b} \right)^{mn}$

c)  $\frac{1}{k} \ln \left( \frac{a}{b} \right)^{m-n}$

d)  $k \ln \left( \frac{a}{b} \right)^{mn}$

15. If  $K \subseteq \mathbb{R}^n$ , which among the following is not correct ?

a) If every sequence in  $K$  has a convergent subsequence, then  $K$  must be bounded.

b) If every sequence in  $K$  has a convergent subsequence whose limit is in  $K$ , then  $K$  must be closed.

c) If  $K$  is compact then any infinite subset  $A \subseteq K$  has a cluster point in  $K$ .

d) None of the above.

16. Which among the following is not necessarily true ?

a) Cancellation laws hold in commutative rings with unity.

b) Non-zero Elements of a Field form a group under multiplication.

c) Every field is isomorphic to its own field of quotients.

d) The only ideals in a field  $F$  are  $0$  and  $F$ .

17. Newton-Gregory Forward interpolation formula can be used :

a) Only for equally spaced intervals.

b) Only for unequally spaced intervals.

c) For both equally and unequally spaced intervals.

d) None of the above.

18. Select the true one for a Boolean Algebra B :

- a) B is a finite, complemented, and distributive lattice.
- b) B is a finite but not complemented lattice.
- c) B is a finite, distributive but not complemented lattice.
- d) B is a non-distributive lattice.

19. Consider an undirected random graph of eight vertices. The probability that there is an edge between a pair of vertices is  $\frac{1}{2}$ . What is the expected number of unordered cycles of length three ?

- a)  $\frac{1}{4}$ .
- b) 6.
- c) 7.
- d) 8.

20. What are the last two (decimal) digits of  $3^{4242}$  ?

- a) 29.
- b) 19.
- c) 09.
- d) 79.

21. The orthogonal matrix associated with a rotation of  $\mathbb{R}^3$  of  $\theta$  degrees about the  $z$ -axis is :

- a)  $\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .
- b)  $\begin{bmatrix} \cos \theta & -\sin \theta & 1 \\ \sin \theta & \cos \theta & 0 \\ 1 & 0 & 0 \end{bmatrix}$ .
- c)  $\begin{bmatrix} -\cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .
- d)  $\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 1 & 0 & 1 \end{bmatrix}$ .

22. The situation of at least one artificial variable is positive in the optimum iteration will not happen in :

- a) Dual simplex method.
- b) Big-M Method.
- c) Two-Phase Method.
- d) All of the above.

**Turn over**

23. The integral equation  $g(s) = s - \frac{1}{6}s^3 + \int_0^s (s-t)g(t)dt$  is :
- Homogeneous linear Fredholm integral equation of the second kind.
  - Non-Homogeneous linear Volterra integral equation of the second kind.
  - Homogeneous linear Volterra integral equation of the second kind.
  - Non-Homogeneous non-linear Fredholm integral equation of the second kind.
24. Poincare polynomial of Klein Bottle is :
- 1
  - $x$ .
  - $1+x$ .
  - $1+x+x^2$ .
25. The functions  $\sinh z$  and  $\cosh z$  are :
- Bounded and entire.
  - Bounded but not entire.
  - Unbounded and not entire.
  - Unbounded and entire.

(25 × 2 = 50 marks)

### Part B

*Answer any ten questions.*

*Each question carries 5 marks.*

26. Find general solution of the differential equations  $(1-x^2)y'' - 2xy' + 2y = 0$  given that one solution is  $y = x$ .
27. Prove that an infinite subset of a compact space has at least one limit point.
28. Prove that  $2^n + 1$  has no prime factors of the form  $8k + 7$ .
29. Find the general solution of the PDE  $(y-z)p + (z-x)q = x-y$ .

30. Give example of a space which is first countable, but not second countable. Also provide justifications for your claim.
31. Suppose  $f$  is bounded on  $[a, b]$  and continuous at  $s \in (a, b)$ . If  $\alpha(x) = \chi_{(0, \infty)}(x - s)$ , prove that

$$\int_a^b f(x) d\alpha(x) = f(s)$$

$$\text{where } \chi_{(0, \infty)}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

32. Show that a Banach space  $X$  is reflexive if and only if its dual space  $X'$  is reflexive.
33. Prove there is no branch of the logarithm defined on  $G = \mathbb{C} \setminus \{0\}$ .
34. Solve the system of congruence equations :

$$2x \equiv 6 \pmod{14}$$

$$3x \equiv 9 \pmod{15}$$

$$5x \equiv 20 \pmod{60}.$$

35. Prove or Disprove : Every path connected space is connected.
36. Solve :

$$z(t) = \alpha \cos(\omega t) + \beta \int_0^t \sin(\omega(t-u)) z(u) du.$$

37. (i) Prove or disprove : Geodesics are having constant speed.  
 (ii) Find velocity and acceleration of the parametrized curve

$$\alpha(t) = (\cos t, \sin t, 2 \cos t, 2 \sin t).$$

38. Show that the torus less one point, the Klein bottle less one point and  $\mathbb{R}^2$  less two points are each homotopy equivalent to the space obtained by gluing two disjoint circles together at a point.

39. Find the maximum and minimum values of the function  $f(x, y) = 3x + y$  subject to the constraint  $x^2 + y^2 = 10$ .

40. Find the Hermite interpolating polynomial  $f(x)$  satisfying

$$f(0) = 2, f'(0) = 1, f(1) = 4, f'(1) = -1, f(3) = 5, f'(3) = -2.$$

(10 × 5 = 50 marks)

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